Hydrodynamic Limit with Geometric Correction in Kinetic Equations

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Simple Model - Neutron Transport Equation

We consider the steady homogeneous isotropic one-speed neutron transport equation in a two-dimensional unit plate. We denote the space variables as $\vec{x} = (x_1, x_2)$ and the velocity variables as $\vec{v} = (v_1, v_2)$. In the space domain $\Omega = \{\vec{x} : \left|\vec{x}\right| \leq 1\}$ and the velocity domain $\Sigma = \{\vec{v} : \vec{v} \in \mathcal{S}^1\}$, the neutron density $u^\epsilon(\vec{x}, \vec{v})$ satisfies

$$\left\{ \begin{array}{rcl} \epsilon\vec{v} \cdot \nabla_x u^\epsilon + u^\epsilon - \bar{u}^\epsilon & = & 0 \ \text{ for } \ \vec{x} \in \Omega, \\ u^\epsilon(\vec{x}_0, \vec{v}) & = & g(\vec{x}_0, \vec{v}) \ \text{ for } \ \vec{v} \cdot \vec{n} < 0 \ \text{ and } \ \vec{x}_0 \in \partial\Omega, \end{array} \right. \tag{1}$$

where

$$\bar{u}^\epsilon(\vec{x}) = \frac{1}{2\pi} \int_{\mathcal{S}^1} u^\epsilon(\vec{x}, \vec{v}) \mathrm{d}\vec{v},$$

with the Knudsen number $0 < \epsilon << 1$ as a parameter. We want to study the behavior of u^ϵ as $\epsilon \to 0$.

Features of the Equation

- Half boundary condition:

$$\begin{cases} v \cdot \partial_x u &= h(x,v) \ \text{ for } \ x \in [0,1], \\ u(0,v) &= g_1(v) \ \text{ for } \ v \in (0,1], \\ u(1,v) &= g_2(v) \ \text{ for } \ v \in [-1,0). \end{cases}$$

- Non-local operator:

$$K[u](\vec{v}) = \int_{\mathcal{S}^1} u(\vec{v}^*) k(\vec{v}, \vec{v}^*) \mathrm{d}\vec{v}^*,$$

with

$$\int_{\mathcal{S}^1} k(\vec{v}, \vec{v}^*) \mathrm{d}\vec{v}^* = 1.$$

Complex Model - Boltzmann Equation near Maxwellian

We consider stationary Boltzmann equation for probability density $F^{\epsilon}(\vec{x},\vec{v})$ in a two-dimensional unit plate $\Omega=\{\vec{x}=(x_1,x_2): \left|\vec{x}\right|\leq 1\}$ with velocity $\Sigma=\{\vec{v}=(v_1,v_2)\in\mathbb{R}^2\}$ as

$$\left\{\begin{array}{rcl}\epsilon\vec{v}\cdot\nabla_x F^{\epsilon} &=& Q[F^{\epsilon},F^{\epsilon}] \ \text{ in } \ \Omega\times\mathbb{R}^2,\\ F^{\epsilon}(\vec{x}_0,\vec{v}) &=& B^{\epsilon}(\vec{x}_0,\vec{v}) \ \text{ for } \ \vec{x}_0\in\partial\Omega \ \text{ and } \ \vec{n}(\vec{x}_0)\cdot\vec{v}<0,\end{array}\right.$$

where $\vec{n}(\vec{x}_0)$ is the outward normal vector at $\vec{x}_0$ and the Knudsen number ϵ satisfies $0<\epsilon<<1$. Here we have

$$Q[F,G] \ = \ \int_{\mathbb{R}^2}\int_{\mathcal{S}^1} q(\vec{\omega},\left|\vec{\mathfrak{u}}-\vec{v}\right|)\bigg(F(\vec{\mathfrak{u}}_*)G(\vec{v}_*)-F(\vec{\mathfrak{u}})G(\vec{v})\bigg)\mathrm{d}\vec{\omega}\mathrm{d}\vec{\mathfrak{u}},$$

with $\vec{\mathfrak{u}}_*=\vec{\mathfrak{u}}+\vec{\omega}\bigg((\vec{v}-\vec{\mathfrak{u}})\cdot\vec{\omega}\bigg)$, $\vec{v}_*=\vec{v}-\vec{\omega}\bigg((\vec{v}-\vec{\mathfrak{u}})\cdot\vec{\omega}\bigg)$, and the hard-sphere collision kernel $q(\vec{\omega},\left|\vec{\mathfrak{u}}-\vec{v}\right|)=q_0\left|\vec{\mathfrak{u}}-\vec{v}\right||\cos\phi|$, for positive constant q_0 related to the size of ball, $\vec{\omega}\cdot(\vec{v}-\vec{\mathfrak{u}})=\left|\vec{v}-\vec{\mathfrak{u}}\right|\cos\phi$ and $0\leq\phi\leq\pi/2$. We intend to study the behavior of F^{ϵ} as $\epsilon\to 0$.

Complex Model - Boltzmann Equation (Cont.)

We assume that the boundary data is as $B^{\epsilon}(\vec{x}_0,\vec{v}) = \mu + \epsilon\mu^{\frac{1}{2}}b(\vec{x}_0,\vec{v})$, where $\mu(\vec{v})$ is the standard Maxwellian $\mu(\vec{v}) = \frac{1}{2\pi}\exp\left(-\frac{\left|\vec{v}\right|^2}{2}\right)$. Then we have$F^{\epsilon}(\vec{x},\vec{v}) = \mu + \epsilon\mu^{\frac{1}{2}}f^{\epsilon}(\vec{x},\vec{v})$, where f^{ϵ} satisfies the equation

$$\begin{cases} \epsilon\vec{v}\cdot\nabla_x f^{\epsilon} + \mathcal{L}[f^{\epsilon}] &= \Gamma[f^{\epsilon},f^{\epsilon}], \\ f^{\epsilon}(\vec{x}_0,\vec{v}) &= b(\vec{x}_0,\vec{v}) \ \text{ for } \ \vec{n}\cdot\vec{v}<0 \ \text{ and } \ \vec{x}_0\in\partial\Omega, \end{cases}$$

for

$$\begin{aligned} \Gamma[f^{\epsilon},f^{\epsilon}] &= \mu^{-\frac{1}{2}}Q[\mu^{\frac{1}{2}}f^{\epsilon},\mu^{\frac{1}{2}}f^{\epsilon}], \\ \mathcal{L}[f^{\epsilon}] &= -2\mu^{-\frac{1}{2}}Q[\mu,\mu^{\frac{1}{2}}f^{\epsilon}] = \nu(\vec{v})f^{\epsilon} - K[f^{\epsilon}], \\ \nu(\vec{v}) &= \int_{\mathbb{R}^2}\int_{\mathcal{S}^1} q(\vec{v}-\vec{u},\vec{\omega})\mu(\vec{u})\mathrm{d}\vec{\omega}\mathrm{d}\vec{u} \\ K[f^{\epsilon}](\vec{v}) &= \int_{\mathbb{R}^2} k(\vec{u},\vec{v})f^{\epsilon}(\vec{u})\mathrm{d}\vec{u} \end{aligned}$$

Background

- Spacial Domain: $\mathbb{R}^n$ or periodic, bounded;
- Temporal Domain: steady, unsteady
- Solution: strong(smooth), weak(renormalized), weighted.

- 1950s - 1970s: CASE K. M., ZWEIFEL P. F. AND LARSEN, E; 1D explicit solution; spectral analysis of kinetic operators; formal asymptotic expansion;
- 1979: BENSOUSSAN, ALAIN, LIONS, JACQUES-L. AND PAPANICOLAOU, GEORGE C.; *Boundary layers and homogenization of transport processes.* Publ. Res. Inst. Math. Sci. 15 (1979), no. 1, 53-157.
- 1984: BARDOS, C., SANTOS, R. AND SENTIS, R.; *Diffusion approximation and computation of the critical size.* Trans. Amer. Math. Soc. 284 (1984), no. 2, 617-649.
- 2002: SONE, Y; *Kinetic theory and fluid dynamics.* Birkhäuser Boston, Inc., Boston, MA.

Asymptotic Analysis - Perturbation Theory

The main goal is to study the behavior of parameterized problems as the parameter goes to a limit.

- Algebraic Equations:
  - Regular: $(x^\epsilon)^2 + \epsilon x^\epsilon - 1 = 0$.
  - Singular: $\epsilon(x^\epsilon)^2 + x^\epsilon - 1 = 0$.
- Differential Equations:
  - Regular: $(y^\epsilon)'' + \epsilon(y^\epsilon)' + y^\epsilon = 1$ with $y^\epsilon(0) = 0$ and $y^\epsilon(1) = 1$.
  - Singular: $\epsilon(y^\epsilon)'' + (y^\epsilon)' + y^\epsilon = 1$ with $y^\epsilon(0) = 0$ and $y^\epsilon(1) = 1$.

Ingredients: interior solution; boundary layer; decay; cut-off function in 1D and 2D.

Hilbert Expansion

The classical method is to introduce a power series in ϵ:

1. Define the formal expansion

$$x^\epsilon \sim \sum_{k=0}^{\infty} \epsilon^k x_k, \qquad y^\epsilon(t) \sim \sum_{k=0}^{\infty} \epsilon^k y_k(t),$$

   where x_k and $y_k(t)$ are independent of ϵ.

2. Then plugging this expansion into the original equations, we obtain a series of relations for x_k and $y_k(t)$, which can be solved or estimated directly.

3. Finally, we can estimate the remainder

$$R_N[x] = x^\epsilon - \sum_{k=0}^{N} \epsilon^k x_k, \qquad R_N[y] = y^\epsilon(t) - \sum_{k=0}^{N} \epsilon^k y_k(t).$$

Hilbert Expansion(Cont.)

- This method can be used to analyzed both the interior solution and boundary(initial) layer.
- The convergence here is different from that of power series.
- Not all asymptotic relations can be expressed in power series with respect to the parameter.
- Hilbert expansion is not the only expansion to analyze asymptotic behaviors.
- This procedure is ideal. We may encounter difficulties in each step. Sometimes a trade-off is inevitable.

We define the interior expansion as follows:

$$U(\vec{x}, \vec{v}) \sim \sum_{k=0}^{\infty} \epsilon^k U_k(\vec{x}, \vec{v}),$$

where U_k can be defined by comparing the order of ϵ via plugging this expansion into the neutron transport equation. Thus, we have

$$\begin{aligned} U_0 - \bar{U}_0 &= 0, \\ U_1 - \bar{U}_1 &= -\vec{v} \cdot \nabla_x U_0, \\ U_2 - \bar{U}_2 &= -\vec{v} \cdot \nabla_x U_1, \\ &\dots \\ U_k - \bar{U}_k &= -\vec{v} \cdot \nabla_x U_{k-1}. \end{aligned}$$

Interior Solution (Cont.)

We can show $U_0(\vec{x}, \vec{v})$ satisfies the equation

$$\begin{cases} U_0(\vec{x}, \vec{v}) & = & \bar{U}_0(\vec{x}), \\ \Delta_x \bar{U}_0 & = & 0. \end{cases}$$

Similarly, we can derive $U_k(\vec{x}, \vec{v})$ for $k \geq 1$ satisfies

$$\begin{cases} U_k & = & \bar{U}_k - \vec{v} \cdot \nabla_x U_{k-1}, \\ \Delta_x \bar{U}_k & = & 0. \end{cases}$$

We need to determine the boundary data of U_k.

Boundary Layer

The boundary layer can be constructed as follows:

- Polar coordinates: $(x_1, x_2) \to (r, \theta)$.
- Boundary layer scaling: $\eta = (1 - r)/\epsilon$.
- We define the boundary layer expansion as follows:

$$\mathscr{U}(\eta, \theta, \vec{v}) \sim \sum_{k=0}^{\infty} \epsilon^k \mathscr{U}_k(\eta, \theta, \vec{v}),$$

  which satisfies

$$(\vec{v} \cdot \vec{n}) \frac{\partial \mathscr{U}}{\partial \eta} + (\vec{v} \cdot \vec{\tau}) \frac{\epsilon}{1 - \epsilon\eta} \frac{\partial \mathscr{U}}{\partial \theta} + \mathscr{U} - \bar{\mathscr{U}} = 0,$$

  where $\vec{n}$ is the outer normal vector and $\vec{\tau}$ is the tangential vector.

Boundary Layer(cont.)

By comparing the order of ϵ, we have the relation

$$\begin{aligned}
(\vec{v}\cdot\vec{n})\frac{\partial \mathscr{U}_0}{\partial \eta} + \mathscr{U}_0 - \bar{\mathscr{U}_0} &= 0,\\
(\vec{v}\cdot\vec{n})\frac{\partial \mathscr{U}_1}{\partial \eta} + \mathscr{U}_1 - \bar{\mathscr{U}_1} &= -(\vec{v}\cdot\vec{\tau})\frac{1}{1-\epsilon\eta}\frac{\partial \mathscr{U}_0}{\partial \theta},\\
&\cdots\\
(\vec{v}\cdot\vec{n})\frac{\partial \mathscr{U}_k}{\partial \eta} + \mathscr{U}_k - \bar{\mathscr{U}_k} &= -(\vec{v}\cdot\vec{\tau})\frac{1}{1-\epsilon\eta}\frac{\partial \mathscr{U}_{k-1}}{\partial \theta},
\end{aligned}$$

in a neighborhood of the boundary.

Matching of Interior Solution and Boundary Layer

We define the boundary layer $\mathscr{U}_0$ as

$$\begin{cases} \mathscr{U}_0 & = & f_0(\eta,\theta,\vec{v}) - f_0(\infty,\theta) \\ (\vec{v}\cdot\vec{n})\dfrac{\partial f_0}{\partial \eta} + f_0 - \bar{f}_0 & = & 0, \\ f_0(0,\theta,\vec{v}) & = & g(\theta,\vec{v}) \;\text{ for }\; \vec{n}\cdot\vec{v} < 0, \\ \lim_{\eta\to\infty} f_0(\eta,\theta,\vec{v}) & = & f_0(\infty,\theta), \end{cases}$$

and the interior solution U_0 as

$$\begin{cases} U_0(\vec{x},\vec{v}) & = & \bar{U}_0(\vec{x}), \\ \Delta_x \bar{U}_0 & = & 0, \\ \bar{U}_0(\vec{x}_0) & = & f_0(\infty,\theta). \end{cases}$$

Good Results

In 1979's and 1984's papers, the author showed that both the interior and boundary layer expansion can be constructed to higher order and then proved the following theorem:

Theorem

Assume $g(\vec{x}_0, \vec{v})$ is sufficiently smooth. Then for the steady neutron transport equation, the unique solution $u^\epsilon(\vec{x}, \vec{v}) \in L^\infty(\Omega \times \mathcal{S}^1)$ satisfies

$$\left\| u^\epsilon - U_0 - \mathscr{U}_0 \right\|_{L^\infty} = O(\epsilon).$$

This is a remarkable result!

Think about it

The proof is based on the following key theorem:

Theorem

Consider the Milne problem

$$\begin{cases} (\vec{v}\cdot\vec{n})\dfrac{\partial f}{\partial\eta}+f-\bar{f} &= S(\eta,\theta,\vec{v}),\\ f(0,\theta,\vec{v}) &= h(\theta,\vec{v}) \ \ \text{for} \ \ \vec{n}\cdot\vec{v}<0,\\ \lim_{\eta\to\infty} f(\eta,\theta,\vec{v}) &= f_\infty(\theta), \end{cases}$$

with

$$\left\|e^{\beta_0\eta}S\right\|_{L^\infty L^\infty} \leq C, \qquad \|h\|_{L^\infty}\leq C.$$

Then for $\beta>0$ sufficiently small, there exists a unique solution $f(\eta,\theta,\vec{v})\in L^\infty$ satisfying

$$\left\|e^{\beta\eta}(f-f_\infty)\right\|_{L^\infty L^\infty}\leq C.$$

In the proof of 1979's and 1984's papers, we have to go to U_1 and $\mathscr{U}_1$ at least. In all the known results, in order to show the L^∞ well-posedness of Milne problem, we need the source term is in L^∞ and exponentially decays. Thus in order to show the well-posedness of $\mathscr{U}_1$, we need

$$(\vec{v}\cdot\vec{\tau})\frac{1}{1-\epsilon\eta}\frac{\partial\mathscr{U}_0}{\partial\theta}\in L^\infty([0,\infty)\times[-\pi,\pi)\times\mathcal{S}^1),$$

which further needs

$$(\vec{v}\cdot\frac{\partial\vec{\tau}}{\partial\theta})\frac{\partial\mathscr{U}_0}{\partial\eta}\in L^\infty([0,\infty)\times[-\pi,\pi)\times\mathcal{S}^1).$$

This is not always true. We have counterexamples to illustrate this fact.

Counterexample

Lemma

For the Milne problem

$$\begin{cases} \sin(\theta+\xi)\dfrac{\partial f}{\partial \eta} + f - \bar{f} &= 0, \\ f(0,\theta,\xi) &= g(\theta,\xi) \;\; \textit{for} \;\; \sin(\theta+\xi) > 0, \\ \lim_{\eta\to\infty} f(\eta,\theta,\xi) &= f(\infty,\theta), \end{cases}$$

if $g(\theta,\xi) = \cos(3(\theta+\xi))$*, then we have*

$$\frac{\partial f}{\partial \eta} \notin L^\infty([0,\infty)\times[-\pi,\pi)\times[-\pi,\pi)).$$

Counterexample (Cont.)

The central idea of the proof is by contradiction:

1. By maximum principle, we have $f(0,\theta,\xi) \leq 1$ for $\sin(\theta+\xi) < 0$. Then this implies

$$\bar{f}(0,\theta) \leq \frac{1}{2}.$$

2. We can obtain $\partial_\eta f(0,\theta,\xi) \in L^\infty[-\pi,\pi) \times [-\pi,\pi)$ is a.e. well-defined and satisfies the formula

$$\partial_\eta f(0,\theta,\xi) = \frac{\bar{f}(0,\theta) - f(0,\theta,\xi)}{\sin(\theta+\xi)}.$$

3. Finally, we can directly estimate

$$\lim_{\xi \to -\theta^+} \frac{\partial f}{\partial \eta}(0,\theta,\xi) = -\infty.$$

which is a contradiction.

Boundary Layer with Geometric Correction

- Polar coordinates: $(x_1, x_2) \to (r, \theta)$.
- Boundary layer scaling: $\eta = (1 - r)/\epsilon$.
- **Change of Variables:** $v_n = \vec{v} \cdot \vec{n}$ and $v_\tau = \vec{v} \cdot \vec{\tau}$.
- We define the boundary layer expansion as follows:

$$\mathscr{U}^\epsilon(\eta, \theta, v_n, v_\tau) \sim \sum_{k=0}^{\infty} \epsilon^k \mathscr{U}_k^\epsilon(\eta, \theta, v_n, v_\tau),$$

which satisfies

$$v_n \frac{\partial \mathscr{U}^\epsilon}{\partial \eta} + \frac{\epsilon}{1 - \epsilon\eta}\left(- v_\tau \frac{\partial \mathscr{U}^\epsilon}{\partial \theta} + v_\tau^2 \frac{\partial \mathscr{U}^\epsilon}{\partial v_n} - v_n v_\tau \frac{\partial \mathscr{U}^\epsilon}{\partial v_\tau}\right) + \mathscr{U}^\epsilon - \bar{\mathscr{U}}^\epsilon = 0$$

Where is the Singularity?

The singular term is decomposed into three terms

$$v_\tau \frac{\partial \mathscr{U}_0^\epsilon}{\partial \theta} - v_\tau^2 \frac{\partial \mathscr{U}_0^\epsilon}{\partial v_n} + v_n v_\tau \frac{\partial \mathscr{U}_0^\epsilon}{\partial v_\tau}.$$

By comparing the order of ϵ, we have the relation

$$\begin{aligned}
v_n \frac{\partial \mathscr{U}_0^\epsilon}{\partial \eta} + \mathscr{U}_0^\epsilon - \bar{\mathscr{U}}_0^\epsilon &= 0, \\
v_n \frac{\partial \mathscr{U}_1^\epsilon}{\partial \eta} + \mathscr{U}_1^\epsilon - \bar{\mathscr{U}}_1^\epsilon &= \frac{1}{1-\epsilon\eta}\left(v_\tau \frac{\partial \mathscr{U}_0^\epsilon}{\partial \theta} - v_\tau^2 \frac{\partial \mathscr{U}_0^\epsilon}{\partial v_n} + v_n v_\tau \frac{\partial \mathscr{U}_0^\epsilon}{\partial v_\tau} \right), \\
&\cdots \\
v_n \frac{\partial \mathscr{U}_k^\epsilon}{\partial \eta} + \mathscr{U}_k^\epsilon - \bar{\mathscr{U}}_k^\epsilon &= \frac{1}{1-\epsilon\eta}\left(v_\tau \frac{\partial \mathscr{U}_{k-1}^\epsilon}{\partial \theta} - v_\tau^2 \frac{\partial \mathscr{U}_{k-1}^\epsilon}{\partial v_n} + v_n v_\tau \frac{\partial \mathscr{U}_{k-1}^\epsilon}{\partial v_\tau} \right),
\end{aligned}$$

in a neighborhood of the boundary.

Boundary Layer with Geometric Correction(cont.)

Putting the singular terms together, we have the relation

$$
\begin{aligned}
v_n \frac{\partial \mathscr{U}_0^\epsilon}{\partial \eta} + \frac{\epsilon}{1-\epsilon\eta}\left(v_\tau^2 \frac{\partial \mathscr{U}_0^\epsilon}{\partial v_n} - v_n v_\tau \frac{\partial \mathscr{U}_0^\epsilon}{\partial v_\tau} \right) + \mathscr{U}_0^\epsilon - \bar{\mathscr{U}}_0^\epsilon &= 0, \\
v_n \frac{\partial \mathscr{U}_1^\epsilon}{\partial \eta} + \frac{\epsilon}{1-\epsilon\eta}\left(v_\tau^2 \frac{\partial \mathscr{U}_1^\epsilon}{\partial v_n} - v_n v_\tau \frac{\partial \mathscr{U}_1^\epsilon}{\partial v_\tau} \right) + \mathscr{U}_1^\epsilon - \bar{\mathscr{U}}_1^\epsilon &= \frac{1}{1-\epsilon\eta} v_\tau \frac{\partial \mathscr{U}_0^\epsilon}{\partial \theta}, \\
&\ldots \\
v_n \frac{\partial \mathscr{U}_k^\epsilon}{\partial \eta} + \frac{\epsilon}{1-\epsilon\eta}\left(v_\tau^2 \frac{\partial \mathscr{U}_k^\epsilon}{\partial v_n} - v_n v_\tau \frac{\partial \mathscr{U}_k^\epsilon}{\partial v_\tau} \right) + \mathscr{U}_k^\epsilon - \bar{\mathscr{U}}_k^\epsilon &= \frac{1}{1-\epsilon\eta} v_\tau \frac{\partial \mathscr{U}_{k-1}^\epsilon}{\partial \theta},
\end{aligned}
$$

in a neighborhood of the boundary.

ϵ-Milne Problem with Geometric Correction

Consider the substitution $v_n = \sin\phi$ and $v_\tau = \cos\phi$. The construction of the boundary layer depends on the properties of the Milne problem for $f^\epsilon(\eta,\theta,\phi)$ in the domain $(\eta,\theta,\phi) \in [0,\infty) \times [-\pi,\pi) \times [-\pi,\pi)$

$$\begin{cases} \sin\phi\dfrac{\partial f^\epsilon}{\partial\eta} + F(\epsilon;\eta)\cos\phi\dfrac{\partial f^\epsilon}{\partial\phi} + f^\epsilon - \bar{f}^\epsilon &= S^\epsilon(\eta,\theta,\phi), \\ f^\epsilon(0,\theta,\phi) &= h^\epsilon(\theta,\phi) \ \text{ for } \ \sin\phi > 0, \\ \lim_{\eta\to\infty} f^\epsilon(\eta,\theta,\phi) &= f^\epsilon_\infty(\theta). \end{cases}$$

where

$$F(\epsilon;\eta) = -\frac{\epsilon\psi(\epsilon\eta)}{1-\epsilon\eta}, \qquad \psi(\mu) = \begin{cases} 1 & 0 \le \mu \le 1/2, \\ 0 & 3/4 \le \mu \le \infty, \end{cases}$$

and

$$\left|h^\epsilon(\theta,\phi)\right| \le C, \qquad \left|S^\epsilon(\eta,\theta,\phi)\right| \le Ce^{-\beta_0\eta},$$

for C and β_0 uniform in ϵ and θ.

ϵ-Milne Problem with Geometric Correction (Cont.)

Theorem

For $\beta > 0$ sufficiently small, there exists a unique solution $f^\epsilon(\eta, \theta, \phi) \in L^\infty$ to the ϵ-Milne problem satisfying

$$\left\| e^{\beta\eta}(f^\epsilon - f^\epsilon_\infty) \right\|_{L^\infty L^\infty} \leq C,$$

where C depends on the data h^ϵ and S^ϵ.

Theorem

The solution $f^\epsilon(\eta, \theta, \phi)$ to the ϵ-Milne problem with $S^\epsilon = 0$ satisfies the maximum principle, i.e.

$$\min_{\sin\phi > 0} h^\epsilon(\theta, \phi) \leq f^\epsilon(\eta, \theta, \phi) \leq \max_{\sin\phi > 0} h^\epsilon(\theta, \phi).$$

ϵ-Milne Problem with Geometric Correction (Cont.)

Basic ideas: penalized finite slab $\rightarrow$ finite slab $\rightarrow$ infinite slab; homogeneous $\rightarrow$ inhomogeneous.

1. Using energy estimate to define f_∞^ϵ and show
$$\left\|f^\epsilon - f_\infty^\epsilon)\right\|_{L^2L^2} \leq C.$$
2. Using the characteristics to get
$$\left\|f^\epsilon - f_\infty^\epsilon)\right\|_{L^\infty L^\infty} \leq C + C\left\|f^\epsilon - f_\infty^\epsilon)\right\|_{L^2L^2}.$$
3. Applying the similar techniques to the equation satisfied by $F^\epsilon = e^{\beta\eta} f^\epsilon$.

Remainder Estimate

Theorem

Assume $f(\vec{x}, \vec{v}) \in L^\infty(\Omega \times \mathcal{S}^1)$ *and* $g(x_0, \vec{v}) \in L^\infty(\Gamma^-)$. *Then for the remainder equation*

$$\begin{cases} \epsilon \vec{w} \cdot \nabla_x R + R - \bar{R} = f(\vec{x}, \vec{v}) \text{ in } \Omega, \\ R(\vec{x}_0, \vec{w}) = g(\vec{x}_0, \vec{w}) \text{ for } \vec{x}_0 \in \partial\Omega \text{ and } \vec{v} \cdot \vec{n} < 0, \end{cases}$$

there exists a unique solution $R(\vec{x}, \vec{v}) \in L^\infty(\Omega \times \mathcal{S}^1)$ *satisfying*

$$\|R\|_{L^\infty(\Omega \times \mathcal{S}^1)} \leq C(\Omega) \left(\frac{1}{\epsilon^{5/2}} \|f\|_{L^\infty(\Omega \times \mathcal{S}^1)} + \|g\|_{L^\infty(\Gamma^-)} \right).$$

Main Theorem

Theorem

Assume $g(\vec{x}_0, \vec{v}) \in C^2(\Gamma^-)$. Then for the steady neutron transport equation (1), the unique solution $u^\epsilon(\vec{x}, \vec{v}) \in L^\infty(\Omega \times \mathcal{S}^1)$ satisfies

$$\left\| u^\epsilon - U_0^\epsilon - \mathscr{U}_0^\epsilon \right\|_{L^\infty} = O(\epsilon)$$

Moreover, if $g(\theta, v_n, v_\tau) = v_\tau$, then there exists a $C > 0$ such that

$$\left\| u^\epsilon - U_0 - \mathscr{U}_0 \right\|_{L^\infty} \geq C > 0$$

when ϵ is sufficiently small.

Remark

The comparison of L^p and L^∞ result.

Main Theorem (Cont.)

Proof of $\|u^\epsilon - U_0 - \mathscr{U}_0\|_{L^\infty} \geq C > 0$ is as follows:

1. The problem can be simplified into the estimate of solutions u in Milne problem and U in ϵ-Milne problems with exactly the same boundary data $v_\tau + 2$.
2. Rewriting the solution along the characteristics, we can obtain the estimate at point $(\eta, \phi) = (n\epsilon, \epsilon)$ as

$$\begin{aligned} u(n\epsilon, \epsilon) &= \bar{u}(0) + e^{-n}(-\bar{u}(0) + 3) + o(\epsilon), \\ U(n\epsilon, \epsilon) &= \bar{U}(0) + e^{1-\sqrt{1+2n}}(-\bar{U}(0) + 3) + o(\epsilon). \end{aligned}$$

3. We can derive $\lim_{\epsilon\to 0} \left\|(-\bar{u}(0) + 3) - (-\bar{U}(0) + 3)\right\|_{L^\infty} = 0.$ and $-\bar{u}(0) + 3 = O(1)$ with $-\bar{U}(0) + 3 = O(1)$. Due to the smallness of ϵ, we can obtain

$$\left|U(n\epsilon, \epsilon) - u(n\epsilon, \epsilon)\right| = O(1).$$

Unsteady Neutron Transport Equation

We consider a homogeneous isotropic unsteady neutron transport equation in a two-dimensional unit disk $\Omega = \{\vec{x} = (x_1, x_2) : \left|\vec{x}\right| \leq 1\}$ with one-speed velocity $\Sigma = \{\vec{v} = (v_1, v_2) : \vec{v} \in \mathcal{S}^1\}$ as

$$\begin{cases} \epsilon^2 \partial_t u^\epsilon + \epsilon \vec{v} \cdot \nabla_x u^\epsilon + u^\epsilon - \bar{u}^\epsilon & = & 0 \ \text{ in } \ [0, \infty) \times \Omega, \\ u^\epsilon(0, \vec{x}, \vec{v}) & = & h(\vec{x}, \vec{v}) \ \text{ in } \ \Omega \\ u^\epsilon(t, \vec{x}_0, \vec{v}) & = & g(t, \vec{x}_0, \vec{v}) \ \text{ for } \ \vec{v} \cdot \vec{n} < 0 \ \text{ and } \ \vec{x}_0 \in \partial\Omega, \end{cases} \tag{2}$$

where

$$\bar{u}^\epsilon(t, \vec{x}) = \frac{1}{2\pi} \int_{\mathcal{S}^1} u^\epsilon(t, \vec{x}, \vec{v}) \mathrm{d}\vec{v}.$$

and $\vec{n}$ is the outward normal vector on $\partial\Omega$, with the Knudsen number $0 < \epsilon << 1$. The initial and boundary data satisfy the compatibility condition

$$h(\vec{x}_0, \vec{v}) = g(0, \vec{x}_0, \vec{v}) \ \text{ for } \ \vec{v} \cdot \vec{n} < 0 \ \text{ and } \ \vec{x}_0 \in \partial\Omega.$$

Remainder Estimate

Theorem

Assume $f(t,\vec{x},\vec{v}) \in L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)$, $h(\vec{x},\vec{v}) \in L^\infty(\Omega\times\mathcal{S}^1)$ *and* $g(t,x_0,\vec{v}) \in L^\infty([0,\infty)\times\Gamma^-)$. *Then for the remainder equation*

$$\begin{cases} \epsilon^2\partial_t R + \epsilon\vec{v}\cdot\nabla_x R + R - \bar{R} &= f(t,\vec{x},\vec{v}) \quad in \ \ [0,\infty)\times\Omega, \\ R(0,\vec{x},\vec{v}) &= h(\vec{x},\vec{v}) \quad in \ \ \Omega \\ R(t,\vec{x}_0,\vec{v}) &= g(t,\vec{x}_0,\vec{v}) \quad for \ \ \vec{v}\cdot\vec{n}<0 \ \ and \ \ \vec{x}_0\in\partial\Omega, \end{cases}$$

there exists a unique solution $R(t,\vec{x},\vec{v}) \in L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)$ *satisfying*

$$\begin{aligned} &\|R\|_{L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)} \\ \leq\ & C(\Omega)\bigg(\frac{1}{\epsilon^{5/2}}\|f\|_{L^\infty([0,\infty)\times\Omega\times\mathcal{S}^1)} + \|h\|_{L^\infty(\Omega\times\mathcal{S}^1)} + \|g\|_{L^\infty([0,\infty)\times\Gamma^-)}\bigg). \end{aligned}$$

Diffusive Limit

Theorem

Assume $g(t, \vec{x}_0, \vec{v}) \in C^2([0,\infty) \times \Gamma^-)$ *and* $h(\vec{x}, \vec{v}) \in C^2(\Omega \times \mathcal{S}^1)$. *Then for the unsteady neutron transport equation (2), the unique solution* $u^\epsilon(t, \vec{x}, \vec{v}) \in L^\infty([0,\infty) \times \Omega \times \mathcal{S}^1)$ *satisfies*

$$\left\| u^\epsilon - U_0^\epsilon - \mathscr{U}_{I,0}^\epsilon - \mathscr{U}_{B,0}^\epsilon \right\|_{L^\infty} = O(\epsilon),$$

for the interior solution U_0^ϵ, *the initial layer* $\mathscr{U}_{I,0}^\epsilon$, *and the boundary layer* $\mathscr{U}_{B,0}^\epsilon$.

Boltzmann Equation near Maxwellian

We turn back to the stationary Boltzmann equation

$$\begin{cases} \epsilon\vec{v}\cdot\nabla_x F^\epsilon &= Q[F^\epsilon, F^\epsilon] \ \text{ in } \ \Omega\times\mathbb{R}^2, \\ F^\epsilon(\vec{x}_0,\vec{v}) &= B^\epsilon(\vec{x}_0,\vec{v}) \ \text{ for } \ \vec{x}_0\in\partial\Omega \ \text{ and } \ \vec{n}(\vec{x}_0)\cdot\vec{v}<0, \end{cases}$$

and

$$F^\epsilon(\vec{x},\vec{v}) = \mu+\epsilon\mu^{\frac{1}{2}}f^\epsilon(\vec{x},\vec{v}),$$

where f^ϵ satisfies the equation

$$\begin{cases} \epsilon\vec{v}\cdot\nabla_x f^\epsilon+\mathcal{L}[f^\epsilon] &= \Gamma[f^\epsilon, f^\epsilon], \\ f^\epsilon(\vec{x}_0,\vec{v}) &= b(\vec{x}_0,\vec{v}) \ \text{ for } \ \vec{n}\cdot\vec{v}<0 \ \text{ and } \ \vec{x}_0\in\partial\Omega, \end{cases}$$

Hydrodynamic Limit of Stationary Boltzmann Equation

Theorem

For given $b(\vec{x}_0, \vec{v})$ sufficiently small and $0 < \epsilon << 1$, there exists a unique positive solution $F^\epsilon = \mu + \epsilon\mu^{\frac{1}{2}} f^\epsilon$ to the stationary Boltzmann equation, where

$$f^\epsilon = \epsilon^3 R_N + \left(\sum_{k=1}^{N} \epsilon^k \mathcal{F}_k^\epsilon\right) + \left(\sum_{k=1}^{N} \epsilon^k \mathscr{F}_k^\epsilon\right),$$

for $N \geq 3$, R_N satisfies the remainder equation, $\mathcal{F}_k^\epsilon$ and $\mathscr{F}_k^\epsilon$ are interior solution and boundary layer. Also, there exists a $C > 0$ such that f^ϵ satisfies

$$\left\| \langle \vec{v} \rangle^{\vartheta} \mathrm{e}^{\zeta |\vec{v}|^2} f^\epsilon \right\|_{L^\infty} \leq C,$$

for any $\vartheta > 2$, $0 \leq \zeta \leq 1/4$.

In particular, the leading order interior solution satisfies

$$\mathcal{F}_1^\epsilon = \sqrt{\mu}\bigg(\rho_1^\epsilon + u_{1,1}^\epsilon v_1 + u_{1,2}^\epsilon v_2 + \theta_1^\epsilon\bigg(\frac{\left|\vec{v}\right|^2 - 2}{2}\bigg)\bigg),$$

with

$$\begin{cases} \nabla_x(\rho_1^\epsilon + \theta_1^\epsilon) &= 0, \\ \vec{u}_1^\epsilon \cdot \nabla_x \vec{u}_1^\epsilon - \gamma_1 \Delta_x \vec{u}_1^\epsilon + \nabla_x P_2^\epsilon &= 0, \\ \nabla_x \cdot \vec{u}_1^\epsilon &= 0, \\ \vec{u}_1^\epsilon \cdot \nabla_x \theta_1^\epsilon - \gamma_2 \Delta_x \theta_1^\epsilon &= 0, \end{cases}$$

and suitable Dirichlet-type boundary conditions.

Ongoing and Future Work

- Steady problem in smooth domain(general smooth convex domain, annulus).
- Detailed structure of boundary layer(How does $\mathscr{U}^{\epsilon}$ depend on ϵ?).
- Higher dimensional problems.
- Boltzmann equation with time.

Thank you for your attention!