Matrix-valued Quantum Boltzmann Methods

joint work with Jianfeng Lu, Martin L.R. Fürst, Herbert Spohn

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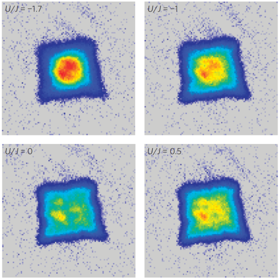
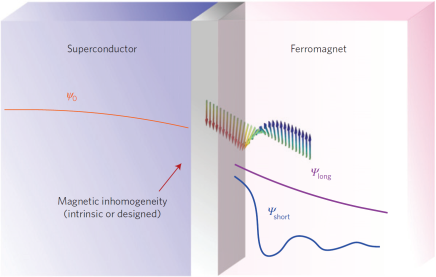
November 29, 2016

KI-Net Workshop
Stochastic and deterministic methods in kinetic theory
Duke University, NC, USA

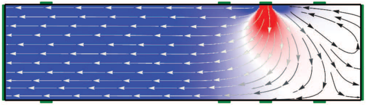
Introduction: Boltzmann equations for quantum systems



(a) ultracold atoms in an optical lattice (Schneider et al., Science 2012)

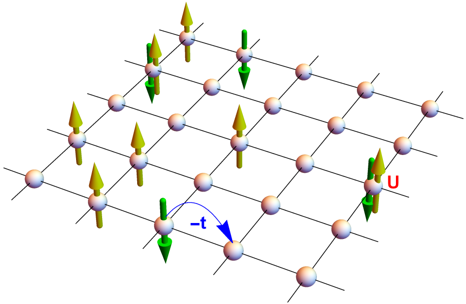


(b) superconducting spintronics (Linder and Robinson, Nat. Phys. 2015)



(c) Viscous electron backflow (Bandurin et al., Science 2016)

Quantum Boltzmann derived from the Hubbard model



Second order time-dependent perturbation theory for $U \ll 1 \rightsquigarrow$

$$\frac{\partial}{\partial t} W(k,t) = \mathcal{C}_{\mathrm{cons}}[W](k,t) + \mathcal{C}_{\mathrm{diss}}[W](k,t) = \mathcal{C}[W](k,t)$$

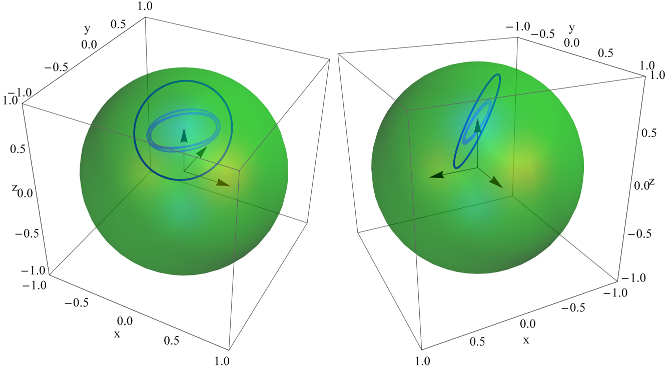
Wigner function $W(k,t)$ is the 2×2 matrix-valued “spin density matrix”.

Fürst, Lukkarinen, Mei, Spohn, J. Phys. A 46, 485002 (2013)

Bloch sphere representation

Bloch sphere representation $\vec{r}(k,t) \in \mathbb{R}^3$ of $W(k,t)$:

$$W(k,t) = \frac{1}{2}\left(\alpha \mathbb{1} + \vec{r}(k,t)\cdot\vec{\sigma}\right), \quad \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z).$$



$k \in \mathbb{T} = [-\frac{1}{2}, \frac{1}{2})$ parametrizes a closed path within the Bloch sphere.

Conservative collision operator $\mathcal{C}_{\mathrm{cons}}[W]$

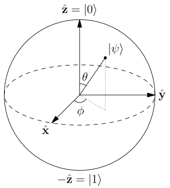
Vlasov type commutator:

$$\mathcal{C}_{\mathrm{cons}}[W](k,t) = -i\,[H_{\mathrm{eff}}(k,t), W(k,t)],$$

with an effective Hamiltonian $H_{\mathrm{eff}}(k,t)$ itself depending on $W(\cdot,t)$.

$\rightsquigarrow$ k-dependent rotations on the Bloch sphere.

Denoted “conservative” since $\mathrm{tr}[W]$
does not change, and entropy does not increase.



Dissipative collision operator $\mathcal{C}_{\mathrm{diss}}[W]$

Notation: $\tilde{W} = \mathbb{1} - W$,
$\underline{k} = k_1 + k_2 - k_3 - k_4 \mod 1$,
$\underline{\omega} = \omega(k_1) + \omega(k_2) - \omega(k_3) - \omega(k_4)$.

$$\mathcal{C}_{\mathrm{diss}}[W](k_1) = \pi \int_{\mathbb{T}^3} \delta(\underline{k})\delta(\underline{\omega})\big(\mathcal{A}[W]_{1234} + \mathcal{A}[W]^{\dagger}_{1234}\big)\,\mathrm{d}k_2\,\mathrm{d}k_3\,\mathrm{d}k_4,$$

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with

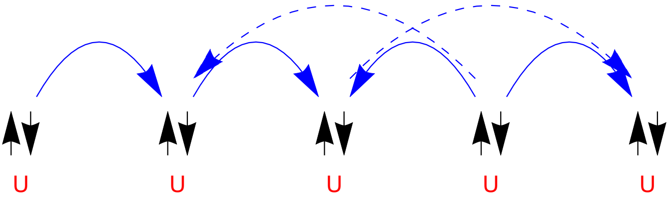
$$\begin{aligned}\mathcal{A}[W]_{1234} = &-W_4\tilde{W}_2W_3 + W_4\ \mathrm{tr}[\tilde{W}_2W_3]\\ &-\big(\tilde{W}_4W_3 - \tilde{W}_4W_2 - \tilde{W}_2W_3 + \tilde{W}_4\ \mathrm{tr}[W_2]\\ &\qquad -\tilde{W}_4\ \mathrm{tr}[W_3] + \mathrm{tr}[W_3\tilde{W}_2]\big)W_1\end{aligned}$$

with gain term (+ h.c.) and loss term (+ h.c.). The gain term is always positive definite since

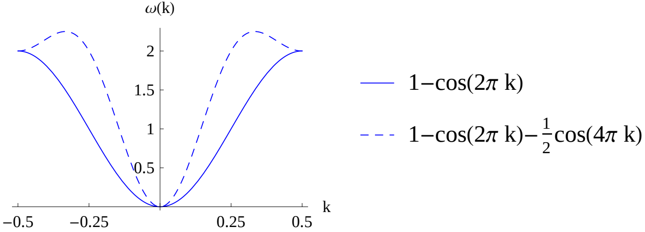
$$A\ \mathrm{tr}[BC] + C\ \mathrm{tr}[BA] - ABC - CBA \geq 0$$

for any positive definite matrices A, B, C.

(Next-)nearest neighbor hopping model (1D)

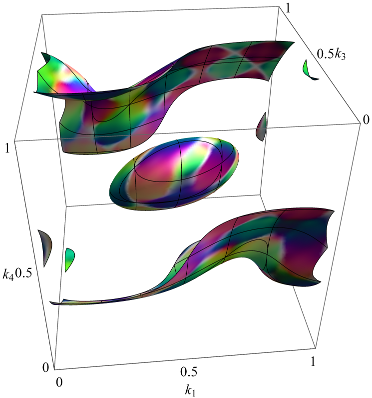


Nearest neighbor hopping: $\omega(k) = 1 - \cos(2\pi k)$
Next-nearest neighbor: $\omega_\eta(k) = 1 - \cos(2\pi k) - \eta \cos(4\pi k)$



Collision channels for next-nearest neighbor hopping (1D)

Simultaneous non-trivial solutions of momentum and energy conservation constraints, here $\omega_\eta(k) = 1 - \cos(2\pi k) - \eta \cos(4\pi k)$ with $\eta = \frac{1}{2}$:



Discretization of the collision manifold which preserves discrete (finite k-grid) conservation laws exactly.

Fürst, Mendl, Spohn, Phys. Rev. E 88, 012108 (2013)

Mollification of $\mathcal{C}_{\mathrm{diss}}[W]$

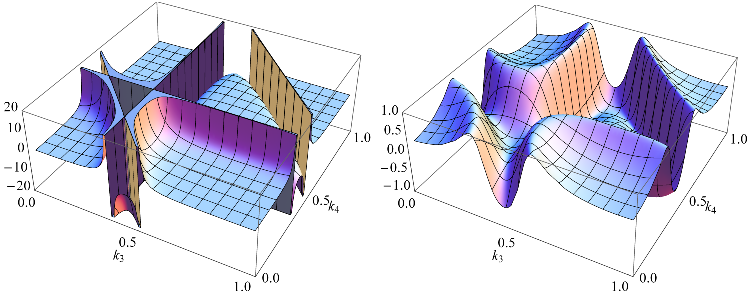
Conservative collision operator

$$\mathcal{C}_{\mathrm{cons}}[W](k,t) = -i\,[H_{\mathrm{eff}}(k,t), W(k,t)],$$

integral defining $H_{\mathrm{eff}}(k,t)$ contains principal value term $\mathcal{P}\left(\frac{1}{\underline{\omega}}\right)$.
"Mollification":

$$\frac{1}{\underline{\omega}} \to \frac{\underline{\omega}}{\underline{\omega}^2 + \epsilon^2}$$

with finite $\epsilon > 0$ (here $\epsilon = \frac{1}{2}$):



Invariances and conservation laws

- SU(2) invariance of H:

$$\mathcal{C}[U^\dagger W U] = U^\dagger \mathcal{C}[W] U$$

  for all $U \in \mathrm{SU}(2)$. Hermiticity is propagated in time
- Fermi property $0 \leq W(k,t) \leq 1$ (eigenvalues) is propagated in time
- spin density conservation

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}^d} W(k,t)\,\mathrm{d}k = 0$$

- energy conservation

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}^d} \omega(k)\ \mathrm{tr}[W(k,t)]\,\mathrm{d}k = 0$$

Entropy and H-theorem

Entropy of a state $W(k)$:

$$S[W] = -\int_{\mathbb{T}^d} \operatorname{tr}[W(k)\log W(k)] + \operatorname{tr}[\tilde{W}(k)\log \tilde{W}(k)]\,\mathrm{d}k,$$

and corresponding entropy production

$$\sigma[W] = \frac{\mathrm{d}}{\mathrm{d}t} S[W] = -\int_{\mathbb{T}^d} \operatorname{tr}\Big[\big(\log W(k) - \log \tilde{W}(k)\big)\,\mathcal{C}[W](k)\Big]\,\mathrm{d}k.$$

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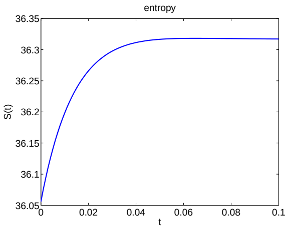
The *H-theorem* asserts that

$$\sigma[W] \geq 0 \qquad \text{for all } W \text{ with } 0 \leq W \leq 1$$

(We proof the H-theorem by writing $\sigma[W]$ as an integral of non-negative terms times $(x-y)\log(x/y) \geq 0$.)

Fürst, Mendl, Spohn, Phys. Rev. E 86, 031122 (2012)



(Thermal) Stationary states

Stationarity $\leftrightarrow \mathcal{C}[W] = 0$

In particular, entropy production $\sigma[W] = 0$. Physically, thermal equilibrium (Fermi-Dirac) states should be stationary:

$$W_{\mathrm{FD}}(k) = \sum_{\sigma \in \{\uparrow,\downarrow\}} \left(\mathrm{e}^{\beta(\omega(k)-\mu_\sigma)} + 1 \right)^{-1} |\sigma\rangle\langle\sigma|,$$

$W_{\mathrm{FD}}(k)$ indeed fulfills $\mathcal{C}[W_{\mathrm{FD}}] = 0$.

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$W_{\mathrm{FD}}(k)$ indeed fulfills $\mathcal{C}[W_{\mathrm{FD}}] = 0$.

Conservation laws uniquely define $W_{\mathrm{FD}}(k)$, can map initial $W(k, 0)$ to asymptotic $W(k, t)$ as $t \to \infty$

All stationary states of this form? $\rightsquigarrow$ depends on dispersion relation!

Simulation example (1D): initial state

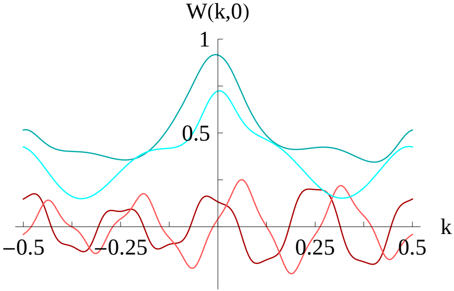
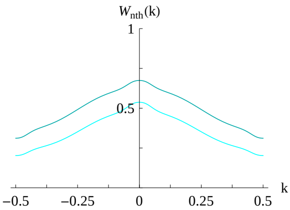


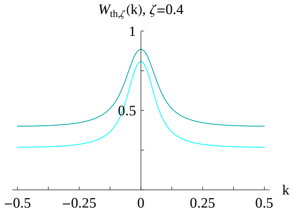
Figure: Cyan (upper) curves: real diagonal entries, darker and lighter red curves: real and imaginary parts of the off-diagonal $|\uparrow\rangle\langle\downarrow|$ entry, respectively

Fürst, Mendl, Spohn, Phys. Rev. E 88, 012108 (2013)

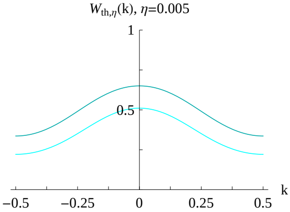
Simulation example (1D): asymptotic ($t \to \infty$) states



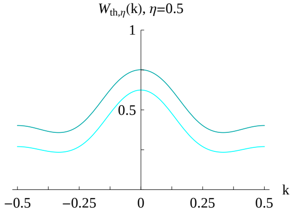
(a) non-thermal stationary state (nearest neighbor)



(b) thermal equilibrium state (exponential, $\zeta = \frac{2}{5}$)



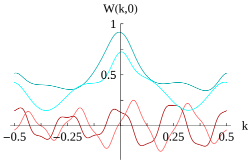
(c) thermal equilibrium state (next-nearest neighbor hopping with $\eta = \frac{1}{200}$)



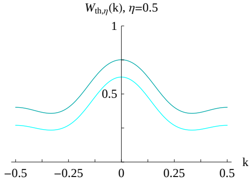
(d) thermal equilibrium state (next-nearest neighbor hopping with $\eta = \frac{1}{2}$)

Simulation example (1D): entropy convergence

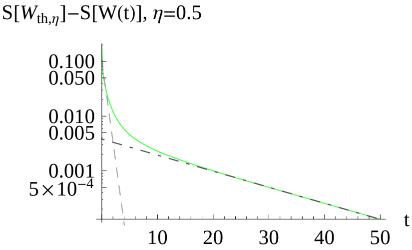
Entropy convergence for next-nearest neighbor hopping with $\eta = \frac{1}{2}$



(a) initial state

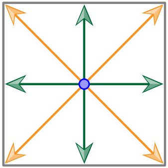


(b) thermal equilibrium

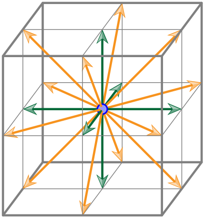


(c) entropy convergence

Quantum lattice Boltzmann methods (LBM)



D2Q9 model



D3Q19 model

C. B. Mendl, Int. J. Mod. Phys. C 26, 1550113 (2015)

Spatially inhomogeneous version, using LBM

Spatially inhomogeneous case: Wigner function depends on position $\boldsymbol{x}$, Boltzmann equation is augmented by transport term:

$$\frac{\partial}{\partial t} W(\boldsymbol{x}, \boldsymbol{p}, t) + \boldsymbol{\nabla}_{\boldsymbol{p}} \omega(\boldsymbol{p}) \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} W(\boldsymbol{x}, \boldsymbol{p}, t) = \mathcal{C}[W](\boldsymbol{x}, \boldsymbol{p}, t).$$

Discretization of momentum by the lattice Boltzmann method (LBM):

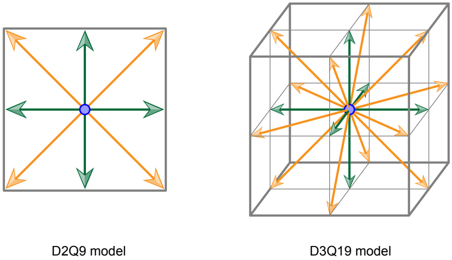


Figure: Illustration of the velocity vectors $\mathbf{e}_i$ of the D2Q9 and D3Q19 models (adapted from Thürey 2006)

Quantum LBM: Quadrature rules

Idea: construct quadrature rules with Fermi-Dirac weight which are compatible with D2Q9 and D3Q19 discretizations:

$$\frac{1}{n_{2,\beta,\mu}} \int_{\mathbb{R}^2} h(\boldsymbol{p}) \left(\mathrm{e}^{\beta\left(\frac{1}{2}\boldsymbol{p}^2 - \mu\right)} + 1 \right)^{-1} \mathrm{d}\boldsymbol{p} \approx \sum_{i_1, i_2 \in \{-1,0,1\}} w_{\boldsymbol{i}}\, h(\xi\, \boldsymbol{i})$$

(analogous in 3D) for smooth functions $h : \mathbb{R}^2 \to \mathbb{R}$, with equality for polynomials up to order 5.

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(analogous in 3D) for smooth functions $h : \mathbb{R}^2 \to \mathbb{R}$, with equality for polynomials up to order 5.
Analytic solution

$$w_{00} = 1 - 5a, \quad w_{01} = a, \quad w_{11} = \frac{a}{4},$$

$$a = \frac{F_1(\beta\mu)^2}{9\, F_0(\beta\mu) F_2(\beta\mu)}, \quad \xi = \left(\frac{3}{\beta} \frac{F_2(\beta\mu)}{F_1(\beta\mu)} \right)^{1/2},$$

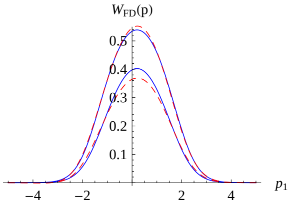
with $F_k(x)$ the complete Fermi-Dirac integral.

Quantum LBM: polynomial expansion of $W_{\mathrm{FD}}(k)$

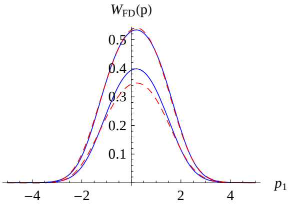
Goal: construct approximate equilibrium distribution function compatible with the quadrature formula, such that its moments agree with the exact Fermi-Dirac density, velocity and energy (analogous to "classical" LBM):

$$W^{(\mathrm{eq})}(\boldsymbol{p}) = \frac{1}{n_{d,\beta_0,0}} \left(\mathrm{e}^{\beta_0 \frac{1}{2}\boldsymbol{p}^2} + 1\right)^{-1} U \begin{pmatrix} n_{d,\beta,\mu_\uparrow} & 0 \\ 0 & n_{d,\beta,\mu_\downarrow} \end{pmatrix} U^*$$
$$\times \underbrace{\left(\alpha_1 + \alpha_2\,\beta_0\,\tfrac{1}{2}\boldsymbol{p}^2 + \alpha_3\,\beta_0\,(\boldsymbol{p}\cdot\boldsymbol{u}) + \alpha_4(\beta_0\,\boldsymbol{p}\cdot\boldsymbol{u})^2 + \alpha_5\,\beta_0\,\tfrac{1}{2}\boldsymbol{u}^2\right)}_{\text{polynomial in } \boldsymbol{p}},$$

with the coefficients α_i to be determined.



(a) $d = 2,\ \beta_0 = 0.8$



(b) $d = 3,\ \beta_0 = 0.8$

Quantum LBM: Algorithm

time step $t \to t + \Delta t$:

- For each cell $\boldsymbol{x}$, compute local average density ρ, velocity $\boldsymbol{u}$ and energy $\varepsilon \rightsquigarrow$ local equilibrium function

$$W_i^{(\mathrm{eq})} = w_i\,\rho\left(\alpha_1+\alpha_2\,\beta_0\,\tfrac{1}{2}\mathbf{e}_i^2+\alpha_3\,\beta_0\,(\mathbf{e}_i\cdot\boldsymbol{u})+\alpha_4(\beta_0\,\mathbf{e}_i\cdot\boldsymbol{u})^2+\alpha_5\,\beta_0\,\tfrac{1}{2}\boldsymbol{u}^2\right)$$

Quantum LBM: Algorithm

time step $t \rightarrow t + \Delta t$:

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- Collision: for each cell $\boldsymbol{x}$, apply discretized BGK collision operator:

$$\mathrm{W}_i^{\mathrm{coll}}(\boldsymbol{x}, t) = \mathrm{W}_i(\boldsymbol{x}, t) + \frac{\Delta t}{\tau}\left(\mathrm{W}_i^{(\mathrm{eq})} - \mathrm{W}_i(\boldsymbol{x}, t)\right)$$

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- For each cell $\boldsymbol{x}$, compute local average density ρ, velocity $\boldsymbol{u}$ and energy $\varepsilon \rightsquigarrow$ local equilibrium function

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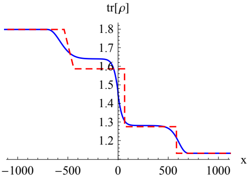
- Streaming:

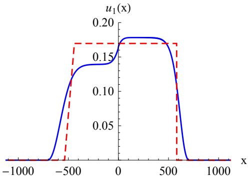
$$\mathrm{W}_i(\boldsymbol{x} + \Delta t\,\mathbf{e}_i, t + \Delta t) = \mathrm{W}_i^{\mathrm{coll}}(\boldsymbol{x}, t)$$

  for all $i = 1, \dots, b$ and cells $\boldsymbol{x}$, to approximate transport term $\boldsymbol{p} \cdot \boldsymbol{\nabla}_x W(\boldsymbol{x}, \boldsymbol{p}, t)$

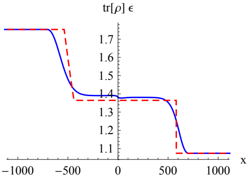
Comparison with analytic Riemann problem solution
(quasi-1D, 2048×2):



(a) density $\mathrm{tr}[\rho]$



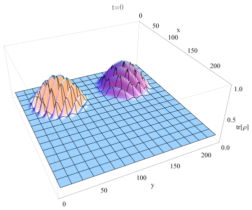
(b) velocity



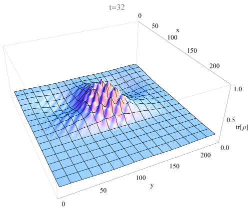
(c) pressure
$P = (\gamma - 1)\,\mathrm{tr}[\rho]\varepsilon$

Adiabatic exponent $\gamma = 1 + \frac{2}{d} = 2$

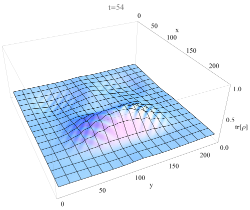
Quantum LBM simulation example



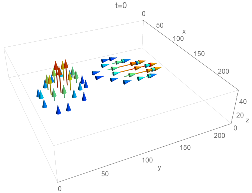
(a) tr[ρ] at $t = 0$



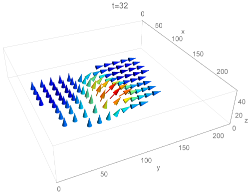
(b) tr[ρ] at time step 32



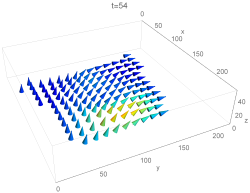
(c) tr[ρ] at time step 54



(d) Bloch vectors at $t = 0$



(e) Bloch vectors at time step 32



(f) Bloch vectors at time step 54

C. B. Mendl, Int. J. Mod. Phys. C 26, 1550113 (2015)

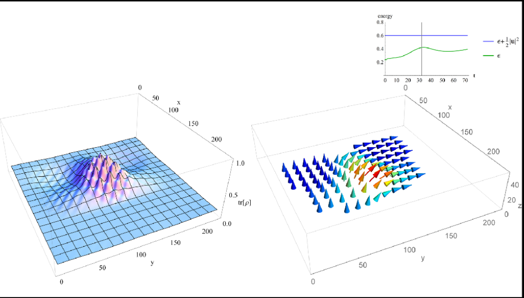
Quantum LBM simulation example (animation)

Matrix-valued Quantum Lattice Boltzmann Method

Time evolution of the spin density matrix ρ(x,t)

Left: trace of density matrix
Right: Bloch vectors
Inset: global energy

Simulation grid: 128 x 128
Relaxation time: 10



Christian B. Mendl
http://christian.mendl.net

Reference: Int. J. Mod. Phys. C 26, 1550113 (2015)

Spectral methods for the quantum Boltzmann equation

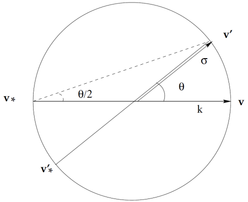
Lu and Mendl, J. Comput. Phys. 291, 303–316 (2015)

Spectral methods for the quantum Boltzmann equation

Idea: use fast Fourier spectral methods to evaluate the collision operators.

Carleman representation: perform a change of variables $p_1 \mapsto v$, $p_3 \mapsto v + u$, $p_4 \mapsto v + u'$, such that

$$p_2 = p_3 + p_4 - p_1 = v + u + u';$$
$$\underline{\omega} = \omega(p_1) + \omega(p_2) - \omega(p_3) - \omega(p_4) = u \cdot u'$$



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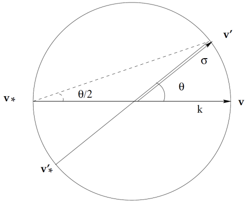

$$\begin{aligned} p_2 &= p_3 + p_4 - p_1 = v + u + u'; \\ \underline{\omega} &= \omega(p_1) + \omega(p_2) - \omega(p_3) - \omega(p_4) = u \cdot u' \end{aligned}$$

$\rightsquigarrow$ integral defining $\mathcal{C}_{\mathrm{cons}}[W]$ can be written as

$$I_1(v) = \int_{B_R} \int_{B_R} \mathcal{P}\left(\frac{1}{u \cdot u'}\right) f(v+u) g(v+u+u') h(v) \, \mathrm{d}u \, \mathrm{d}u'$$

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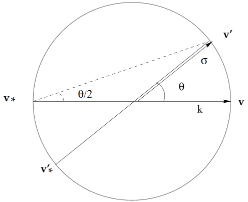

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Next step: Fourier representation

Spectral methods: Fourier representation

Fourier representation:

$$I_1(v) = \int_{B_R}\int_{B_R} \mathcal{P}\left(\frac{1}{u \cdot u'}\right) f(v+u)g(v+u+u')h(v)\,\mathrm{d}u\,\mathrm{d}u'$$
$$\approx \sum_{\chi,\eta,\zeta} \hat{f}(\chi)\hat{g}(\eta)\hat{h}(\zeta)\mathrm{e}^{i\frac{\pi}{L}v\cdot(\chi+\eta+\zeta)}G(\chi+\eta,\eta)$$

with

$$G(\xi,\chi) = -\int_{S^1}\int_{S^1} \mathcal{P}\left(\frac{1}{\theta\cdot\theta'}\right)\phi_R(\xi\cdot\theta)\phi_R(\chi\cdot\theta')\,\mathrm{d}\theta\,\mathrm{d}\theta'.$$

Spectral methods: Fourier representation

Fourier representation:

$$I_1(v) = \int_{B_R}\int_{B_R} \mathcal{P}\left(\frac{1}{u\cdot u'}\right) f(v+u)g(v+u+u')h(v)\,\mathrm{d}u\,\mathrm{d}u'$$
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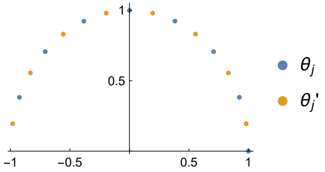
Use numerical quadrature with uniform grid to approximate angular integrals:

$$G(\xi,\chi) \approx \sum_{j=1}^{J}\sum_{j'=1}^{J}\omega_{G,j,j'}\phi_{R,j}(\xi)\phi'_{R,j'}(\chi)$$



In summary, obtain the approximation

$$\hat{I}_1(\xi) \approx \sum_{\substack{\chi,\eta,\zeta,\\ \chi+\eta+\zeta=\xi}} \sum_{j,j'} \omega_{G,j,j'} \, \hat{f}(\chi) \, \hat{g}(\eta) \, \hat{h}(\zeta) \, \phi_{R,j}(\chi+\eta) \, \phi'_{R,j'}(\eta)$$

$$= \sum_{j,j'} \omega_{G,j,j'} \sum_{\zeta} \Big[\sum_{\eta} \hat{f}(\xi-\zeta-\eta) \left(\phi'_{R,j'}(\eta) \, \hat{g}(\eta) \right) \Big] \phi_{R,j}(\xi-\zeta) \, \hat{h}(\zeta).$$

Overall cost: $\mathcal{O}(J^2 N^2 \log N)$

In summary, obtain the approximation

$$\hat{I}_1(\xi) \approx \sum_{\substack{\chi,\eta,\zeta,\\ \chi+\eta+\zeta=\xi}} \sum_{j,j'} \omega_{G,j,j'}\, \hat{f}(\chi)\, \hat{g}(\eta)\, \hat{h}(\zeta)\, \phi_{R,j}(\chi+\eta)\, \phi'_{R,j'}(\eta)$$

$$= \sum_{j,j'} \omega_{G,j,j'} \sum_{\zeta} \Big[\sum_{\eta} \hat{f}(\xi-\zeta-\eta)\, \big(\phi'_{R,j'}(\eta)\, \hat{g}(\eta)\big) \Big] \phi_{R,j}(\xi-\zeta)\, \hat{h}(\zeta).$$

Overall cost: $\mathcal{O}(J^2 N^2 \log N)$

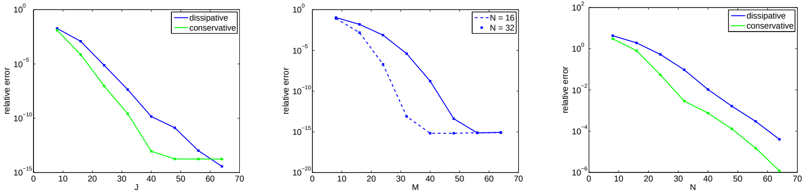
Similar approach for conservative collision operator, combined with

$$\frac{2}{\pi}\frac{\sin(\pi R \theta)}{\theta} = \int_{-R}^{R} \mathrm{e}^{i\pi\theta\rho}\, \mathrm{d}\rho \approx \sum_{m=1}^{M} \omega_m\, \mathrm{e}^{i\pi\theta\rho_m}$$

with $(\rho_m, \omega_m)_m$ the points and weights of a quadrature rule.

Hu and Ying, Commun. Math. Sci. 10, 989–999 (2012)

Spectral methods: Convergence test



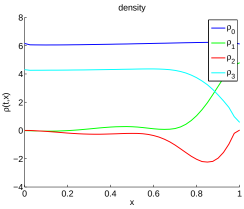
(a) convergence with J (b) convergence with M (c) convergence with N

Figure: Exponential convergence of the dissipative $\mathcal{C}_{\text{diss}}$ and conservative $\mathcal{C}_{\text{cons}}$ collision operator calculation with respect to J, M and N

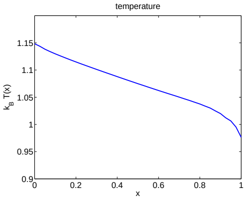
J: number of angular grid points
M: number of Gauss-Legendre quadrature points on $[-R, R]$
N: Fourier grid size in each direction

Spectral methods: Simulation example

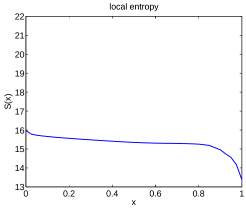
Dirichlet boundary conditions:



(a) stationary density



(b) $k_\mathrm{B} T$



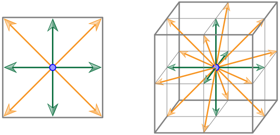
(c) local entropy

Summary and conclusions

- Quantum Boltzmann equation derived from the Hubbard model, 2×2 matrix-valued due to spin
- Thermal equilibrium (Fermi-Dirac) states are stationary,

$$W_{\mathrm{FD}}(k) = \sum_{\sigma \in \{\uparrow,\downarrow\}} \left(\mathrm{e}^{\beta(\omega(k)-\mu_\sigma)} + 1\right)^{-1} |\sigma\rangle\langle\sigma|,$$

- Numerically observe exponentially fast convergence to the predicted stationary state in 1D
- Generalization of the classical lattice Boltzmann method (LBM) to the quantum setting
- Fast evaluation of collision operator by spectral methods



D2Q9 model

D3Q19 model

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Additional conservation laws for nearest neighbor hopping

Generalized energy conservation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} \mathrm{d}k\, g(k)\ \mathrm{tr}[W(k)] = 0$$

for any $g : \mathbb{T} \to \mathbb{R}$ with $g(k) = -g(\frac{1}{2} - k)$. It follows that

$$h(k) = \mathrm{tr}[W(k)] - \mathrm{tr}[W(\tfrac{1}{2} - k)]$$

is pointwise constant for each $k \in \mathbb{T}$. The additional conservation laws render the Hamiltonian *integrable*. By adding next-nearest neighbor hopping terms, these conservation laws disappear, and Hamiltonian becomes *non-integrable*.

Stationary states for nearest neighbor hopping (1D)

For pure nearest neighbor hopping, the general stationary states are

$$W_{\mathrm{st}}(k) = \sum_{\sigma\in\{\uparrow,\downarrow\}} \left(\mathrm{e}^{f(k)-a_\sigma} + 1\right)^{-1} |\sigma\rangle\langle\sigma|, \quad \text{with}$$

$$f(k) = -f(\tfrac{1}{2} - k).$$

Adding higher order hopping terms destroys integrability, and only thermal (Fermi-Dirac) equilibrium states remain:

$$W_{\mathrm{FD}}(k) = \sum_{\sigma\in\{\uparrow,\downarrow\}} \left(\mathrm{e}^{\beta(\omega(k)-\mu_\sigma)} + 1\right)^{-1} |\sigma\rangle\langle\sigma|$$

Remark: conservation laws uniquely define stationary state, can map initial $W(k,0)$ to asymptotic $W(k,t)$ as $t \to \infty$

Map from initial $W(0)$ to W_{st} (1D)

Generalized energy conservation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}} g(k)\,\mathrm{tr}[W(k)]\,\mathrm{d}k = 0$$

for any $g : \mathbb{T} \to \mathbb{R}$ with $g(k) = -g(\frac{1}{2} - k)$. It follows that

$$h(k) = \mathrm{tr}[W(k)] - \mathrm{tr}[W(\tfrac{1}{2} - k)]$$

is pointwise constant for each $k \in \mathbb{T}$. Together with the spin conservation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d} W(k,t)\,\mathrm{d}k = 0$$

these conservation laws uniquely determine f and a_σ defining a stationary state. (The main ingredient of the proof is the Legendre transform.)