Landau damping of inhomogeneous states in the Kuramoto model

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Joint work with Bastien Fernandez and David Gérard-Varet

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Aim: Model synchronisation behaviour of oscillators

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- When does this coupling synchronise the system?

Mean-field limit

We study the mean-field (continuum) limit as $N \to \infty$:

- Describe the state by the probability density $\rho(t, \cdot, \cdot)$, i.e.

$$\rho(t, \omega, \theta)\mathrm{d}\omega\mathrm{d}\theta$$

  is the proportion of oscillators at time t with natural frequency within $[\omega, \omega + \mathrm{d}\omega]$ and phase angle within $[\theta, \theta + \mathrm{d}\theta]$.
- Evolution is given by the PDE

$$\begin{cases} \partial_t \rho(t, \theta, \omega) + \partial_\theta \left[\left(\omega + \frac{K}{2\mathrm{i}} (\eta(t)\, \mathrm{e}^{-\mathrm{i}\theta} - \overline{\eta(t)}\, \mathrm{e}^{\mathrm{i}\theta}) \right) \rho(t, \theta, \omega) \right] = 0, \\ \eta(t) = \int_{\theta=0}^{2\pi} \mathrm{e}^{\mathrm{i}\theta} \int_{\mathbb{R}} \rho(t, \theta, \omega) \mathrm{d}\omega \mathrm{d}\theta, \end{cases}$$

  where $\eta(t)$ is the order parameter.
- As kinetic equation θ is the position and ω is the velocity.

Homogeneous state

A spatial homogeneous state $\rho(\theta, \omega) = (2\pi)^{-1} g(\omega)$ is a stationary solution with order parameter $\eta = 0$.

Questions

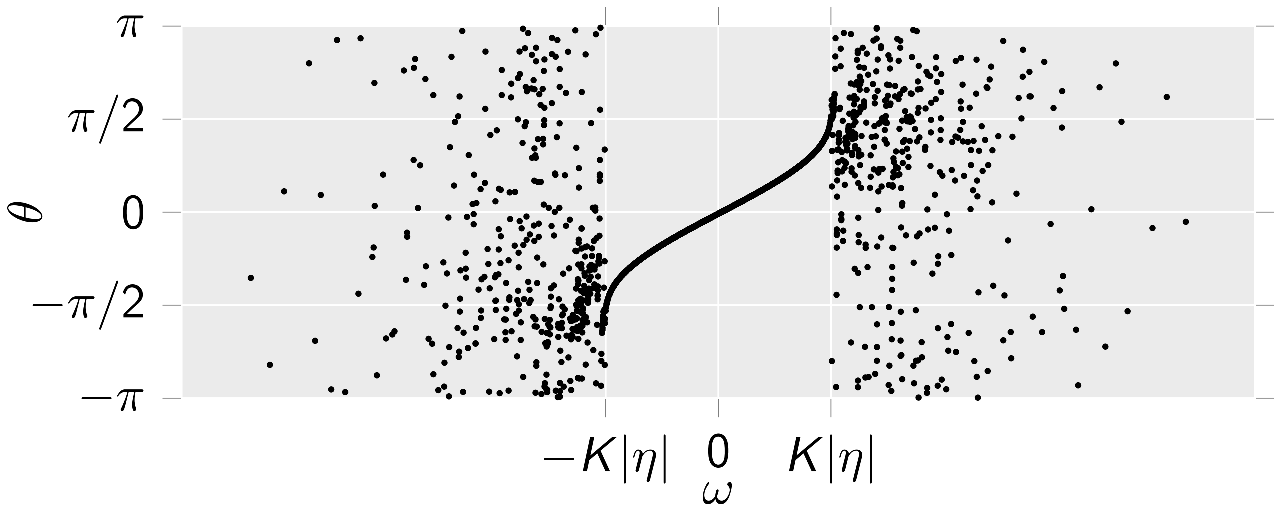
- Is it stable?
- How is the phase transition as the order parameter increases?

Inhomogeneous state

If we look at a stationary solution with order parameter $\eta \neq 0$:

- Oscillators with $|\omega| \leq K|\eta|$ are trapped
- Oscillators with $|\omega| > K|\eta|$ are moving around with varying velocity

These states are called *partially locked states* and we ask again whether they are stable.



Question

When are these states stable?

Intuitive picture

Two competing mechanisms

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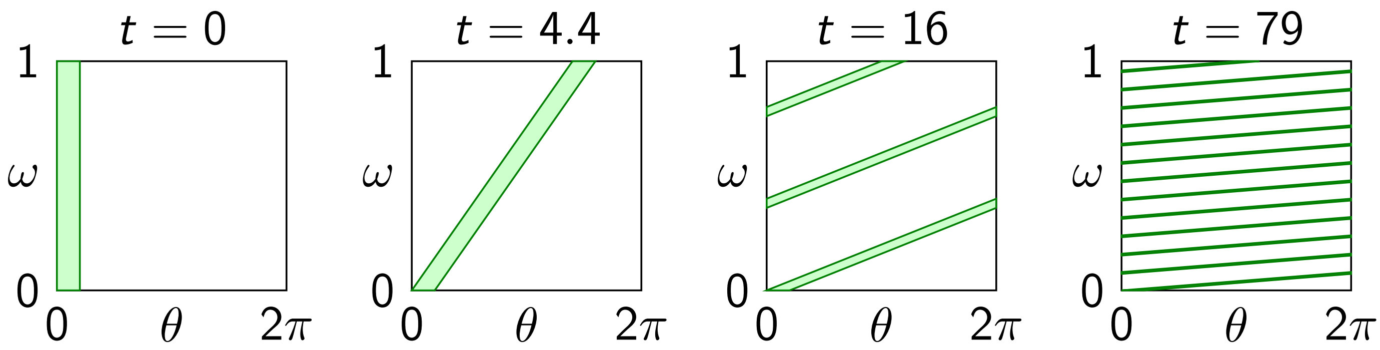
Two competing mechanisms

- Averaging through the free transport $\partial_t \rho + \partial_\theta[\omega\rho] = 0$: The heterogeneity of the natural frequencies ω mixes the distribution in phase space



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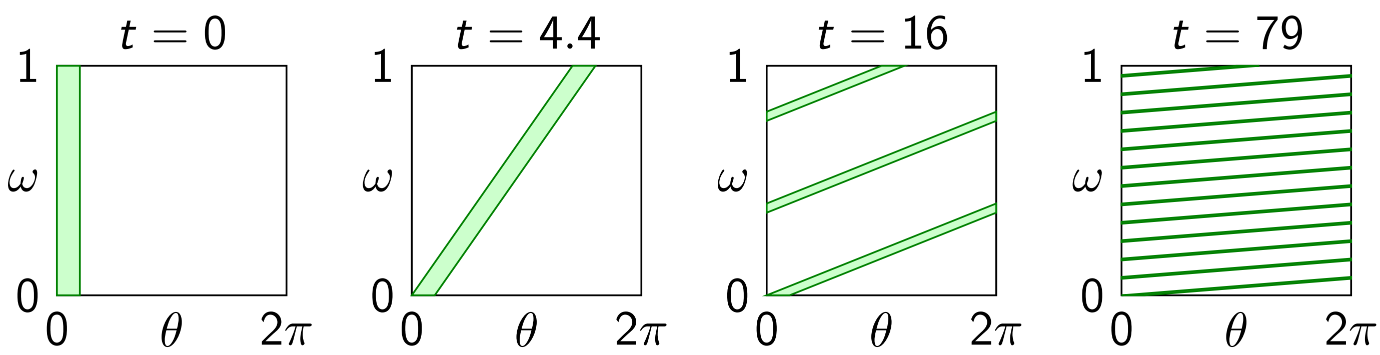

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Challenge

Find norms that capture the spreading of the free transport

Capturing Landau damping

Idea

Capture Landau damping by focusing on macroscopic quantities η.

Here η is just the order paramter. For the Vlasov–Poisson equation take the modes of the electric field.

Linearised behaviour

A perturbation u of a stationary state has the linear evolution operator $L = L_1 + L_2$ with

- L_1 is the transport operator under the stationary state.
- L_2 is a bounded operator depending only on $\eta[u]$ and models the interaction of the perturbation on the background state.

Volterra equation

By Duhamel's principle we find

$$u(t) = \mathrm{e}^{tL_1} u_{\mathrm{in}} + \int_0^t \mathrm{e}^{(t-s)L_1} L_2 u(s) \mathrm{d}s.$$

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Computing η from u gives that $\eta(t) = \eta[u(t)]$ satisfies the Volterra equation

$$\eta(t) + \int_0^t k(t-s)\eta(s)\mathrm{d}s = F(t),$$

where

- k is the interaction kernel,
- F is the forcing.

Resolvent

The solution to the Volterra equation

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The resolvent is the unique solution to

$$r = k - k * r = k - r * k.$$

Stability (Paley-Wiener, Gel'fand)

The resolvent r has the same weighted integrability as k apart from eigenmodes with eigenvalue z solving

$$(\mathcal{L}k)(z) = \int_0^\infty k(t)e^{-tz}\mathrm{d}t = -\frac{K}{2}\int_0^\infty \hat{g}(t)e^{-tz}\mathrm{d}t = -1.$$

Localise energy in Fourier

Observations

- The spatial mode $l = 0$ is the distribution of the natural frequencies and constant
- The positive modes $l \geq 1$ decouple from the negative modes $l \leq -1$

Take the Fourier transformation $\theta \to l$ and $\omega \to \xi$. The transform $\rho \to u$ evolves by

$$\partial_t u(t, 1, \xi) = \partial_\xi u(t, 1, \xi) + \frac{K}{2} \left[\eta(t)\, \hat{g}(\xi) - \overline{\eta(t)}\, u(t, 2, \xi) \right]$$

and for $l \geq 2$

$$\partial_t u(t, l, \xi) = l \partial_\xi u(t, l, \xi) + \frac{Kl}{2} \left[\eta(t)\, u(t, l-1, \xi) - \overline{\eta(t)}\, u(t, l+1, \xi) \right]$$

and the coupling is modulated by the order parameter $\eta(t) = u(t, 1, 0)$

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Global stability result of the homogeneous state

Theorem (Global stability)

Let

$$K_{ec} = \frac{2}{\int_{\xi=0}^{\infty} |\hat{g}(\xi)| \mathrm{d}\xi}.$$

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Remark: For a Gaussian distribution $K_{ec} = K_c$
Proof: Use energy functional

$$I(t) = \int_{\xi=0}^{\infty} \sum_{l \geq 1} \frac{1}{l} |u(t,l,\xi)|^2 \phi(\xi) \mathrm{d}\xi,$$

where ϕ is increasing. Under this most coupling terms vanish due to the skew-symmetry.

Linearised system of the homogeneous state

In the linear setting only the first mode is interesting:

$$\partial_t u(t,1,\xi) = \partial_\xi u(t,1,\xi) + \frac{K}{2}\eta(t)\,\hat{g}(\xi)$$

Here $\hat{g}(\xi)$ is the constant $u(0,\xi)$ function.
For $\eta(t) = u(t,1,0)$, find the Volterra equation (Duhamel's principle)

$$\eta(t) + (k * \eta)(t) = u_{in}(1,t)$$

with the convolution kernel

$$k(t) = -\frac{K}{2}\hat{g}(t)$$

Stability (Paley-Wiener)

If k is sufficiently decaying, the Volterra equation is stable iff

$$(\mathcal{L}k)(z) = \int_0^\infty k(t)e^{-tz}\mathrm{d}t = -\frac{K}{2}\int_0^\infty \hat{g}(t)e^{-tz}\mathrm{d}t \neq -1 \quad \forall \Re z \geq 0.$$

Stability of incoherent state

Linear stability

If the linear stability condition is satisfied, then we have decay as expected from the linear transport:

- If $|u(1,\xi)| \leq \mathrm{e}^{-ax}$, then η decays as e^{-at}
- If $|u(1,\xi)| \leq (1+t)^{-k}$, then η decays as $(1+t)^{-k}$

Nonlinear stability

Can propagate control in

$$\sup_{l\geq 1}\sup_{\xi\in\mathbb{R}} |u(t,l,\xi)|\mathrm{e}^{a(\xi+tl/2)}$$

and

$$\sup_{l\geq 1}\sup_{\xi\in\mathbb{R}^+} |u(t,l,\xi)|\frac{(1+\xi+t)^b}{(1+t)^{\alpha(l-1)}}.$$

Center-manifold reduction

Eigenmodes

In the case the linear stability condition is violated, we have discrete eigenmodes, while the remainder decays as the free transport.

Aim: Reduce the dynamics to these eigenmodes for

- understanding the bifurcation behaviour
- (later) handle the rotation invariance of the partially locked states

Center manifold reduction

Can reduce the dynamics to the amplitude along the eigenvector with nonlinear correction around the bifurcation.

Stability of the partially inhomogeneous states

We now study the stability of partially locked states.

- Partially locked states have a rotation symmetry and thus we cannot expect decay to the same state.

Theorem (Stability)

If a partially locked states is linearly stable, then perturbed initial data will converge to the initial data up to a possible small rotation.

Duhamel reduction

Recall the evolution equation

$$\begin{cases} \partial_t \rho(t,\theta,\omega) + \partial_\theta \left[\left(\omega + \frac{K}{2\mathrm{i}} (\eta(t)\, \mathrm{e}^{-\mathrm{i}\theta} - \overline{\eta(t)}\, \mathrm{e}^{\mathrm{i}\theta}) \right) \rho(t,\theta,\omega) \right] = 0, \\ \eta(t) = \int_{\theta=0}^{2\pi} \mathrm{e}^{\mathrm{i}\theta} \int_{\mathbb{R}} \rho(t,\theta,\omega) \mathrm{d}\omega \mathrm{d}\theta \end{cases}$$

or in Fourier

$$\partial_t u(t,l,\xi) = l \partial_\xi u(t,l,\xi) + \frac{Kl}{2} \left[\eta(t)\, u(t,l-1,\xi) - \overline{\eta(t)}\, u(t,l+1,\xi) \right].$$

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The modes are not decoupled anymore, however, the reduction to a Volterra equation still works!
In order to find a *complex* linear equation, consider η and $\overline{\eta}$ as independent. We then find a *matrix* Volterra equation

$$\begin{pmatrix} \eta \\ \overline{\eta} \end{pmatrix} + k * \begin{pmatrix} \eta \\ \overline{\eta} \end{pmatrix} = F.$$

Linear stability

The kernel $k(t)$ consists of entries like

$$\int_0^{2\pi} \int_{\mathbb{R}} \left[\mathrm{e}^{tL_1} \partial_\theta (f_{\mathrm{st}} \mathrm{e}^{\pm \mathrm{i}\theta}) \right] (\theta, \omega) \mathrm{e}^{\pm \mathrm{i}\theta} \mathrm{d}\theta \mathrm{d}\omega.$$

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Can be explicitly expressed using integrals!
Eigenmodes z at

$$\det \left[1 + (\mathcal{L}k)(z) \right] = 0.$$

Linear analysis (analytic regularity)

For the perturbation u in Fourier space show decay in norms like

$$\|u\|_{a,k} = \left(\sum_{l \in \mathbb{N}} \int_{\mathbb{R}} \mathrm{e}^{2a\xi} l^{2k} \left(|u_l(\xi)|^2 + |\partial_\xi u_l(\xi)|^2 \right) \mathrm{d}\xi \right)^{1/2}.$$

Strategy

- Split the linear evolution operator $L = L_1 + L_2$ where
  - L_1 is the transport term under a fixed external forcing (matching the free transport operator)
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Strategy

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  - L_1 is the transport term under a fixed external forcing (matching the free transport operator)
  - L_2 is the finite-rank operator corresponding to the coupling
- For L_1 replace the explicit solution formula with resolvent estimates in suitable Hilbert spaces
- For L_2 add a complexification by treating $\overline{\eta}$ as independent variable.

Nonlinear stability

Remove eigenmode from rotation symmetry

Express the solution as

$$f = R_{\Theta}(f_{\mathrm{st}} + u)$$

where R is the rotation and the angle Θ is continuously chosen such that u is in the stable subspace. Then

$$\partial_t u = Lu + P_s Q' u \text{ where } Q'u = Qu - \frac{2\Re\langle Qu, u^*\rangle_{a,0}}{1 + 2\Re\langle \mathrm{D}\hat{R}u, u^*\rangle_{a,0}} \mathrm{D}\hat{R}u.$$

Close the estimate

Using the regularisation effect of the linear evolution between $\|\cdot\|_{a,-1}$ and $\|\cdot\|_{a,0}$, we can close the estimates.

Sobolev regularity

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Solution

Adapt the splitting and perturb the Volterra equation.

Finally

Thank you for listening!