Growth and Singularity in 2D Fluids

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Euler equations in 2D

The (incompressible) Euler equations are

$$\begin{aligned} u_t + (u \cdot \nabla) u + \nabla p &= 0 \\ \nabla \cdot u &= 0 \end{aligned}$$

on $D \times (0, T)$ for some domain $D \subseteq \mathbb{R}^d$ and time $T \leq \infty$, with

$$u \cdot n = 0$$

on $\partial D \times (0, T)$ (no-flow boundary condition) and given $u(\cdot, 0)$.

In 2D, their vorticity form is the active scalar equation

$$\omega_t + u \cdot \nabla \omega = 0$$

with vorticity $\omega := \nabla \times u = -(u_1)_{x_2} + (u_2)_{x_1} \in \mathbb{R}$ and

$$u = \nabla^\perp \Delta^{-1} \omega$$

Here Δ is the Dirichlet Laplacian (no-flow boundary condition).

Growth of solutions to the 2D Euler equations

Solutions of any transport equation

$$\omega_t + u \cdot \nabla\omega = 0$$

are uniformly bounded, so blow-up might only be possible in the derivatives of ω (loss of regularity).

- Wolibner (1933) and Hölder (1933) showed that solutions remain regular, with the double-exponential bound

$$\|\nabla\omega(\cdot, t)\|_{L^\infty} \leq Ce^{e^{Ct}}$$

- Examples with unbounded (up to super-linear) growth by Yudovich (1974), Nadirashvili (1991), Denissov (2009).
- Kiselev-Šverák (2014) proved existence of solutions on a disc with double-exponential growth (on the boundary).
- Z. (2015) proved existence of at least exponential growth for $\omega(\cdot, 0) \in C^{1,1-}(\mathbb{T}^2) \cap C^\infty(\mathbb{T}^2 \setminus \{0\})$ (hence $\partial D = \emptyset$). Double-exponential growth on $\mathbb{R}^2$ and $\mathbb{T}^2$ is still open.
- Kiselev-Z. (2015) showed finite time blow-up on a domain with (two) singular points.

SQG and modified SQG equations

Double-exponential (i.e., fast) growth for the 2D Euler equations suggests that they could be critical in the sense that finite time blow-up could happen for more singular models. Particularly interesting is the surface quasi-geostrophic (SQG) equation

$$\omega_t + u \cdot \nabla \omega = 0$$
$$u = -\nabla^{\perp}(-\Delta)^{-1/2}\omega$$

It is used in atmospheric science models and was first rigorously studied by Constantin-Majda-Tabak (1994).

2D Euler and SQG are extremal members of the natural family

$$\omega_t + u \cdot \nabla \omega = 0$$
$$u = -\nabla^{\perp}(-\Delta)^{-1+\alpha}\omega$$

of modified SQG (m-SQG) equations, with parameter $\alpha \in [0, \frac{1}{2}]$.

The regularity/blow-up question remains open for all $\alpha > 0$.

Patch solutions

I will talk about the corresponding patch problem (Bertozzi, Chemin, Constantin, Córdoba, Denissov, Depauw, Gancedo, Rodrigo, Yudovich,...) on the half-plane $D = \mathbb{R} \times \mathbb{R}^+$. Here

$$\omega(\cdot, t) = \sum_{n=1}^{N} \theta_n \chi_{\Omega_n(t)}$$

with $\theta_n \in \mathbb{R} \setminus \{0\}$, and each patch $\Omega_n(t) \subseteq D$ is a bounded open set advected by $u = -\nabla^\perp(-\Delta)^{-1+\alpha}\omega$ (see later). For the half-plane D, this is (with $\bar{y} = (y_1, -y_2)$ and some $c_\alpha > 0$)

$$u(x,t) = -c_\alpha \int_D \left(\frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} - \frac{(x-\bar{y})^\perp}{|x-\bar{y}|^{2+2\alpha}} \right) \omega(y,t) dy$$

We require patch-like initial data with some regularity:

- Patches do not touch each other or themselves:
  - $\overline{\Omega_n(0)} \cap \overline{\Omega_m(0)} = \emptyset$ for $n \neq m$
  - each $\partial\Omega_n(0)$ is a simple closed curve
- All $\partial\Omega_n(0)$ have certain prescribed regularity.

Blow-up happens if one of these fails at some time $t > 0$.

Global regularity of $C^{1,\gamma}$ Euler patches on $\mathbb{R} \times \mathbb{R}^+$

Theorem (Kiselev-Ryzhik-Yao-Z., 2015)

Let $\alpha = 0$ and $\gamma \in (0, 1]$. Then for each $C^{1,\gamma}$ patch-like initial data $\omega(\cdot, 0)$, there exists a unique global $C^{1,\gamma}$ patch solution ω.

- The same whole-plane result for a single patch was proved by Chemin (1993). Our proof is motivated by an alternative approach by Bertozzi-Constantin (1993).
- Specifically, each patch boundary is the zero-level set of a function which is advected by u. The rates of change of their $C^{1,\gamma}$ norms, of their gradients on their zero-level sets, and of the distances of their zero-level sets are controlled.
- Previously Depauw (1999) proved local regularity on the half-plane (and global if patches do not touch ∂D initially).
- A result of Dutrifoy (2003) implies global existence in $C^{1,s}$ for some $s < \gamma$.

Blow-up of H^3 patches on $\mathbb{R} \times \mathbb{R}^+$ for small $\alpha > 0$

Theorem (Kiselev-Yao-Z., 2015)

Let $\alpha \in (0, \frac{1}{24})$. Then for each H^3 patch-like initial data $\omega(\cdot, 0)$, there exists a unique local H^3 patch solution ω.
Moreover, if the maximal time T_ω of existence of ω is finite, then at T_ω either two patches touch, or a patch boundary touches itself, or a patch boundary loses H^3 regularity (i.e., blow-up).

Local existence on the whole plane was proved for $\alpha \in (0, \frac{1}{2})$ by Gancedo (2008). We can prove uniqueness and the last claim.

Theorem (Kiselev-Ryzhik-Yao-Z., 2015)

Let $\alpha \in (0, \frac{1}{24})$. Then there are H^3 patch-like initial data $\omega(\cdot, 0)$ for which the solution ω blows up in finite time (i.e., $T_\omega < \infty$).

To the best of our knowledge, this is the first rigorous result proving finite time blow-up in this type of fluid dynamics models.

Definition of patch solutions

In the Euler case one usually requires that $\Phi_t : \bar{D} \to \bar{D}$ given by

$$\frac{d}{dt}\Phi_t(x) = u(\Phi_t(x), t) \qquad \text{and} \qquad \Phi_0(x) = x$$

preserves each patch: $\Phi_t(\Omega_n(0)) = \Omega_n(t)$ for each $t \in (0, T)$.
However, the map Φ_t need not be uniquely defined for $\alpha > 0$.

Definition

A patch-like (i.e., no touches of patches at any $t \in [0, T)$ plus continuity of each $\partial\Omega_n(t)$ in time w.r.t Hausdorff distance)

$$\omega(\cdot, t) = \sum_{n=1}^{N} \theta_n \chi_{\Omega_n(t)}$$

is a patch solution to m-SQG on $[0, T)$ if for each t, n we have

$$\lim_{h \to 0} \frac{d_H\Big(\partial\Omega_n(t+h), X^h_{u(\cdot,t)}[\partial\Omega_n(t)]\Big)}{h} = 0,$$

with d_H Hausdorff distance and $X^h_u[A] = \{x + hu(x) \,|\, x \in A\}$.

Properties of patch solutions

Denote $\Omega(t) = \bigcup_n \Omega_n(t)$. The definition shows that:

- $\partial\Omega(t)$ is moving with velocity $u(x,t)$ at $x \in \partial\Omega(t)$.
- Patch solutions to m-SQG are also weak solutions (and weak solutions with C^1 boundaries which move with some continuous velocity are patch solutions).
- In the Euler case it is equivalent to the definition via Φ.
- It is also essentially equivalent to the definition via Φ in the case of H^3 patch solutions to m-SQG with $\alpha < \frac{1}{4}$ [KYZ].
- In fact, $\Phi_t(x)$ is uniquely defined for $x \in \overline{D} \setminus \partial\Omega(0)$, and

$$\Phi_t : \Omega_n(0) \to \Omega_n(t) \qquad \text{and} \qquad \Phi_t : \left[\overline{D} \setminus \overline{\Omega(0)}\right] \to \left[\overline{D} \setminus \overline{\Omega(t)}\right].$$

  Also, these maps are measure preserving bijections and we have $\Phi_t(\partial\Omega_n(0)) = \partial\Omega_n(t)$ in an appropriate sense.

  - This uses that the normal component of u (w.r.t. $\partial\Omega(t)$) is Lipschitz in the normal direction if $\alpha < \frac{1}{4}$.

Local H^3 regularity: The contour equation

For simplicity assume a single patch. Parametrize $\partial\Omega(t)$ by $z(\cdot,t) \in H^3(\mathbb{T})$. Then for any $x = z(\xi,t) \in \partial\Omega(t)$ we obtain

$$u(x,t) = \frac{c_\alpha \theta}{2\alpha} \sum_{i=1}^{2} \int_{\mathbb{T}} \frac{-\partial_\xi z^i(\xi-\eta,t)}{|z(\xi,t) - z^i(\xi-\eta,t)|^{2\alpha}} d\eta$$

with

$$z^1(\xi,t) := z(\xi,t) \qquad \text{and} \qquad z^2(\xi,t) := \bar{z}(\xi,t)$$

Next add a multiple of the tangent vector $\partial_\xi z(\xi,t)$ so that the integrand becomes more regular, and get the contour equation

$$\partial_t z(\xi,t) = \frac{c_\alpha \theta}{2\alpha} \sum_{i=1}^{2} \int_{\mathbb{T}} \frac{\partial_\xi z(\xi,t) - \partial_\xi z^i(\xi-\eta,t)}{|z(\xi,t) - z^i(\xi-\eta,t)|^{2\alpha}} d\eta$$

Gancedo proves local regularity for the contour equation in $\mathbb{R}^2$ (which has only $i = 1$, and also a single patch) for any $\alpha < \frac{1}{2}$.

Local H^3 regularity: Existence of a patch solution

We prove local regularity on $D = \mathbb{R} \times \mathbb{R}^+$ for $\alpha < \frac{1}{24}$, via

$$\frac{d}{dt} \|\|z(\cdot, t)\|\| \leq C(\alpha)\theta \|\|z(\cdot, t)\|\|^8$$

where $\|\| \cdot \|\| = \|z(\cdot, t)\|_{H^3}$ + inverse Lipschitz norm of $z(\cdot, t)$ (+ distance of patches when $N \geq 2$). Quite a bit more involved...

- The method does not seem to work for Hölder norms.

Limitation on α is essentially due to insufficient bounds on the tangential velocity. Where a patch departs x_1-axis, tangential velocity generated by its reflection might deform it excessively.

- Most of the proof works for $\alpha < \frac{1}{4}$.

This local contour solution z then yields a patch solution ω.

Local H^3 regularity: Independence of parametrization

Proving uniqueness via some version of Gronwall difficult:

$$|u(x) - \tilde{u}(x)| \lesssim d_H(\partial\Omega, \partial\tilde{\Omega})^{1-2\alpha}.$$

- Gronwall does apply to $\|z - \tilde{z}\|_{L^2}$ but $z, \tilde{z}$ might not exist.

First step towards uniqueness is showing independence of the "contour" patch from parametrization of $\partial\Omega(0)$.

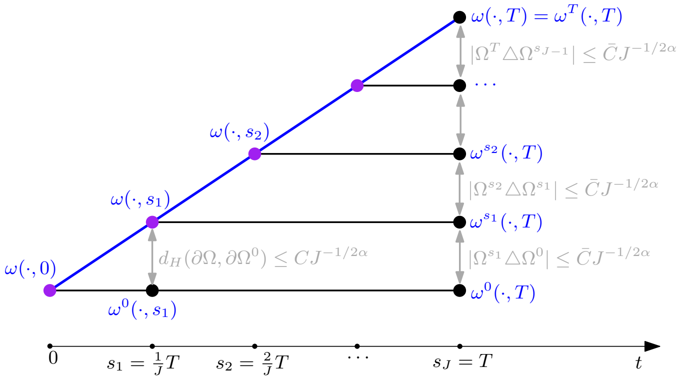
- Regularize:

$$u^\varepsilon(x,t) = -c_\alpha \int_D \left(\frac{(x-y)^\perp}{(|x-y|^2+\varepsilon^2)^{1+\alpha}} - \frac{(x-\bar{y})^\perp}{(|x-\bar{y}|^2+\varepsilon^2)^{1+\alpha}} \right) \omega(y,t)dy$$

- Show uniqueness of patch solution ω_ε (e.g., via Gronwall). Then any contour solutions $z_\varepsilon, \tilde{z}_\varepsilon$ which parametrize the same initial patch must yield the same ω_ε.
- Show $z_\varepsilon \to z$ if they have the same initial parametrization. Similarly $\tilde{z}_\varepsilon \to \tilde{z}$, hence $z, \tilde{z}$ must yield the same ω.

Local H^3 regularity: Uniqueness of the patch solution

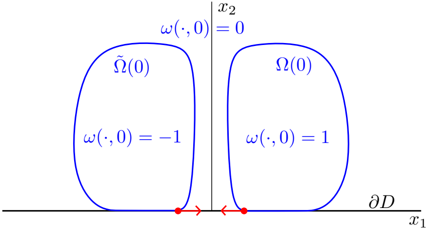
Let ω be any patch solution and ω^s the "contour" patch solution with $\omega^s(\cdot, s) = \omega(\cdot, s)$ (ω^s is unique). For small $T > 0$ and $J \in \mathbb{N}$:



Successive estimation of the rates of change of $d_H(\partial\Omega, \partial\tilde{\Omega})$ and $\|z - \tilde{z}\|_{L^2}$ and telescoping give $|\Omega(T) \triangle \Omega^0(T)| \lesssim J^{1-1/2\alpha}$. Then take $J \to \infty$ and get $\Omega = \Omega^0$ on $[0, T]$.

Finite time blow-up in H^3: Initial data and symmetry

Our initial data will be made of two patches and odd in x_1.



$$\omega(\cdot,0) = \chi_{\Omega(0)} - \chi_{\tilde{\Omega}(0)}$$

Then local uniqueness shows that before blow-up we have

$$\omega(\cdot,t) = \chi_{\Omega(t)} - \chi_{\tilde{\Omega}(t)}$$

with $\Omega(t) \subseteq D^+ = (\mathbb{R}^+)^2$ and $\tilde{y} = (-y_1, y_2)$. Then (let $c_\alpha = 1$)

$$u(x,t) = -\int_{\Omega(t)} H(x,y)dy$$

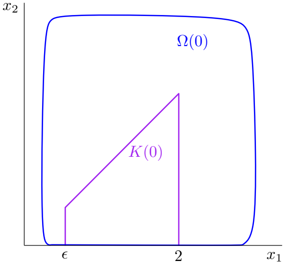
$$H(x,y) = \frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} - \frac{(x-\bar{y})^\perp}{|x-\bar{y}|^{2+2\alpha}} - \frac{(x-\tilde{y})^\perp}{|x-\tilde{y}|^{2+2\alpha}} + \frac{(x+y)^\perp}{|x+y|^{2+2\alpha}}$$

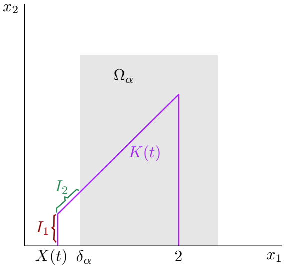
Finite time blow-up in H^3: A barrier argument

Goal: show that if $\Omega(0) \supseteq [\varepsilon, 3] \times [0, 3]$ and $\varepsilon > 0$ is small, then

$$\Omega(t) \supseteq K(t) = \{X(t) < x_1 < 2\} \cap \{0 < x_2 < x_1\}$$

until blow-up, where $X(0) = \varepsilon$ and $X'(t) = -\frac{1}{100\alpha} X(t)^{1-2\alpha}$.
This gives blow-up because $X(50\varepsilon^{2\alpha}) = 0$.





If $t < 50\varepsilon^{2\alpha}$ is the first time with $\overline{D^+ \setminus \Omega(t)} \cap \overline{K(t)} \neq \emptyset$, then by

$$\|u\|_{L^\infty} \leq C_1 \|\omega(\cdot, 0)\|_{L^\infty} + C_2 \|\omega(\cdot, 0)\|_{L^1} \leq C$$

the touch can only be on $I_1 \cup I_2$ (since $\Omega(t) \supseteq \Omega_\alpha$ by ε small).
Also uses that the patch cannot separate from the x_1-axis...

Finite time blow-up in H^3: Estimates on the flow

We have $u_1(x,t) = -\int_{\Omega(t)} H_1(x,y)dy$, where

$$H_1(x,y) = \frac{y_2 - x_2}{|x-y|^{2+2\alpha}} - \frac{y_2 - x_2}{|x-\tilde{y}|^{2+2\alpha}} + \frac{y_2 + x_2}{|x-\bar{y}|^{2+2\alpha}} - \frac{y_2 + x_2}{|x+y|^{2+2\alpha}}$$

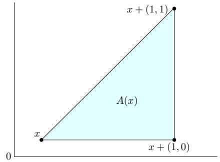
Then $|x-\bar{y}| \le |x+y|$ on $\Omega(t) \subseteq D^+$ gives

$$u_1(x,t) \le -\int_{\Omega(t)} \underbrace{\left(\frac{y_2 - x_2}{|x-y|^{2+2\alpha}} - \frac{y_2 - x_2}{|x-\tilde{y}|^{2+2\alpha}}\right)}_{G(x,y)} dy$$

From $K(t) \subseteq \Omega(t)$ we have for $x \in K(t) \cap \{x_1 \le 1\}$

$$u_1(x,t) \le \int_{\mathbb{R}\times(0,x_2)} |G(x,y)|dy - \int_{A(x)} G(x,y)dy$$

because $\mathrm{sgn}(G(x,y)) = \mathrm{sgn}(y_2 - x_2)$.



Small α is crucial for $A(x)$ to compensate limited control near x.
Blow-up may be easier to prove in slightly super-critical models.

Finite time blow-up in H^3: Conclusion of the proof

A computation and cancellations yield for $x_2 \leq x_1 \leq \delta_\alpha$ (> 0)

$$\int_{\mathbb{R}\times(0,x_2)} |G(x,y)|dy \leq \frac{1}{\alpha}\left(\frac{1}{1-2\alpha} - 2^{-\alpha}\right) x_1^{1-2\alpha}$$

$$-\int_{A(x)} G(x,y)dy \leq -\frac{1}{\alpha}\left(\frac{1}{6\cdot 20^\alpha}\right) x_1^{1-2\alpha}$$

and we get for small α and $x \in I_1 \cup I_2$ (using $x_1 \geq X(t)$)

$$u_1(x,t) \leq -\frac{1}{50\alpha} x_1^{1-2\alpha} < -\frac{1}{100\alpha} X(t)^{1-2\alpha} = X'(t)$$

So touch cannot happen on I_1.

Similarly, for small α and $x \in I_2$

$$u_2(x,t) \geq \frac{1}{50\alpha} x_2^{1-2\alpha} > 0$$

so touch cannot happen on I_2.