Recent Results for the 3D Quasi-Geostrophic System

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3D Quasi-Geostrophic Flow

Physical Model

- QG - a model for large time-scale, rotating oceanic/atmospheric flows
- Derivation from Navier-Stokes/Euler equations with Boussinesq approximation and Coriolis force. See Bourgeois-Beale (94), Desjardins-Grenier (98)
- The Rossby number and the geostrophic balance - wind velocity is orthogonal to the gradient of the pressure in the asymptotic limit

The Equations

- $\Psi(t,x,y,z) : [0,T] \times \Omega \times [0,\infty) \to \mathbb{R} \qquad (\Omega \subset \mathbb{R}^2)$
- The velocity $(u,v,0)$ is stratified and verifies

$$(u,v,0) = (-\partial_y \Psi, \partial_x \Psi, 0) = \overline{\nabla}^{\perp} \Psi.$$

- Notations -

$$\overline{\nabla} = (\partial_x, \partial_y, 0), \quad \partial_\nu = -\partial_z|_{z=0}, \quad \overline{\Delta} = \partial_{xx} + \partial_{yy}.$$

- Viscosity parameter $r \in \{0,1\}$ - inviscid model / viscous model

$$\begin{aligned}
&(\partial_t + \overline{\nabla}^{\perp}\Psi \cdot \overline{\nabla})(\Delta\Psi) = 0 && [0,T] \times \Omega \times (0,\infty) \\
&(\partial_t + \overline{\nabla}^{\perp}\Psi \cdot \overline{\nabla})(\partial_\nu\Psi) = r\overline{\Delta}\Psi && [0,T] \times \Omega \times \{z=0\} \\
&\Psi(0,x,y,z) = \Psi^0 && t = 0.
\end{aligned}$$

Main Results

Theorem (N., '17)

Choose an initial value $\nabla\Psi^0$ *with* $\Delta\Psi_0 \in L^q(\mathbb{R}^3_+)$*for* $q \in (\frac{6}{5}, 3]$, $\partial_\nu\Psi^0 \in L^p(\mathbb{R}^2)$ *for* $p \in (\frac{4}{3}, \infty]$. *Then there exists a global in time weak solution such that* $\nabla\Psi \in L^\infty_t(L^{\frac{3p}{2}} + L^{\frac{3q}{3-q}}(\mathbb{R}^3_+))$.

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- Challenge is for small p and small q - how to define $\overline{\nabla}^\perp\Psi \cdot \overline{\nabla}(\partial_\nu\Psi)$ and $\overline{\nabla}^\perp\Psi \cdot \overline{\nabla}(\Delta\Psi)$?

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- Need the right notion of "weak" solution

Further Properties of Weak Solutions

2D SQG - A simplified model

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$$\partial_t(\partial_\nu \Psi) + \overline{\nabla}^\perp \Psi \cdot \overline{\nabla}(\partial_\nu \Psi) = 0 \quad \Rightarrow \quad \partial_t \theta + u \cdot \overline{\nabla}\theta = 0$$

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Theorem (N., 17)

1. *When $\Delta\Psi = 0$, weak solutions to SQG are "weak solutions" to 3D QG and vice versa*
2. *Under appropriate assumptions on p and q, "weak solutions" to 3D QG satisfy the transport equations in the usual weak sense.*

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Theorem (N., 17)

When $\nabla\Psi \in C\left([0,T);L^2(\mathbb{R}^3_+)\right) \cap L^\infty\left([0,T)\times[0,\infty);\mathring{B}^\alpha_{3,\infty}(\mathbb{R}^2)\right)$ *for* $\alpha > \frac{1}{3}$,

$$\frac{\partial}{\partial t}\|\nabla\Psi(t)\|_{L^2(\mathbb{R}^3_+)} = 0$$

The Inviscid Case for Bounded Domains

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Theorem (N.-Vasseur, '18)

The natural lateral boundary conditions are

- $\Psi(t, x, y, z)|_{\partial\Omega \times [0,\infty)} = c(t, z)$
- $\frac{\partial}{\partial t} \int_{\partial\Omega \times \{z\}} \overline{\nabla}\Psi(z) \cdot \nu_s = 0$

With these boundary conditions, there exists a global weak solutions to inviscid QG posed on $[0, \infty) \times \Omega \times [0, \infty)$.

The Case with Dissipation

Theorem (N.-Vasseur, ('17))

Consider dissipative (QG) (diffusive term $\overline{\Delta}\Psi$ *at* $z = 0$*) supplemented with an initial value* $\nabla\Psi^0 \in H^s(\mathbb{R}^3_+)$ *with* $s \geq 3$*. Then there exists a unique, global in time solution* $\nabla\Psi$ *such that for every* $T > 0$*,*
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- In particular, if the initial data is smooth (C^∞), the unique solution is smooth
- Pure transport allows for propagation of regularity but no smoothing

Inviscid Models

A Priori Estimates

$$\begin{aligned} &(\partial_t + \overline{\nabla}^\perp \Psi \cdot \overline{\nabla})(\Delta \Psi) = 0 && [0,T] \times \Omega \times (0,\infty) \\ &(\partial_t + \overline{\nabla}^\perp \Psi \cdot \overline{\nabla})(\partial_\nu \Psi) = 0 && [0,T] \times \Omega \times \{z=0\} \\ &\Psi(0,x,y,z) = \Psi^0 && t = 0. \end{aligned}$$

- For any $p \in [1,\infty]$ and $q \in [1,\infty]$, integrating by parts and using the divergence free property yields

$$\|\boldsymbol{\Delta\Psi}(t)\|_{L^p(\Omega\times(0,\infty))} \leq \|\boldsymbol{\Delta\Psi}^0\|_{L^p(\Omega\times(0,\infty))}$$

$$\|\partial_\nu \boldsymbol{\Psi}(t)\|_{L^q(\Omega)} \leq \|\boldsymbol{\Delta\Psi}^0\|_{L^q(\Omega\times(0,\infty))}$$

- Lack of compactness at $z = 0$ - no strong convergence for $\overline{\nabla}^\perp \boldsymbol{\Psi}|_{z=0}$ or $\partial_\nu \boldsymbol{\Psi}$

The Reformulated Problem

$$
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$$(\partial_t + \overline{\nabla}^{\perp}\Psi \cdot \overline{\nabla})(\nabla\Psi) = \nabla \times (Q)$$
- Weak solutions are defined for $\nabla\Psi$ - compactness available

Boundary Conditions When $\Omega \neq \mathbb{R}^2$

- Back to inviscid SQG - $\partial_\nu \Psi = \theta = (-\overline{\Delta})^{\frac{1}{2}} \Psi$, $u = \overline{\nabla}^{\perp} \Psi = \mathcal{R}^{\perp} \theta$

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  - $\frac{\partial}{\partial t}\int_{\partial\Omega\times\{z\}} \overline{\nabla}\Psi(z)\cdot\nu = 0$
- Our solutions do not coincide with those of Constantin-Nguyen

Viscous Model

Regularity for 2D Critical SQG

- Critical SQG - $\partial_\nu \Psi = \theta = (-\overline{\Delta})^{\frac{1}{2}}\Psi$, $u = \overline{\nabla}^\perp \Psi = \mathcal{R}^\perp \theta$, $\overline{\Delta}\Psi = -(-\overline{\Delta})^{\frac{1}{2}}\theta$

$$\partial_t \theta + u \cdot \overline{\nabla}\theta + (-\overline{\Delta})^{\frac{1}{2}}\theta = 0$$

- Global regularity for L^2 initial data established by Caffarelli-Vasseur ('10). Several other proofs by Kiselev-Nazarov-Volberg, Constantin-Vicol, Constantin-Vicol-Tarfulea

Difficulties in 3 Dimensions

- The transport equation for $\Delta\Psi$ is hyperbolic - no regularization
- Beale-Kato-Majda criterion is necessary ($\overline{\nabla}^{\perp}\Psi$ is a log-Lipschitz velocity field)
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- Showing that $\theta \in L_t^\infty\left(\mathring{B}^1_{\infty,\infty}\right)$ requires a combination of De Giorgi, potential theory, Littlewood-Paley techniques

Ongoing Work and Future Directions

Ongoing Work

Theorem (N.)

Let $\alpha < \frac{1}{5}$. Then weak solutions to inviscid QG on the torus $\mathbb{T}^3$ in the class $C^{\alpha}_{t,x}$ are not unique and may dissipate energy.

- Recall energy is conserved when $\alpha > \frac{1}{3}$ (N., '17). This is referred to as rigidity. Conversely, when $\alpha < \frac{1}{5}$, this theorem demonstrates flexibility.

Future Directions

- Smooth solutions to the inviscid model on bounded domains and the validity of our boundary conditions
- Blow-up on bounded domains?
- Non-uniqueness in other regularity classes

Thank you

Thanks for your attention!