Log-log blow up solutions at m points

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Model

Focusing mass critical nonlinear Schrodinger equations.

$$\begin{cases} iu_t + \Delta u = -|u|^{4/d}u, \\ u(0) = u_0, \end{cases} \tag{1}$$

We will focus on the case $d = 2$.
Physical interest: many body quantum system; optical collapsing.
For mathematicians: an important model for dispersive equations.

Conservation law

- **Mass**:

$$M(u(t,x)) := \int |u(t,x)|^2 dx, \tag{2}$$

- **Energy**:

$$E(u(t,x)) := \frac{1}{2}\int |\nabla u(t,x)|^2 dx - \frac{1}{2+\frac{4}{d}}\int |u(t,x)|^{2+\frac{4}{d}} dx, \tag{3}$$

- Momentum:

$$P(u(t,x)) := \Im(\int \nabla u(t,x)\overline{u(t,x)})dx, \tag{4}$$

Symmetry

- Space-time translation: $u(t,x) \Longrightarrow u(t-t_0, x-x_0)$
- Phase transformation: $u(t,x) \Longrightarrow u(t,x)e^{i\theta_0}$
- Galilean transformation: If $u(t,x) \Longrightarrow u(t, x-\beta t)e^{i\frac{\beta}{2}(x-\frac{\beta}{2}t)}$
- **Scaling**: $u(t,x) \longrightarrow u_\lambda(t,x) := \frac{1}{\lambda^{\frac{d}{2}}}u(\frac{t}{\lambda^2}, \frac{x}{\lambda})$
- Pseudo-conformal transformation:
  $u(t,x) \Longrightarrow \frac{1}{t^{\frac{d}{2}}}\bar{u}(\frac{1}{t}, \frac{x}{t})e^{i\frac{|x|^2}{4t}}$

Basic questions

- Local theory : classical unless you want to work on (**very**) low regularity initial data.
- Is the solution global?
- If the solution is global, what is the asymptotic behavior in long time ?
- If the solution is not global (i.e. the solution blows up in finite time), what is the mechanism for singularity formation?

Defocusing vs Focusing

- Defocusing problem finally turns out to be linear. (Dodson)
- Easy to see nonlinear objects for focusing problem (solitary wave Qe^{it}, Q is the ground state).
- Exists blow up dynamic for focusing problem
  - The dynamic is very rich
  - Very few can be said for general data

Blow up solution exists!

There are a lot of blow up solutions for focusing problem.
Existence of blow up solutions: Virial identity (Glassey)

$$\partial_t^2 \int |x|^2|u|^2 = 4\partial_t \Im\left(\int x\nabla u\bar{u}\right) = 16E(u_0). \tag{5}$$

- Good, but has limit, does not directly explain the mechanism of singularity formation.
- Still, almost every long time results for mass critical problem somehow relies on this.

Explicit blow up solutions

Explicit blow up solution given by pesudocomformal symmetry

$$S(t,x) := \frac{1}{|t|^{\frac{d}{2}}} Q(\frac{x}{t}) e^{-\frac{i|x|^2}{4t} + \frac{i}{t}}. \tag{6}$$

- minimal mass blow up solution
- no blow up below $\|Q\|_2$, (Weinstein, Dodson)
- the only minimal mass blow up (Merle)
- the blow up rate is $1/t$
- btw,$E(S) > 0$, cannot be understood as an example of Virial Identiy

Log-log blow up solutions

There are solutions with different blow up rates.

$$\begin{aligned} u(t,x) &\sim \frac{1}{\lambda^{d/2}(t)}(Q+\epsilon)\left(\frac{x-x(t)}{\lambda(t)}\right)e^{i\gamma(t)}, \\ \frac{1}{\lambda(t)} &\sim \sqrt{\frac{\ln\ln(T-t)}{T-t}} \end{aligned} \tag{7}$$

Landman, Papanocolaou, Sulem, Sulem
Perelman
Merle and Raphael, and many others.

Solutions blow up at more than one point

In 1990, Merle constructed k points blow up solutions.

$$u(t,x) \sim \sum_{i=1}^{k} S\left(\frac{t-T}{\lambda_i^2}, \frac{x-x_i}{\lambda_i}\right), \lambda_i > 0, x_i \neq x_j, \forall i \neq j. \quad (8)$$

- Note infinity speed of propagation of the system, unlike wave.
- Two difficulty: decouple the bubbles and prescribe the blow up points.
- Idea: integration from infinity, relying on infinity decay.
- This idea later have been applied to construct blow up solution on torus (note no Virial identity on torus.)

Log-log blow up solution at *m* points

Our theorem in words.

Theorem (Fan)

We construct log-log blow up solutions blow up at m prescribed points.

This should be understood as an analogue of Merle's result. The difference is that we use log-log blow up solutions rather than minimal mass blow up solutions as basic building blocks.

Log-log blow up solution at m points

Precise statement.

Theorem (Fan)

For $d = 1, 2$, for each positive integer m, and given any m different points $x_{1,\infty}, ...x_{m,\infty}$ in $\mathbb{R}^d$, there exists a solution u blows up in finite time T, and for t close enough to T,

$$u(t,x) = \sum_{j=1}^{m} \frac{1}{\lambda_j^{\frac{d}{2}}(t)} Q(\frac{x - x_{j,\infty}}{\lambda_j(t)}) e^{-i\gamma_j(t)} + \Xi(t,x). \tag{9}$$

where, for $j = 1, \dots, m$,

$$\frac{1}{\lambda_j(t)} \sim \sqrt{\frac{\ln|\ln T - t|}{T - t}}, \quad \text{and} \quad \lambda_j \|\Xi(t)\|_{H^1} \xrightarrow{t \to T} 0, \tag{10}$$

A few remarks

- Since the m given points are arbitrary, the solutions do not necessarily have any symmetry restriction.
- The construction also works on $\mathbb{T}^d$, $d = 1, 2$.
- The construction gives example of log-log blow up solutions with large mass.
- The construction in 1d essentially gives examples of the so-called multiple-standing ring blow up solution.

More remarks

- We evolve data forward rather than integrate from infinity, due to the remainder term Ξ.
- And the remainder term cannot vanish asymptotically in L^2 due to blow up mechanism of log-log blow up.
- The point of the Theorem is not only construct solution blow up simultaneously according to the log-log law, but also prescribe the blow up point.
- The current construction needs the data to be very smooth.

General strategy

- Modulation analysis: reduce the PDE problem into the evolution of the parameters
- Use conservation law to control interaction
- Idea behind: Elliptic PDEs and ODEs are well studied, we want use them to analyze dispersive system

Difficulties and solutions

- decouple different bubbles —- use extra regularity
- balance different bubbles to make them blow up simultaneously
- And furthermore, prescribe the blow up points—- use soft topological argument

Decouple the bubbles

- Log-log blow up solutions propagate out partial regularity, (Raphael-Stzeftel, see also Zwiers, Holmer-Roudenko)
- Log-log blow up solutions finally falls into some trapping set, (Merle-Raphael)
- A good bootstrap structure for the evolution of data in the trapping set, (Raphael-Planchon)

Formal loop argument

- Prepare *m* separate bubbles
- If those *m* bubbles does not see each other, they blow up according to log-log law
- For each bubble, since it blows up according to log-log law, it propagate out partial regularity
- Thus, the interaction between different bubbles are not strong enough to change the log-log dynamic
- Thus, those *m* bubble "does not see each other", they blow up according to log-log law

Balance different bubbles

- It is known topological argument can help, (Merle; Raphael-Planchon; Cote-Martel-Merle)
- All topological argument finally reduce to homology group of S^n, which is well-known
- Different problems needs different topological argument
- Making bubbles blow up simultaneously are relatively easy
- Prescribing blow up points are more tricky, relies on sharp dynamic of log-log blow up

Thanks!