On numerical approximation algorithms for high-dimensional nonlinear PDEs, nonlinear SDEs, and high-dimensional nonlinear FBSDEs

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Joint works with

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Introduction

Consider $T > 0$, $d \in \mathbb{N}$, $\xi \in \mathbb{R}^d$ and sufficiently regular $f\colon \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $g\colon \mathbb{R}^d \to \mathbb{R}$, $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$, $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$, $u\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}$ such that $u(T,x) = g(x)$ and

$$\tfrac{\partial}{\partial t}u(t,x) + f\big(x, u(t,x), (\nabla_x u)(t,x)\big) + \langle \mu(x), (\nabla_x u)(t,x)\rangle_{\mathbb{R}^d} + \tfrac{1}{2}\operatorname{Trace}_{\mathbb{R}^d}\big(\sigma(x)\,[\sigma(x)]^*\,(\operatorname{Hess}_x u)(t,x)\big) = 0.$$

for $(t,x) \in [0,T) \times \mathbb{R}^d$. **Goal:** Compute $u(0,\xi)$ approximately.

Application: Pricing of financial derivatives

Approximations methods such as finite element methods, finite differences, sparse grids suffer under the curse of dimensionality.

Consider probability space $(\Omega, \mathcal{F}, \mathbb{P})$, Brownian motion $W\colon [0,T] \times \Omega \to \mathbb{R}^d$, and for every $s \in [0,T]$, $x \in \mathbb{R}^d$ a solution process $X^{s,x}\colon [s,T] \times \Omega \to \mathbb{R}^d$ of

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$$u(s,x) = \mathbb{E}\big[g(X^{s,x}_T)\big] + \int_s^T \mathbb{E}\big[f\big(t, X^{s,x}_t, u(t,X^{s,x}_t), (\nabla_x u)(t, X^{s,x}_t)\big)\big]\,dt.$$

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Linear pricing models

$f = 0$

- **Black-Scholes model** Consider $T, \beta > 0$, $\alpha \in \mathbb{R}$ and

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for $t \in [0, T]$, where $(W_t)_{t \in [0,T]}$ is a one-dimensional Brownian motion.

- **Heston model** Consider $\alpha, \gamma \in \mathbb{R}$, $\beta, \delta, X_0^{(1)}, X_0^{(2)} > 0$, $\rho \in [-1, 1]$ and

$$\frac{\partial}{\partial t} X_t^{(1)} = \alpha\, X_t^{(1)} + \sqrt{X_t^{(2)}}\, X_t^{(1)}\, \frac{\partial}{\partial t} W_t^{(1)}$$

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Theorem (Hairer, Hutzenthaler, & J 2015 AOP)

Let $T \in (0,\infty)$, $d \in \{4,5,\dots\}$, $\xi \in \mathbb{R}^d$. Then there exist globally bounded $\mu, \sigma \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$ such that for every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every Brownian motion $W \colon [0,T] \times \Omega \to \mathbb{R}$, every solution $X \colon [0,T] \times \Omega \to \mathbb{R}^d$ of

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every $Y^N \colon \{0,1,\dots,N\} \times \Omega \to \mathbb{R}^4$, $N \in \mathbb{N}$, with $\forall\, N \in \mathbb{N}, n \in \{0,1,\dots,N-1\} \colon\ Y^N_0 = X_0$ and

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(Euler-Maruyama approximations), and every $\alpha \in [0,\infty)$ we have

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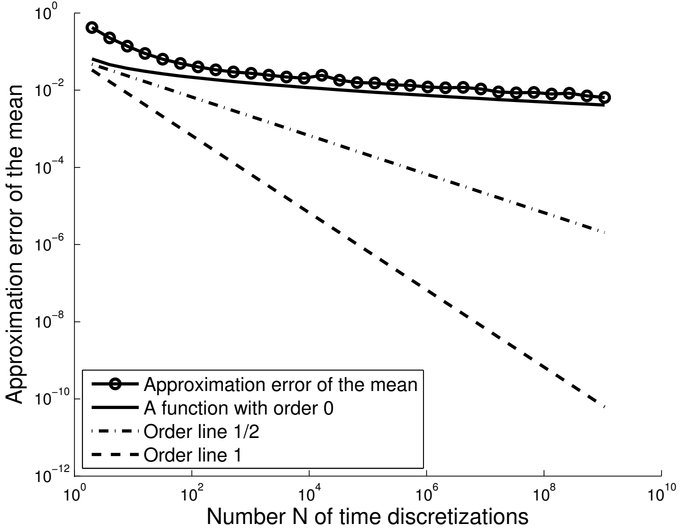
Plot of $\left\| \mathbb{E}\left[X_T\right] - \mathbb{E}\left[Y_N^N\right] \right\|$ for $T = 2$ and $N \in \{2^1, 2^2, \ldots, 2^{30}\}$.



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$$\inf_{\substack{u \colon \mathbb{R}^N \to \mathbb{R} \\ \text{measurable}}} \mathbb{E}\Big[\Big|X_T - u\big(W_{\frac{T}{N}}, W_{\frac{2T}{N}}, \ldots, W_T\big)\Big|\Big] \geq c \cdot N^{-\min\{1, \frac{2\delta}{\beta^2}\}}. \tag{*}$$

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Theorem (Hefter & J 2017)

Let $T, \delta, \beta \in (0, \infty)$, $\gamma, \xi \in [0, \infty)$, *let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space, let* $W\colon [0, T] \times \Omega \to \mathbb{R}$ *be a Brownian motion, let* $X\colon [0, T] \times \Omega \to \mathbb{R}$ *be a solution of*

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Nonlinear pricing models

$f \neq 0$

Assume $\forall\, x \in \mathbb{R}^d\colon\ \mu(x) = 0, \sigma(x) = \mathrm{Id}_{\mathbb{R}^d}$, assume $f\colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, let $\Theta = \cup_{n\in\mathbb{N}}\mathbb{R}^n$, let $W^\theta\colon [0,T]\times\Omega \to \mathbb{R}^d$, $\theta \in \Theta$, be independent Brownian motions, define $\Delta W^\theta_{s,t} = W^\theta_t - W^\theta_s$ and note $\forall\, s \in [0,T), x \in \mathbb{R}^d$:

$$u(s,x) = g(x) + \mathbb{E}\Big[\Big(g(x + \Delta W^0_{s,T}) - g(x)\Big)\Big] + \int_s^T \mathbb{E}\big[f\big(x + \Delta W^0_{s,t}, u(t, x + \Delta W^0_{s,t})\big)\big]\, dt\,.$$

Full history recursive multilevel Picard approximations For all $\theta \in \Theta$, $k, \rho \in \mathbb{N}$, $s \in [0,T)$, $x \in \mathbb{R}^d$ define $\mathbf{U}^\theta_{0,\rho,s}(x) = 0$ and

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$$+ \sum_{l=0}^{k-1} \sum_{i=1}^{m_{k-l,\rho}} \sum_{t\in(s,T]} \frac{q^{k-l,\rho}_s(t)}{m_{k-l,\rho}} \Big[f\Big(x + \Delta W^{(\theta,l,i)}_{s,t}, \mathbf{U}^{(\theta,l,i,t)}_{l,\rho,t}\big(x + \Delta W^{(\theta,l,i)}_{s,t}\big)\Big)$$
$$- \mathbb{1}_{\mathbb{N}}(l)\, f\Big(x + \Delta W^{(\theta,l,i)}_{s,t}, \mathbf{U}^{(\theta,-l,i,t)}_{[l-1]^+,\rho,t}\big(x + \Delta W^{(\theta,l,i)}_{s,t}\big)\Big)\Big].$$

Assume $\forall\, x \in \mathbb{R}^d\colon \mu(x) = 0, \sigma(x) = \mathrm{Id}_{\mathbb{R}^d}$, assume $f\colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, let $\Theta = \cup_{n\in\mathbb{N}} \mathbb{R}^n$, let $W^\theta\colon [0,T]\times\Omega \to \mathbb{R}^d$, $\theta\in\Theta$, be independent Brownian motions, define $\Delta W^\theta_{s,t} = W^\theta_t - W^\theta_s$ and note $\forall\, s\in[0,T), x\in\mathbb{R}^d\colon$

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$$\mathbf{U}^\theta_{k,\rho,s}(x) = g(x) + \sum_{i=1}^{m_{k,\rho}} \frac{g\big(x + \Delta W^{(\theta,0,-i)}_{s,T}\big) - g(x)}{m_{k,\rho}} + \sum_{l=0}^{k-1} \sum_{i=1}^{m_{k-l,\rho}} \sum_{t\in(s,T]} \frac{q^{k-l,\rho}_s(t)}{m_{k-l,\rho}} \Big[f\Big(x + \Delta W^{(\theta,l,i)}_{s,t}, \mathbf{U}^{(\theta,l,i,t)}_{l,\rho,t}\big(x + \Delta W^{(\theta,l,i)}_{s,t}\big)\Big) - \mathbb{1}_{\mathbb{N}}(l)\, f\Big(x + \Delta W^{(\theta,l,i)}_{s,t}, \mathbf{U}^{(\theta,-l,i,t)}_{[l-1]^+,\rho,t}\big(x + \Delta W^{(\theta,l,i)}_{s,t}\big)\Big)\Big].$$

Assume $\forall\, x \in \mathbb{R}^d\colon \mu(x) = 0, \sigma(x) = \mathrm{Id}_{\mathbb{R}^d}$, assume $f\colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, let $\Theta = \cup_{n\in\mathbb{N}}\mathbb{R}^n$, let $W^\theta\colon [0,T] \times \Omega \to \mathbb{R}^d$, $\theta \in \Theta$, be independent Brownian motions, define $\Delta W^\theta_{s,t} = W^\theta_t - W^\theta_s$ and note $\forall\, s \in [0,T), x \in \mathbb{R}^d$:

$$u(s,x) = g(x) + \mathbb{E}\Big[\Big(g(x + \Delta W^0_{s,T}) - g(x)\Big)\Big] + \int_s^T \mathbb{E}\big[f\big(x + \Delta W^0_{s,t}, u(t, x + \Delta W^0_{s,t})\big)\big]\, dt\,.$$

Full history recursive multilevel Picard approximations For all $\theta \in \Theta$, $k, \rho \in \mathbb{N}$, $s \in [0,T)$, $x \in \mathbb{R}^d$ define $\mathbf{U}^\theta_{0,\rho,s}(x) = 0$ and

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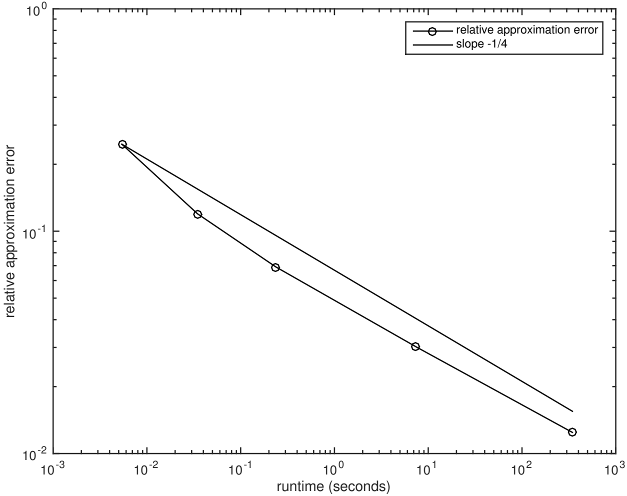
Allen-Cahn equation $T = 1, \xi = 0 \in \mathbb{R}^1$, $u(T, x) = \frac{1}{1+\|x\|_\infty}$, and

$$\frac{\partial}{\partial t}u(t,x) + u(t,x) - [u(t,x)]^3 + \frac{1}{2}(\Delta_x u)(t,x) = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^1.$$

Relative errors $\frac{1}{10|\mathrm{v}|}\sum_{i=1}^{10} |\mathbf{U}^i_{\rho,\rho}(0,\xi) - \mathrm{v}|$ for $\rho \in \{1, 2, \ldots, 5\}$ against runtime; $u(0,\xi) \approx \mathrm{v} = 0.905$. Simulations: MATLAB, Intel i7 CPU, 2.8 GHz, 16 GB RAM.



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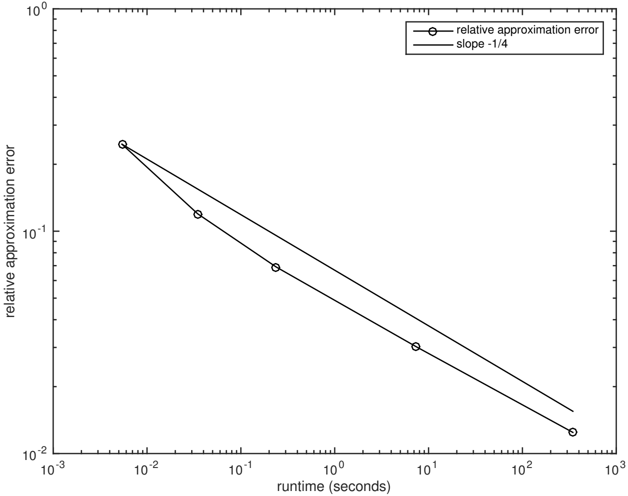

Allen-Cahn equation $T = 1$, $\xi = 0 \in \mathbb{R}^1$, $u(T,x) = \frac{1}{1+\|x\|_\infty}$, and

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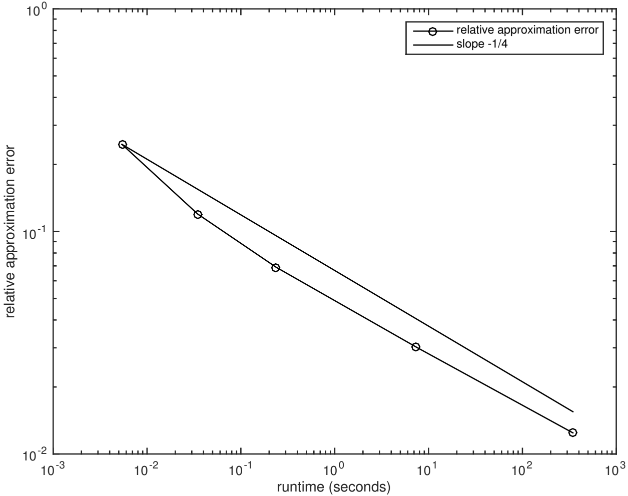
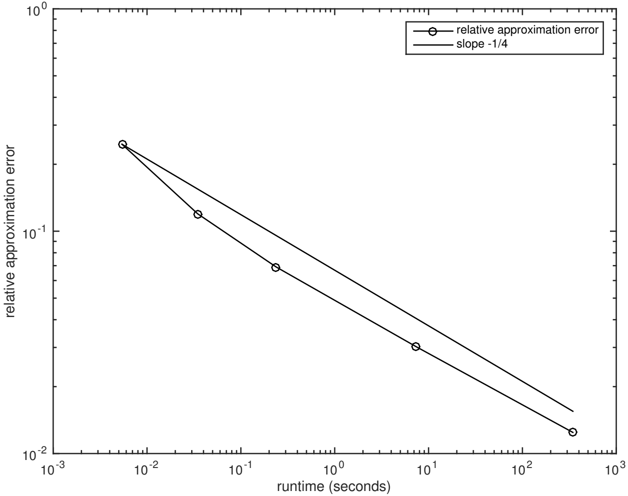
Relative errors $\frac{1}{10|\mathrm{v}|}\sum_{i=1}^{10}|\mathbf{U}^i_{\rho,\rho}(0,\xi)-\mathrm{v}|$ for $\rho \in \{1,2,\ldots,5\}$ against runtime; $u(0,\xi) \approx \mathrm{v} = 0.905$. Simulations: MATLAB, Intel i7 CPU, 2.8 GHz, 16 GB RAM.



Allen-Cahn equation $T = 1$, $\xi = 0 \in \mathbb{R}^1$, $u(T, x) = \frac{1}{1+\|x\|_\infty}$, and

$$\frac{\partial}{\partial t}u(t,x) + u(t,x) - [u(t,x)]^3 + \frac{1}{2}(\Delta_x u)(t,x) = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^1.$$

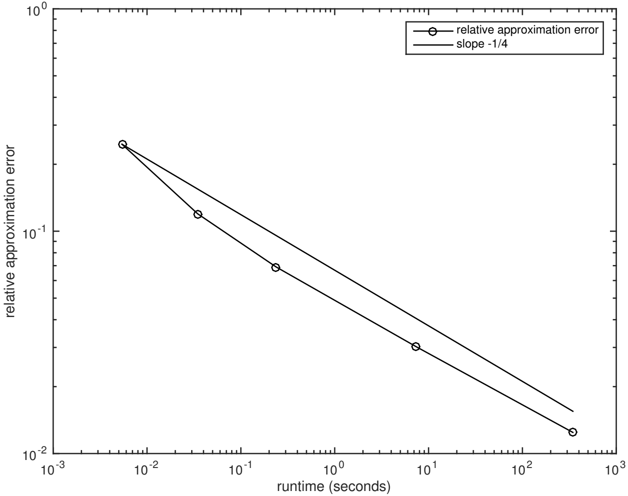
Relative errors $\frac{1}{10|\mathrm{v}|}\sum_{i=1}^{10} |\mathbf{U}^i_{\rho,\rho}(0,\xi) - \mathrm{v}|$ for $\rho \in \{1, 2, \ldots, 5\}$ against runtime; $u(0,\xi) \approx \mathrm{v} = 0.905$. Simulations: MATLAB, Intel i7 CPU, 2.8 GHz, 16 GB RAM.



Allen-Cahn equation $T = 1$, $\xi = 0 \in \mathbb{R}^1$, $u(T, x) = \frac{1}{1+\|x\|_\infty}$, and

$$\frac{\partial}{\partial t} u(t,x) + u(t,x) - [u(t,x)]^3 + \frac{1}{2}(\Delta_x u)(t,x) = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^1.$$

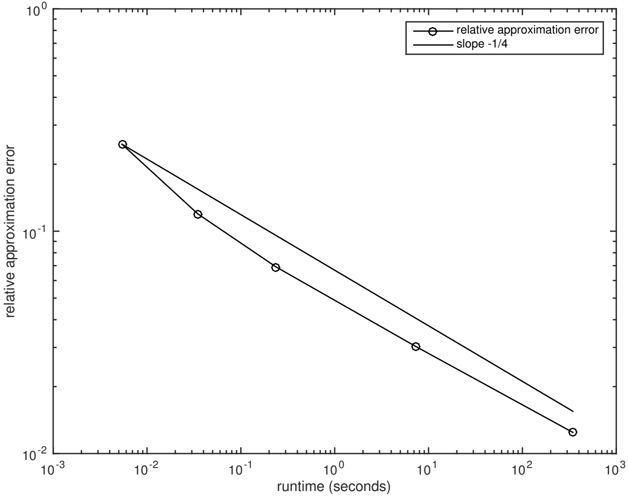
Relative errors $\frac{1}{10|\mathrm{v}|} \sum_{i=1}^{10} |\mathbf{U}^i_{\rho,\rho}(0,\xi) - \mathrm{v}|$ for $\rho \in \{1, 2, \ldots, 5\}$ against runtime; $u(0,\xi) \approx \mathrm{v} = 0.905$. Simulations: MATLAB, Intel i7 CPU, 2.8 GHz, 16 GB RAM.



Allen-Cahn equation $T = 1$, $\xi = 0 \in \mathbb{R}^1$, $u(T,x) = \frac{1}{1+\|x\|_\infty}$, and

$$\frac{\partial}{\partial t}u(t,x) + u(t,x) - [u(t,x)]^3 + \frac{1}{2}(\Delta_x u)(t,x) = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^1.$$

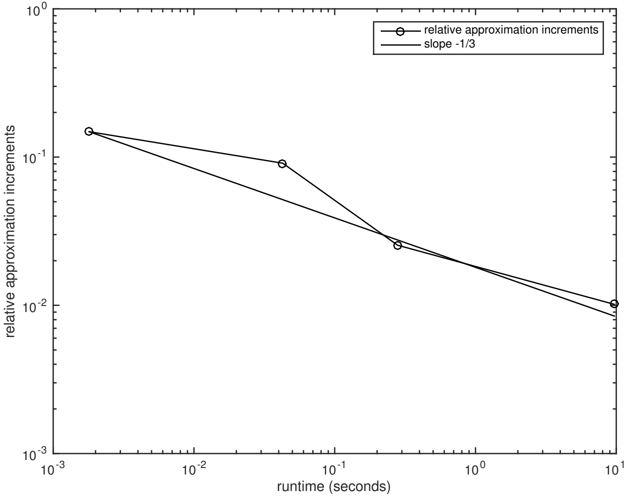
Relative errors $\frac{1}{10|\mathrm{v}|}\sum_{i=1}^{10} |\mathbf{U}^i_{\rho,\rho}(0,\xi) - \mathrm{v}|$ for $\rho \in \{1, 2, \dots, 5\}$ against runtime; $u(0,\xi) \approx \mathrm{v} = 0.905$. Simulations: MATLAB, Intel i7 CPU, 2.8 GHz, 16 GB RAM.



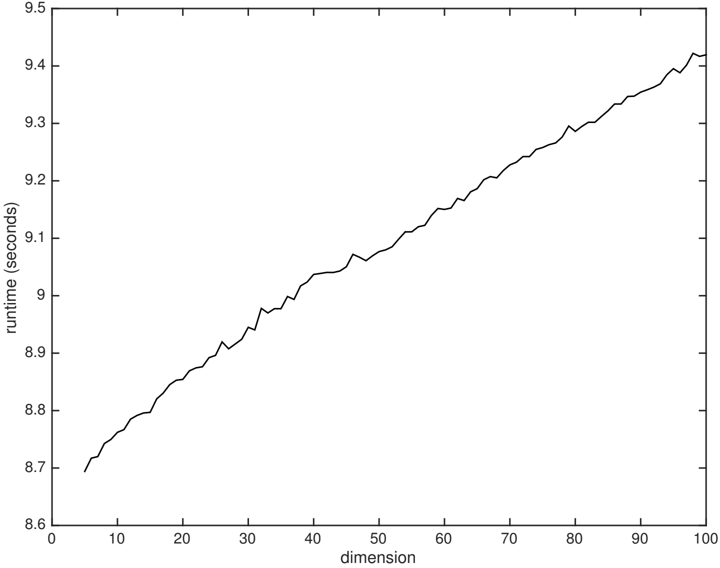
Allen-Cahn equation $T = 1$, $\xi = (0, 0, \ldots, 0) \in \mathbb{R}^{100}$, $u(T, x) = \frac{1}{1+\|x\|_\infty}$, and

$$\frac{\partial}{\partial t}u(t,x) + u(t,x) - [u(t,x)]^3 + \frac{1}{2}(\Delta_x u)(t,x) = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^{100}.$$

Relative increments $\left[\frac{1}{10}\sum_{i=1}^{10}|\mathbf{U}^i_{\rho+1,\rho+1}(0,\xi) - \mathbf{U}^i_{\rho,\rho}(0,\xi)|\right] \Big/ \left[\frac{1}{10}|\sum_{i=1}^{10}\mathbf{U}^i_{5,5}(0,\xi)|\right]$ for $\rho \in \{1, 2, 3, 4\}$ against runtime; $u(0,\xi) \approx 0.317$.



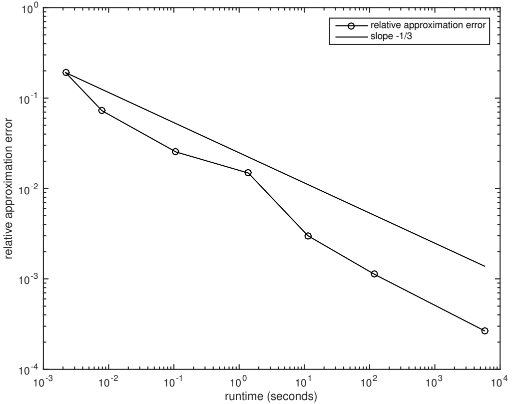
Allen-Cahn equation Runtime for one realization of $\mathbf{U}^1_{4,4}(0,\xi)$ against dimension $d \in \{5, 6, \ldots, 100\}$.



Pricing with default risk (Duffie, Schroder, & Skiadas 1996 AAP, Bender, Schweizer, & Zhuo 2015 MF) $T = 1$, $d = 1$, $\xi = (100, \ldots, 100) \in \mathbb{R}^d$, $u(T, x) = \min_{1 \le i \le d} x_i$,

$$\tfrac{\partial}{\partial t} u(t,x) + f(x, u(t,x)) + \bar{\mu} \sum_{i=1}^{d} x_i \big(\tfrac{\partial}{\partial x_i} u\big)(t,x) + \tfrac{\bar{\sigma}^2}{2} \sum_{i=1}^{d} |x_i|^2 \big(\tfrac{\partial^2}{\partial x_i^2} u\big)(t,x) = 0$$

for $(t, x) \in [0, T) \times \mathbb{R}^d$. Relative errors $\frac{1}{10|\mathrm{v}|} \sum_{i=1}^{10} |\mathbf{U}^i_{\rho,\rho}(0,\xi) - \mathrm{v}|$ for $\rho \in \{1, 2, \ldots, 7\}$ against runtime; $u(0, \xi) \approx \mathrm{v} = 97.705$.



Pricing with default risk (Duffie, Schroder, & Skiadas 1996 AAP, Bender, Schweizer, & Zhuo 2015 MF) $T = 1$, $d = 1$, $\xi = (100, \ldots, 100) \in \mathbb{R}^d$, $u(T, x) = \min_{1 \le i \le d} x_i$,

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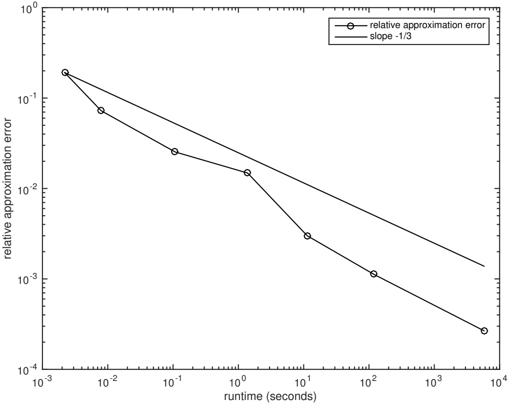
for $(t,x) \in [0,T) \times \mathbb{R}^d$. Relative errors $\frac{1}{10|\mathrm{v}|} \sum_{i=1}^{10} |\mathbf{U}^i_{\rho,\rho}(0,\xi) - \mathrm{v}|$ for $\rho \in \{1, 2, \ldots, 7\}$ against runtime; $u(0,\xi) \approx \mathrm{v} = 97.705$.



Pricing with default risk (Duffie, Schroder, & Skiadas 1996 AAP, Bender, Schweizer, & Zhuo 2015 MF) $T = 1$, $d = 1$, $\xi = (100, \ldots, 100) \in \mathbb{R}^d$, $u(T, x) = \min_{1 \le i \le d} x_i$,

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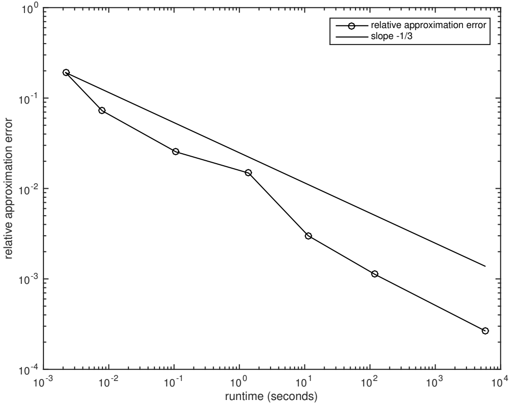
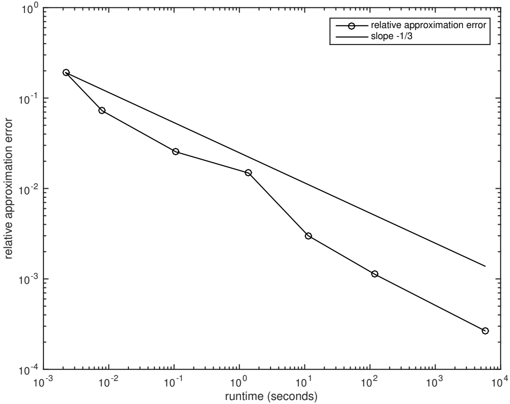
for $(t,x) \in [0,T) \times \mathbb{R}^d$. Relative errors $\frac{1}{10|\mathrm{v}|} \sum_{i=1}^{10} |\mathbf{U}^i_{\rho,\rho}(0,\xi) - \mathrm{v}|$ for $\rho \in \{1, 2, \ldots, 7\}$ against runtime; $u(0,\xi) \approx \mathrm{v} = 97.705$.



Pricing with default risk (Duffie, Schroder, & Skiadas 1996 AAP, Bender, Schweizer, & Zhuo 2015 MF) $T = 1$, $d = 1$, $\xi = (100, \dots, 100) \in \mathbb{R}^d$, $u(T, x) = \min_{1 \le i \le d} x_i$,

$$\tfrac{\partial}{\partial t} u(t,x) + f(x, u(t,x)) + \bar{\mu} \sum_{i=1}^{d} x_i \big(\tfrac{\partial}{\partial x_i} u\big)(t,x) + \tfrac{\bar{\sigma}^2}{2} \sum_{i=1}^{d} |x_i|^2 \big(\tfrac{\partial^2}{\partial x_i^2} u\big)(t,x) = 0$$

for $(t, x) \in [0, T) \times \mathbb{R}^d$. Relative errors $\frac{1}{10|\mathrm{v}|} \sum_{i=1}^{10} |\mathbf{U}^i_{\rho,\rho}(0,\xi) - \mathrm{v}|$ for $\rho \in \{1, 2, \dots, 7\}$ against runtime; $u(0, \xi) \approx \mathrm{v} = 97.705$.



Pricing with default risk (Duffie, Schroder, & Skiadas 1996 AAP, Bender, Schweizer, & Zhuo 2015 MF) $T = 1$, $d = 1$, $\xi = (100, \ldots, 100) \in \mathbb{R}^d$, $u(T, x) = \min_{1 \leq i \leq d} x_i$,

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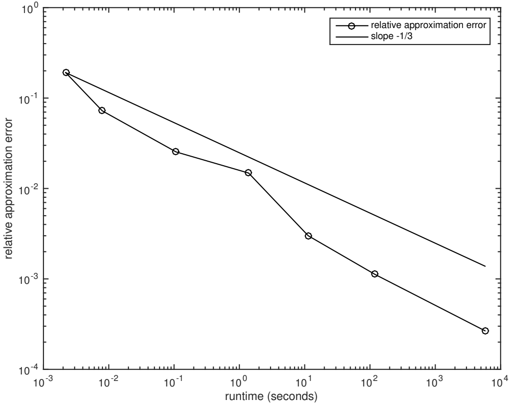
for $(t, x) \in [0, T) \times \mathbb{R}^d$. Relative errors $\frac{1}{10|\mathrm{v}|} \sum_{i=1}^{10} |\mathbf{U}^i_{\rho,\rho}(0,\xi) - \mathrm{v}|$ for $\rho \in \{1, 2, \ldots, 7\}$ against runtime; $u(0, \xi) \approx \mathrm{v} = 97.705$.



Pricing with default risk (Duffie, Schroder, & Skiadas 1996 AAP, Bender, Schweizer, & Zhuo 2015 MF) $T = 1$, $d = 1$, $\xi = (100, \ldots, 100) \in \mathbb{R}^d$, $u(T, x) = \min_{1 \le i \le d} x_i$,

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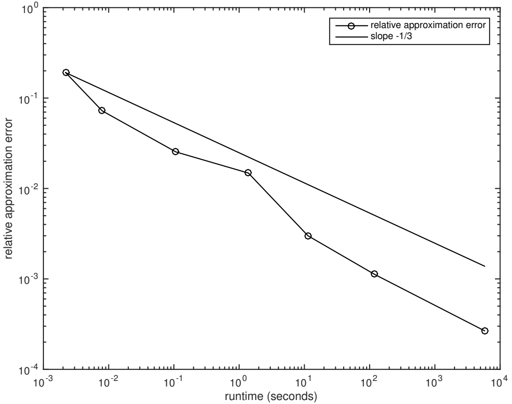
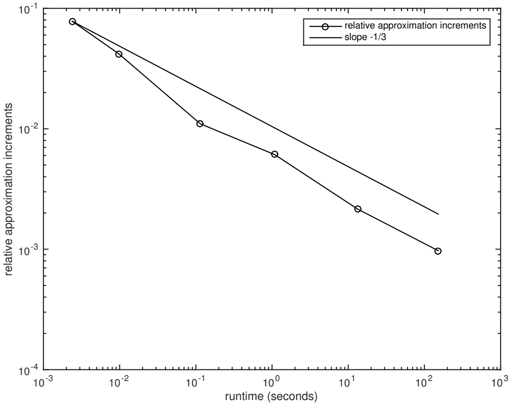
for $(t, x) \in [0, T) \times \mathbb{R}^d$. Relative errors $\frac{1}{10|\mathrm{v}|} \sum_{i=1}^{10} |\mathbf{U}^i_{\rho,\rho}(0,\xi) - \mathrm{v}|$ for $\rho \in \{1, 2, \ldots, 7\}$ against runtime; $u(0, \xi) \approx \mathrm{v} = 97.705$.



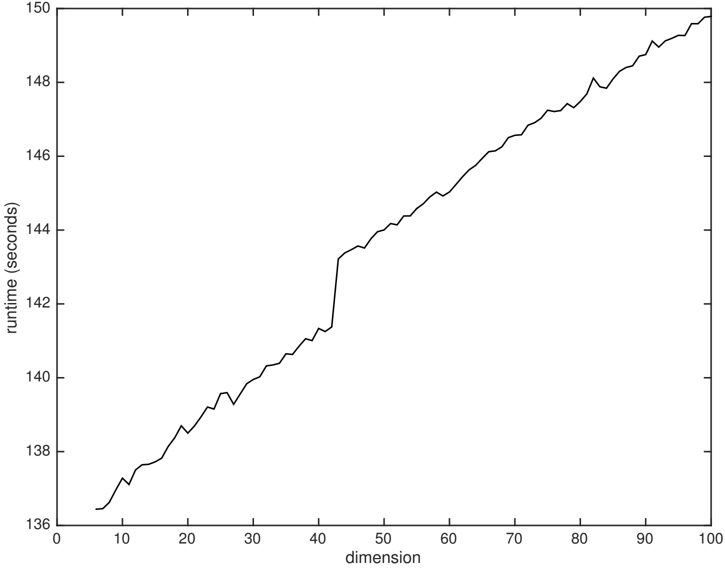
Pricing with default risk (Duffie, Schroder, & Skiadas 1996 AAP, Bender, Schweizer, & Zhuo 2015 MF) $T = 1$, $d = 100$, $\xi = (100, \dots, 100) \in \mathbb{R}^d$, $u(T, x) = \min_{1 \leq i \leq d} x_i$,

$$\tfrac{\partial}{\partial t} u(t, x) + f(x, u(t, x)) + \bar{\mu} \sum_{i=1}^{d} x_i \big(\tfrac{\partial}{\partial x_i} u\big)(t, x) + \tfrac{\bar{\sigma}^2}{2} \sum_{i=1}^{d} |x_i|^2 \big(\tfrac{\partial^2}{\partial x_i^2} u\big)(t, x) = 0$$

for $(t, x) \in [0, T) \times \mathbb{R}^d$. $\big[\frac{1}{10} \sum_{i=1}^{10} |\mathbf{U}^i_{\rho+1,\rho+1}(0,\xi) - \mathbf{U}^i_{\rho,\rho}(0,\xi)|\big] \big/ \big[\frac{1}{10} |\sum_{i=1}^{10} \mathbf{U}^i_{7,7}(0,\xi)|\big]$ for $\rho \in \{1, 2, \dots, 6\}$ against runtime; $u(0, \xi) \approx 58.113$.



Pricing with default risk Runtime for one realization of $\mathbf{U}^1_{6,6}(0,\xi)$ against dimension $d \in \{5, 6, \ldots, 100\}$.



Thanks for your attention!

Thanks for your attention!

Pricing with default risk (Duffie, Schroder, & Skiadas 1996 AAP, Bender, Schweizer, & Zhuo 2015 MF)

Consider $\delta = \frac{2}{3}$, $R = \frac{2}{100}$, $\gamma^h = \frac{2}{10}$, $\gamma^l = \frac{2}{100}$, $\bar{\mu} = \frac{2}{100}$, $\bar{\sigma} = \frac{2}{10}$, $v^h, v^l \in (0, \infty)$ satisfy $v^h < v^l$, and assume for all $x \in \mathbb{R}^d$, $y \in \mathbb{R}$ that

$$\mu(x) = \bar{\mu}x, \qquad \sigma(x) = \bar{\sigma}\,\mathrm{diag}(x),$$

and

$$f(x,y) = -(1-\delta)\,y\Big[\gamma^h\,\mathbb{1}_{(-\infty,v^h)}(y) + \gamma^l\,\mathbb{1}_{[v^l,\infty)}(y) + \Big[\tfrac{(\gamma^h-\gamma^l)}{(v^h-v^l)}\big(y - v^h\big) + \gamma^h\Big]\,\mathbb{1}_{[v^h,v^l)}(y)\Big] - Ry.$$

- We consider $v^h = 50$, $v^l = 120$ in the case $d = 1$.
- Bender et al. consider $v^h = 54$, $v^l = 90$ in the case $d = 5$.
- We consider $v^h = 47$, $v^l = 65$ in the case $d = 100$.