# On numerical approximation algorithms for high-dimensional nonlinear PDEs, nonlinear SDEs, and high-dimensional nonlinear FBSDEs 

Arnulf Jentzen (ETH Zurich, Switzerland) Joint works with<br>Weinan E (Princeton University, USA \& Beijing University, China),<br>Mate Gerencsér (IST Austria, Austria),<br>Martin Hairer (University of Warwick, UK), Mario Hefter (University of Kaiserslautern, Germany),<br>Martin Hutzenthaler (University of Duisburg-Essen, Germany),<br>Thomas Kruse (University of Duisburg-Essen, Germany),<br>Thomas Müller-Gronbach (University of Passau, Germany),<br>Diyora Salimova (ETH Zurich, Switzerland), and<br>Larisa Yaroslavtseva (University of Passau, Germany)<br>KI-Net Conference Selected topics in transport phenomena: deterministic and probabilistic aspects, University of Maryland, USA

Wednesday, 19 April 2017

Introduction

Consider $T>0, d \in \mathbb{N}, \xi \in \mathbb{R}^{d}$ and sufficiently regular $f: \mathbb{R}^{d} \times \mathbb{R} \times \mathbb{R}^{d} \rightarrow \mathbb{R}$, $g: \mathbb{R}^{d} \rightarrow \mathbb{R}, \mu: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}, \sigma: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d \times d}, u:[0, T] \times \mathbb{R}^{d} \rightarrow \mathbb{R}$ such that $u(T, x)=g(x)$ and

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\begin{aligned}
& \frac{\partial}{\partial t} u(t, x)+t\left(x, u(t, x),\left(\nabla_{x} u\right)(t, x)\right)+\left\langle\mu(x),\left(\nabla_{x} u\right)(t, x)\right\rangle_{\mathbb{R}^{d}} \\
& +\frac{1}{2} \operatorname{Trace}_{R^{d}}\left(\sigma(x)[\sigma(x)]^{*}\left(\text { Hess }_{x} u\right)(t, x)\right)=0 .
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for $(t, x) \in[0, T) \times \mathbb{R}^{d}$. Goal: Compute $u(0, \xi)$ approximatively.

## Application: Pricing of financial derivatives

Approximations methods such as finite element methods, finite differences, sparse grids suffer under the curse of dimensionality.

Consider probability space $(\Omega, \mathcal{F}, \mathbb{P})$, Brownian motion $W:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$, and for every $s \in[0, T], x \in \mathbb{R}^{d}$ a solution process $X^{s, x}:[s, T] \times \Omega \rightarrow \mathbb{R}^{d}$ of

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\frac{\partial}{\partial t} x^{s, x}=\mu\left(x^{s, x}\right)+\sigma\left(x^{s, x}\right) \frac{\partial}{\partial t} M N_{2}, \quad t \in[s, T], \quad x_{s}^{s, x}=x
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Feynman-Kac formula $\forall s \in[0, T], x \in \mathbb{R}^{d}$ :

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u(s, x)=\mathbb{E}\left[g\left(X_{T}^{s, x}\right)\right]+\int_{s}^{T} \mathbb{E}\left[f\left(t, X_{t}^{s, x}, u\left(t, X_{t}^{s, x}\right),\left(\nabla_{x} u\right)\left(t, X_{t}^{s, x}\right)\right)\right] d t
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Approximations methods such as finite element methods, finite differences, sparse grids suffer under the curse of dimensionality. Consider probability space $(\Omega, \mathcal{F}, \mathbb{P})$, Brownian motion $W:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$, and for every $s \in[0, T], x \in \mathbb{R}^{d}$ a solution process $X^{s, x}:[s, T] \times \Omega \rightarrow \mathbb{R}^{d}$ of

Feynman-Kac formula $\forall s \in[0, T], x \in \mathbb{R}^{d}$

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\begin{aligned}
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& +\frac{1}{2} \operatorname{Trace}_{\mathbb{R}^{d}}\left(\sigma(x)[\sigma(x)]^{*}\left(\operatorname{Hess}_{x} u\right)(t, x)\right)=0 .
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Feynman-Kac formula

Consider $T>0, d \in \mathbb{N}, \xi \in \mathbb{R}^{d}$ and sufficiently regular $f: \mathbb{R}^{d} \times \mathbb{R} \times \mathbb{R}^{d} \rightarrow \mathbb{R}$, $g: \mathbb{R}^{d} \rightarrow \mathbb{R}, \mu: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}, \sigma: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d \times d}, u:[0, T] \times \mathbb{R}^{d} \rightarrow \mathbb{R}$ such that $u(T, x)=g(x)$ and

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Feynman-Kac formula $\forall s \in[0, T], x \in \mathbb{R}^{d}$ :

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$$

Feynman-Kac formula $\forall s \in[0, T], x \in \mathbb{R}^{d}$ :

$$
u(s, x)=\mathbb{E}\left[g\left(X_{T}^{s, x}\right)\right]+\int_{s}^{T} \mathbb{E}\left[f\left(t, X_{t}^{s, x}, u\left(t, X_{t}^{s, x}\right),\left(\nabla_{x} u\right)\left(t, X_{t}^{s, x}\right)\right)\right] d t .
$$

Linear pricing models

$$
f=0
$$

- Black-Scholes model Consider $T, \beta>0, \alpha \in \mathbb{R}$ and

$$
\frac{\partial}{\partial t} X_{t}=\alpha X_{t}+\beta X_{t} \frac{\partial}{\partial t} d W_{t}
$$

for $t \in[0, T]$, where $\left(W_{t}\right)_{t \in[0, T]}$ is a one-dimensional Brownian motion.

- Heston model Consider $\alpha, \gamma \in \mathbb{R}, \beta, \delta, x_{0}^{(1)}, x_{0}^{(2)}>0, \rho \in[-1,1]$ and

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- Heston model Consider $\alpha, \gamma \in \mathbb{R}, \beta, \delta, x_{0}^{(1)}, x_{0}^{(2)}>0, \rho \in[-1,1]$ and

$$
\begin{aligned}
& \frac{\partial}{\partial t} x_{t}^{(1)}=\alpha X_{t}^{(1)}+\sqrt{x_{t}^{(2)}} X_{t}^{(1)} \frac{\partial}{\partial t} W_{t}^{(1)} \\
& \frac{\partial}{\partial t} x_{t}^{(2)}=\delta-\gamma X_{t}^{(2)}+\beta \sqrt{X_{t}^{(2)}}\left(\rho \frac{\partial}{\partial t} W_{t}^{(1)}+\sqrt{1-\rho^{2}} \frac{\partial}{\partial t} W_{t}^{(2)}\right)
\end{aligned}
$$

for $t \in[0, T]$, where $\left(W_{t}\right)_{t \in[0, T]}=\left(\left(W_{t}^{(1)}, W_{t}^{(2)}\right)\right)_{t \in[0, T]}$ is a two-dim. BM.

## Theorem (Hairer, Hutzenthaler, \& J 2015 AOP)

Let $T \in(0, \infty), d \in\{4,5, \ldots\}, \xi \in \mathbb{R}^{d}$. Then there exist globally bounded $\mu, \sigma \in \mathcal{C}^{\infty}\left(\mathbb{R}^{d}, \mathbb{R}^{d}\right)$ such that for every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every Brownian motion W : $[0, T] \times \Omega \rightarrow \mathbb{R}$, every solution $X:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$ of

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\frac{\partial}{\partial t} X_{t}=\mu\left(X_{t}\right)+\sigma\left(X_{t}\right) \frac{\partial}{\partial t} W_{t}, \quad t \in[0, T], \quad X_{0}=\xi
$$

every $Y^{N}:\{0,1, \ldots, N\} \times \Omega \rightarrow \mathbb{R}^{4}, N \in \mathbb{N}$, with $\forall N \in \mathbb{N}, n \in\{0,1, \ldots, N-1\}: Y_{0}^{N}=X_{0}$ and

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Y_{n+1}^{N}=Y_{n}^{N}+\mu\left(Y_{n}^{N}\right) \frac{T}{N}+\sigma\left(Y_{n}^{N}\right)\left(W_{\frac{(n+1) T}{N}}-W_{\frac{n T}{N}}\right)
$$

(Euler-Maruyama approximations), and every $\alpha \in[0, \infty)$ we have

$$
\lim _{N \rightarrow \infty}\left(N^{\alpha}\left\|\mathbb{E}\left[X_{T}\right]-\mathbb{E}\left[Y_{N}^{N}\right]\right\|\right)= \begin{cases}0 & : \alpha=0 \\ \infty & : \alpha>0\end{cases}
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$\forall N \in \mathbb{N}, n \in\{0,1, \ldots, N-1\}: Y_{0}^{N}=X_{0}$ and

$$
Y_{n+1}^{N}=Y_{n}^{N}+\mu\left(Y_{n}^{N}\right) \frac{T}{N}+\sigma\left(Y_{n}^{N}\right)\left(W_{\frac{(n+1) \tau}{N}}-W_{\frac{n T}{N}}\right)
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(Euler-Maruyama approximations), and every $\alpha \in[0, \infty)$ we have

$$
\lim _{N \rightarrow \infty}\left(N^{\alpha}\left\|\mathbb{E}\left[X_{T}\right]-\mathbb{E}\left[Y_{N}^{N}\right]\right\|\right)=
$$

## Theorem (Hairer, Hutzenthaler, \& J 2015 AOP)

Let $T \in(0, \infty), d \in\{4,5, \ldots\}, \xi \in \mathbb{R}^{d}$. Then there exist globally bounded $\mu, \sigma \in \mathcal{C}^{\infty}\left(\mathbb{R}^{d}, \mathbb{R}^{d}\right)$ such that for every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every Brownian motion W: $[0, T] \times \Omega \rightarrow \mathbb{R}$, every solution $X:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$ of

$$
\frac{\partial}{\partial t} X_{t}=\mu\left(X_{t}\right)+\sigma\left(X_{t}\right) \frac{\partial}{\partial t} W_{t}, \quad t \in[0, T], \quad X_{0}=\xi,
$$

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\end{array}\right.
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$$

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$$

Plot of $\left\|\mathbb{E}\left[X_{T}\right]-\mathbb{E}\left[Y_{N}^{N}\right]\right\|$ for $T=2$ and $N \in\left\{2^{1}, 2^{2}, \ldots, 2^{30}\right\}$.


## Theorem (Gerencsér, J, \& Salimova 2017)

Lot $T \subset(0, \infty), d \in\{2,3,1, \ldots\}, \delta \subset \mathbb{D} C,\left(a_{N}\right)_{N \in \mathbb{N}} \subseteq \mathbb{R}$ satisfy $\lim _{N \rightarrow \infty} a_{N}=0$. Then there exist globally bounded $\mu, \sigma \in \mathcal{C} \infty\left(\mathbb{R}^{d}, \mathbb{R}^{d}\right)$ such that for every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every Brownian motion $W:[0, T] \times \Omega \rightarrow \mathbb{R}$, every solution $X:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$ of

$$
\frac{\partial}{\partial t} X_{t}=\mu\left(X_{t}\right)+\sigma\left(X_{t}\right) \frac{\partial}{\partial t} W_{t}, \quad t \in[0, T], \quad X_{0}=\xi
$$

and every $N \in \mathbb{N}$ we have

- Dimension $d \geq$ 4: J, Müller-Gronbach \& Yaroslavtseva 2016 CMS
- Weak convergence and $d>4$ : Müller-Gronbach \& Yaroslavtseva 2016 SAA (to appear)
- Adaptive approximations and $d \geq 4$ : Yaroslavtseva 2016


## Theorem (Gerencsér, J, \& Salimova 2017)

Let $T \in(0, \infty), d \in\{2,3,4, \ldots\}, \xi \in \mathbb{R}^{d},\left(a_{N}\right)_{N \in N} \subseteq \mathbb{R}$ satisfy
$\lim _{N \rightarrow \infty} a_{N}=0$. Then there exist globally bounded $\mu, \sigma \in \mathcal{C}^{\infty}\left(\mathbb{R}^{d}, \mathbb{R}^{d}\right)$ such that for every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every Brownian motion $W:[0, T] \times \Omega \rightarrow \mathbb{R}$, every solution $X:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$ of

$$
\frac{\partial}{\partial t} X_{t}=\mu\left(X_{t}\right)+\sigma\left(X_{t}\right) \frac{\partial}{\partial t} W_{t}, \quad t \in[0, T], \quad X_{0}=\xi,
$$

and every $N \in \mathbb{N}$ we have


- Dimension $d \geq$ 4: J, Müller-Gronbach \& Yaroslavtseva 2016 CMS
- Weak convergence and $d \geq$ 4: Müller-Gronbach \& Varoslavtseva 2016 SAA (to appear)
- Adaptive approximations and $d \geq$ 4: Yaroslavtseva 2016


## Theorem (Gerencsér, J, \& Salimova 2017)

Let $T \in(0, \infty), d \in\{2,3,4, \ldots\}, \xi \in \mathbb{R}^{d},\left(a_{N}\right)_{N \in \mathbb{N}} \subseteq \mathbb{R}$ satisfy
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$$
\frac{\partial}{\partial t} X_{t}=\mu\left(X_{t}\right)+\sigma\left(X_{t}\right) \frac{\partial}{\partial t} W_{t}, \quad t \in[0, T], \quad X_{0}=\xi,
$$

and every $N \in \mathbb{N}$ we have


- Dimension $d \geq$ 4: J, Müller-Gronbach \& Yaroslavtseva 2016 CMS
- Weak convergence and $d \geq$ 4: Müller-Gronbach \& Varoslavtseva 2016 SAA (to appear)
- Adaptive approximations and $d \geq$ 4: Yaroslavtseva 2016


## Theorem (Gerencsér, J, \& Salimova 2017)

Let $T \in(0, \infty), d \in\{2,3,4, \ldots\}, \xi \in \mathbb{R}^{d},\left(a_{N}\right)_{N \in \mathbb{N}} \subseteq \mathbb{R}$ satisfy $\lim _{N \rightarrow \infty} a_{N}=0$. Then there exist globally bounded $\mu, \sigma \in C^{\infty}\left(\mathbb{R}^{d}, \mathbb{R}^{d}\right)$ such that for every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every Brownian motion $W:[0, T] \times \Omega \rightarrow \mathbb{R}$, every solution $X:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$ of

$$
\frac{\partial}{\partial t} x_{t}=\mu\left(X_{t}\right)+\sigma\left(X_{t}\right) \frac{\partial}{\partial t} W_{t}, \quad t \in[0, T], \quad x_{0}=\xi
$$

and every $N \in \mathbb{N}$ we have


- Dimension $d \geq$ 4: J, Müller-Gronbach \& Yaroslavtseva 2016 CMS
- Weak convargence and $d \geq$ 4. Müller-Gronbach \& Varoslavtseva 2016 SAA (to appear)
- Adaptive approximations and $d \geq$ 4: Yaroslavtseva 2016


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Let $T \in(0, \infty), d \in\{2,3,4, \ldots\}, \xi \in \mathbb{R}^{d},\left(a_{N}\right)_{N \in \mathbb{N}} \subseteq \mathbb{R}$ satisfy $\lim _{N \rightarrow \infty} a_{N}=0$. Then there exist globally bounded $\mu, \sigma \in \mathcal{C}^{\infty}\left(\mathbb{R}^{d}, \mathbb{R}^{d}\right)$ such that for every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every Brownian motion $W:[0, T] \times \Omega \rightarrow \mathbb{R}$, every solution $X:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$ of

$$
\frac{\partial}{\partial t} x_{t}=\mu\left(X_{t}\right)+\sigma\left(X_{t}\right) \frac{\partial}{\partial t} W_{t}, \quad t \in[0, T], \quad x_{0}=\xi_{1}
$$

and every $N \in \mathbb{N}$ we have


- Dimension $d \geq$ 4: J, Müller-Gronbach \& Yaroslavtseva 2016 CMS
- Weak convergence and $d \geq 1$. Müller-Gronbach \& Varoslavtseva 2016 SAA (to appear)
- Adaptive approximations and $d \geq$ 4: Yaroslavtseva 2016


## Theorem (Gerencsér, J, \& Salimova 2017)

Let $T \in(0, \infty), d \in\{2,3,4, \ldots\}, \xi \in \mathbb{R}^{d},\left(a_{N}\right)_{N \in \mathbb{N}} \subseteq \mathbb{R}$ satisfy $\lim _{N \rightarrow \infty} a_{N}=0$. Then there exist globally bounded $\mu, \sigma \in \mathcal{C}^{\infty}\left(\mathbb{R}^{d}, \mathbb{R}^{d}\right)$ such that for every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every Brownian motion $W:[0, T] \times \Omega \rightarrow \mathbb{R}$,

and every $N \in \mathbb{N}$ we have


- Dimension $d \geq$ 4: J, Müller-Gronbach \& Yaroslavtseva 2016 CMS
- Weak convergence and $d \geq$ 4. Müller-Gronbach \& Yaroslavtseva 2016 SAA (to appear)
- Adaptive approximations and $d \geq$ 4: Yaroslavtseva 2016


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every solution $X:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$ of

$$
\frac{\partial}{\partial t} X_{t}=\mu\left(X_{t}\right)+\sigma\left(X_{t}\right) \frac{\partial}{\partial t} W_{t}, \quad t \in[0, T], \quad X_{0}=\xi,
$$

and every $N \in \mathbb{N}$ we have


- Dimension $d \geq$ 4: J, Müller-Gronbach \& Yaroslavtseva 2016 CMS
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## Theorem (Gerencsér, J, \& Salimova 2017)

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and every $N \in \mathbb{N}$ we have
measurable

- Dimension $d \geq$ 4: J, Müller-Gronbach \& Yaroslavtseva 2016 CMS
- Weak convergence and $d \geq$ 4: Müller-Gronbach \& Varoslavtseva 2016 SAA (to appear)
- Adaptive approximations and $d \geq$ 4: Yaroslavtseva 2016


## Theorem (Gerencsér, J, \& Salimova 2017)

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$$
\frac{\partial}{\partial t} X_{t}=\mu\left(X_{t}\right)+\sigma\left(X_{t}\right) \frac{\partial}{\partial t} W_{t}, \quad t \in[0, T], \quad X_{0}=\xi
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and every $N \in \mathbb{N}$ we have
measurable

- Dimension $d \geq$ 4: J, Müller-Gronbach \& Yaroslavtseva 2016 CMS
- Weak convergence and $d \geq$ 4: Müller-Gronbach \& Yaroslavtseva 2016 SAA (to appear)
- Adaptive approximations and $d \geq$ 4: Yaroslavtseva 2016


## Theorem (Gerencsér, J, \& Salimova 2017)

Let $T \in(0, \infty), d \in\{2,3,4, \ldots\}, \xi \in \mathbb{R}^{d},\left(a_{N}\right)_{N \in \mathbb{N}} \subseteq \mathbb{R}$ satisfy $\lim _{N \rightarrow \infty} a_{N}=0$. Then there exist globally bounded $\mu, \sigma \in \mathcal{C}^{\infty}\left(\mathbb{R}^{d}, \mathbb{R}^{d}\right)$ such that for every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every Brownian motion $W:[0, T] \times \Omega \rightarrow \mathbb{R}$, every solution $X:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$ of

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\frac{\partial}{\partial t} X_{t}=\mu\left(X_{t}\right)+\sigma\left(X_{t}\right) \frac{\partial}{\partial t} W_{t}, \quad t \in[0, T], \quad X_{0}=\xi
$$

and every $N \in \mathbb{N}$ we have


- Dimension $d>4$ : J, Müller-Gronbach \& Yaroslavtseva 2016 CMS
- Weak convergence and $d \geq$ 4: Müller-Gronbach \& Yaroslavtseva 2016 SAA (to appear)
- Adaptive approximations and $d \geq$ 4: Yaroslavtseva 2016


## Theorem (Gerencsér, J, \& Salimova 2017)

Let $T \in(0, \infty), d \in\{2,3,4, \ldots\}, \xi \in \mathbb{R}^{d},\left(a_{N}\right)_{N \in \mathbb{N}} \subseteq \mathbb{R}$ satisfy $\lim _{N \rightarrow \infty} a_{N}=0$. Then there exist globally bounded $\mu, \sigma \in \mathcal{C}^{\infty}\left(\mathbb{R}^{d}, \mathbb{R}^{d}\right)$ such that for every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every Brownian motion $W:[0, T] \times \Omega \rightarrow \mathbb{R}$, every solution $X:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$ of

$$
\frac{\partial}{\partial t} X_{t}=\mu\left(X_{t}\right)+\sigma\left(X_{t}\right) \frac{\partial}{\partial t} W_{t}, \quad t \in[0, T], \quad X_{0}=\xi
$$

and every $N \in \mathbb{N}$ we have

- Dimension $d \geq$ 4: J, Müller-Gronbach \& Yaroslavtseva 2016 CMS
- Weak convergence and $d \geq$ 4. Müller-Gronbach \& Yaroslavtseva 2016 SAA (to appear)
- Adaptive approximations and $d \geq$ 4: Yaroslavtseva 2016


## Theorem (Gerencsér, J, \& Salimova 2017)

Let $T \in(0, \infty), d \in\{2,3,4, \ldots\}, \xi \in \mathbb{R}^{d},\left(a_{N}\right)_{N \in \mathbb{N}} \subseteq \mathbb{R}$ satisfy $\lim _{N \rightarrow \infty} a_{N}=0$. Then there exist globally bounded $\mu, \sigma \in \mathcal{C}^{\infty}\left(\mathbb{R}^{d}, \mathbb{R}^{d}\right)$ such that for every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every Brownian motion $W:[0, T] \times \Omega \rightarrow \mathbb{R}$, every solution $X:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$ of

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- Dimension $d \geq$ 4: J, Müller-Gronbach \& Yaroslavtseva 2016 CMS
- Weak convergence and $d \geq$ 4: Müller-Gronbach \& Yaroslavtseva 2016 SAA (to appear)
- Adaptive approximations and $d \geq$ 4: Yaroslavtseva 2016


## Theorem (Gerencsér, J, \& Salimova 2017)

Let $T \in(0, \infty), d \in\{2,3,4, \ldots\}, \xi \in \mathbb{R}^{d},\left(a_{N}\right)_{N \in \mathbb{N}} \subseteq \mathbb{R}$ satisfy $\lim _{N \rightarrow \infty} a_{N}=0$. Then there exist globally bounded $\mu, \sigma \in \mathcal{C}^{\infty}\left(\mathbb{R}^{d}, \mathbb{R}^{d}\right)$ such that for every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every Brownian motion $W:[0, T] \times \Omega \rightarrow \mathbb{R}$, every solution $X:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$ of

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\frac{\partial}{\partial t} X_{t}=\mu\left(X_{t}\right)+\sigma\left(X_{t}\right) \frac{\partial}{\partial t} W_{t}, \quad t \in[0, T], \quad X_{0}=\xi
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- Dimension $d \geq$ 4: J, Müller-Gronbach \& Yaroslavtseva 2016 CMS
- Weak convergence and $d \geq$ 4: Müller-Gronbach \& Yaroslavtseva 2016 SAA (to appear)
- Adaptive approximations and $d \geq$ 4: Yaroslavtseva 2016


## Theorem (Gerencsér, J, \& Salimova 2017)

Let $T \in(0, \infty), d \in\{2,3,4, \ldots\}, \xi \in \mathbb{R}^{d},\left(a_{N}\right)_{N \in \mathbb{N}} \subseteq \mathbb{R}$ satisfy $\lim _{N \rightarrow \infty} a_{N}=0$. Then there exist globally bounded $\mu, \sigma \in \mathcal{C}^{\infty}\left(\mathbb{R}^{d}, \mathbb{R}^{d}\right)$ such that for every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every Brownian motion $W:[0, T] \times \Omega \rightarrow \mathbb{R}$, every solution $X:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}$ of

$$
\frac{\partial}{\partial t} X_{t}=\mu\left(X_{t}\right)+\sigma\left(X_{t}\right) \frac{\partial}{\partial t} W_{t}, \quad t \in[0, T], \quad X_{0}=\xi
$$

and every $N \in \mathbb{N}$ we have

- Dimension $d \geq$ 4: J, Müller-Gronbach \& Yaroslavtseva 2016 CMS
- Weak convergence and $d \geq$ 4: Müller-Gronbach \& Yaroslavtseva 2016 SAA (to appear)
- Adaptive approximations and $d \geq 4$ : Yaroslavtseva 2016


## Theorem (Hefter \& J 2017)

Let $T, \delta, \beta \in(0, \infty), \gamma, \xi \in[0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let
$W:[0, T] \times \Omega \rightarrow \mathbb{R}$ be a Brownian motion, let $X:[0, T] \times \Omega \rightarrow \mathbb{R}$ be a solution of

$$
\frac{\partial}{\partial t} X_{t}=\left(\delta-\gamma X_{t}\right)+\beta \sqrt{X_{t}} \frac{\partial}{\partial t} W_{t}, \quad t \in[0, T], \quad X_{0}=\xi .
$$

Then there exists a c $\in(0, \infty)$ such that for all $N \in \mathbb{N}$ we have

$$
\begin{equation*}
\inf _{\mathbb{R}^{N} \rightarrow \mathbb{R}}\left[X_{T}-u\left(W_{\frac{T}{N}}, W_{\frac{2 T}{N}}, \ldots, W_{T}\right)\right]^{]} \geq c \cdot N^{-\operatorname{mn}\left\{1, \frac{25}{\beta^{2}}\right\}} \tag{*}
\end{equation*}
$$

measurable
Deelstra \& Delbaen 1998 Appl Stoch Models Data Anal: Let
$Y^{N}:\{0,1, \ldots, N\} \times \Omega \rightarrow \mathbb{R}, N \in \mathbb{N}$, satisfy for all $N \in \mathbb{N}, n \in\{0,1, \ldots, N-1\}$
that $Y_{0}^{N}=\xi$ and

$$
Y_{n+1}^{N}=Y_{n}^{N}+\left(\delta-\gamma Y_{n}^{N}\right) \frac{T}{N}+\beta \sqrt{\left[Y_{n}^{N}\right]^{+}}\left(W_{\frac{(n+1) T}{N}}-W_{\frac{n T}{N}}\right) .
$$

There exists a $c \in \mathbb{R}$ such that for all $N \in \mathbb{N}$ we have

$$
\begin{equation*}
\mathbb{E}\left[\left|X_{T}-Y_{N}^{N}\right|\right] \leq c \cdot N^{-\frac{1}{2}} . \tag{}
\end{equation*}
$$

$\left(^{*}\right)$ disproves (**).

## Theorem (Hefter \& J 2017)

Let $T, \delta, \beta \in(0, \infty), \gamma, \xi \in[0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let
W: $[0, T] \times \Omega \rightarrow \mathbb{R}$ be a Brownian motion, let $X:[0, T] \times \Omega \rightarrow \mathbb{R}$ be a solution of

$$
\frac{\partial}{\partial t} X_{t}=\left(\delta-\gamma X_{t}\right)+\beta \sqrt{X_{t}} \frac{\partial}{\partial t} W_{t}, \quad t \in[0, T], \quad X_{0}=\xi .
$$

Then there exists a $c \in(0, \infty)$ such that for all $N \in \mathbb{N}$ we have

$$
\begin{equation*}
\inf _{u: \mathbb{R}^{N} \rightarrow \mathbb{R}} \mathbb{E}\left[\left\lvert\, X_{T}-u\left(W_{\frac{T}{N}}, W_{\frac{2 T}{N}}, \ldots, W_{T}\right)\right.\right] \geq c \cdot N^{-\operatorname{mn}}\left\{1, \frac{28}{B^{2}}\right\} . \tag{*}
\end{equation*}
$$

Deelstra \& Delbaen 1998 Appl Stoch Models Data Anal: Let $Y^{N}:\{0,1, \ldots, N\} \times \Omega \rightarrow \mathbb{R}, N \in \mathbb{N}$, satisfy for all $N \in \mathbb{N}, n \in\{0,1, \ldots, N-1\}$ that $Y_{0}^{N}=\xi$ and

$$
Y_{n+1}^{N}=Y_{n}^{N}+\left(\delta-\gamma Y_{n}^{N}\right) \frac{T}{N}+\beta \sqrt{\left[Y_{n}^{N}\right]^{+}}\left(W_{\frac{(n+1) T}{N}}-W_{\frac{n}{N}}\right) .
$$

There exists a $c \in \mathbb{R}$ such that for all $N \in \mathbb{N}$ we have

$$
\begin{equation*}
\mathbb{E}\left[\left|X_{T}-V_{N}^{N}\right|\right] \leq c \cdot N^{-\frac{1}{2}} \tag{}
\end{equation*}
$$

(*) disproves (**).

## Theorem (Hefter \& J 2017)

Let $T, \delta, \beta \in(0, \infty), \gamma, \xi \in[0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space,
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$$

Then there exists a $c \in(0, \infty)$ such that for all $N \in \mathbb{N}$ we have $\inf _{u: \mathbb{N}_{N}} \mathbb{E}\left[\left\lvert\, X_{T}-u\left(W_{\frac{T}{N}}, W_{\frac{2 T}{N}}, \ldots, W_{T}\right)\right.\right] \geq 0 \cdot N^{-\operatorname{mn}}\left\{1, \frac{25}{B^{2}}\right\}$
measurable
Deelstra \& Delbaen 1998 Appl Stoch Models Data Anal: Let
$Y^{N}:\{0,1, \ldots, N\} \times \Omega \rightarrow \mathbb{R}, N \in \mathbb{N}$, satisfy for all $N \in \mathbb{N}, n \in\{0,1, \ldots, N-1\}$
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There exists a $c \in \mathbb{R}$ such that for all $N \in \mathbb{N}$ we have

(*) disproves (**).

## Theorem (Hefter \& J 2017)

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\inf _{u: \mathbb{R}^{N} \rightarrow \mathbb{R}} \mathbb{E}\left[\left|X_{T}-u\left(W_{\frac{T}{N}}, W_{\frac{2 T}{N}}, \ldots, W_{T}\right)\right|\right] \geq c \cdot N^{-\min \left\{1, \frac{2 \delta}{\beta^{2}}\right\}} .
$$

## Deelstra \& Delbaen 1998 Appl Stoch Models Data Anal: Let

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$$

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$$

There exists a $c \in \mathbb{R}$ such that for all $N \in \mathbb{N}$ we have

$$
\begin{equation*}
\mathbb{E}\left[\left|X_{T}-Y_{N}^{N}\right|\right] \leq c \cdot N^{-\frac{1}{2}} \tag{**}
\end{equation*}
$$

## Theorem (Hefter \& J 2017)

Let $T, \delta, \beta \in(0, \infty), \gamma, \xi \in[0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let
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\end{equation*}
$$

${ }^{(*)}$ disproves (**).

Nonlinear pricing models

$$
f \neq 0
$$

Assume $\forall x \in \mathbb{R}^{d}: \mu(x)=0, \sigma(x)=\operatorname{ld}_{\mathbb{R}^{d}}$, assume $f: \mathbb{R}^{d} \times \mathbb{R} \rightarrow \mathbb{R}$, let
$\theta=\cup_{n \in \mathbb{N}} \mathbb{R}^{n}$, let $W^{\theta}:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}, \theta \in \Theta$, be independent Brownian motions, define $\Delta W_{s, t}^{\theta}=W_{t}^{\theta}-W_{s}^{\theta}$ and note $\forall s \in[0, T), x \in \mathbb{R}^{d}$ :

$$
\begin{aligned}
u(s, x)= & g(x)+\mathbb{E}\left[\left(g\left(x+\Delta W_{s, T}^{0}\right)-g(x)\right)\right] \\
& +\int_{s}^{T} \mathbb{E}\left[f\left(x+\Delta W_{s, t}^{0}, u\left(t, x+\Delta W_{s, t}^{0}\right)\right)\right] d t .
\end{aligned}
$$

Full history recursive multilevel Picard approximations For all $\theta \in \Theta, k, \rho \in \mathbb{N}$, $s \in[0, T), x \in \mathbb{R}^{d}$ define $U_{0 . \rho, s}^{\theta}(x)=0$ and


Assume $\forall x \in \mathbb{R}^{d}: \mu(x)=0, \sigma(x)=\operatorname{ld}_{\mathbb{R}^{d}}$, assume $f: \mathbb{R}^{d} \times \mathbb{R} \rightarrow \mathbb{R}$, let $\Theta=\cup_{n \in \mathbb{N}} \mathbb{R}^{n}$, let $W^{\theta}:[0, T] \times \Omega \rightarrow \mathbb{R}^{d}, \theta \in \Theta$, be independent Brownian motions, define $\Delta W_{s, t}^{\theta}=W_{t}^{\theta}-W_{s}^{\theta}$ and note $\forall s \in[0, T), x \in \mathbb{R}^{d}$ :

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Full history recursive multilevel Picard approximations

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$$
\begin{aligned}
& \mathbf{U}_{k, \rho, s}^{\theta}(x)=g(x)+\sum_{i=1}^{m_{k, \rho}} \frac{g\left(x+\Delta W_{s, T}^{(\theta, 0,-i)}\right)-g(x)}{m_{k, \rho}} \\
& +\sum_{l=0}^{k-1} \sum_{i=1}^{m_{k-1, \rho}} \sum_{t \in(s, T]} \frac{q_{s}^{k-l, \rho}(t)}{m_{k-l, \rho}}\left[f\left(x+\Delta W_{s, t}^{(\theta, l, i)}, \mathbf{U}_{l, \rho, t}^{(\theta, l, t)}\left(x+\Delta W_{s, t}^{(\theta, l, i)}\right)\right)\right. \\
& \left.-\mathbb{1}_{\mathbb{N}}(I) f\left(x+\Delta W_{s, t}^{(\theta, l, i)}, \mathbf{U}_{[l-1]^{+}, \rho, t}^{(\theta,-l, i)}\left(x+\Delta W_{s, t}^{(\theta, l, i)}\right)\right)\right] .
\end{aligned}
$$

## Allen-Cahn equation

$$
\frac{\partial}{\partial t} u(t, x)+u(t, x)-[u(t, x)]^{3}+\frac{1}{2}\left(\Delta_{x} u\right)(t, x)=0, \quad(t, x) \in[0, T) \times \mathbb{R}^{1}
$$ $u(0, \xi) \approx \mathrm{v}=0.905$. Simulations: MATLAB, Intel i7 CPU, 2.8 GHz, 16 GB RAM.



## Allen-Cahn equation $T=1, \xi=0 \in \mathbb{R}^{1}, u(T, x)=\frac{1}{1+\|x\|_{\infty}}$, and

$$
\frac{\partial}{\partial t} u(t, x)+u(t, x)-[u(t, x)]^{3}+\frac{1}{2}\left(\Delta_{x} u\right)(t, x)=0, \quad(t, x) \in[0, T) \times \mathbb{R}^{1}
$$

## Relative errors $\frac{1}{10|\mathrm{v}|} \sum_{i=1}^{10}\left|\mathbf{U}_{\rho, \rho}^{i}(0, \xi)-\mathrm{v}\right|$ for $\rho \in\{1,2, \ldots, 5\}$ against runtime;

 $u(0, \xi) \approx v=0.905$. Simulations: MATLAB, Intel i7 CPU, 2.8 GHz, 16 GB RA .

Allen-Cahn equation $T=1, \xi=0 \in \mathbb{R}^{1}, u(T, x)=\frac{1}{1+\|x\|_{\infty}}$, and

$$
\frac{\partial}{\partial t} u(t, x)+u(t, x)-[u(t, x)]^{3}+\frac{1}{2}\left(\Delta_{x} u\right)(t, x)=0, \quad(t, x) \in[0, T) \times \mathbb{R}^{1} .
$$

## Relative errors $\frac{1}{10|\mathrm{v}|} \sum_{i=1}^{10}\left|u_{\rho, \rho}^{i}(0, \xi)-\mathrm{v}\right|$ for $\rho \in\{1,2, \ldots, 5\}$ against runtime;



Allen-Cahn equation $T=1, \xi=0 \in \mathbb{R}^{1}, u(T, x)=\frac{1}{1+\|x\|_{\infty}}$, and

$$
\frac{\partial}{\partial t} u(t, x)+u(t, x)-[u(t, x)]^{3}+\frac{1}{2}\left(\Delta_{x} u\right)(t, x)=0, \quad(t, x) \in[0, T) \times \mathbb{R}^{1} .
$$

Relative errors $\frac{1}{10|v|} \sum_{i=1}^{10}\left|\mathbf{U}_{\rho, \rho}^{i}(0, \xi)-v\right|$ for $\rho \in\{1,2, \ldots, 5\}$ against runtime;


Allen-Cahn equation $T=1, \xi=0 \in \mathbb{R}^{1}, u(T, x)=\frac{1}{1+\|x\|_{\infty}}$, and

$$
\frac{\partial}{\partial t} u(t, x)+u(t, x)-[u(t, x)]^{3}+\frac{1}{2}\left(\Delta_{x} u\right)(t, x)=0, \quad(t, x) \in[0, T) \times \mathbb{R}^{1} .
$$

Relative errors $\frac{1}{10|v|} \sum_{i=1}^{10}\left|\mathbf{U}_{\rho, \rho}^{i}(0, \xi)-v\right|$ for $\rho \in\{1,2, \ldots, 5\}$ against runtime; $u(0, \xi) \approx \mathrm{v}=0.905$.


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$$

Relative errors $\frac{1}{10|v|} \sum_{i=1}^{10}\left|\mathbf{U}_{\rho, \rho}^{i}(0, \xi)-v\right|$ for $\rho \in\{1,2, \ldots, 5\}$ against runtime; $u(0, \xi) \approx \mathrm{v}=0.905$. Simulations: MATLAB, Intel i7 CPU, $2.8 \mathrm{GHz}, 16 \mathrm{~GB}$ RAM.


Allen-Cahn equation $T=1, \xi=(0,0, \ldots, 0) \in \mathbb{R}^{100}, u(T, x)=\frac{1}{1+\|x\|_{\infty}}$, and

$$
\frac{\partial}{\partial t} u(t, x)+u(t, x)-[u(t, x)]^{3}+\frac{1}{2}\left(\Delta_{x} u\right)(t, x)=0, \quad(t, x) \in[0, T) \times \mathbb{R}^{100}
$$

Relative increments $\left[\frac{1}{10} \sum_{i=1}^{10}\left|\mathbf{U}_{\rho+1, \rho+1}^{i}(0, \xi)-\mathbf{U}_{\rho, \rho}^{i}(0, \xi)\right|\right] /\left[\frac{1}{10}\left|\sum_{i=1}^{10} \mathbf{U}_{5,5}^{i}(0, \xi)\right|\right]$ for $\rho \in\{1,2,3,4\}$ against runtime $; u(0, \xi) \approx 0.317$.


Allen-Cahn equation Runtime for one realization of $\mathbf{U}_{4.4}^{1}(0, \xi)$ against dimension $d \in\{5,6, \ldots, 100\}$.


## Pricing with default risk

\& Zhuo 2015 MF$) T=1, d=1, \xi=(100$,
$\frac{\partial}{\partial t} u(t, x)+f(x, u(t, x))+\bar{\mu} \sum_{i=1}^{d} x_{i}\left(\frac{\partial}{\partial x_{i}} u\right)(t, x)+\frac{\bar{\sigma}^{2}}{2} \sum_{i=1}^{d}\left|x_{i}\right|^{2}\left(\frac{\partial^{2}}{\partial x_{i}^{2}} u\right)(t, x)=0$ for $(t, x) \in[0, T) \times \mathbb{R}^{d}$. Relative arrors $\frac{1}{10} \sum^{10}\left|w_{p, \rho}(0,5)-\cdots\right|$ for $\rho \in\{1,2, \ldots, 7\}$ against runtime; $u(0, \xi) \approx \mathrm{v}=97.705$.


Pricing with default risk (Duffie, Schroder, \& Skiadas 1996 AAP, Bender, Schweizer, \& Zhuo 2015 MF)
for $(t, x) \in[0, T) \times \mathbb{R}^{d}$. Relative errors $\frac{1}{10|v|} \sum_{i=1}^{10}\left|u_{\rho, \rho}^{i}(0, \xi)-v\right|$ for $\rho \in\{1,2, \ldots, 7\}$ against runtime; $u(0, \xi) \approx \mathrm{v}=97.705$.


Pricing with default risk (Duffie, Schroder, \& Skiadas 1996 AAP, Bender, Schweizer, \& Zhuo 2015 MF) $T=1, d=1, \xi=(100, \ldots, 100) \in \mathbb{R}^{d}, u(T, x)=\min _{1 \leq i \leq d} x_{i}$, against runtime; $u(0, \xi) \approx \mathrm{v}=97.705$.


Pricing with default risk (Duffie, Schroder, \& Skiadas 1996 AAP, Bender, Schweizer, \& Zhuo 2015 MF) $T=1, d=1, \xi=(100, \ldots, 100) \in \mathbb{R}^{d}, u(T, x)=\min _{1 \leq i \leq d} x_{i}$,

$$
\frac{\partial}{\partial t} u(t, x)+f(x, u(t, x))+\bar{\mu} \sum_{i=1}^{d} x_{i}\left(\frac{\partial}{\partial x_{i}} u\right)(t, x)+\frac{\bar{\sigma}^{2}}{2} \sum_{i=1}^{d}\left|x_{i}\right|^{2}\left(\frac{\partial^{2}}{\partial x_{i}^{2}} u\right)(t, x)=0
$$ for $(t, x) \in[0, T) \times \mathbb{R}^{d}$.

against runtime; $u(0, \xi) \approx \mathrm{v}=97.705$.


Pricing with default risk (Duffie, Schroder, \& Skiadas 1996 AAP, Bender, Schweizer, \& Zhuo 2015 MF) $T=1, d=1, \xi=(100, \ldots, 100) \in \mathbb{R}^{d}, u(T, x)=\min _{1 \leq i \leq d} x_{i}$,

$$
\frac{\partial}{\partial t} u(t, x)+f(x, u(t, x))+\bar{\mu} \sum_{i=1}^{d} x_{i}\left(\frac{\partial}{\partial x_{i}} u\right)(t, x)+\frac{\bar{\sigma}^{2}}{2} \sum_{i=1}^{d}\left|x_{i}\right|^{2}\left(\frac{\partial^{2}}{\partial x_{i}^{2}} u\right)(t, x)=0
$$ for $(t, x) \in[0, T) \times \mathbb{R}^{d}$. Relative errors $\frac{1}{10|v|} \sum_{i=1}^{10}\left|\mathbf{U}_{\rho, \rho}^{i}(0, \xi)-\mathrm{v}\right|$ for $\rho \in\{1,2, \ldots, 7\}$ against runtime;



Pricing with default risk (Duffie, Schroder, \& Skiadas 1996 AAP, Bender, Schweizer, \& Zhuo 2015 MF) $T=1, d=1, \xi=(100, \ldots, 100) \in \mathbb{R}^{d}, u(T, x)=\min _{1 \leq i \leq d} x_{i}$,

$$
\frac{\partial}{\partial t} u(t, x)+f(x, u(t, x))+\bar{\mu} \sum_{i=1}^{d} x_{i}\left(\frac{\partial}{\partial x_{i}} u\right)(t, x)+\frac{\bar{\sigma}^{2}}{2} \sum_{i=1}^{d}\left|x_{i}\right|^{2}\left(\frac{\partial^{2}}{\partial x_{i}^{2}} u\right)(t, x)=0
$$

for $(t, x) \in[0, T) \times \mathbb{R}^{d}$. Relative errors $\frac{1}{10|v|} \sum_{i=1}^{10}\left|\mathbf{U}_{\rho, \rho}^{i}(0, \xi)-\mathrm{v}\right|$ for $\rho \in\{1,2, \ldots, 7\}$ against runtime; $u(0, \xi) \approx \mathrm{v}=97.705$.


Pricing with default risk (Duffie, Schroder, \& Skiadas 1996 AAP, Bender, Schweizer, \& Zhuo 2015 MF) $T=1, d=100, \xi=(100, \ldots, 100) \in \mathbb{R}^{d}, u(T, x)=\min _{1 \leq i \leq d} x_{i}$,

$$
\begin{aligned}
& \frac{\partial}{\partial t} u(t, x)+f(x, u(t, x))+\bar{\mu} \sum_{i=1}^{d} x_{i}\left(\frac{\partial}{\partial x_{i}} u\right)(t, x)+\frac{\bar{\sigma}^{2}}{2} \sum_{i=1}^{d}\left|x_{i}\right|^{2}\left(\frac{\partial^{2}}{\partial x_{i}^{2}} u\right)(t, x)=0 \\
& \text { for }(t, x) \in[0, T) \times \mathbb{R}^{d} .\left[\frac{1}{10} \sum_{i=1}^{10}\left|\mathbf{U}_{\rho+1, \rho+1}^{i}(0, \xi)-\mathbf{U}_{\rho, \rho}^{i}(0, \xi)\right|\right] /\left[\frac{1}{10}\left|\sum_{i=1}^{10} \mathbf{U}_{7,7}^{i}(0, \xi)\right|\right] \text { for } \\
& \rho \in\{1,2, \ldots, 6\} \text { against runtime; } u(0, \xi) \approx 58.113 .
\end{aligned}
$$



Pricing with default risk Runtime for one realization of $\mathbf{U}_{6.6}^{1}(0, \xi)$ against dimension $d \in\{5,6, \ldots, 100\}$.


Thanks for your attention!

Thanks for your attention!

Pricing with default risk (Duffie, Schroder, \& Skiadas 1996 AAP, Bender, Schweizer, \& Zhuo 2015 MF)

Consider $\delta=\frac{2}{3}, R=\frac{2}{100}, \gamma^{h}=\frac{2}{10}, \gamma^{\prime}=\frac{2}{100}, \bar{\mu}=\frac{2}{100}, \bar{\sigma}=\frac{2}{10}, v^{h}, v^{\prime} \in(0, \infty)$ satisfy $v^{h}<v^{\prime}$, and assume for all $x \in \mathbb{R}^{d}, y \in \mathbb{R}$ that

$$
\mu(x)=\bar{\mu} x, \quad \sigma(x)=\bar{\sigma} \operatorname{diag}(x)
$$

and

$$
\begin{aligned}
& f(x, y)=-(1-\delta) y\left[\gamma^{h} \mathbb{1}_{\left(-\infty, v^{h}\right)}(y)+\gamma^{\prime} \mathbb{1}_{\left[v^{\prime}, \infty\right)}(y)\right. \\
&\left.+\left[\frac{\left(\gamma^{h}-\gamma^{\prime}\right)}{\left(v^{h}-v^{\prime}\right)}\left(y-v^{h}\right)+\gamma^{h}\right] \mathbb{1}_{\left[v^{h}, v^{\prime}\right)}(y)\right]-R y .
\end{aligned}
$$

- We consider $v^{h}=50, v^{\prime}=120$ in the case $d=1$.
- Bender et al. consider $v^{h}=54, v^{\prime}=90$ in the case $d=5$.
- We consider $v^{h}=47, v^{\prime}=65$ in the case $d=100$.

