Hydrodynamic flocking model with external potential forcing

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The model

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = \int \phi(\|\mathbf{x} - \mathbf{y}\|)(\mathbf{u}(\mathbf{y}, t) - \mathbf{u}(\mathbf{x}, t))\rho(\mathbf{y})\, \mathrm{d}\mathbf{y} - \nabla \Psi(\mathbf{x}) \end{cases}$$

pressureless Euler equations

Cucker-Smale interaction (non-local alignment)

external forcing

- ρ, $\mathbf{u}$: density and velocity field of a continuum of agents
- ϕ : Cucker-Smale interaction potential
- Ψ : external potential

Motivation

- Particle Cucker-Smale model

$$\begin{cases} \dot{\mathbf{x}}_i = \mathbf{v}_i \\ \dot{\mathbf{v}}_i = \dfrac{1}{N} \displaystyle\sum_{j \neq i} \phi(\|\mathbf{x}_i - \mathbf{x}_j\|)(\mathbf{v}_j - \mathbf{v}_i) \end{cases} \qquad i = 1, \ldots, N$$

- **Flocking**: under suitable assumptions on ϕ, there holds

$$\|\mathbf{v}_i(t) - \mathbf{v}_j(t)\| \to 0, \qquad \text{as } t \to \infty$$

- Pairwise interaction —> global velocity alignment

Cucker-Smale (2007), Ha-Tadmor (2008), Ha-Liu (2009)

Motivation

particle model

$$\begin{cases} \dot{\mathbf{x}}_i = \mathbf{v}_i \\ \dot{\mathbf{v}}_i = \frac{1}{N} \sum_{j \neq i} \phi(\|\mathbf{x}_i - \mathbf{x}_j\|)(\mathbf{v}_j - \mathbf{v}_i) \end{cases} \qquad i = 1, \dots, N$$

mean field limit ↓

kinetic model

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \nabla_{\mathbf{v}} \cdot \left(f \int \phi(\|\mathbf{x} - \mathbf{y}\|)(\mathbf{w} - \mathbf{v}) f(\mathbf{y}, \mathbf{w}, t) \, \mathrm{d}\mathbf{w} \, \mathrm{d}\mathbf{y} \right) = 0$$

mono-kinetic ansatz ↓

$f(\mathbf{x}, \mathbf{v}, t) = \rho(\mathbf{x}, t) \delta(\mathbf{v} - \mathbf{u}(\mathbf{x}, t))$

(other ansatz (e.g. Maxwellian) leads to Euler equations with pressure)

hydrodynamic model

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = \int \phi(\|\mathbf{x} - \mathbf{y}\|)(\mathbf{u}(\mathbf{y}, t) - \mathbf{u}(\mathbf{x}, t)) \rho(\mathbf{y}) \, \mathrm{d}\mathbf{y} \end{cases}$$

Ha-Tadmor (2008)

Motivation

- Another aspect for the hydrodynamic model: existence of global smooth solutions — **critical thresholds**.
- Cucker-Smale interaction tends to **suppress** the finite time blow-up of the pressureless Euler equations.
- 1d: Tadmor-Tan (2014), Carrillo-Choi-Tadmor-Tan (2016), Shvydkoy-Tadmor (2017, 2018), Do-Kiselev-Ryzhik-Tan (2018)
- 2d: Tadmor-Tan (2014), He-Tadmor (2017)

Motivation

- In reality, moving agents are subject to pairwise interaction forces as well as external forces.

- These forces may **compete** with the alignment forces, make it harder to achieve the flocking state.

- In this talk we focus on the **external potential forces**. Its particle model counterpart is studied in Ha-Shu (2018), mainly in one spatial dimension.

- Potential forces is possibly the easiest type of external force to study, since there still holds the energy dissipation.

Pairwise interaction: Carrillo-Choi-Tadmor-Tan (2016), Carrillo-Choi-Tse (2018), …

Main results

- 'Smooth solutions must flock':
  - (1) Harmonic potential $\Psi(\mathbf{x}) = \frac{a}{2}\|\mathbf{x}\|^2$, general ϕ
  - (2) General convex potential, constant and large ϕ
- Method: hypocoercivity (two different types)

Main results

- Existence of global smooth solutions: critical thresholds
  - (1) 1d: thresholds for global smooth solutions and blow-up
  - (2) 2d, harmonic potential (similar to He-Tadmor)
  - (3) 2d, general potential (including those without a flocking estimate!)
- All thresholds depend on **the size of** $\nabla^2\Psi$: one reason why harmonic potential is special.
- Method: characteristics + spectral dynamics + L^∞ estimates

Flocking: harmonic potential

Theorem 2.2. *Let $\Psi(\mathbf{x}) = \frac{a}{2}\|\mathbf{x}\|^2$ be the harmonic potential. Let $(\rho, \mathbf{u})$ be a global smooth solution to (1.1). Assume ρ_{in} has compact support. Then there holds the flocking estimate at exponential rate in both velocity and position:*

$$\mathcal{E}(t) := \int\int (\|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})\|^2 + a\|\mathbf{x} - \mathbf{y}\|^2)\rho(\mathbf{x})\rho(\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \le C\mathcal{E}(0)e^{-\lambda t}, \tag{2.3}$$

where $\lambda > 0$ depends on a, ϕ_-, ϕ_+, m_0, and $C > 0$ is an absolute constant.

- Velocity alignment happens together with spatial concentration! (because spatial deviation induces velocity deviation, if Ψ is convex.)

- In the limit $a \to 0$, one has $\lambda = O(a)$. This means the strength of external potential may have big influence on flocking rate!

- Reduce to the case: mean velocity/position zero
- Energy estimate

$$\partial_t \int (\frac{1}{2}\|\mathbf{u}(\mathbf{x},t)\|^2 + \frac{a}{2}\|\mathbf{x}\|^2)\rho(\mathbf{x},t)\,\mathrm{d}\mathbf{x} \le -m_0\phi_- \int \|\mathbf{u}\|^2\rho\,\mathrm{d}\mathbf{x}$$

- Cross term

$$\partial_t \int \mathbf{u}(\mathbf{x},t)\cdot\mathbf{x}\rho(\mathbf{x},t)\,\mathrm{d}\mathbf{x} \le -\frac{a}{2}\int \|\mathbf{x}\|^2\rho\,\mathrm{d}\mathbf{x} + (1+\frac{m_0^2\phi_+^2}{a})\int \|\mathbf{u}\|^2\rho\,\mathrm{d}\mathbf{x}$$

- Use hypocoercivity to get exponential decay

Flocking: general convex potential

Theorem 2.3. *Let* $\Psi(\mathbf{x})$ *be strictly convex with bounded Hessian:*

$$a\|\mathbf{y}\|^2 \le \mathbf{y}^T \nabla^2 \Psi(\mathbf{x}) \mathbf{y} \le A\|\mathbf{y}\|^2, \quad \forall \mathbf{y} \neq 0, \quad 0 < a < A \tag{2.4}$$

and ϕ *is constant, and satisfies*

$$m_0 \phi > \frac{A}{\sqrt{a}} \tag{2.5}$$

Let $(\rho, \mathbf{u})$ *be a global smooth solution to (1.1). Assume* ρ_{in} *has compact support. Then there holds the flocking at exponential rate in both velocity and position:*

$$\mathcal{E}(t) := \int\int (\|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})\|^2 + a\|\mathbf{x} - \mathbf{y}\|^2)\rho(\mathbf{x})\rho(\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \le C\mathcal{E}(0)e^{-\tilde{\lambda}t}, \tag{2.6}$$

where $\tilde{\lambda} > 0$ *depends on* $a, A, \phi_-, \phi_+, m_0$*, and* $C > 0$ *depends on* $a, A, m_0\phi$.

- Cannot reduce to ‘mean-zero’ case
- Total energy does NOT measure position concentration: cannot apply hypocoercivity directly!

- A new Lyapunov functional

$$F(t) = \frac{K}{2}|||\mathbf{x}-\mathbf{y}|||^2 + \langle \mathbf{x}-\mathbf{y}, \mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})\rangle + \frac{\beta}{2}|||\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})|||^2$$

$K = m_0\phi$

$$\langle f(\mathbf{x},\mathbf{y}), g(\mathbf{x},\mathbf{y})\rangle := \int\int f(\mathbf{x},\mathbf{y})\cdot g(\mathbf{x},\mathbf{y})\rho(\mathbf{x})\rho(\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}, \quad |||f(\mathbf{x},\mathbf{y})|||^2 := \langle f(\mathbf{x},\mathbf{y}), f(\mathbf{x},\mathbf{y})\rangle$$

$$\begin{aligned}\frac{\mathrm{d}F}{\mathrm{d}t} &= \int\int [-(K\beta-1)\|\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y})\|^2 - (\mathbf{x}-\mathbf{y})\cdot(\nabla\Psi(\mathbf{x})-\nabla\Psi(\mathbf{y})) \\ &\quad - \beta(\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{y}))\cdot(\nabla\Psi(\mathbf{x})-\nabla\Psi(\mathbf{y}))]\rho(\mathbf{x})\rho(\mathbf{y})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\end{aligned}$$

- The second term is a good term if Ψ is convex.

- If K is large enough, then one can choose a proper β to absorb the bad term (the last term)

Regularity: 1d

Theorem 2.7. *Let the space dimension $d = 1$. Assume Ψ'' is bounded below, and bounded above by:*

$$\Psi''(x) \leq A, \quad \forall x \in \Omega \tag{2.12}$$

with A being a constant satisfying

$$A < \frac{(m_0\phi_-)^2}{4} \tag{2.13}$$

Further assume that

$$\max_{x \in \text{supp } \rho_{in}} (\partial_x u_{in}(x) + (\phi * \rho_{in})(x)) > \underline{\frac{m_0\phi_-}{2} - \sqrt{\frac{(m_0\phi_-)^2}{4} - A}} \tag{2.14}$$

then (1.1) admits global smooth solution.

positive, if A > 0;
negative, if A < 0.

- Positive Ψ'' induces blow-up; negative Ψ'' suppresses.
- The assumptions do NOT imply flocking.

Theorem 2.8. *Assume*

$$\Psi''(x) \geq B, \quad \forall x \in \Omega$$

- *If* B *is so large that*

$$B > \frac{(m_0\phi_+)^2}{4}$$

  holds, then $\partial_x u$ *blows up to* $-\infty$ *in finite time for any initial data.*

- *If (2.18) does not hold but* $B > 0$*, then* $\partial_x u$ *blows up to* $-\infty$ *in finite time if*

$$\partial_x u_{in}(x) + (\phi * \rho_{in})(x) < \frac{m_0\phi_+}{2} - \sqrt{\frac{(m_0\phi_+)^2}{4} - B}$$

  for some x*. (notice that in this condition* $RHS > 0$*)*

- *If* $B \leq 0$*, then* $\partial_x u$ *blows up to* $-\infty$ *in finite time if*

$$\partial_x u_{in}(x) + (\phi * \rho_{in})(x) < \frac{m_0\phi_-}{2} - \sqrt{\frac{(m_0\phi_-)^2}{4} - B}$$

  for some x*. (notice that in this condition* $RHS \leq 0$*)*

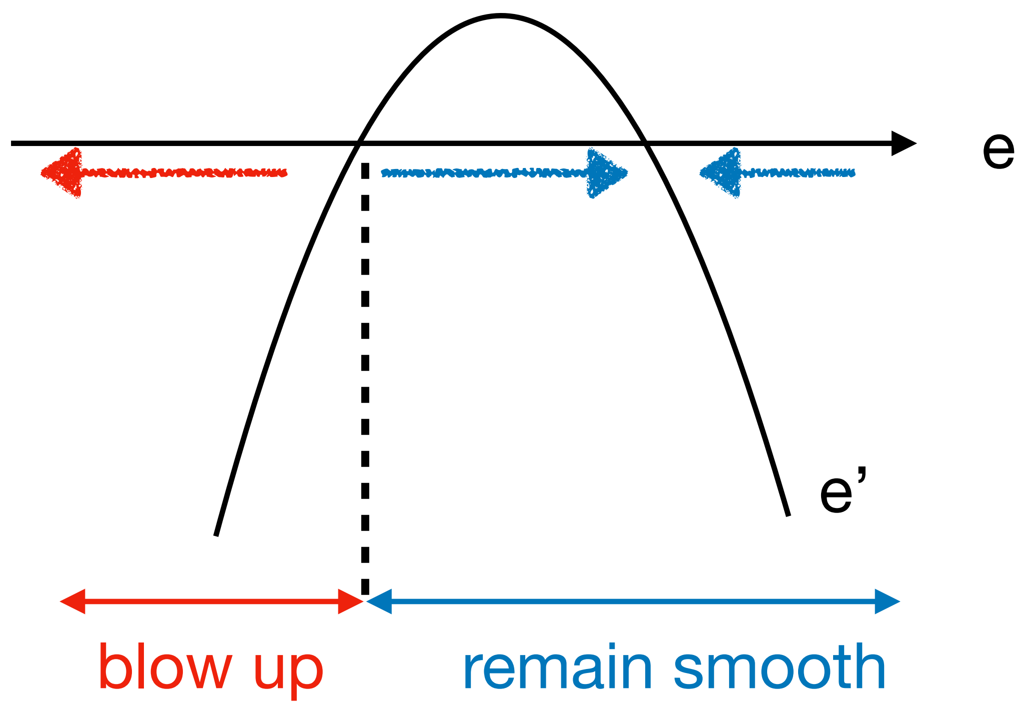
- The 'magic quantity' $\mathsf{e} = \partial_x u + \phi * \rho$

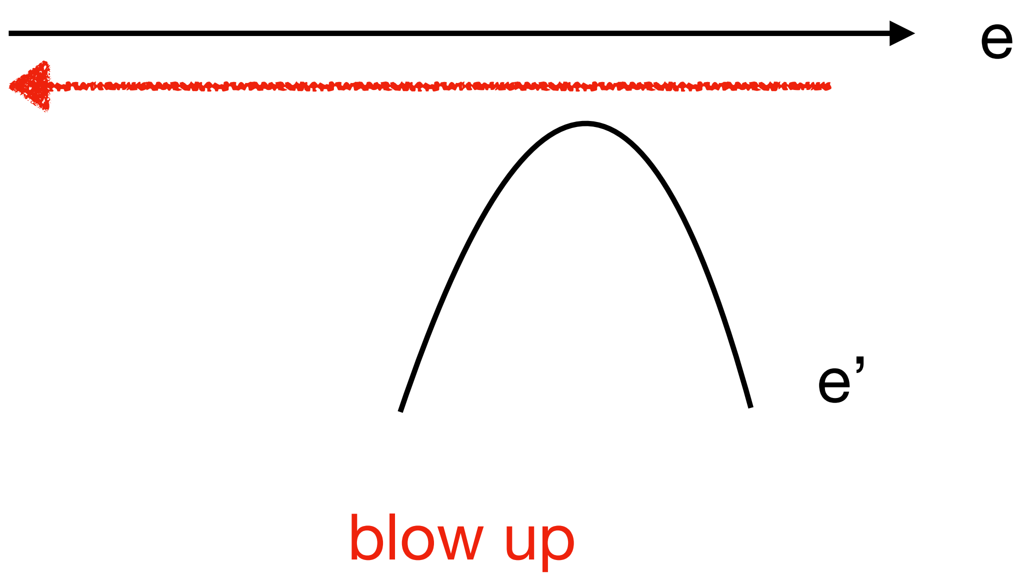
$$\rho' = -\rho(\mathsf{e} - \phi * \rho)$$

$$\mathsf{e}' = \boxed{-\mathsf{e}(\mathsf{e} - \phi * \rho) - \Psi''}$$

time derivative along characteristics

quadratic form in e





Carrillo-Choi-Tadmor-Tan, Shvydkoy-Tadmor

Regularity: 2d, harmonic potential

Theorem 2.9. *Let the space dimension $d = 2$, and $\Psi(\mathbf{x}) = \frac{a}{2}\|\mathbf{x}\|^2$. There exists a positive constant C_1, depending on m_0, ϕ_-, ϕ_+, a and $|\phi'|_\infty$ (the L^∞ norm of ϕ'), such that the following holds: Assume*

$$c_1^2 := m_0^2\phi_-^2 - \left(\max_{\mathbf{x}\in \text{supp } \rho_{in}} |(\eta_S)_{in}(\mathbf{x})| + C_1\mathcal{E}_\infty(0)\right)^2 - 4a > 0 \tag{2.21}$$

where η_S is the difference between the two eigenvalues of the symmetric matrix $(\nabla\mathbf{u}+(\nabla\mathbf{u})^T)/2$. Assume

$$\max_{\mathbf{x}\in \text{supp } \rho_{in}} (\nabla\cdot\mathbf{u}_{in}(\mathbf{x}) + (\phi * \rho_{in})(\mathbf{x})) \geq 0 \tag{2.22}$$

then (1.1) admits global smooth solution.

- The estimate is good only when the **size of a is moderate**: large a will blow up the 4a term (the same reason as 1d), and small a will blow up the C_1 term (because flocking estimate is bad).

$$\partial_t M + \mathbf{u}\cdot\nabla M + M^2 = -(\phi * \rho)M + R - \nabla^2\Psi \qquad M_{ij} = \partial_j u_i$$

$$R_{ij} = \partial_j\phi * (\rho u_i) - u_i(\partial_j\phi * \rho)$$

- Spectral dynamics

$$\mathsf{e}' = \frac{1}{2}(4\omega^2 + (\phi * \rho)^2 - \eta_S^2 - \mathsf{e}^2 - 2\Delta\Psi) \qquad \mathsf{e} = \nabla\cdot\mathbf{u} + \phi * \rho$$

$$\omega = (\partial_1 u_2 - \partial_2 u_1)/2$$

$$\eta_S' + \mathsf{e}\eta_S = q := \langle \mathbf{s}_2, R_{sym}\mathbf{s}_2\rangle - \langle \mathbf{s}_1, R_{sym}\mathbf{s}_1\rangle - \langle \mathbf{s}_2, \nabla^2\Psi\mathbf{s}_2\rangle + \langle \mathbf{s}_1, \nabla^2\Psi\mathbf{s}_1\rangle$$

$\mathbf{s}_i$: (orthonormal) eigenvectors of S, the symmetric part of M

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H. Liu-Tadmor (2002), He-Tadmor

- For harmonic potential,

$$\eta_S' + \mathrm{e}\eta_S = q := \langle \mathbf{s}_2, R_{sym}\mathbf{s}_2\rangle - \langle \mathbf{s}_1, R_{sym}\mathbf{s}_1\rangle - \langle \mathbf{s}_2, \nabla^2\Psi\mathbf{s}_2\rangle + \langle \mathbf{s}_1, \nabla^2\Psi\mathbf{s}_1\rangle$$

- Flocking estimate —> exponential decay of R

- If e>=0, then

$$|\eta_S(t)| \le |(\eta_S)_{in}| + \int_0^\infty |R(s)|\mathrm{d}s, \quad \forall t$$

- Use this to propagate e>=0

require pointwise (L^\infty) flocking estimate: this can be done by similar hypocoercivity on characteristics

Similar to He-Tadmor

Regularity: 2d, general potential

Theorem 2.10. *Let the space dimension $d = 2$. Assume $\nabla^2\Psi$ is bounded:*

$$\|\nabla^2\Psi(\mathbf{x})\| \le A$$

Assume that there is an a priori estimate

$$\max_{t\ge 0,\, \mathbf{x}\in \text{supp}} \|\mathbf{u}(\mathbf{x},t)\| \le u_{max}$$

for some constant u_{max}. If there hold

$$C_2 := 8|\phi'|_\infty m_0 u_{max} + 2A < m_0^2\phi_-^2/2 - 2A$$

and

$$\max_{\mathbf{x}\in \text{supp}\ \rho_{in}} |(\eta_S)_{in}(\mathbf{x})| \le \sqrt{(m_0^2\phi_-^2/2 - 2A) + \sqrt{(m_0^2\phi_-^2/2 - 2A)^2 - C_2^2}}$$

and

$$\max_{\mathbf{x}\in \text{supp}\ \rho_{in}} (\nabla\cdot\mathbf{u}_{in}(\mathbf{x}) + (\phi * \rho_{in})(\mathbf{x})) > \boxed{\sqrt{(m_0^2\phi_-^2/2 - 2A) - \sqrt{(m_0^2\phi_-^2/2 - 2A)^2 - C_2^2}}}$$

then (1.1) admits global smooth solution.

require strictly positive lower bound

- For general potential,

$$\eta_S' + \mathsf{e}\eta_S = q := \langle \mathbf{s}_2, R_{sym}\mathbf{s}_2\rangle - \langle \mathbf{s}_1, R_{sym}\mathbf{s}_1\rangle - \langle \mathbf{s}_2, \nabla^2\Psi\mathbf{s}_2\rangle + \langle \mathbf{s}_1, \nabla^2\Psi\mathbf{s}_1\rangle$$

- No flocking estimate: R does not decay. The best we can do is a uniform-in-time bound: the u_{max} assumption
- $\nabla^2\Psi$ is not identity matrix —> last two terms are O(1)
- **New idea:** make use of the good term $\mathsf{e}\eta_S$
- Need to propagate a **positive** lower bound of e

L^\infty estimate

This allows non-convex potentials
(even with multiple local minima)

Proposition 2.11. *Assume that there exists constant $A, a > 0$, $X_0 > 0$ such that*

$$\frac{a}{2}\|\mathbf{x}\|^2 \le \Psi(\mathbf{x}) \le \frac{A}{2}\|\mathbf{x}\|^2, \quad a\|\mathbf{x}\| \le \|\nabla\Psi(\mathbf{x})\| \le A\|\mathbf{x}\|, \quad \forall \mathbf{x} \in \Omega,\ \|\mathbf{x}\| \ge X_0 \tag{2.29}$$

Then there exists a constant u_{max}, depending on $a, A, \phi_-, \phi_+, m_0, E(0)$, where $E(t)$ is the total energy

$$E(t) = \int (\frac{1}{2}\|\mathbf{u}(\mathbf{x},t)\|^2 + \Psi(\mathbf{x}))\rho(\mathbf{x},t)\,\mathrm{d}\mathbf{x} \tag{2.30}$$

such that

$$\max_{t\ge 0,\, \mathbf{x}\in \mathrm{supp}\ \rho_{in}} \|\mathbf{u}(\mathbf{x},t)\| \le u_{max} \tag{2.31}$$

- Method: hypocoercivity along characteristics

$$F(\mathbf{x},t) = \frac{1}{2}\|\mathbf{u}(\mathbf{x},t)\|^2 + \Psi(\mathbf{x}) + c\mathbf{u}(\mathbf{x},t)\cdot\nabla\Psi(\mathbf{x})$$

Conclusion

- 'Smooth solutions must flock':
  - (1) Harmonic potential $\Psi(\mathbf{x}) = \frac{a}{2}\|\mathbf{x}\|^2$, general ϕ
  - (2) General convex potential, constant and large ϕ
- Existence of global smooth solutions: critical thresholds
  - (1) 1d: thresholds for global smooth solutions and blow-up
  - (2) 2d, harmonic potential (similar to He-Tadmor)
  - (3) 2d, general potential (including those without a flocking estimate!)