Monte Carlo method with negative particles for binary collisions

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The long range Coulomb collisions in plasma can be modeled by Landau-Fokker-Planck (LFP) equation

$$\frac{\partial f}{\partial t} = Q_{LFP}(f,f),$$

with binary collision term

$$Q_{LFP}(f,f) = \frac{1}{4}\frac{\partial}{\partial v_i}\int_{\mathbb{R}^3} u^{-3}(u^2\delta_{ij} - u_i u_j)\left(\frac{\partial}{\partial v_j} - \frac{\partial}{\partial w_j}\right) f(\mathbf{w}) f(\mathbf{v})\, \mathrm{d}\mathbf{w}.$$

- Bilinear;
- Conserves density, momentum and energy;
- Dissipates entropy. $f \to m$ as $t \to \infty$. $Q_{LFP}(m,m) = 0$.

Probabilistic methods– Direct Simulation Monte Carlo

DSMC for binary collisions

- Rarefied gas: Bird 76, Nanbu-Babovsky 83
- Coulomb gas: Takizuka-Abe 77, Nanbu 97

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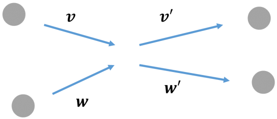
How? Initially sample particles from $f(v, t = 0)$, then in each time step,

- Randomly pick up N_c pairs of particles.
- For each pair ($\mathbf{v}$ and $\mathbf{w}$),
  - Sample a collision angle $\mathbf{n}$.
  - Update $\mathbf{v}, \mathbf{w} \to \mathbf{v}', \mathbf{w}'$.



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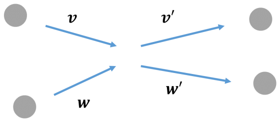


- For rarefied gas (charge neutral), $N_c = O(\Delta t N)$.
- For Coulomb gas (charged), $N_c = N/2$.

Problem in DSMC

Near fluid regime, where $f \approx m$,

- Most computation is spent on the collision between particles sampled from m.
- $Q_{LFP}(m, m) = 0$. The major part of collisions has no net effect.

Highly inefficient!

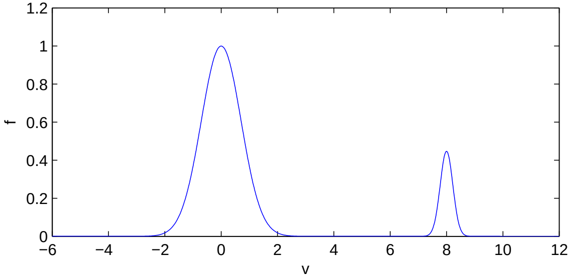
Hybrid methods

Apply splitting

$$f(\mathbf{v}) = m(\mathbf{v}) + f_p(\mathbf{v}),$$

- Equilibrium $m(\mathbf{v})$: evolved according to a fluid equation – cheap
- Deviation $f_p(\mathbf{v}) \geq 0$: represented by particles – expensive

Example: an energetic particle stream injected in a plasma



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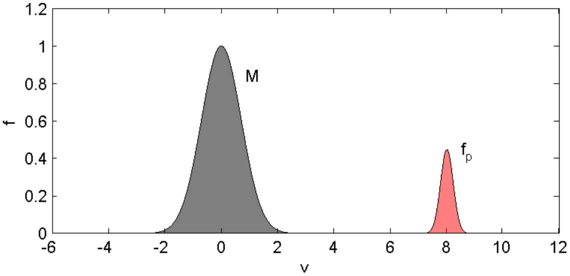

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Collision type	#	
P-P	$\frac{N_p^2}{2N_{tot}}$	
P-M	$\frac{N_p N_m}{N_{tot}}$	
M-M	$\frac{N_m^2}{2N_{tot}}$	Omitted

with $N_{tot} = N_m + N_p$.

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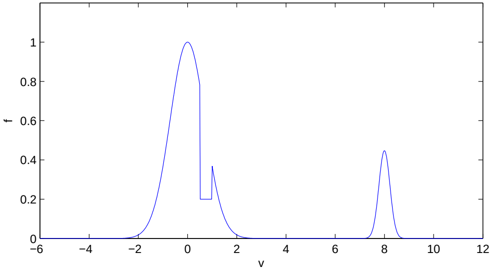
However, Maxwellian part might have defect.



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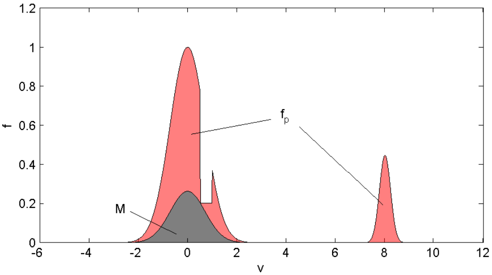

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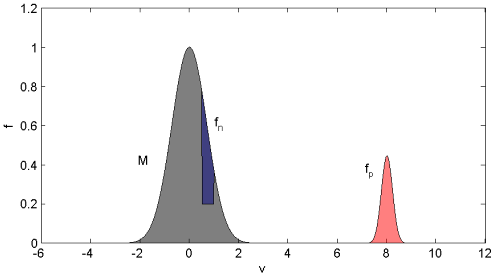

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To minimize the deviation part, we allow $f_p(\mathbf{v}) < 0$. Write

$$f(\mathbf{v}) = m(\mathbf{v}) + f_p(\mathbf{v}) - f_n(\mathbf{v}),$$

with $f_p(\mathbf{v}) \geq 0, f_n(\mathbf{v}) \geq 0$.

We introduce "negative particles" to represent f_n.

- f_p and f_n are represented by P and N particles.
- Q_{LFP} is bilinear $\Rightarrow$ Need to perform P-P, P-N, N-N, P-M and N-M collisions.

Negative particle methods for rarefied gas

(Hadjiconstantinou 05)

One negative particle means the number of particle is -1.

- An N particle cancels a P particle with the same velocity

$$\mathbf{w}_+ + \mathbf{w}_- = 0 \text{ particle.}$$

- A P-N collision cancels a regular P-P collision

$$\begin{aligned} \text{P-P:} \quad & \mathbf{v}_+, \mathbf{w}_+ \to \mathbf{v}'_+, \mathbf{w}'_+, \\ \text{P-N:} \quad & \mathbf{v}_+, \mathbf{w}_- \to 2\mathbf{v}_+, \mathbf{v}'_-, \mathbf{w}'_-. \end{aligned}$$

This can be derived from the Boltzmann equation.

Collision rules with negative particles

$$
\begin{aligned}
\text{P-P:} \quad & \mathbf{v}_+, \mathbf{w}_+ \rightarrow \mathbf{v}'_+, \mathbf{w}'_+, \\
\text{P-N:} \quad & \mathbf{v}_+, \mathbf{w}_- \rightarrow 2\mathbf{v}_+, \mathbf{v}'_-, \mathbf{w}'_-, \\
\text{N-N:} \quad & \mathbf{v}_-, \mathbf{w}_- \rightarrow 2\mathbf{v}_-, 2\mathbf{w}_-, \mathbf{v}'_+, \mathbf{w}'_+, \\
\text{P-M:} \quad & m, \mathbf{v}_+ \rightarrow m, \mathbf{w}_-, \mathbf{v}'_+, \mathbf{w}'_+, \\
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Problem: particle number increases!

Collision with negative particles

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In rarefied gas (charge free),

- short range collision $\Rightarrow$ # collisions in one time step = $\mathcal{O}(\Delta t)$
- The particle number grows in the physical scale

$$(N_p + N_n)\Big|_{t+\Delta t} = (1 + c\Delta t)\,(N_p + N_n)\Big|_t .$$

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- OK...

In Coulomb gas (charged),

- long range collision ⇒ # collisions in one time step = N
- The particle number grows in the numerical scale in Coulomb collisions!

$$(N_p + N_n)\Big|_{t+\Delta t} = \left(1 + \frac{N_m + 2N_n}{N_m + N_p - N_n}\right)(N_p + N_n)\Big|_t.$$

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- Not OK!

- The existing method
  - designed for collisions in rarefied gas
  - does not apply on Coulomb collision
- We develop a new negative particle method
  - for general binary collisions
  - can be applied to Coulomb collision

Combine collisions to reduce new particles

- Problem: too many collisions.
- Some collisions can be “combined”. For example,

$$\text{N-P:} \quad \mathbf{w}_-, \mathbf{v}_+ \to 2\mathbf{v}_+, \mathbf{v}'_-, \mathbf{w}'_-$$

can be combined with

$$\text{P-P:} \quad \mathbf{w}_+, \mathbf{v}_+ \to \mathbf{v}'_+, \mathbf{w}'_+,$$

or

$$\text{M-P:} \quad m, \mathbf{v}_+ \to (m, \mathbf{w}_-, \mathbf{w}_+, \mathbf{v}_+) \to m, \mathbf{w}_-, \mathbf{v}'_+, \mathbf{w}'_+.$$

- Collide first vs “combine” first.

Key idea # 1: Combine collisions

The notation

Bilinear operator $Q(f,g)$: the change in g due to collisions with f.

Ex:

- Boltzmann

$$Q_B(f,g) = \iint_{\mathbb{R}^3\times\mathbb{S}^2} B(|\mathbf{v}-\mathbf{v}_*|,\cos\theta)(f'_*g' - f_*g)\,\mathrm{d}\mathbf{v}_*\,\mathrm{d}\sigma$$

- Landau

$$Q_{LFP}(f,g) = \frac{1}{4}\frac{\partial}{\partial v_i}\int_{\mathbb{R}^3} u^{-3}(u^2\delta_{ij} - u_iu_j)\left(\frac{\partial}{\partial v_j} - \frac{\partial}{\partial w_j}\right) f(\mathbf{w})g(\mathbf{v})\,\mathrm{d}\mathbf{w}.$$

Key idea # 1: Combine collisions

For $f = m + f_p - f_n$, the equation $\partial_t f = Q(f,f)$ is reformulated

$$\begin{aligned}\partial_t f = Q(f,f) &= Q(f,f_p) - Q(f,f_n) + Q(f,m)\\ &= Q(f,f_p) - Q(f,f_n) + Q(f_p - f_n, m) + \underbrace{Q(m,m)}_{=0},\end{aligned}$$

and split:

$$\begin{aligned}\partial_t m &= Q(m,m) = 0,\\ \partial_t f_p &= Q(f,f_p) + (Q(f_p - f_n, m))_+,\\ \partial_t f_n &= Q(f,f_n) + (Q(f_p - f_n, m))_-.\end{aligned}$$

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P - P	P - N	M - M
N - P	N - N	N - M
M - P	M - N	P - M

Key idea # 1: Combine collisions

Apply a forward Euler method in time,

$$\begin{cases} m(t+\Delta t) = m, \\ f_p(t+\Delta t) = f_p + \Delta t Q(f, f_p) + \Delta t (Q(f_p - f_n, m))_+, \\ f_n(t+\Delta t) = f_n + \Delta t Q(f, f_n) + \Delta t (Q(f_p - f_n, m))_-. \end{cases}$$

The dependence on t is omitted in notations.

A Monte Carlo method can be designed accordingly.

Key idea # 1: Combine collisions

A Monte Carlo method

$$f_p(t+\Delta t) = \underbrace{f_p + \Delta t Q(f, f_p)}_{\swarrow} + \underbrace{\Delta t (Q(f_p - f_n, m))_+}_{\searrow} .$$

regular collisions between f and f_p, N_p not change

source term, N_p increases by $O\left(\Delta t (N_p + N_n)\right)$

The particle number grows in the physical scale for *any* binary collisions

$$(N_p + N_n)\Big|_{t+\Delta t} = (1 + c\Delta t)\,(N_p + N_n)\Big|_t .$$

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Summary: Combine collisions to reduce the total number of collisions.

Step 1, collisions between f and f_p

$$f_p(t+\Delta t) = f_p + \Delta t Q(f, f_p) + \Delta t(Q(f_p - f_n, m))_+.$$

Sample a particle from f and collide with a P particle.

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$$\underbrace{\text{Sample a particle from } f}_{\text{How?}} \text{ and collide with a P particle.}$$

- $f = m + f_p - f_n$.
- Need to recover the distributions f_p and f_n from P and N particles $\Rightarrow$ computationally expensive and inaccurate.

Step 1, collisions between f and f_p

We introduce F particles

- give a solution to the original equation $\partial_t f = Q(f, f)$.
  - Initially sampled from $f(v, t = 0)$ directly. Then perform regular DSMC method.
- To sample a particle from f, just randomly pick one sample from F particles.

One only needs

$$\#(\text{F particles}) \geq N_p + N_n.$$

Hence

- F particles give a coarse approximation of f.
- P and N particles are finer approximation of $f - m$.

The new method

A new Monte Carlo method with negative particles

$$\begin{cases} \partial_t m = 0, \\ \partial_t f_p = Q(f, f_p) + (Q(f_p - f_n, m))_+, \\ \partial_t f_n = Q(f, f_n) + (Q(f_p - f_n, m))_-. \end{cases}$$

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$$\begin{cases} \partial_t \tilde{f} = Q(\tilde{f}, \tilde{f}), \\ \partial_t m = 0, \\ \partial_t f_p = Q(\tilde{f}, f_p) + (Q(f_p - f_n, m))_+, \\ \partial_t f_n = Q(\tilde{f}, f_n) + (Q(f_p - f_n, m))_-. \end{cases}$$

- $\tilde{f}$: coarse solution. Simulated by F particles.
- $f = m + f_p - f_n$: finer solution, the desired result. Simulated by P and N particles.

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Step 2, Sample from the source term

$$\begin{cases} \tilde{f}(t+\Delta t) = \tilde{f} + \Delta t Q(\tilde{f},\tilde{f}), \\ m(t+\Delta t) = m, \\ f_p(t+\Delta t) = f_p + \Delta t Q(\tilde{f}, f_p) + \Delta t(Q(f_p - f_n, m))_+, \\ f_n(t+\Delta t) = f_n + \Delta t Q(\tilde{f}, f_n) + \Delta t(Q(f_p - f_n, m))_-. \end{cases}$$

Source term

$$Q(f_p - f_n, m) = N_{\text{eff}} \sum_{\mathbf{v}_p} Q\left(\delta(\mathbf{v} - \mathbf{v}_p), m(\mathbf{v})\right) - N_{\text{eff}} \sum_{\mathbf{v}_n} Q\left(\delta(\mathbf{v} - \mathbf{v}_n), m(\mathbf{v})\right)$$

Need to know how to sample from $Q\left(\delta(\mathbf{v} - \mathbf{v}_p), m(\mathbf{v})\right)$.

- $Q\left(\delta(\mathbf{v} - \mathbf{v}_p), m(\mathbf{v})\right)$ exhibits singularities at $\mathbf{v} = \mathbf{v}_p$.
- For different particle interaction, the singularity behaves differently.
- Later we show how to sample when Q represents the Coulomb collision.

Apply to Coulomb collision

Apply the previous ideas

- combine collisions
- approximate f by F particles

to Bobylev-Nanbu's formulation of Coulomb collision,

$$\begin{cases} \tilde{f}(\mathbf{v}, t+\Delta t) = \frac{1}{\rho} \iint D \tilde{f}(\mathbf{w}') \tilde{f}(\mathbf{v}') \, \mathrm{d}\mathbf{w} \, \mathrm{d}\mathbf{n}, & \text{regular collisions} \\ m(\mathbf{v}, t+\Delta t) = m(\mathbf{v}, t), & m \text{ not changed} \\ f_p(\mathbf{v}, t+\Delta t) = \frac{1}{\rho} \iint D \tilde{f}(\mathbf{w}') f_p(\mathbf{v}') \, \mathrm{d}\mathbf{w} \, \mathrm{d}\mathbf{n} + (\Delta m(\mathbf{v}))_+ \,, & \text{F-P, F-N collisions} \\ f_n(\mathbf{v}, t+\Delta t) = \frac{1}{\rho} \iint D \tilde{f}(\mathbf{w}') f_n(\mathbf{v}') \, \mathrm{d}\mathbf{w} \, \mathrm{d}\mathbf{n} + (\Delta m(\mathbf{v}))_- \,. & \Delta m = \Delta t Q(f_p - f_n, m) \end{cases}$$

Coulomb collision – Sample from the source term

The source term

$$\Delta t Q_{LFP}(f_p - f_n, m) = N_{\text{eff}} \sum_{\mathbf{v}_p} \delta m(\mathbf{v}; \mathbf{v}_p) - N_{\text{eff}} \sum_{\mathbf{v}_n} \delta m(\mathbf{v}; \mathbf{v}_n),$$

where $\delta m(\mathbf{v}; \mathbf{v}_1)$ describes the change in m due to collisions with particles with velocity $\mathbf{v}_1$.

$\delta m(\mathbf{v}; \mathbf{v}_1)$ is a $5D$ integral, can be simplified to $2D$, then approximated by a $1D$ integral. The upper bounds:

$$0 \le \delta m_+(\mathbf{v}; \mathbf{v}_1) \le \alpha_u \frac{m(\mathbf{v})}{|\mathbf{v} - \mathbf{v}_1|^2},$$

$$0 \le \delta m_-(\mathbf{v}; \mathbf{v}_1) \le \alpha_l m(\mathbf{v}).$$

with two constants $\alpha_u, \alpha_l \ge 0$.

To summarize

- Combine collisions to reduce the total number of collisions.
- Use F particles to perform the combined collisions.

$$\begin{cases} \partial_t \tilde{f} = Q(\tilde{f}, \tilde{f}), \\ \partial_t m = 0, \\ \partial_t f_p = Q(\tilde{f}, f_p) + (Q(f_p - f_n, m))_+, \\ \partial_t f_n = Q(\tilde{f}, f_n) + (Q(f_p - f_n, m))_-. \end{cases}$$

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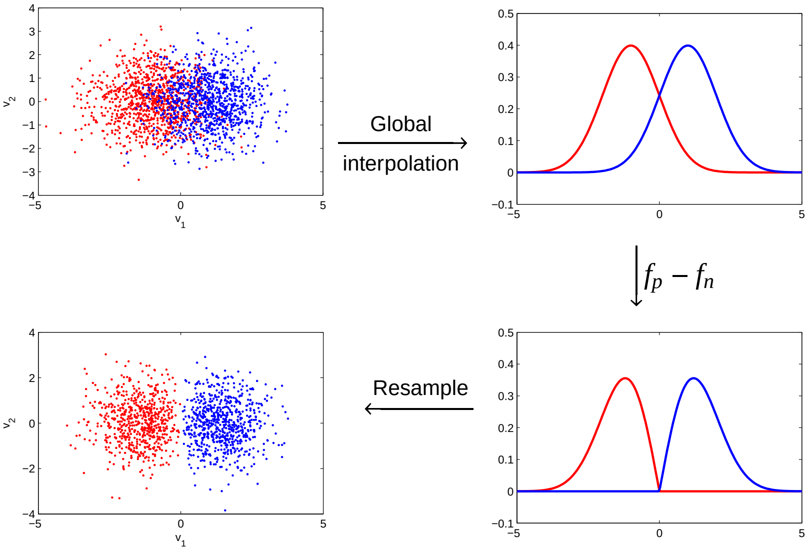
Notice that

- We need $N_{\tilde{f}} \geq (N_p + N_n)$. However N_p and N_n grow with time, while $N_{\tilde{f}}$ is constant.
- $f_p \to m$ and $f_n \to m$ as time evolves, hence they have overlap.

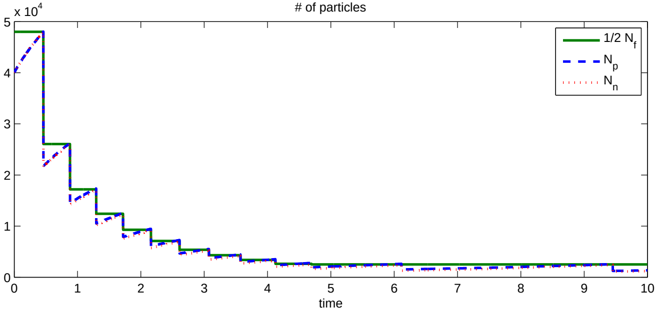
Reducing N_p and N_n is both necessary and efficient.

An extra step: Particle Resampling



Control of particle number – Particle Resampling

Evolution of Particle Numbers



- Particle resampling is accurate but expensive. But it is only needed whenever $N_{\tilde{f}} \geq (N_p + N_n)$ is violated.
- After resampling, only need to keep a subset of the F particles.

Bump on Tail problem

The initial value

$$f^I(\mathbf{v}) = \underbrace{\frac{\beta\rho}{(2\pi T)^{3/2}} \exp\left(-\frac{|\mathbf{v}|^2}{2T}\right)}_{\text{a central Maxwellian}} + \underbrace{\frac{(1-\beta)\rho}{(2\pi T_b)^{3/2}} \exp\left(-\frac{|\mathbf{v}-\mathbf{u}_b|^2}{2T_b}\right)}_{\text{a small bump with high energy}},$$

where

$$\rho = 1, \quad \beta = 0.9, \quad T = 1, \quad T_b = 0.01, \quad \mathbf{u}_b = [5, 0, 0].$$

Bump on Tail problem

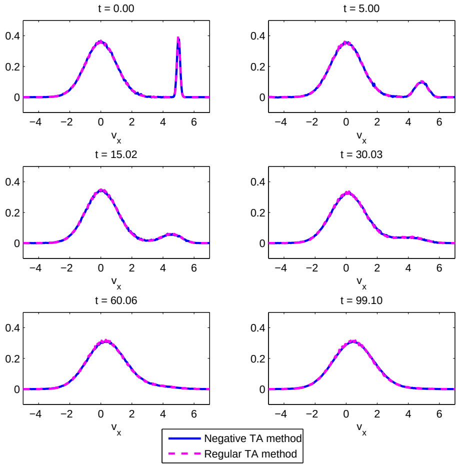


Figure : The snaps of time evolution of marginal distribution $\int f(v_x, v_y, v_z)\, dv_y\, dv_z$ in Bump-on-Tail problem.

Bump on Tail problem

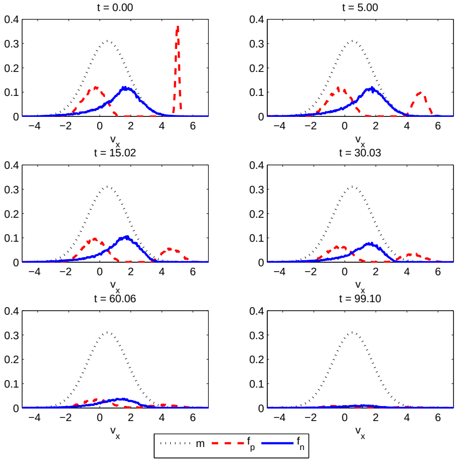


Figure : The snaps of time evolution of the components m, f_p and f_n in Bump-on-Tail problem.

Rosenbluth’s problem

Volcano-like initial data:

$$f^I(\mathbf{v}) = 0.01 \exp\left(-10(|\mathbf{v}| - 1)^2\right).$$

The distribution f stays radially symmetric for all time.

Rosenbluth's problem

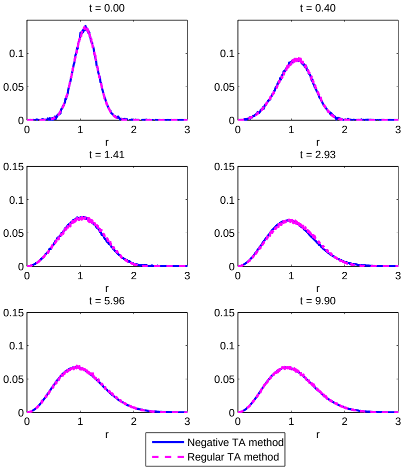


Figure : The snaps of time evolution of the radial symmetric distribution $r^2 f(r)$ in Rosenbluth's test problem.

Rosenbluth's problem

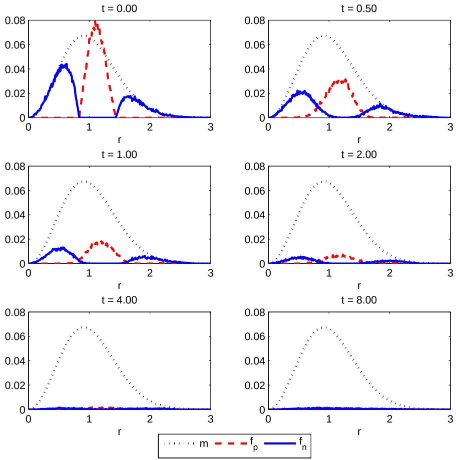
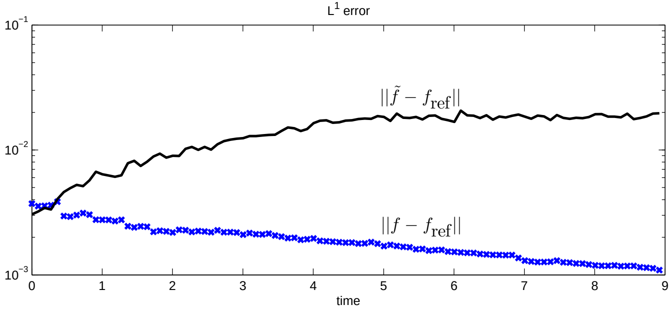


Figure : The snaps of time evolution of the components $r^2 m$, $r^2 f_p$ and $r^2 f_n$ in Rosenbluth's test problem.

Convergence test

coarse solution: $$\|\tilde{f} - f_{\text{ref}}\| \sim \frac{\rho_f}{\sqrt{N_{\tilde{f}}}}$$

fine solution: $$\|f - f_{\text{ref}}\| \sim \frac{\rho_p}{\sqrt{N_{\tilde{p}}}} + \frac{\rho_n}{\sqrt{N_{\tilde{n}}}}$$



Convergence test

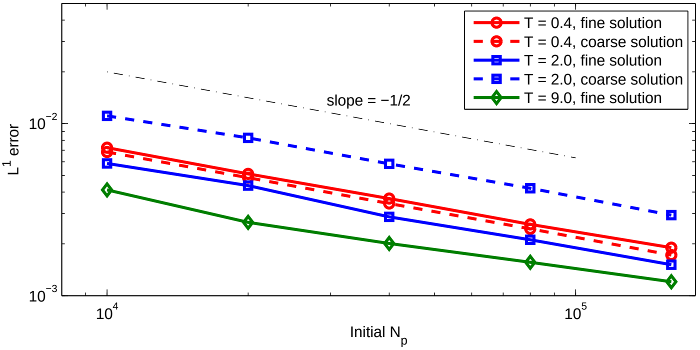


Figure : The convergence rate for fine solution f and coarse solution $\tilde{f}$ at different times. Test on Rosenbluth's problem.

Efficiency test

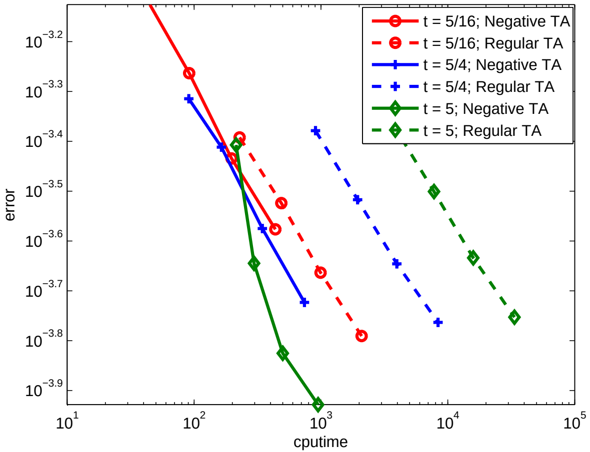


Figure : The efficiency test on Rosenbluth's problem.

Summarize

For $f = m + f_p - f_n$, the equation $\partial_t f = Q(f, f)$ is reformulated

$$\begin{cases} \partial_t \tilde{f} = Q(\tilde{f}, \tilde{f}), \\ \partial_t m = 0, \\ \partial_t f_p = Q(\tilde{f}, f_p) + (Q(f_p - f_n, m))_+, \\ \partial_t f_n = Q(\tilde{f}, f_n) + (Q(f_p - f_n, m))_-. \end{cases}$$

Summarize

For $f = m + f_p - f_n$, the equation $\partial_t f = Q(f, f)$ is reformulated

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A compact form:

For $f = m + f_d$, we can solve

$$\begin{cases} \partial_t \tilde{f} = Q(\tilde{f}, \tilde{f}), \\ \partial_t m = 0, \\ \partial_t f_d = Q(\tilde{f}, f_d) + Q(f_d, m), \end{cases}$$

Future work

- Multi-component plasma.
- Evolve Maxwellian part m to further improve the efficiency.
- General non-linear operators.
- Spatial inhomogeneous. Design a hybrid method which uses very few particles in the fluid regime.

Inhomogeneous system

Let $f = m + f_d$, the Vlasov-Poisson-Landau system

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = Q(f, f), \\ E = \nabla_x \Phi, \quad -\Delta_x \Phi = \int f \, \mathrm{d}\mathbf{v} \end{cases}$$

can be rewritten (with negative particle method, micro-macro, PIC)

$$\begin{cases} \partial_t \tilde{f} + \mathcal{T} \tilde{f} = Q(\tilde{f}, \tilde{f}), \\ \partial_t \langle f \phi \rangle + \nabla_x \cdot \langle v M \phi \rangle + \nabla_x \cdot \langle v f_d \phi \rangle = (0, \rho E, \rho u \cdot E)^T, \\ \partial_t f_d + \mathcal{T} f_d = Q(\tilde{f}, f_d) + S, \\ E = \nabla_x \Phi, \quad -\Delta_x \Phi = \langle m + f_d \rangle \end{cases}$$

with

$$\mathcal{T} = v \cdot \nabla_x + E \cdot \nabla_v$$

$$S = -(I - \Pi_M)(\mathcal{T} M) + \Pi_M(\mathcal{T} f_d) + Q(f_d, m).$$

Negative particle methods for rarefied gas

Insert the splitting $f(\mathbf{v}) = m(\mathbf{v}) - f_n(\mathbf{v}) + f_p(\mathbf{v})$,

$$
\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}t} &= Q(m,m) && M-M \\
&\quad +\Big(Q^+(f_p,f_p) - Q^-(f_p)f_p\Big) && P-P \\
&\quad +\Big(-Q^+(f_n,f_p) - Q^+(f_p,f_n) + Q^-(f_n)f_p + Q^-(f_p)f_n\Big) && P-N \\
&\quad +(Q^+(f_n,f_n) - Q^-(f_n)f_n) && N-N \\
&\quad +\Big(Q^+(f_p,m) + Q^+(m,f_p) - Q^-(f_p)m - Q^-(m)f_p\Big) && P-M \\
&\quad +(-Q^+(f_n,m) - Q^+(m,f_n) + Q^-(f_n)m + Q^-(m)f_n) && N-M
\end{aligned}
$$

Negative particle methods for rarefied gas

Insert the splitting $f(\mathbf{v}) = m(\mathbf{v}) - f_n(\mathbf{v}) + f_p(\mathbf{v})$,

$$
\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}t} = {} & Q(m,m) && M-M \\
& + \left(Q^+(f_p,f_p) - Q^-(f_p)f_p\right) && P-P \\
& + \left(-Q^+(f_n,f_p) - Q^+(f_p,f_n) + Q^-(f_n)f_p + Q^-(f_p)f_n\right) && P-N \\
& + (Q^+(f_n,f_n) - Q^-(f_n)f_n) && N-N \\
& + \left(Q^+(f_p,m) + Q^+(m,f_p) - Q^-(f_p)m - Q^-(m)f_p\right) && P-M \\
& + (-Q^+(f_n,m) - Q^+(m,f_n) + Q^-(f_n)m + Q^-(m)f_n) && N-M
\end{aligned}
$$

Negative particle methods for rarefied gas

Insert the splitting $f(\mathbf{v}) = m(\mathbf{v}) - f_n(\mathbf{v}) + f_p(\mathbf{v})$,

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}t} &= Q(m,m) && M-M \\
&+\left(Q^+(f_p,f_p) - Q^-(f_p)f_p\right) && P-P \\
&+\left(-Q^+(f_n,f_p) - Q^+(f_p,f_n) + Q^-(f_n)f_p + Q^-(f_p)f_n\right) && P-N \\
&+\left(Q^+(f_n,f_n) - Q^-(f_n)f_n\right) && N-N \\
&+\left(Q^+(f_p,m) + Q^+(m,f_p) - Q^-(f_p)m - Q^-(m)f_p\right) && P-M \\
&+\left(-Q^+(f_n,m) - Q^+(m,f_n) + Q^-(f_n)m + Q^-(m)f_n\right) && N-M
\end{aligned}$$

Reorganize,

$$\begin{aligned}
\frac{\mathrm{d}m}{\mathrm{d}t} &= Q(m,m) = 0, \\
\frac{\mathrm{d}f_p}{\mathrm{d}t} &= \left(Q^+(m,f_p) + Q^+(f_p,m) + Q^+(f_p,f_p) + Q^+(f_n,f_n)\right) \\
&\qquad - \left(Q^-(m) + Q^-(f_p) - Q^-(f_n)\right)f_p + Q^-(f_n)m, \\
\frac{\mathrm{d}f_n}{\mathrm{d}t} &= \left(Q^+(m,f_n) + Q^+(f_n,m) + Q^+(f_p,f_n) + Q^+(f_n,f_p)\right) \\
&\qquad - \left(Q^-(m) + Q^-(f_p) - Q^-(f_n)\right)f_n + Q^-(f_p)m.
\end{aligned}$$

Bobylev-Nanbu approximation

Not solving the LFP equation directly, but the Bobylev-Nanbu approximation (2000'),

$$f(\mathbf{v}, t + \Delta t) = \frac{1}{\rho} \int_{\mathbb{R}^3 \times \mathbb{S}^2} D\left(\frac{\mathbf{u} \cdot \mathbf{n}}{u}, A\frac{\Delta t}{u^3}\right) f(\mathbf{v}')f(\mathbf{w}') \,\mathrm{d}\mathbf{w} \,\mathrm{d}\mathbf{n},$$

with $A = c_{FP}$, and

$$D(\mu, \tau) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} P_l(\mu) \exp(-l(l+1)\tau).$$

This is a first order approximation (in Δt) of LFP equation.

- Takizuka and Abe (TA) 1977'

$$D_{TA}(\mu, \tau) = \frac{1}{2\pi} \frac{1}{\sqrt{2\pi\tau}} e^{-\zeta^2/2\tau} \frac{\mathrm{d}\zeta}{\mathrm{d}\mu},$$

  with $\mu = \cos(2 \arctan \zeta)$.

- Nanbu 1997'

$$D_{Nanbu}(\mu, \tau) = \frac{A}{4\pi \sinh A} e^{\mu A},$$

  where A is defined by $\coth A - \frac{1}{A} = e^{-2\tau}$.

Apply to Coulomb collision

Apply the negative particle method to Bobylev-Nanbu's reformulation,

$$f(\mathbf{v}, t+\Delta t) = \frac{1}{\rho} \iint_{\mathbb{R}^3\times\mathbb{S}^2} D\left(\frac{\mathbf{u}\cdot\mathbf{n}}{u}, A\frac{\Delta t}{u^3}\right) f(\mathbf{w}', t) f(\mathbf{v}', t)\, \mathrm{d}\mathbf{w}\, \mathrm{d}\mathbf{n} \doteq \frac{1}{\rho} P(f, f).$$

Split as

$$m(\mathbf{v}, t+\Delta t) = \frac{\rho_m}{\rho} m(\mathbf{v}, t),$$

$$f_p(\mathbf{v}, t+\Delta t) = \frac{\rho_p^2}{\rho} P(\hat{f}_p, \hat{f}_p) + \frac{\rho_n^2}{\rho} P(\hat{f}_n, \hat{f}_n) + \frac{\rho_m \rho_p}{\rho} P(\hat{m}, \hat{f}_p) + \frac{\rho_p \rho_m}{\rho} P(\hat{f}_p, \hat{m}),$$

$$f_n(\mathbf{v}, t+\Delta t) = \frac{\rho_p \rho_n}{\rho} P(\hat{f}_p, \hat{f}_n) + \frac{\rho_n \rho_p}{\rho} P(\hat{f}_n, \hat{f}_p) + \frac{\rho_m \rho_n}{\rho} P(\hat{m}, \hat{f}_n) + \frac{\rho_n \rho_m}{\rho} P(\hat{f}_n, \hat{m}),$$

with

$$\rho = \int f(\mathbf{v}, t)\, \mathrm{d}\mathbf{v}, \quad \rho_m = \int m(\mathbf{v}, t)\, \mathrm{d}\mathbf{v}, \quad \rho_p = \int f_p(\mathbf{v}, t)\, \mathrm{d}\mathbf{v}, \quad \rho_n = \int f_n(\mathbf{v}, t)\, \mathrm{d}\mathbf{v},$$

$$\hat{f} = \frac{f}{\rho}, \quad \hat{m} = \frac{m}{\rho_m}, \quad \hat{f}_p = \frac{f_p}{\rho_p}, \quad \hat{f}_n = \frac{f_n}{\rho_n}.$$

Error analysis

What's the error by approximating $Q(f, f_p)$ and $Q(f, f_n)$ with $Q(\tilde{f}, f_p)$ and $Q(\tilde{f}, f_n)$?

$$\begin{aligned} \partial_t f_p &= Q(\tilde{f}, f_p) + (Q(f_p - f_n, m))_+ \\ &= \underbrace{Q(f, f_p) + (Q(f_p - f_n, m))_+}_{\text{original equation}} + \underbrace{Q(\tilde{f} - f, f_p)}_{\text{drift term}}, \end{aligned}$$

- Solving the original equation with N_p P-particles introduces a statistical error $\mathcal{O}\left(\rho_p (N_p)^{-1/2}\right)$.
- The drift term is $\mathcal{O}\left(\rho_p (N_f)^{-1/2}\right)$, since $\tilde{f}(t) = f(t) + \mathcal{O}\left(\rho (N_f)^{-1/2}\right)$.
- As long as $N_f \geq N_p$, one has

$$\mathcal{O}\left(\rho_p (N_f)^{-1/2}\right) \leq \mathcal{O}\left(\rho_p (N_p)^{-1/2}\right).$$

Rosenbluth's problem

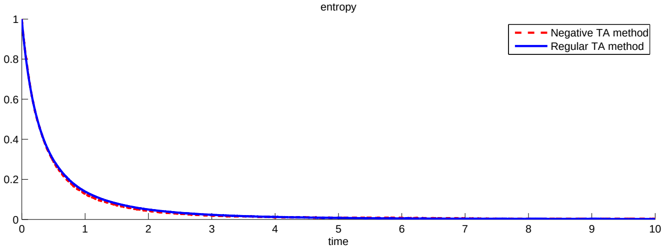

Figure : Time evolution of entropy $H(t)/H(0)$ in Rosenbluth's test problem. Blue solid line: regular TA method with 10^6 particles. Red dashed line: negative particle method with $N_p = 40000$ initially.