Hybrid methods with Deviational Particles for spatial inhomogeneous plasma

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The non-equilibrium plasma can be modeled by Vlasov-Poisson-Landau (VPL) system

$$\begin{cases} \partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f - \mathbf{E} \cdot \nabla_{\mathbf{v}} f = Q_L(f, f), \\ -\nabla_{\mathbf{x}} \cdot \mathbf{E} = \rho(t, \mathbf{x}) = \int f(t, \mathbf{x}, \mathbf{v}) \, \mathrm{d}\mathbf{v}, \end{cases}$$

where the Landau (or Landau-Fokker-Planck) operator

$$Q_L(g, f) = \frac{A}{4} \frac{\partial}{\partial v_i} \int_{\mathbb{R}^3} u \sigma_{tr}(u)(u^2 \delta_{ij} - u_i u_j) \left(\frac{\partial}{\partial v_j} - \frac{\partial}{\partial w_j} \right) g(\mathbf{w}) f(\mathbf{v}) \, \mathrm{d}\mathbf{w}$$

models the binary collisions due to the long range Coulomb interaction.

- Q_L is bilinear.
- $Q_L(g, f)$ is asymmetric. It describes the change in f due to collisions with g.
- Conserve density, momentum and energy;
- Dissipate entropy. $f \to M$ as $t \to \infty$. $Q_L(M, M) = 0$.

Probabilistic methods

The PIC-MCC (or PIC-DSMC) method is widely used in plasma simulation

- Particle-In-Cell method (PIC) for collisionless plasma. Dawson 83, Birdsall-Langdon 85 . . .
- Direct Simulation Monte Carlo (DSMC) for binary collisions. Takizuka-Abe 77, Nanbu 97, Bobylev-Nanbu 2000.

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Problem: near fluid regime, where $f \approx M$,

- Most computation is spent on the collisions between particles sampled from M.
- $Q_L(M, M) = 0$. Collisions of M have no net effect.

Highly inefficient!

Hybrid methods

Apply decomposition

$$f(t, \mathbf{x}, \mathbf{v}) = M(t, \mathbf{x}, \mathbf{v}) + f_d(t, \mathbf{x}, \mathbf{v}),$$

- Equilibrium $M(t, \mathbf{x}, \mathbf{v})$: evolved according to a fluid equation – cheap
- Deviation $f_d(t, \mathbf{x}, \mathbf{v})$: represented by particles – expensive

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To minimize the deviation part, we allow $f_d(t, \mathbf{x}, \mathbf{v}) < 0$.

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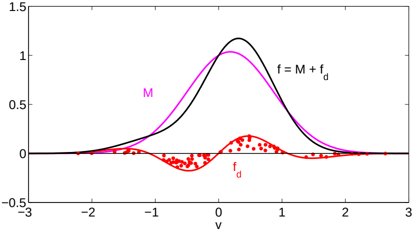

$(f_d)_+$ and $(f_d)_-$ are represented by positive and negative deviational particles.

Hybrid methods

To solve the VPL system:

- Need to evolve M and f_d in advection.
- Q_L is bilinear $\Rightarrow$ Need to consider P-P, P-N, N-N, P-M and N-M collisions. M-M collisions are omitted.

Hybrid methods

Related works,

- Caflisch-Wang-Dimarco-Cohen-Dimits (2008), Ricketson-Rosin-Caflisch-Dimits (2013)
- Hadjiconstantinou et.al. (2005)
- Crestetto-Crouseilles-Lemou (2012)
- δf methods (1988)

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Our goal: to design a hybrid method with deviational particles for the spatial inhomogeneous VPL system

- much more efficient than PIC-MCC near fluid regime
- applicable to all regimes

In our methods:

- Deviational particles have 0 density, momentum and energy in each spatial cell, i.e. for $\phi = 1, \mathbf{v}, |\mathbf{v}|^2/2$,

$$\langle \phi f_d(t, \mathbf{x}_k, \mathbf{v}) \rangle = N_{\text{eff}} \sum_{\mathbf{v}_p \in C_k} \phi(\mathbf{v}_p) - N_{\text{eff}} \sum_{\mathbf{v}_n \in C_k} \phi(\mathbf{v}_n) = 0,$$

  in each cell C_k.
- N_{eff}, the effective number of deviational particles, is a prescribed constant.

These restrictions can be relaxed in applications.

First, apply an operator splitting.

- Collision substep: in each cell,

$$\partial_t f = Q_L(f).$$

  M is invariant in this substep.

- Advection substep:

$$\begin{cases} \partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f - \mathbf{E} \cdot \nabla_{\mathbf{v}} f = 0, \\ -\nabla_{\mathbf{x}} \cdot \mathbf{E} = \rho(t, \mathbf{x}). \end{cases}$$

Deviational particles in Collisions

Collisions – Landau

Q_L is bilinear $\Rightarrow$ Need to perform P-P, P-N, N-N, P-M and N-M collision.

A straightforward method: perform each type of collision individually.

[1] Hadjiconstantinou 2005, for rarefied gas

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A straightforward method: perform each type of collision individually.

Consider the P-N collisions. Plug $f = M + f_p - f_n$ into $\partial_t f = Q(f, f)$.

$$\begin{cases} \partial_t(+f_p) = Q(-f_n, +f_p) + \dots \\ \partial_t(-f_n) = Q(+f_p, -f_n) + \dots \end{cases}$$

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$$\begin{cases} \partial_t f_p = -Q(f_n, f_p) + \dots \\ \partial_t f_n = Q(f_p, f_n) + \dots \end{cases}$$

Collision rules[1]

$$\begin{aligned} \text{P-P:} \quad & \mathbf{v}_+, \mathbf{w}_+ \rightarrow \mathbf{v}'_+, \mathbf{w}'_+, \\ \text{P-N:} \quad & \mathbf{v}_+, \mathbf{w}_- \rightarrow 2\mathbf{v}_+, \mathbf{v}'_-, \mathbf{w}'_-. \end{aligned}$$

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$$\text{P-N:} \quad \mathbf{v}_+, \mathbf{w}_- \to 2\mathbf{v}_+, \mathbf{v}'_-, \mathbf{w}'_-.$$

Particle number increases!
#(particle created) $\propto$ #(collision).

[1] Hadjiconstantinou 2005, for rarefied gas

Collisions – Landau

In rarefied gas (charge free),

- short range collision ⇒ # collisions in one time step = $\mathcal{O}(\Delta t)$
- The particle number grows in the physical scale

$$N_d\Big|_{t+\Delta t} = (1 + c\Delta t)\, N_d\Big|_t .$$

In Coulomb gas (charged),

- long range collision ⇒ # collisions in one time step = N
- The particle number grows in the numerical scale in Coulomb collisions!

$$N_d\Big|_{t+\Delta t} = \left(1 + \frac{N_m + 2N_n}{N_m + N_p - N_n}\right) N_d\Big|_t .$$

Collisions – Landau

A new method: group the collisions.

For a general bilinear collision operator

$$Q(f,f) = Q(f,f_d) + Q(f,M) = Q(f,f_d) + Q(f_d,M) + \underbrace{Q(M,M)}_{=0}.$$

$\partial_t f = Q(f,f)$ becomes

$$\partial_t f_d = Q(f,f_d) + Q(f_d,M).$$

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$Q(f,f_d) = Q(f,f_p) - Q(f,f_n)$.
$Q(f,f_p)$ groups three terms: $Q(M,f_p)$, $Q(f_p,f_p)$, $-Q(f_n,f_p)$.

Collisions – Landau

A Monte Carlo method

$$f_d(t+\Delta t) = \underbrace{f_d + \Delta t Q(f, f_d)}_{\swarrow} + \underbrace{\Delta t Q(f_d, M)}_{\searrow}.$$

regular collisions between f and f_d, N_d not change

source term, N_d increases by $O(\Delta t N_d)$

The particle number grows in the physical scale for *any* binary collisions

$$N_d\Big|_{t+\Delta t} = (1 + c\Delta t)\, N_d\Big|_t.$$

Collisions – Landau

$$f_d(t+\Delta t) = f_d + \Delta t Q(f, f_d) + \Delta t Q(f_d, M).$$

Sample a particle from f and collide with a deviational particle.

Collisions – Landau

$$f_d(t+\Delta t) = f_d + \Delta t Q(f, f_d) + \Delta t Q(f_d, M).$$

$\underbrace{\text{Sample a particle from } f}_{\text{How?}}$ and collide with a deviational particle.

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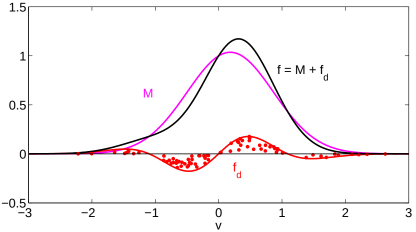
$$f_d(t+\Delta t) = f_d + \Delta t Q(f, f_d) + \Delta t Q(f_d, M).$$

$\underbrace{\text{Sample a particle from } f}_{\text{How?}}$ and collide with a deviational particle.

- $f = M + f_d$.
- Need to recover the distribution f_d from deviational particles $\Rightarrow$ computationally expensive and inaccurate.



Collisions – Landau

We introduce coarse particles

- give an approximation to f.
  - Initially sampled from $f(v, t = 0)$ directly.
  - Then perform regular PIC-MCC method.
- To sample a particle from f, just randomly pick one sample from coarse particles.

Collisions – Landau

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- give an approximation to f.
  - Initially sampled from $f(v, t = 0)$ directly.
  - Then perform regular PIC-MCC method.
- To sample a particle from f, just randomly pick one sample from coarse particles.

One only needs, in each cell

$$N_c \geq N_d. \qquad \text{A small number!}$$

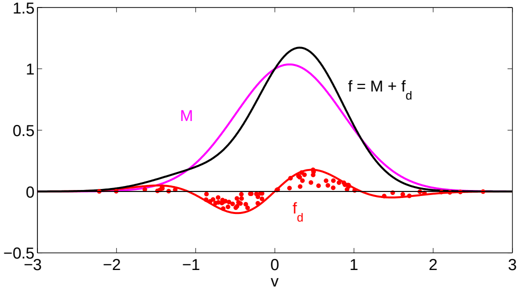
Hence

- Coarse particles give a coarse, direct approximation of f.
- Deviational particles give a finer, deviational approximation of $(f - M)$.

Collisions – Landau

The collision step is solved by[2]

$$\partial_t f_d = Q(f\ , f_d) + Q(f_d, M),$$



[2] Yan-Caflisch, J. Comput. Phys. 2015

Collisions – Landau

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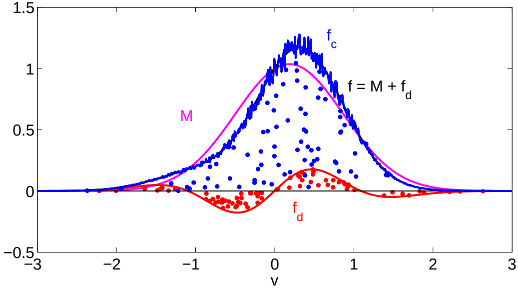

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Collisions – Landau

Step 2, Sample from the source term.

$$\begin{cases} f_d(t+\Delta t) = f_d + \Delta t Q(f_c, f_d) + \Delta t Q(f_d, M), \\ f_c(t+\Delta t) = f_c + \Delta t Q(f_c, f_c). \end{cases}$$

Source term

$$Q(f_d, M) = N_{\text{eff}} \sum_{\mathbf{v}_p} Q\left(\delta(\mathbf{v}-\mathbf{v}_p), M(\mathbf{v})\right) - N_{\text{eff}} \sum_{\mathbf{v}_n} Q\left(\delta(\mathbf{v}-\mathbf{v}_n), M(\mathbf{v})\right)$$

models the change on Maxwellian due to collisions with deviational particles.

Need to know how to sample from $Q\left(\delta(\mathbf{v}-\mathbf{v}_p), M(\mathbf{v})\right)$.

- $Q\left(\delta(\mathbf{v}-\mathbf{v}_p), M(\mathbf{v})\right)$ exhibits singularities at $\mathbf{v} = \mathbf{v}_p$.
- We derived an efficient approximation for the Landau operator $Q_L(\delta, M)$.

Collisions – summarize

For Landau operator,

$$\begin{cases} \partial_t f_d = Q(f_c, f_d) + Q(f_d, M), \\ \partial_t f_c = Q(f_c, f_c). \end{cases}$$

- Both N_p and N_n grow due to the source $Q(f_d, M)$.
- But their distributions approach each other due to the collisions with f_c, i.e. $Q(f_c, f_d)$.
- As a result, $f_d = (f_d)_+ - (f_d)_-$ decays.

Deviational particles in Advection

Advection of Maxwellian

Apply the macro-micro decomposition[3] to evolve M and f_d.

Advection of M:

With $\phi = 1, \mathbf{v}, |\mathbf{v}|^2/2$, take moments of

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f - \mathbf{E} \cdot \nabla_{\mathbf{v}} f = 0,$$

one has

$$\frac{\partial}{\partial t} \langle M\phi \rangle + \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} M \phi \rangle + \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} f_d \phi \rangle = (0, -\rho \mathbf{E}, -\rho \mathbf{u} \cdot \mathbf{E})^T.$$

[3] Bennoune-Lemou-Mieussens 2008

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The moments of M can be updated by a compressible Euler system + electric field terms + corrections from deviational particles.

[3] Bennoune-Lemou-Mieussens 2008

Advection of deviational particles

Denote $\mathcal{T} = \mathbf{v} \cdot \nabla_{\mathbf{x}} - \mathbf{E} \cdot \nabla_{\mathbf{v}}$. Rewrite $\partial_t f + \mathcal{T} f = 0$ as

$$\partial_t f_d + \mathcal{T} f_d = -\partial_t M - \mathcal{T} M. \tag{1}$$

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To remove $\partial_t M$ term, define a projection operator Π_M by

$$\Pi_M \psi = \frac{M}{\rho_M} \left[\langle \psi \rangle + \frac{(\mathbf{v} - \mathbf{u}_M) \cdot \langle (\mathbf{v} - \mathbf{u}_M) \psi \rangle}{T_M} + \frac{1}{2d} \left(\frac{|\mathbf{v} - \mathbf{u}_M|^2}{T_M} - d \right) \left\langle \left(\frac{|\mathbf{v} - \mathbf{u}_M|^2}{T_M} - d \right) \psi \right\rangle \right],$$

$\Pi_M \psi$ is in the form of $M(\mathbf{v}) P_2(\mathbf{v})$, and has same moments of ψ.

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Note

$$\langle \phi f_d \rangle = 0 \quad \Rightarrow \quad \Pi_M f_d = \Pi_M(\partial_t f_d) = (I - \Pi_M)(\partial_t M) = 0.$$

Apply $(I - \Pi_M)$ on both sides of (1),

$$\partial_t f_d + \mathcal{T} f_d = -(I - \Pi_M)(\mathcal{T} M) + \Pi_M(\mathcal{T} f_d).$$

Advection of deviational particles

Denote $\mathcal{T} = \mathbf{v} \cdot \nabla_{\mathbf{x}} - \mathbf{E} \cdot \nabla_{\mathbf{v}}$. Rewrite $\partial_t f + \mathcal{T} f = 0$ as

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- Left: advection of deviational particles, as in PIC.
- Right: source term. The Maxwellian form cannot be preserved in advection. $\mathcal{O}\left(\Delta t\left(N_{\text{eff}}^{-1} + N_d\right)\right)$ deviational particles are created.

summarize – HDP methods

Hybrid methods with Deviational Particles[4] for VPL. Coarse particles are needed.

Collision step:

$$\begin{cases} \partial_t f_c = Q(f_c, f_c), \\ \partial_t M = 0, \\ \partial_t f_d = Q(f_c, f_d) + Q(f_d, M). \end{cases}$$

Advection step:

$$\begin{cases} \partial_t f_c + \mathcal{T} f_c = 0, \\ \dfrac{\partial}{\partial t} \langle M\phi \rangle + \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} M\phi \rangle + \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} f_d \phi \rangle = (0, -\rho \mathbf{E}, -\rho \mathbf{u} \cdot \mathbf{E})^T, \\ \partial_t f_d + \mathcal{T} f_d = -(I - \Pi_M)(\mathcal{T} M) + \Pi_M(\mathcal{T} f_d), \\ -\nabla_{\mathbf{x}} \cdot \mathbf{E} = \rho_M(\mathbf{x}). \end{cases}$$

This method is also applied on VP-BGK system. Coarse particles are not needed in VP-BGK.

4 Yan. arXiv:1510.03893

Resample Deviational and Coarse Particles

Resample Particles

How?

- Recover f_d from deviational particles
- Discard old deviational and/or coarse particles, sample new ones from f_d and/or $M + f_d$

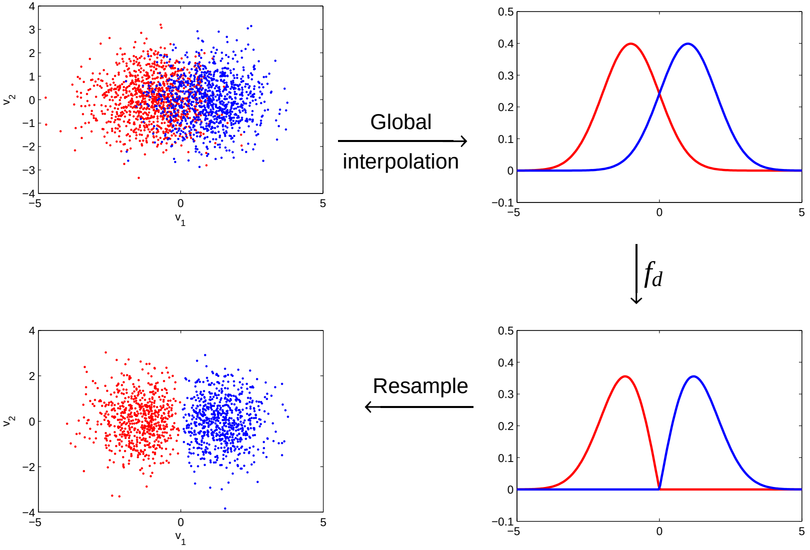
Why do we need to resample particles?
One needs in each cell,

$$N_c \geq N_d.$$

However N_d grows with time, while N_c is constant. Whenever this condition fails, two options:

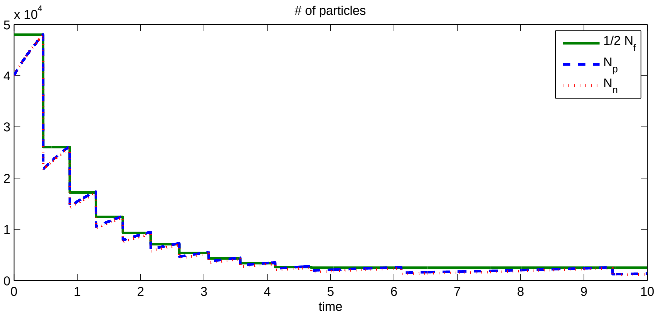
- Reduce N_d. $\Leftarrow$ Resample deviational particles.
- Increase N_c. $\Leftarrow$ Resample coarse particles.

Resample deviational particles



Resample deviational particles

Evolution of Particle Numbers in homogeneous case,



Particle resampling is accurate but expensive. But it is only needed whenever $N_c \geq N_d$ is violated.

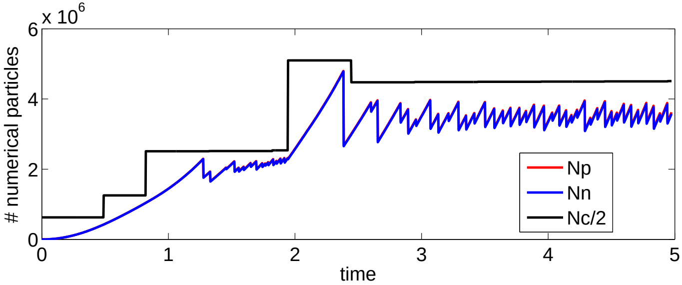
Resample coarse particles

When do we need to resample coarse particles?

- Resampling of deviational particles may fail to reduce N_d.
  - N_d increases in the advection
    $\Rightarrow$ small overlap between $(f_d)_+$ and $(f_d)_-$
    $\Rightarrow$ Increase N_c to satisfy $N_c \geq N_d$.
- After a relatively long time, f_c is not a "good" coarse approximation of f.
  $\Rightarrow$ Need to be refreshed.

Resample coarse particles

Evolution of Particle Numbers in HDP method for inhomogeneous VPL system:



Numerical Tests

Homogeneous tests

Solve the homogeneous equation $\partial_t f = Q_L(f, f)$ by

$$\begin{cases} \partial_t f_c = Q_L(f_c, f_c), \\ \partial_t f_d = Q_L(f_c, f_d) + Q_L(f_d, M). \end{cases}$$

We take $\mathbf{v} \in \mathbb{R}^3$.

Bump on Tail problem

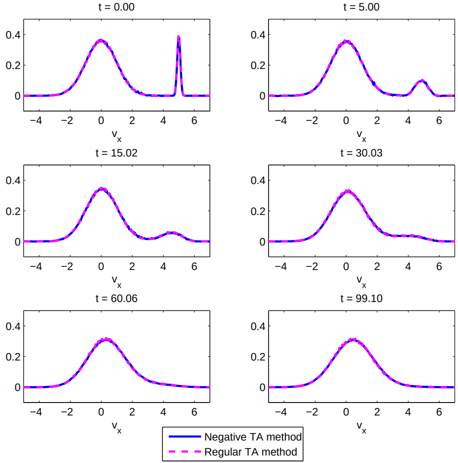


Figure : $\iint f(v_x, v_y, v_z)\, \mathrm{d}v_y\, \mathrm{d}v_z$ in Bump-on-Tail problem.

Bump on Tail problem

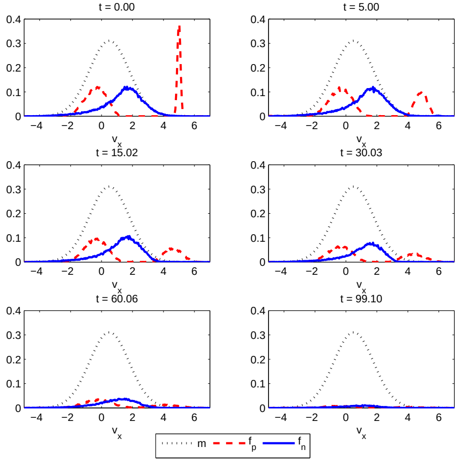


Figure : M, $(f_d)_+$ and $(f_d)_-$ in Bump-on-Tail problem.

Efficiency test

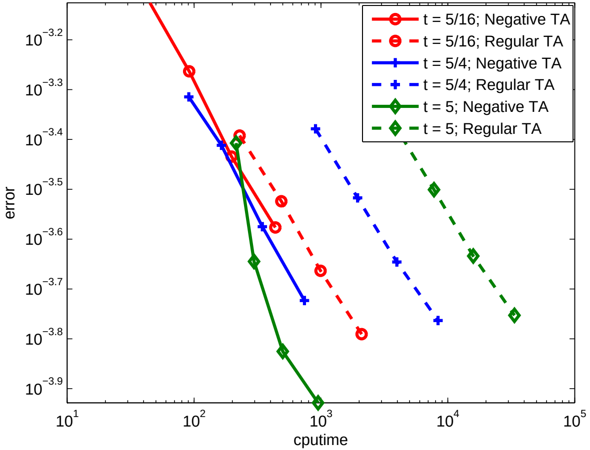


Figure : The efficiency test on Rosenbluth's problem.

Inhomogeneous tests

Apply HDP methods on spatial inhomogeneous VPL systems.

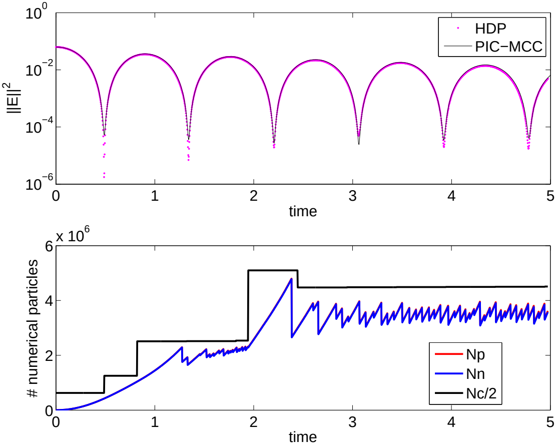
Test on the Landau damping problems, with $f(t=0,x,\mathbf{v}) = M(t=0,x,\mathbf{v})$,

$$\begin{cases} \rho(t=0,x) = 1 + \alpha \sin(x), \\ \mathbf{u}(t=0,x) = 0, \\ T(t=0,x) = 1, \end{cases}$$

with $x \in [0, 4\pi]$, $\mathbf{v} \in \mathbb{R}^3$.

Linear Landau damping, $\alpha = 0.01$
Nonlinear Landau damping, $\alpha = 0.4$

Linear Landau dampling in VPL system



Linear Landau dampling in VPL system

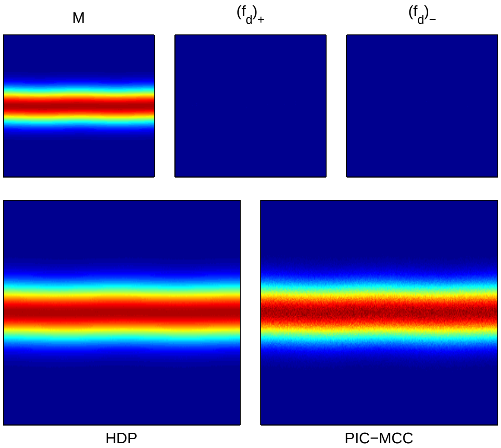
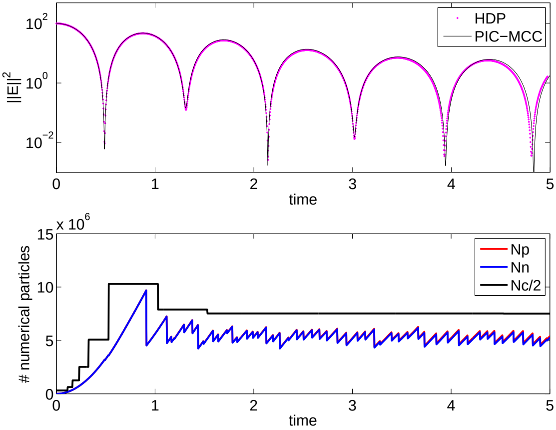


Figure : The distribution in the $x - v_1$ phase space at time $t = 1.25$ in the linear Landau damping problem of the VPL system.

Nonlinear Landau dampling in VPL system



Nonlinear Landau dampling in VPL system

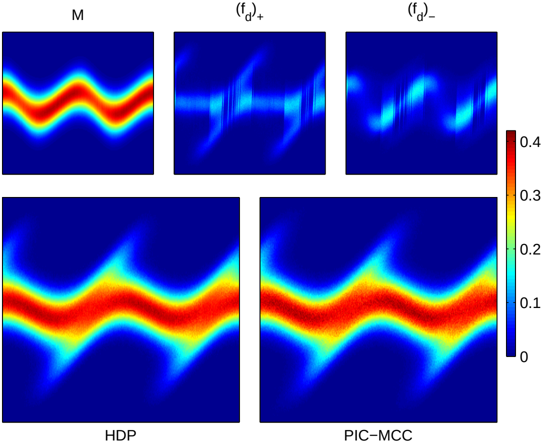


Figure : The distribution in the $x - v_1$ phase space at time $t = 1.25$ in the nonlinear Landau damping problem of the VPL system.

Convergence tests on VPL system

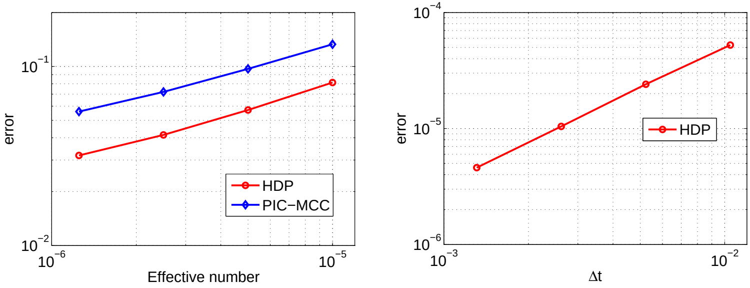


Figure : Left: half order convergence in effective number in nonlinear Landau damping. Right: first order convergence in Δt in linear Landau damping for the HDP method.

Efficiency test on VPL system

$$\rho(t=0,x) = 1 + \alpha \sin(x)$$

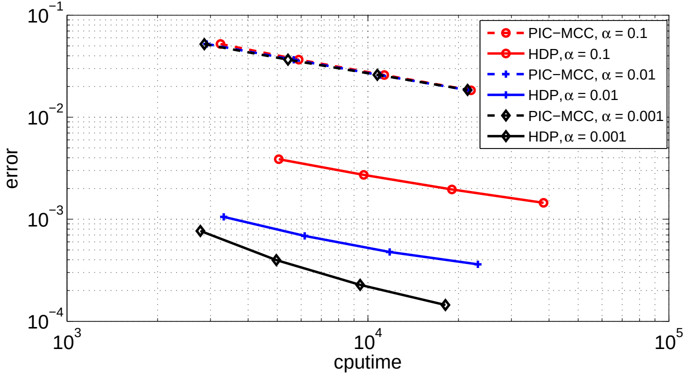


Figure : The efficiency test of the HDP method on the VPL system for different α in the initial density.

Conclusion and Future work

We have designed a hybrid method with deviational particles for the VP-BGK and VPL system,

- much more efficient than PIC-MCC near the fluid regime,
- applicable to kinetic regime.
    - Even in the *worst* case, at least we have a solution from PIC-MCC method, that is the coarse particles.

Next,

- Multi-component plasma.
- Other effective ways to reduce particle number.
- Mixed regimes.

Backup slides

Negative particle methods for rarefied gas

Insert the splitting $f(\mathbf{v}) = m(\mathbf{v}) - f_n(\mathbf{v}) + f_p(\mathbf{v})$,

$$
\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}t} &= Q(m,m) && M-M \\
&\quad +\left(Q^+(f_p,f_p) - Q^-(f_p)f_p\right) && P-P \\
&\quad +\left(-Q^+(f_n,f_p) - Q^+(f_p,f_n) + Q^-(f_n)f_p + Q^-(f_p)f_n\right) && P-N \\
&\quad +(Q^+(f_n,f_n) - Q^-(f_n)f_n) && N-N \\
&\quad +\left(Q^+(f_p,m) + Q^+(m,f_p) - Q^-(f_p)m - Q^-(m)f_p\right) && P-M \\
&\quad +(-Q^+(f_n,m) - Q^+(m,f_n) + Q^-(f_n)m + Q^-(m)f_n) && N-M
\end{aligned}
$$

Negative particle methods for rarefied gas

Insert the splitting $f(\mathbf{v}) = m(\mathbf{v}) - f_n(\mathbf{v}) + f_p(\mathbf{v})$,

$$
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\frac{\mathrm{d}f}{\mathrm{d}t} = {} & Q(m,m) && M-M \\
& + \left(Q^+(f_p,f_p) - Q^-(f_p)f_p\right) && P-P \\
& + \left(-Q^+(f_n,f_p) - Q^+(f_p,f_n) + Q^-(f_n)f_p + Q^-(f_p)f_n\right) && P-N \\
& + \left(Q^+(f_n,f_n) - Q^-(f_n)f_n\right) && N-N \\
& + \left(Q^+(f_p,m) + Q^+(m,f_p) - Q^-(f_p)m - Q^-(m)f_p\right) && P-M \\
& + \left(-Q^+(f_n,m) - Q^+(m,f_n) + Q^-(f_n)m + Q^-(m)f_n\right) && N-M
\end{aligned}
$$

Negative particle methods for rarefied gas

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&\quad + \left(-Q^+(f_n,f_p) - Q^+(f_p,f_n) + Q^-(f_n)f_p + Q^-(f_p)f_n\right) && P-N \\
&\quad + \left(Q^+(f_n,f_n) - Q^-(f_n)f_n\right) && N-N \\
&\quad + \left(Q^+(f_p,m) + Q^+(m,f_p) - Q^-(f_p)m - Q^-(m)f_p\right) && P-M \\
&\quad + \left(-Q^+(f_n,m) - Q^+(m,f_n) + Q^-(f_n)m + Q^-(m)f_n\right) && N-M
\end{aligned}
$$

Reorganize,

$$
\begin{aligned}
\frac{\mathrm{d}m}{\mathrm{d}t} &= Q(m,m) = 0, \\
\frac{\mathrm{d}f_p}{\mathrm{d}t} &= \left(Q^+(m,f_p) + Q^+(f_p,m) + Q^+(f_p,f_p) + Q^+(f_n,f_n)\right) \\
&\qquad - \left(Q^-(m) + Q^-(f_p) - Q^-(f_n)\right)f_p + Q^-(f_n)m, \\
\frac{\mathrm{d}f_n}{\mathrm{d}t} &= \left(Q^+(m,f_n) + Q^+(f_n,m) + Q^+(f_p,f_n) + Q^+(f_n,f_p)\right) \\
&\qquad - \left(Q^-(m) + Q^-(f_p) - Q^-(f_n)\right)f_n + Q^-(f_p)m.
\end{aligned}
$$

Collision rules with negative particles

$$
\begin{aligned}
\text{P-P:} \quad & \mathbf{v}_+, \mathbf{w}_+ \to \mathbf{v}'_+, \mathbf{w}'_+, \\
\text{P-N:} \quad & \mathbf{v}_+, \mathbf{w}_- \to 2\mathbf{v}_+, \mathbf{v}'_-, \mathbf{w}'_-, \\
\text{N-N:} \quad & \mathbf{v}_-, \mathbf{w}_- \to 2\mathbf{v}_-, 2\mathbf{w}_-, \mathbf{v}'_+, \mathbf{w}'_+, \\
\text{P-M:} \quad & m, \mathbf{v}_+ \to m, \mathbf{w}_-, \mathbf{v}'_+, \mathbf{w}'_+, \\
\text{N-M:} \quad & m, \mathbf{v}_- \to m, \mathbf{w}_+, \mathbf{v}'_-, \mathbf{w}'_-.
\end{aligned}
$$

Collision rules with negative particles

$$
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\text{P-P:} &\quad \mathbf{v}_+, \mathbf{w}_+ \to \mathbf{v}'_+, \mathbf{w}'_+, \\
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\text{N-N:} &\quad \mathbf{v}_-, \mathbf{w}_- \to 2\mathbf{v}_-, 2\mathbf{w}_-, \mathbf{v}'_+, \mathbf{w}'_+, \\
\text{P-M:} &\quad m, \mathbf{v}_+ \to m, \mathbf{w}_-, \mathbf{v}'_+, \mathbf{w}'_+, \\
\text{N-M:} &\quad m, \mathbf{v}_- \to m, \mathbf{w}_+, \mathbf{v}'_-, \mathbf{w}'_-.
\end{aligned}
$$

Problem: particle number increases!

Bobylev-Nanbu approximation

Not solving the LFP equation directly, but the Bobylev-Nanbu approximation (2000'),

$$f(\mathbf{v}, t+\Delta t) = \frac{1}{\rho}\int_{\mathbb{R}^3\times\mathbb{S}^2} D\left(\frac{\mathbf{u}\cdot\mathbf{n}}{u}, A\frac{\Delta t}{u^3}\right) f(\mathbf{v}')f(\mathbf{w}')\,\mathrm{d}\mathbf{w}\,\mathrm{d}\mathbf{n},$$

with $A = c_{FP}$, and

$$D(\mu,\tau) = \sum_{l=0}^{\infty}\frac{2l+1}{4\pi}P_l(\mu)\exp(-l(l+1)\tau).$$

This is a first order approximation (in Δt) of LFP equation.

- Takizuka and Abe (TA) 1977'

$$D_{TA}(\mu,\tau) = \frac{1}{2\pi}\frac{1}{\sqrt{2\pi\tau}}e^{-\zeta^2/2\tau}\frac{\mathrm{d}\zeta}{\mathrm{d}\mu},$$

  with $\mu = \cos(2\arctan\zeta)$.

- Nanbu 1997'

$$D_{Nanbu}(\mu,\tau) = \frac{A}{4\pi\sinh A}e^{\mu A},$$

  where A is defined by $\coth A - \frac{1}{A} = e^{-2\tau}$.

Apply to Coulomb collision

Apply the negative particle method to Bobylev-Nanbu's reformulation,

$$f(\mathbf{v}, t+\Delta t) = \frac{1}{\rho} \iint_{\mathbb{R}^3\times\mathbb{S}^2} D\left(\frac{\mathbf{u}\cdot\mathbf{n}}{u}, A\frac{\Delta t}{u^3}\right) f(\mathbf{w}', t) f(\mathbf{v}', t)\, \mathrm{d}\mathbf{w}\, \mathrm{d}\mathbf{n} \doteq \frac{1}{\rho} P(f, f).$$

Split as

$$\begin{aligned}
m(\mathbf{v}, t+\Delta t) &= \frac{\rho_m}{\rho} m(\mathbf{v}, t),\\
f_p(\mathbf{v}, t+\Delta t) &= \frac{\rho_p^2}{\rho} P(\hat{f}_p, \hat{f}_p) + \frac{\rho_n^2}{\rho} P(\hat{f}_n, \hat{f}_n) + \frac{\rho_m\rho_p}{\rho} P(\hat{m}, \hat{f}_p) + \frac{\rho_p\rho_m}{\rho} P(\hat{f}_p, \hat{m}),\\
f_n(\mathbf{v}, t+\Delta t) &= \frac{\rho_p\rho_n}{\rho} P(\hat{f}_p, \hat{f}_n) + \frac{\rho_n\rho_p}{\rho} P(\hat{f}_n, \hat{f}_p) + \frac{\rho_m\rho_n}{\rho} P(\hat{m}, \hat{f}_n) + \frac{\rho_n\rho_m}{\rho} P(\hat{f}_n, \hat{m}),
\end{aligned}$$

with

$$\rho = \int f(\mathbf{v}, t)\, \mathrm{d}\mathbf{v}, \quad \rho_m = \int m(\mathbf{v}, t)\, \mathrm{d}\mathbf{v}, \quad \rho_p = \int f_p(\mathbf{v}, t)\, \mathrm{d}\mathbf{v}, \quad \rho_n = \int f_n(\mathbf{v}, t)\, \mathrm{d}\mathbf{v},$$

$$\hat{f} = \frac{f}{\rho}, \quad \hat{m} = \frac{m}{\rho_m}, \quad \hat{f}_p = \frac{f_p}{\rho_p}, \quad \hat{f}_n = \frac{f_n}{\rho_n}.$$

Error analysis

What's the error by approximating $Q(f, f_p)$ and $Q(f, f_n)$ with $Q(f_c, f_p)$ and $Q(f_c, f_n)$?

$$\begin{aligned}\partial_t f_p &= Q(f_c, f_p) + (Q(f_p - f_n, m))_+ \\ &= \underbrace{Q(f, f_p) + (Q(f_p - f_n, m))_+}_{\text{original equation}} + \underbrace{Q(f_c - f, f_p)}_{\text{drift term}},\end{aligned}$$

- Solving the original equation with N_p P-particles introduces a statistical error $\mathcal{O}\left(\rho_p (N_p)^{-1/2}\right)$.
- The drift term is $\mathcal{O}\left(\rho_p (N_f)^{-1/2}\right)$, since $f_c(t) = f(t) + \mathcal{O}\left(\rho (N_f)^{-1/2}\right)$.
- As long as $N_f \geq N_p$, one has

$$\mathcal{O}\left(\rho_p (N_f)^{-1/2}\right) \leq \mathcal{O}\left(\rho_p (N_p)^{-1/2}\right).$$

Rosenbluth's problem

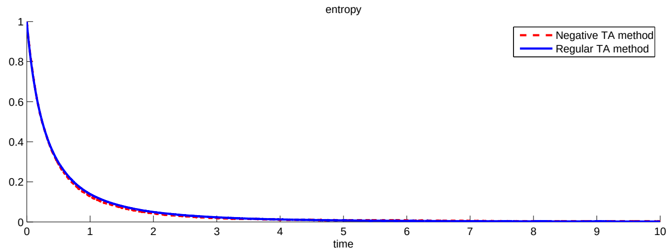


Figure : Time evolution of entropy $H(t)/H(0)$ in Rosenbluth's test problem. Blue solid line: regular TA method with 10^6 particles. Red dashed line: negative particle method with $N_p = 40000$ initially.

Apply to Coulomb collision

Apply the previous ideas

- combine collisions
- approximate f by F particles

to Bobylev-Nanbu's formulation of Coulomb collision,

$$
\begin{cases}
f_c(\mathbf{v}, t+\Delta t) = \dfrac{1}{\rho} \displaystyle\iint D\, f_c(\mathbf{w}')\, f_c(\mathbf{v}')\, \mathrm{d}\mathbf{w}\, \mathrm{d}\mathbf{n}, & \text{regular collisions} \\
M(\mathbf{v}, t+\Delta t) = M(\mathbf{v}, t), & M \text{ not changed} \\
f_d(\mathbf{v}, t+\Delta t) = \dfrac{1}{\rho} \displaystyle\iint D\, f_c(\mathbf{w}')\, f_d(\mathbf{v}')\, \mathrm{d}\mathbf{w}\, \mathrm{d}\mathbf{n} + \Delta M(\mathbf{v}), & \text{C-D collisions}
\end{cases}
$$

$$\Delta M = \Delta t Q(f_d, M).$$

Apply to Coulomb collision

Bobylev-Nanbu's formulation

$$f(\mathbf{v}, t+\Delta t) = \frac{1}{\rho}\iint_{\mathbb{R}^3\times\mathbb{S}^2} D\left(\frac{\mathbf{u}\cdot\mathbf{n}}{u}, A\frac{\Delta t}{u^3}\right) f(\mathbf{w}', t) f(\mathbf{v}', t)\,\mathrm{d}\mathbf{w}\,\mathrm{d}\mathbf{n} \doteq \frac{1}{\rho}P(f,f),$$

can be split by plug in $f = M + f_p - f_n$,

$$\begin{aligned} f(\mathbf{v}, t+\Delta t) &= \frac{1}{\rho}P(M,M) + \frac{1}{\rho}P(f,f_p) - \frac{1}{\rho}P(f,f_n) + \frac{1}{\rho}P(f_p - f_n, M) \\ &= M(\mathbf{v}) + \frac{1}{\rho}P(f,f_p) - \frac{1}{\rho}P(f,f_n) + \left(\frac{1}{\rho}P(f_p - f_n, M) - \frac{\rho_p - \rho_n}{\rho}M(\mathbf{v})\right), \end{aligned}$$

Apply to Coulomb collision

Apply the previous method

$$
\begin{aligned}
f_c(\mathbf{v}, t+\Delta t) &= \frac{1}{\rho}\iint D\, f_c(\mathbf{w}')\, f_c(\mathbf{v}')\, \mathrm{d}\mathbf{w}\, \mathrm{d}\mathbf{n}, \\
M(\mathbf{v}, t+\Delta t) &= M(\mathbf{v}, t), \\
f_p(\mathbf{v}, t+\Delta t) &= \frac{1}{\rho}\iint D\, f_c(\mathbf{w}')\, f_p(\mathbf{v}')\, \mathrm{d}\mathbf{w}\, \mathrm{d}\mathbf{n} + (\Delta M(\mathbf{v}))_+ , \\
f_n(\mathbf{v}, t+\Delta t) &= \frac{1}{\rho}\iint D\, f_c(\mathbf{w}')\, f_n(\mathbf{v}')\, \mathrm{d}\mathbf{w}\, \mathrm{d}\mathbf{n} + (\Delta M(\mathbf{v}))_- ,
\end{aligned}
$$

with

$$
\Delta M(\mathbf{v}) = \frac{1}{\rho}\iint D\left(f_p(\mathbf{w}') - f_n(\mathbf{w}')\right) M(\mathbf{v}')\, \mathrm{d}\mathbf{w}\, \mathrm{d}\mathbf{n} - \frac{\rho_p - \rho_n}{\rho} M(\mathbf{v}).
$$

Hybrid methods with Deviational Particles for VP-BGK

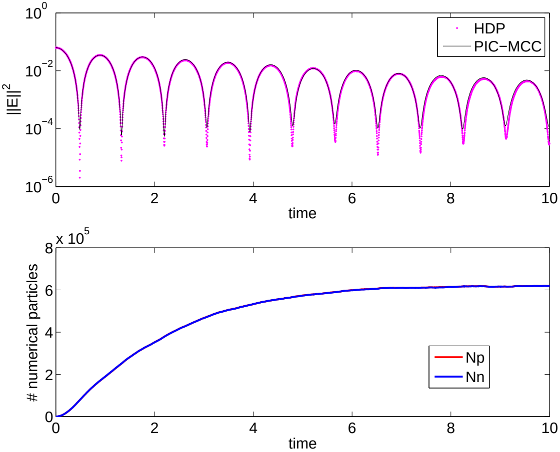
Collision step:

$$\begin{cases} \partial_t M = 0, \\ \partial_t f_d = -\mu f_d. \end{cases}$$

Advection step:

$$\begin{cases} \dfrac{\partial}{\partial t} \langle M\phi\rangle + \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} M\phi\rangle + \nabla_{\mathbf{x}} \cdot \langle \mathbf{v} f_d \phi\rangle = (0, -\rho\mathbf{E}, -\rho\mathbf{u}\cdot\mathbf{E})^T, \\ \partial_t f_d + \mathcal{T} f_d = -(I - \Pi_M)(\mathcal{T} M) + \Pi_M(\mathcal{T} f_d). \end{cases}$$

Linear Landau dampling in VP-BGK system



Nonlinear Landau dampling in VP-BGK system