MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Model Order Reduction for Networked Control Systems

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Collective dynamics, control, and imaging
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Magdeburg

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DFG-Graduiertenkolleg
MATHEMATISCHE KOMPLEXITÄTSREDUKTION

Outline

1. Model Order Reduction
   - Model Order Reduction Problem
   - $\mathcal{H}_2$-Optimal Model Order Reduction
2. Multi-Agent Systems
   - Consensus and Synchronization
   - Examples
3. Model Order Reduction by Clustering
   - $\mathcal{H}_2$-Optimal Clustering Problem
   - Numerical Examples
   - Possible Extensions
   - A Priori Error Bounds
4. Agent Reduction
5. Conclusion

Full Order Model

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned}$$

with:

- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t) \in \mathbb{R}^p$.

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Reduced Order Model

$$\widehat{E}\dot{\widehat{x}}(t) = \widehat{A}\widehat{x}(t) + \widehat{B}u(t),$$
$$\widehat{y}(t) = \widehat{C}\widehat{x}(t),$$

with:

- states $\widehat{x}(t) \in \mathbb{R}^r$, $r \ll n$,
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $\widehat{y}(t) \in \mathbb{R}^p$.

Model Order Reduction

Full Order Model

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Model Order Reduction Problem

Find matrices $\widehat{E}$, $\widehat{A}$, $\widehat{B}$, and $\widehat{C}$ such that $y \approx \widehat{y}$ for all u. More precisely, we want an error bound of the form $\|y - \widehat{y}\| \leq \mathrm{tol} \cdot \|u\|$.

$\mathcal{H}_2$-Optimal MOR

Denoting $S(s) = C(sE - A)^{-1}B$, $\widehat{S}(s) = \widehat{C}(s\widehat{E} - \widehat{A})^{-1}\widehat{B}$, solve

$$\underset{\widehat{A},\widehat{B},\widehat{C}}{\text{minimize}} \quad \|S - \widehat{S}\|_{\mathcal{H}_2} .$$

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$\mathcal{H}_2$-norm

$$\|S\|_{\mathcal{H}_2} = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \|S(i\omega)\|_F^2 \, \mathrm{d}\omega}$$

Model Order Reduction

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Output Error Bound

$$\|y - \widehat{y}\|_{\mathcal{L}_\infty} \leq \|S - \widehat{S}\|_{\mathcal{H}_2} \|u\|_{\mathcal{L}_2}$$

Theorem ($\mathcal{H}_2$ Necessary Optimality Conditions [Gugercin et al. '08])

Let $\widehat{S}(s) = \sum_{i=1}^{r} \frac{c_i b_i^T}{s-\lambda_i}$ be the $\mathcal{H}_2$-optimal reduced model for S. Then

$$\begin{aligned} S(-\lambda_i)b_i &= \widehat{S}(-\lambda_i)b_i, \\ c_i^T S(-\lambda_i) &= c_i^T \widehat{S}(-\lambda_i), \\ c_i^T S'(-\lambda_i)b_i &= c_i^T \widehat{S}'(-\lambda_i)b_i, \end{aligned} \tag{1}$$

for $i = 1, 2, \ldots, r$.

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for $i = 1, 2, \ldots, r$.

Tangential Hermite Interpolation

If $\widehat{E} = W^T EV$, $\widehat{A} = W^T AV$, $\widehat{B} = W^T B$, $\widehat{C} = CV$ where

$$(-\lambda_i E - A)^{-1} B b_i \in \text{range}(V), \quad (-\lambda_i E - A)^{-T} C^T c_i \in \text{range}(W),$$

for $i = 1, 2, \ldots, r$, then $\widehat{S}(s) = \widehat{C}(s\widehat{E} - \widehat{A})^{-1}\widehat{B}$ satisfies (1).

Iterative Rational Krylov Algorithm (IRKA) [Gugercin et al. '08]

- Fixed-point iteration algorithm, based on interpolation-based $\mathcal{H}_2$ necessary optimality conditions.

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- In every iteration, $2r$ linear systems with $sE - A$ as the system matrix need to be solved.

Model Order Reduction

Iterative Rational Krylov Algorithm (IRKA) [Gugercin et al. '08]

- Fixed-point iteration algorithm, based on interpolation-based $\mathcal{H}_2$ necessary optimality conditions.
- In every iteration, $2r$ linear systems with $sE - A$ as the system matrix need to be solved.
- Convergence proved for state-space symmetric systems ($A = A^T$, $B = C^T$). [Flagg/Beattie/Gugercin '12]

Applications

Multi-Agent Systems

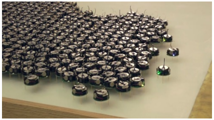
Applications

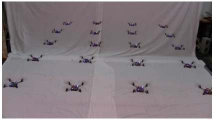
- biological swarms

Multi-Agent Systems

Applications

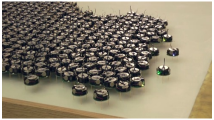
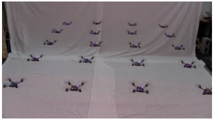
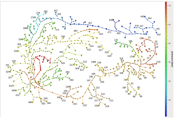
- biological swarms
- robot swarms






Applications

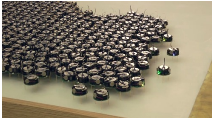
- biological swarms
- robot swarms
- energy transportation networks

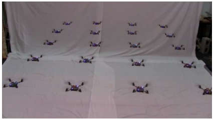




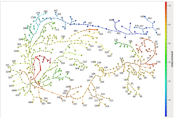

Multi-Agent Systems

Applications

- biological swarms
- robot swarms
- energy transportation networks
- opinion dynamics







General Description

Agents

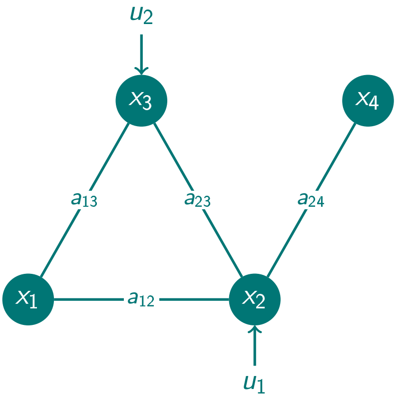
$$\dot{x}_i(t) = f(x_i(t), z_i(t), u_{k(i)}(t))$$

Interconnections

$$z_i(t) = \sum_{j=1}^{N} a_{ij} g(x_i(t), x_j(t))$$

Output

$$y(t) = h(x_1(t), x_2(t), \ldots, x_N(t))$$

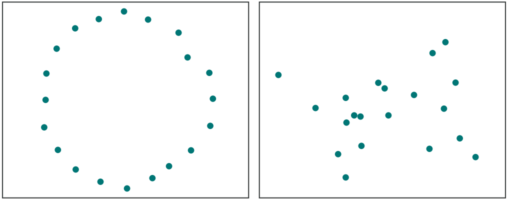


Consensus and Synchronization

Consensus

We say agents achieve consensus if, for any initial condition and zero input, the states $x_i(t)$ (or some value of interest) converge to the same vector c as t tends to infinity, i.e.

$$\lim_{t\to\infty} F(x_i(t)) = c.$$

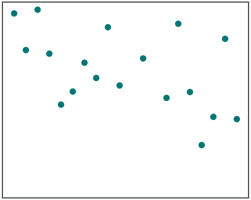


Consensus and Synchronization

Synchronization

We say agents achieve synchronization if, for any initial condition and zero input, the states $x_i(t)$ (or some value of interest) converge to the same trajectory as t tends to infinity, i.e.

$$\lim_{t \to \infty} [F(x_i(t)) - F(x_j(t))] = 0.$$



Consensus and Synchronization

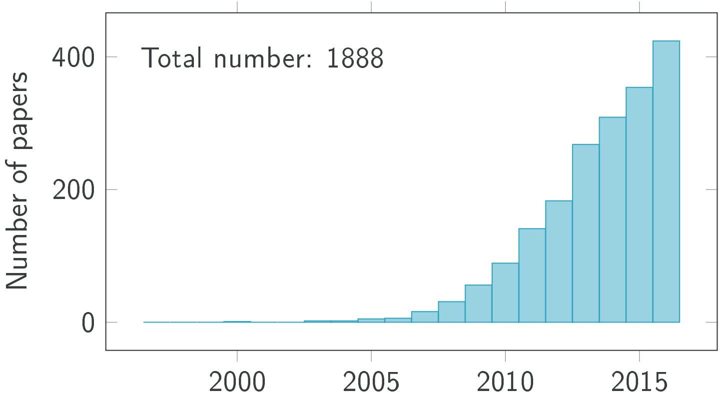


Figure: Number of papers per year (until 2016) containing "multi-agent systems" and "consensus" or "synchronization" in the title (source: Scopus).

Examples (Linear Multi-Agents Systems)

Agents

$$\dot{x}_i(t) = z_i(t) + u_{k(i)}(t)$$

Interconnections

$$z_i(t) = \sum_{j=1}^{N} a_{ij}(x_j(t) - x_i(t))$$

Applications in:

- mass-damper systems
- opinion dynamics

Multi-Agent Systems

Examples (Linear Multi-Agents Systems)

Agents

$$\ddot{x}_i(t) + d\dot{x}_i(t) + kx_i(t) = z_i(t) + u_{k(i)}(t)$$

Interconnections

$$z_i(t) = \sum_{j=1}^{N} a_{ij}(x_j(t) - x_i(t))$$

Applications in:

- power systems (frequency synchronization).

Examples (Attraction and Repulsion)

Agents

$$\dot{x}_i(t) = z_i(t) + u_{k(i)}(t)$$

Interconnections [Shu/Zheng/Shao '09]

$$z_i(t) = \sum_{j=1}^{N} a_{ij} \frac{\|x_j(t) - x_i(t)\| - d}{\|x_j(t) - x_i(t)\|} (x_j(t) - x_i(t))$$

Examples (Chemical Reaction Networks)

Agents

$$\dot{x}_i(t) = z_i(t) + u_{k(i)}(t)$$

Interconnections [Chatterjee et al. '10]

$$z_i(t) = \sum_{j \in \text{reactions}} a_{ij} k_j \prod_{k \in \text{species}} x_k(t)^{\alpha_{kj}}$$

$$a_{ij} \in \{0, \pm 1\}, \quad k_j > 0, \quad \alpha_{kj} \in \{0, 1\}$$

Examples (Mobile Robot Swarms)

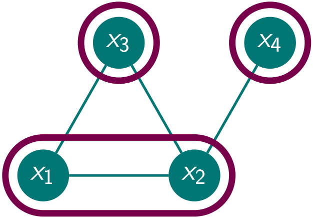
Agents

$$\dot{x}_i(t) = (v_i(t) + d_i(t))\cos\theta_i(t)$$
$$\dot{y}_i(t) = (v_i(t) + d_i(t))\sin\theta_i(t)$$
$$\dot{\theta}_i(t) = \omega_i(t) + \sigma_i(t)$$

Interconnections [Ajorlou/Asadi/Aghdam/Blouin '15]

$r_{ix}(t) = \sum_{j=1}^{N} a_{ij}(x_j(t) - x_i(t)),\ r_{iy}(t) = \sum_{j=1}^{N} a_{ij}(y_j(t) - y_i(t)),$
$\theta_i^*(t) = \operatorname{atan2}(r_{iy}(t), r_{ix}(t)),$

$$v_i(t) = v_i^M$$
$$\omega_i(t) = \dot{\theta}_i^*(t) - \kappa_i(\theta_i(t) - \theta_i^*(t))$$



Galerkin projection

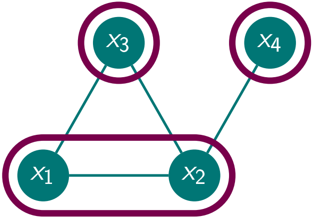
For first-order agents:

$$V = W = P(\pi) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For higher-order agents:

$$V = W = P(\pi) \otimes I.$$

MOR by Clustering



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For higher-order agents:

$$V = W = P(\pi) \otimes I.$$

Motivation

The approximation $x(t) \approx V\widehat{x}(t)$ means that agents in the same cluster are approximated as being equal.

MOR by Clustering

Literature Review

[Ishizaki/Kashima/Imura '14, '15, '16]

- for linear multi-agent systems, on undirected or certain directed graphs, where the output is the complete state
- the method consists of clustering rows in a low-rank factor of the controllability Gramian
- extension to higher-order agents

[Cheng/Kawano/Scherpen '16, '17]

- for linear multi-agent systems, with first-order of second-order agents, where the output is the complete state
- the method consists of clustering rows in a low-rank factor of the controllability Gramian

MOR by Clustering

Literature Review

[Besselink/Sandberg/Johansson '16]

- for linear multi-agent systems, where agents are passive systems, and the graph is a tree (can be directed)
- limited to clustering neighboring agents
- developed an a priori $\mathcal{H}_\infty$-error bound
- the method consists of finding diagonal generalized Gramians of the edge system (solutions of linear matrix inequalities of the size of the number of agents minus one)

[Xue/Chakrabortty '16]

- LQR controller reduction by clustering

MOR by Clustering

$\mathcal{H}_2$-Optimal Clustering Problem

Motivated by IRKA, we consider:

$$\underset{\substack{\pi \in \Pi \\ |\pi| = r}}{\text{minimize}} \quad \|S - \widehat{S}\|_{\mathcal{H}_2}$$

MOR by Clustering

$\mathcal{H}_2$-Optimal Clustering Problem

$$\begin{aligned} \underset{V,W\in\mathbb{R}^{N\times r}}{\text{minimize}} \quad & \|S-\widehat{S}\|_{\mathcal{H}_2} \\ \text{subject to} \quad & V = P(\pi) \\ & W = P(\pi) \\ & \pi \in \Pi,\ |\pi| = r \end{aligned}$$

MOR by Clustering

$\mathcal{H}_2$-Optimal Clustering Problem

$$
\begin{aligned}
\underset{V,W\in\mathbb{R}^{N\times r}}{\text{minimize}} \quad & \|S - \widehat{S}\|_{\mathcal{H}_2} \\
\text{subject to} \quad & \operatorname{range}(V) = \operatorname{range}(P(\pi)) \\
& \operatorname{range}(W) = \operatorname{range}(P(\pi)) \\
& \pi \in \Pi,\ |\pi| = r
\end{aligned}
$$

MOR by Clustering

$\mathcal{H}_2$-Optimal Clustering Problem

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Finding π

- In [Benner/Grundel/Mlinarić '15], we proposed using IRKA on S to find V and the QR decomposition-based clustering from [Zha et al. '08] on the rows of V to find π.

MOR by Clustering

Finding π

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- We observed that for a fixed r, IRKA gives a much better reduced order model than clustering, in terms of the relative $\mathcal{H}_2$-error (a few orders of magnitude difference).

MOR by Clustering

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- We observed that for a fixed r, IRKA gives a much better reduced order model than clustering, in terms of the relative $\mathcal{H}_2$-error (a few orders of magnitude difference).
- Furthermore, we observed very slow convergence of IRKA for large r (e.g. $r = 100$).

MOR by Clustering

Finding π

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- We observed that for a fixed r, IRKA gives a much better reduced order model than clustering, in terms of the relative $\mathcal{H}_2$-error (a few orders of magnitude difference).
- Furthermore, we observed very slow convergence of IRKA for large r (e.g. $r = 100$).
- For these reasons, we were looking for a way to cluster the rows of V into more than r clusters.

MOR by Clustering

K-Means

Theorem ([Beattie/Gugercin/Wyatt '12])

$$\frac{\|H_1 - H_2\|_{\mathcal{H}_\infty}}{\frac{1}{2}(\|H_1\|_{\mathcal{H}_\infty} + \|H_2\|_{\mathcal{H}_\infty})} \leq M \max(\sin\Theta(V_1, V_2), \sin\Theta(W_1, W_2))$$

MOR by Clustering

K-Means

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Motivation to Use K-Means

If V_1 and V_2 are orthonormal, then

$$\sin\Theta(V_1, V_2) = \left\| \left(I - V_1 V_1^T \right) V_2 \right\|_2 \leq \left\| \left(I - V_1 V_1^T \right) V_2 \right\|_F .$$

MOR by Clustering

K-Means

Theorem ([Beattie/Gugercin/Wyatt '12])

$$\frac{\|H_1 - H_2\|_{\mathcal{H}_\infty}}{\frac{1}{2}(\|H_1\|_{\mathcal{H}_\infty} + \|H_2\|_{\mathcal{H}_\infty})} \leq M \max(\sin\Theta(V_1, V_2), \sin\Theta(W_1, W_2))$$

Motivation to Use K-Means

If V_1 and V_2 are orthonormal, then

$$\sin\Theta(V_1, V_2) = \left\| \left(I - V_1 V_1^T\right) V_2 \right\|_2 \leq \left\| \left(I - V_1 V_1^T\right) V_2 \right\|_F .$$

If also $V_1 = P(\pi)(P(\pi)^T P(\pi))^{-\frac{1}{2}}$, then

$$\left\| \left(I - V_1 V_1^T\right) V_2 \right\|_F^2 = \text{k-means objective functional.}$$

MOR by Clustering

Algorithm

Algorithm 1 Clustering-Based Model Order Reduction Method

Input: Full order model (A, B, C).

Output: Clustered reduced order model $(\widehat{A}, \widehat{B}, \widehat{C})$.

1: Project the system to the stable subspace, using sparse matrices V_{stab} and W_{stab}.
2: Reduce the stable model using IRKA, where the resulting projection matrices are V_{IRKA} and W_{IRKA}.
3: Compute V by concatenating $V_{\text{stab}} V_{\text{IRKA}}$ and the unstable subspace.
4: Apply k-means to the rows (or block-rows, if agents are not of first-order) of V to find a partition π.
5: Project the original model by clustering with partition π.

Numerical Examples (Linear Consensus, $x_i(t) \in \mathbb{R}$, $y(t) = x(t)$)

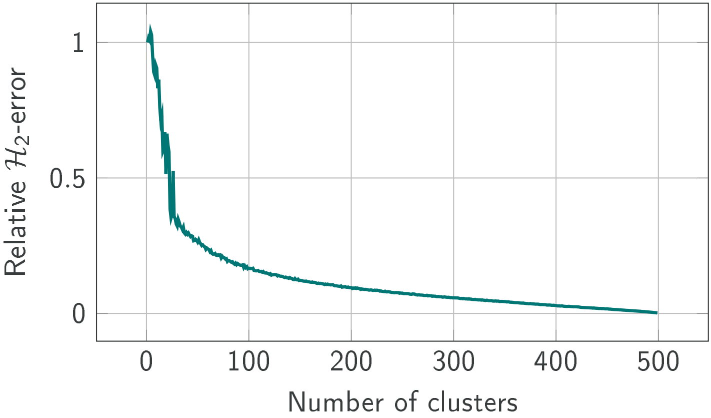

Figure: Relative $\mathcal{H}_2$-errors when clustering a linear multi-agent system evolving on a randomly generated small-world network with 500 nodes.

MOR by Clustering

Nonlinear Multi-Agent Systems

Observation

- In our clustering algorithm, it is possible to replace IRKA by any projection-based MOR method.

Nonlinear Multi-Agent Systems

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- In our clustering algorithm, it is possible to replace IRKA by any projection-based MOR method.
- In particular, this includes MOR methods for nonlinear systems, e.g. Proper Orthogonal Decomposition (POD).

MOR by Clustering

Nonlinear Multi-Agent Systems

Observation

- In our clustering algorithm, it is possible to replace IRKA by any projection-based MOR method.
- In particular, this includes MOR methods for nonlinear systems, e.g. Proper Orthogonal Decomposition (POD).
- For certain classes of nonlinear multi-agent systems, e.g.

$$\dot{x}_i(t) = f(x_i(t)) + z_i(t) + u_{k(i)}(t),$$

$$z_i(t) = \sum_{j=1}^{N} a_{ij} g(x_i(t), x_j(t)),$$

the reduced clustered model is obtained by projecting the adjacency matrix $\mathcal{A} = [a_{ij}]$ to a reduced adjacency matrix $\widehat{\mathcal{A}} = W^T \mathcal{A} V$.

Numerical Examples (Lorenz Systems as Agents)

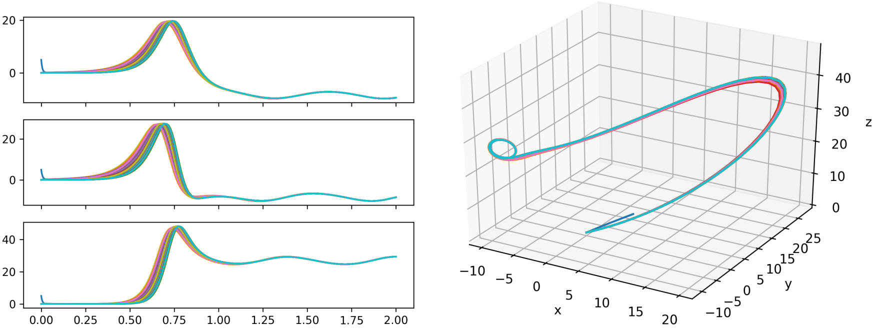

Figure: Impulse response of the full order multi-agent system with 500 Lorenz systems as agents.

Numerical Examples (Lorenz Systems as Agents)

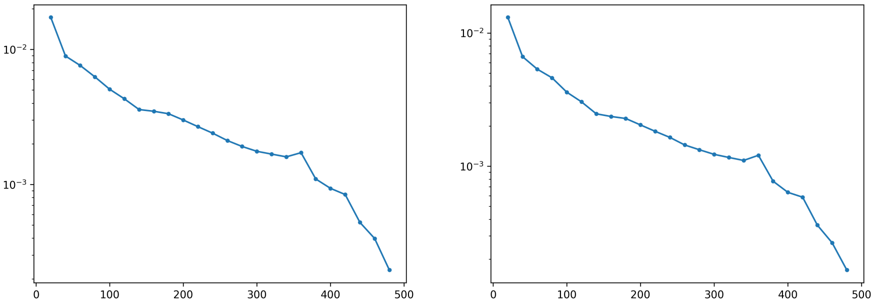

Figure: Time domain L^2 errors for the impulse ($u(t) = \delta(t)$) and rectangle ($u(t) = \chi_{[0,1]}(t)$) response for different number of clusters. In both cases, 10 dominant POD modes from impulse response snapshots were taken to define V.

Possible Extensions

Directed Graphs

- Our approach is still applicable.
- But, it is still an open question what are the simplest sufficient conditions for preserving consensus or synchronization.

Non-Identical Agents

- If there is a small number of types of agents in the network, we can easily modify the clustering to only allow clustering of the agents of the same type.
- Otherwise, we could allow clustering of different agents. The clustered agent would then have an averaged dynamics of the agents in the cluster.

A Priori Error Bounds

Agents

We focus on linear agents

$$\dot{x}_i = Ax_i + Bz_i + Eu_{k(i)},$$

with $x_i(t) \in \mathbb{R}^n$ $(i = 1, 2, \ldots, N)$ and $u_j(t) \in \mathbb{R}^\mu$ $(j = 1, 2, \ldots, m)$.

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A Priori Error Bounds

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Interconnections

For an undirected, weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ with adjacency matrix $\mathcal{A} = [a_{ij}]$ define

$$z_i = \sum_{j=1}^{N} a_{ij}(x_j - x_i).$$

Multi-Agent System Dynamics

$$\dot{x} = (I_N \otimes A - L \otimes B)x + (M \otimes E)u,$$

where

- $L \in \mathbb{R}^{N \times N}$ is the Laplacian matrix,
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Multi-Agent System Dynamics and Output

$$\dot{x} = (I_N \otimes A - L \otimes B)x + (M \otimes E)u,$$

$$y = \begin{cases} \left(W^{\frac{1}{2}} R^T \otimes I_n\right) x, \\ (L \otimes I_n)x, \end{cases}$$

where

- $L \in \mathbb{R}^{N \times N}$ is the Laplacian matrix,
- $M \in \mathbb{R}^{N \times m}$ is the input matrix,
- $R \in \mathbb{R}^{N \times p}$ is the incidence matrix,
- $W \in \mathbb{R}^{p \times p}$ is the edge weights matrix.

Special Case: Single Integrator Agents

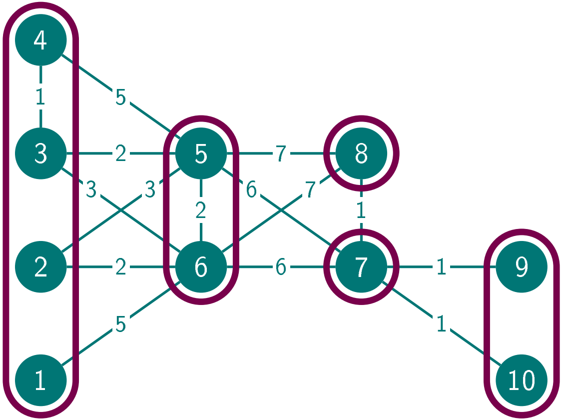
For $A = 0$, $B = 1$, and $E = 1$, agents are

$$\dot{x}_i = z_i + u_{k(i)}$$

with $x_i, z_i, u_j \in \mathbb{R}$.
The multi-agent system dynamics becomes

$$\dot{x} = -Lx + Mu.$$

Definition of Almost Equitable Partitions (Graphically)



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Definition of Almost Equitable Partitions

Definition (Almost Equitable Partition)

Partition $\pi = \{C_1, C_2, \dots, C_r\}$ of a graph $\mathcal{G}$ is almost equitable if

$$(\forall\, p, q \in \{1, 2, \dots, r\},\ p \neq q)(\exists\, c_{pq} \in \mathbb{R})(\forall\, i \in C_p) \sum_{j \in C_q} a_{ij} = c_{pq}.$$

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Example (Trivial Almost Equitable Partitions)

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Simpler, equivalent condition?

Equivalent Condition for Almost-Equitableness

Theorem ([Cardoso/Delorme/Rama '07])

Partition π is almost equitable for $\mathcal{G}$ if and only if $\operatorname{range}(P(\pi))$ *is L-invariant.*

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Example

$$L = \begin{pmatrix} 5 & & & & & -5 & & & & \\ & 5 & & & -3 & -2 & & & & \\ & & 6 & -1 & -2 & -3 & & & & \\ & & -1 & 6 & -5 & & & & & \\ & -3 & -2 & -5 & 25 & -2 & -6 & -7 & & \\ -5 & -2 & -3 & & -2 & 25 & -6 & -7 & & \\ & & & & -6 & -6 & 15 & -1 & -1 & -1 \\ & & & & -7 & -7 & -1 & 15 & & \\ & & & & & & -1 & & 1 & \\ & & & & & & -1 & & & 1 \end{pmatrix}, \quad P(\pi) = \begin{pmatrix} 1 & & & & \\ 1 & & & & \\ 1 & & & & \\ 1 & & & & \\ & 1 & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \\ & & & & 1 \end{pmatrix}$$

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$$LP(\pi) = \begin{pmatrix} 5 & -5 & 0 & 0 & 0 \\ 5 & -5 & 0 & 0 & 0 \\ 5 & -5 & 0 & 0 & 0 \\ 5 & -5 & 0 & 0 & 0 \\ -10 & 23 & -6 & -7 & 0 \\ -10 & 23 & -6 & -7 & 0 \\ 0 & -12 & 15 & -1 & -2 \\ 0 & -14 & -1 & 15 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix} = P(\pi) \begin{pmatrix} 5 & -5 & 0 & 0 & 0 \\ -10 & 23 & -6 & -7 & 0 \\ 0 & -12 & 15 & -1 & -2 \\ 0 & -14 & -1 & 15 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix}$$

AEPs and Controllability

Theorem ([Zhang/Cao/Çamlibel '14])

Let π be an almost equitable partition (AEP) such that

$$\{v_1\}, \{v_2\}, \ldots, \{v_m\} \in \pi.$$

Then

$$\mathcal{K} \subset \mathrm{range}(P(\pi) \otimes I_n),$$

where $\mathcal{K}$ is the controllable subspace of the multi-agent system.

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$\{\{1\}, \{2\}, \dots, \{N\}\}$ always satisfies the above assumptions. Existence of a coarser partition implies uncontrollability (good for model reduction).

Expression for the $\mathcal{H}_2$-Error (Single Integrator Agents and AEPs)

Theorem ([Monshizadeh/Trentelman/Çamlibel '14])

Assume agents are single integrators and output is $y = W^{\frac{1}{2}} R^T x$. *Let* $\pi = \{C_1, C_2, \ldots, C_r\}$ *be an AEP of* $\mathcal{G}$. *Then*

$$\|S - \widehat{S}\|_{\mathcal{H}_2}^2 = \frac{1}{2} \sum_{i=1}^{m} \left(1 - \frac{1}{|C_{k_i}|}\right),$$

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where $k_i \in \{1, 2, \ldots, r\}$ *is such that cluster* C_{k_i} *contains leader* v_i.

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Generalization to arbitrary agents? Arbitrary graph partitions?

Synchronization

Definition

Multi-agent system $\dot{x} = (I_N \otimes A - L \otimes B)x$ is synchronized if

$$x_i(t) - x_j(t) \to 0 \text{ as } t \to \infty,$$

for all $i, j \in \mathcal{V}$ and any initial condition $x(0) = x_0 \in \mathbb{R}^{Nn}$.

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MOR by Clustering

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Synchronization is preserved in the reduced multi-agent system when clustering by an AEP or if the agents are single integrators.

MOR by Clustering

Auxiliary Systems

Denote by S_λ the transfer function of

$$\dot{\xi} = (A - \lambda B)\xi + Ed,$$
$$z = \lambda\xi,$$

i.e.

$$S_\lambda(s) = \lambda(sI_n - A + \lambda B)^{-1}E.$$

MOR by Clustering

Theorem ($\mathcal{H}_2$-Error Bounds for AEPs)

Assume that the multi-agent system is synchronized and $y = (L \otimes I_n)x$. Let π be an AEP of $\mathcal{G}$. Then

$$\|S - \widehat{S}\|_{\mathcal{H}_2}^2 \leq S_{\max,\mathcal{H}_2}^2 \sum_{i=1}^{m} \left(1 - \frac{1}{|C_{k_i}|}\right),$$

$$\frac{\|S - \widehat{S}\|_{\mathcal{H}_2}^2}{\|S\|_{\mathcal{H}_2}^2} \leq \frac{S_{\max,\mathcal{H}_2}^2}{S_{\min,\mathcal{H}_2}^2} \frac{\sum_{i=1}^{m} \left(1 - \frac{1}{|C_{k_i}|}\right)}{m\left(1 - \frac{1}{N}\right)},$$

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$\mathcal{H}_\infty$-norm

$$\|S\|_{\mathcal{H}_\infty} = \sup_{\omega \in \mathbb{R}} \|S(i\omega)\|_2$$

MOR by Clustering

Theorem ($\mathcal{H}_\infty$-Error Bounds for AEPs)

Assume that the multi-agent system is synchronized, $y = (L \otimes I_n)x$, *and matrices* A *and* B *are symmetric. Let* π *be an AEP of* $\mathcal{G}$. *Then*

$$\|S - \widehat{S}\|_{\mathcal{H}_\infty}^2 \leq \begin{cases} S_{\max,\mathcal{H}_\infty}^2 \max\limits_{1 \leq i \leq m} \left(1 - \frac{1}{|C_{k_i}|}\right) & \text{if the leaders are in different clusters,} \\ S_{\max,\mathcal{H}_\infty}^2 & \text{otherwise,} \end{cases}$$

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Generalizing to Arbitrary Graph Partitions

Idea: Change the Graph

For a given partition π of the graph $\mathcal{G}$, find a graph $\mathcal{G}_{\mathsf{AEP}}$ such that

- π is an AEP for $\mathcal{G}_{\mathsf{AEP}}$,
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Then use triangle inequality to obtain a bound

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Ideas How to Change the Graph

1. add/remove edges $\rightsquigarrow$ [Jongsma/Trentelman/Çamlibel '15]
2. modify edge weights $\rightsquigarrow$ [Jongsma/Mlinarić/Grundel/Benner/Trentelman '16]

MOR by Clustering

A Priori Error Bounds

We restrict to multi-agent systems with single-integrator agents

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We have the following transfer functions:

$$S(s) = L(sI + L)^{-1} M, \qquad \widehat{S}(s) = LP(\pi)(sI + \widehat{L})^{-1} \widehat{M},$$
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Idea

Find L_{AEP} close to L.

Optimization Problem

Denote $\mathcal{P} := P(\pi)\left(P(\pi)^T P(\pi)\right)^{-1} P(\pi)^T$, the orthogonal projector onto $\mathrm{range}(P(\pi))$.

MOR by Clustering

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This is a quadratic programming problem, which can be solved efficiently.

MOR by Clustering

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What if we ignore the inequality constraints?

Solving the (Relaxed) Optimization Problem

The unique solution to

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MOR by Clustering

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L_{AEP} can have positive off-diagonal elements, i.e. $\mathcal{G}_{\mathrm{AEP}}$ can have negative weights.
But, L_{AEP} is symmetric positive semi-definite and $\ker L_{\mathrm{AEP}} = \operatorname{span}\{\mathbb{1}_N\}$.

MOR by Clustering

$\mathcal{G}$:

MOR by Clustering

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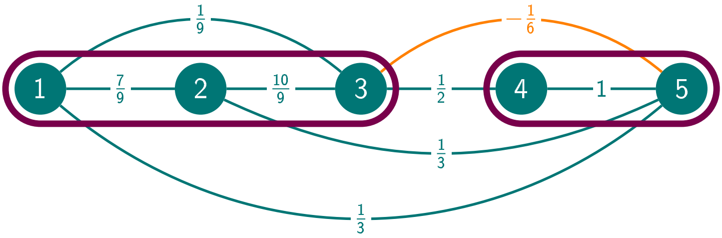
$\mathcal{G}_{\text{AEP}}$:



Error Bounds (Single Integrator Agents and Arbitrary Partition)

$$\|S - S_{\mathsf{AEP}}\|_{\mathcal{H}_p} \leq 2 \left\|(L - L_{\mathsf{AEP}})(sI + L)^{-1} M\right\|_{\mathcal{H}_p}$$

$$\|\widehat{S}_{\mathsf{AEP}} - \widehat{S}\|_{\mathcal{H}_p} \leq \left\|(L - L_{\mathsf{AEP}}) P(\pi)(sI + \widehat{L})^{-1} \widehat{M}\right\|_{\mathcal{H}_p}$$

MOR by Clustering

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$$\|S - S_{\mathsf{AEP}}\|_{\mathcal{H}_p} \leq 2\, \left\|(L - L_{\mathsf{AEP}})(sI + L)^{-1} M\right\|_{\mathcal{H}_p}$$

$$\|\widehat{S}_{\mathsf{AEP}} - \widehat{S}\|_{\mathcal{H}_p} \leq \left\|(L - L_{\mathsf{AEP}}) P(\pi)(sI + \widehat{L})^{-1} \widehat{M}\right\|_{\mathcal{H}_p}$$

"Almost Almost Equitable Partitions"

If π is such that $L_{\mathsf{AEP}} \approx L$, then $\|S - \widehat{S}\|_{\mathcal{H}_p}$ is 'small'.

Agent Reduction

Linear Multi-Agent System

$$\dot{x}(t) = (I \otimes A - L \otimes BC)x(t) + (G \otimes B)u(t)$$
$$y(t) = (H \otimes C)x(t)$$

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Reduced System

$$\dot{\widehat{x}}(t) = (I \otimes \widehat{A} - L \otimes \widehat{B}\widehat{C})\,\widehat{x}(t) + (G \otimes \widehat{B})\,u(t)$$
$$\widehat{y}(t) = (H \otimes \widehat{C})\,\widehat{x}(t)$$

Linear Multi-Agent System

$$\begin{aligned} \dot{x}(t) &= (I \otimes A - L \otimes BC)x(t) + (G \otimes B)u(t) \\ y(t) &= (H \otimes C)x(t) \end{aligned}$$

Reduced System

$$\begin{aligned} \dot{\widehat{x}}(t) &= (I \otimes \widehat{A} - L \otimes \widehat{B}\widehat{C})\,\widehat{x}(t) + (G \otimes \widehat{B})\,u(t) \\ \widehat{y}(t) &= (H \otimes \widehat{C})\,\widehat{x}(t) \end{aligned}$$

$\mathcal{H}_2$-Optimality Conditions

Differentiate

$$\|S - \widehat{S}\|_{\mathcal{H}_2}^2$$

with respect to $\widehat{A}$, $\widehat{B}$, and $\widehat{C}$.

Agent Reduction

Necessary Optimality Conditions for $\mathcal{H}_2$-Optimal Agent Reduction

The necessary optimality conditions are

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\widetilde{Q}_{ji}^{T}A\widetilde{P}_{ji}-\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\ell_{jk}\widetilde{Q}_{ji}^{T}BC\widetilde{P}_{ki}+\sum_{i=1}^{N}\sum_{j=1}^{N}\widehat{Q}_{ij}\widehat{A}\widehat{P}_{ji}-\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\ell_{jk}\widehat{Q}_{ij}\widehat{B}\widehat{C}\widehat{P}_{ki}=0,$$

$$-\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\ell_{ik}\widetilde{Q}_{ji}^{T}\widetilde{P}_{jk}\widehat{C}^{T}-\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\ell_{ik}\widehat{Q}_{ji}^{T}\widehat{P}_{jk}\widehat{C}^{T}+\sum_{i=1}^{N}\sum_{j=1}^{N}[GG^{T}]_{ji}\widetilde{Q}_{ji}^{T}B+\sum_{i=1}^{N}\sum_{j=1}^{N}[GG^{T}]_{ji}\widehat{Q}_{ji}^{T}\widehat{B}=0,$$

$$-\widehat{B}^{T}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\ell_{ik}\widetilde{Q}_{ji}^{T}\widetilde{P}_{jk}-\widehat{B}^{T}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\ell_{ik}\widehat{Q}_{ji}^{T}\widehat{P}_{jk}-C\sum_{j=1}^{N}\sum_{k=1}^{N}[H^{T}H]_{jk}\widetilde{P}_{jk}+\widehat{C}\sum_{j=1}^{N}\sum_{k=1}^{N}[H^{T}H]_{jk}\widehat{P}_{jk}=0,$$

where $\widetilde{P}=[\widetilde{P}_{ij}]$ and $\widehat{P}=[\widehat{P}_{ij}]$ are the upper-right and lower-right block in the controllability Gramian of the error system, and similarly $\widetilde{Q}=[\widetilde{Q}_{ij}]$ and $\widehat{Q}=[\widehat{Q}_{ij}]$ are blocks in the observability Gramian.

Conclusion

Summary

- $\mathcal{H}_2$-quasi-optimal clustering-based MOR method combining IRKA and k-means.
- Extension to nonlinear network systems, using a nonlinear MOR method and k-means.
- $\mathcal{H}_2$-optimality conditions for agent reduction.

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- $\mathcal{H}_2$-quasi-optimal clustering-based MOR method combining IRKA and k-means.
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Outlook

- Error bounds for the clustering method.
- Efficient implementation of $\mathcal{H}_2$-optimal agent reduction MOR method.

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