Mathematical Imaging with Optical Coherence Tomography and Photoacoustics

Peter Elbau[1], Otmar Scherzer[1,2], Cong Shi[1]

[1]Computational Science Center, University Vienna, Austria
&
[2] Radon Institute of Computational and Applied Mathematics (RICAM), Linz, Austria

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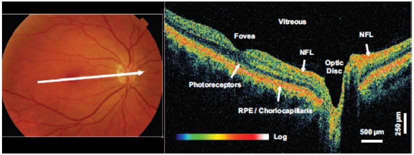
Outline

OCT Images



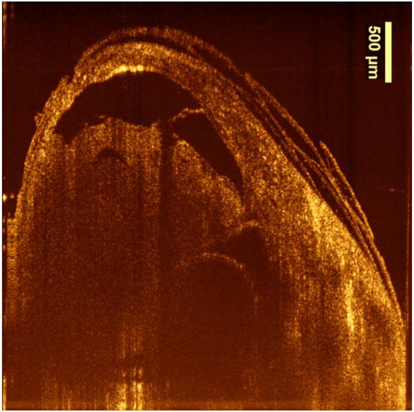


Figure: Data B. Zebhian, W. Drexler: Medical University of Vienna

Physical Scheme of OCT Setup

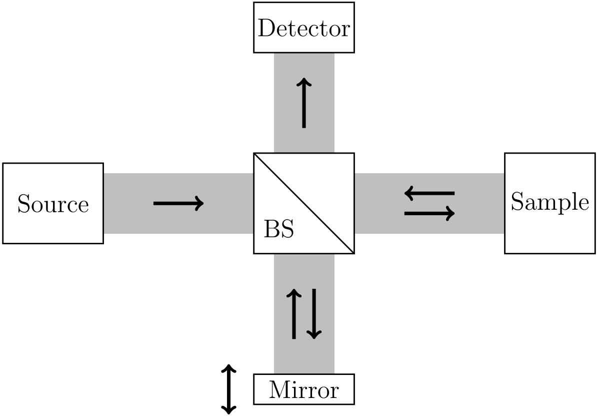


Figure: OCT System

Basics

- Illumination of a specimens with some a short laser pulse

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- Measurements of the backscattered wave: Coherence of the scattered light with the undisturbed original laser pulse

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- Illumination of a specimens with some a short laser pulse
- Measurements of the backscattered wave: Coherence of the scattered light with the undisturbed original laser pulse
- Visualization of backscattered light

Reconstruction parameters depend on modeling:

reconstruction parameter	model
scattering coefficient	Boltzmann equation
refractive index	geometric optics
susceptibility	Maxwell's equations

Photoacoustic setup

Figure: Photoacoustic setup: Robert Nuster and Günther Paltauf (University Graz)

Applications: Microscopy

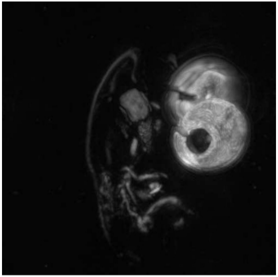

Figure: Chicken embryo (5 days old)(1.2 mm) and mouse heart. Data by Berooz Zebian and Wolfgang Drexler, Robert Nuster et al.

Photoacoustic imaging – “Lightning and Thunder” (L.H. Wang)

- Specimen is uniformly illuminated by a short/pulsed electromagnetic pulse (visible or near infrared light - Photoacoustics, microwaves - Thermoacoustics)

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- Two-step conversion process: Absorbed EM energy is converted into heat ⇒ Material reacts with expansion ⇒ Expansion produces pressure waves
- Imaging: Pressure waves are detected at the boundary of the object (over time) and are used for reconstruction of conversion parameter (EM energy into expansion/ waves)

The method in between

DOT (diffuse optical tomography), OCT (optical coherence tomography), SPIM (single plane imaging). MAD (mandibular advancement device), 2P (2 photon),...

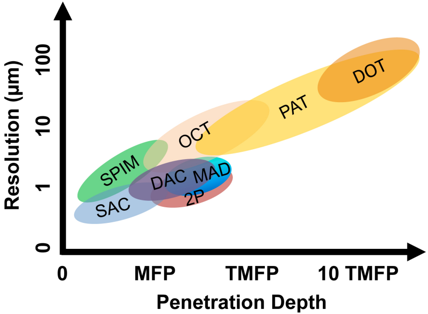

Fig. 1

Citation
Steven Y. Leigh, Ye Chen, Jonathan T.C. Liu, "Modulated-alignment dual-axis (MAD) confocal microscopy for deep optical sectioning in tissues," Biomed. Opt. Express **5,** 1709-1720 (2014);
http://www.opticsinfobase.org/boe/abstract.cfm?URI=boe-5-6-1709

Schematic representation

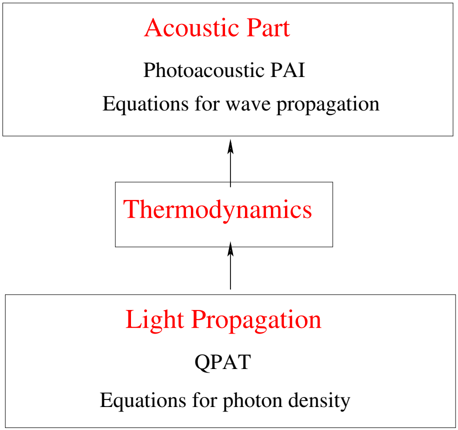

Figure: Three kind of problems

Governing equation of photoacoustics: Standard

Wave equation for the pressure

$$\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2}(x,t) - \Delta p(x,t) = \frac{dj}{dt}(t)\mathcal{H}(x)$$

Parameters and functions:

- $\mathcal{H}$ absorbed electromagnetic energy
- c_0 speed of sound - in most mathematical studies normalized to constant 1
- $j(t)$ can be considered to approximate a δ-impulse (Lightning)
- Alternative if $j = \delta$: Initial value problem for homogeneous wave equation

$$p(x,0) = \mathcal{H}(x), \quad p_t(x,0) = 0$$

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Inverse problem of photoacoustics

- measurement data: $p(x,t)$ for $x \in \mathcal{M}$, $t > 0$ measurement region
- Reconstruction of $\mathcal{H}(x)$ for $x \in \Omega$

Exact Reconstruction formulae for $\mathcal{H}$ depending on measurement geometry

- Sphere, Cylinder, Plane [Xu, Wang, 2002], [Finch, Patch, Rakesh, 2004]
- Circle [Finch, Haltmeier, Rakesh, 2005]
- Universal Backprojection [Wang et al, 2005, Natterer 2012]
- Palamodov 2014
- ...

Reconstruction parameters depend on modeling

Model	reconstruction parameter	model
PAT	absorption density	Wave equation
QPAT	absorption coefficient	Diffusion or Boltzmann equation
QPAT	susceptibility	Maxwell's equations

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Medical applications

3. Human Studies, PAT-OCT

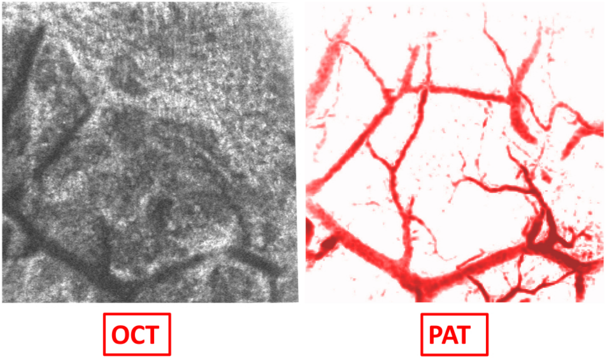

Figure: Human studies of combined OCT (optical coherence tomography) and PAI: data by Boris Hermann and Wolfgang Drexler

Modeling aspects

Standard Photoacoustics does not model variable sound speed, attenuation and variable illumination and does not recover physical parameters

1. Quantitative Photoacoustics (separation of parameters in $\mathcal{H}$) [Arridge, Bal, Ken, Scotland, Uhlmann,...]
2. Sound speed variations: [Agranovsky, Hristova, Kuchment, Stefanov, Uhlmann,...]
3. Attenuation and dispersion [Anastasio, Patch, Riviere, Burgholzer, Kowar, S., Ammari, Wahab, ...]
4. Variable Illumination [Wang, Bal,...]
5. Finite bandwidth detectors [Haltmeier et al]

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Attenuation

Attenuation: Model equation

$$\mathcal{A}_\kappa p(x,t) - \Delta p(x,t) = \frac{d\delta}{dt}(t)\mathcal{H}(x)$$
$$p(x,t) = 0 \text{ for } t < 0$$

- L: pseudo-differential operator w.r.t. t, with symbol $-\kappa^2(\omega)$
- $\Im\kappa(\omega)$ attenuation law

Solution of forward problem

Inverse Fourier-transform with respect to t:

$$\check{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \mathrm{e}^{\mathrm{i}\omega t} dt$$

Solution of attenuated equation:

$$\check{p}(\omega; x) = -\int_{\mathbb{R}^3} \frac{\mathrm{i}\omega}{4\pi\sqrt{2\pi}} \frac{\mathrm{e}^{\mathrm{i}\kappa(\omega)|x-y|}}{|x-y|} \mathcal{H}(y) dy$$

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Object of interest: Time-integrated photoacoustic operator in frequency domain:

$$\check{\mathcal{P}}_\kappa \mathcal{H}(\omega; x) = \frac{1}{4\pi\sqrt{2\pi}} \int_{\mathbb{R}^3} \frac{\mathrm{e}^{\mathrm{i}\kappa(\omega)|x-y|}}{|x-y|} \mathcal{H}(y) dy$$

Formal definition

of attenuation operator

$$\mathcal{A}_\kappa : \mathcal{S}'(\mathbb{R} \times \mathbb{R}^3) \to \mathcal{S}'(\mathbb{R} \times \mathbb{R}^3)$$

is defined by the action on the tensor products $\phi \otimes \psi \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^3)$, given by $(\phi \otimes \psi)(t, x) = \phi(t)\psi(x)$:

$$\langle \mathcal{A}_\kappa u, \phi \otimes \psi \rangle_{\mathcal{S}',\mathcal{S}} = -\left\langle u, (\mathcal{F}^{-1}\kappa^2\mathcal{F}\phi) \otimes \psi \right\rangle_{\mathcal{S}',\mathcal{S}},$$

where

$$\mathcal{F} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R}), \quad \mathcal{F}\phi(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(t) \mathrm{e}^{-\mathrm{i}\omega t} dt$$

denotes the Fourier transform

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Assumptions on κ

$\kappa \in C^\infty(\mathbb{R};\overline{\mathbb{H}})$ with $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$ is called attenuation coefficient if

1. all derivatives of κ are polynomially bounded:

$$\forall_{\ell \in \mathbb{N}_0} \exists_{\kappa_1 > 0, N \in \mathbb{N}} \quad |\kappa^{(\ell)}(\omega)| \leq \kappa_1 (1 + |\omega|)^N$$

2. there exists a continuous continuation $\tilde{\kappa} : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ of κ such that $\tilde{\kappa} : \mathbb{H} \to \overline{\mathbb{H}}$ is holomorphic. Moreover,

$$\exists_{\tilde{\kappa}_1 > 0, N \in \mathbb{N}} \quad |\tilde{\kappa}(z)| \leq \tilde{\kappa}_1 (1 + |z|)^{\tilde{N}} \quad \forall z \in \overline{\mathbb{H}}$$

3. $\kappa(-\omega) = -\overline{\kappa(\omega)}$ for all $\omega \in \mathbb{R}$

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Consequences of these assumptions

1. $\mathcal{A}_\kappa$ is well-defined: $\kappa^2 u \in \mathcal{S}'(\mathbb{R}) \quad \forall u \in \mathcal{S}'(\mathbb{R})$
2. $p \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}^3)$ is causal: that is $\operatorname{supp}(p) \subseteq [0, \infty) \times \mathbb{R}^3$
3. $\mathcal{A}_\kappa$ maps real-valued distributions onto itself

Finite speed of propagation

$p \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}^3)$ propagates with finite speed $c > 0$ if

$$\operatorname{supp} p \subset \{(t,x) \in \mathbb{R} \times \mathbb{R}^3 : |x| \le ct + R\} \quad \forall_h \operatorname{supp} h \subset B_R(0)$$

Lemma

Let κ be an attenuation coefficient with the holomorphic extension $\tilde{\kappa} : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$.
Then, p propagates with finite speed if and only if

$$\lim_{\omega \to \infty} \frac{\tilde{\kappa}(\mathrm{i}\omega)}{\mathrm{i}\omega} > 0$$

In this case, it propagates with the speed $c = \lim_{\omega \to \infty} \frac{\mathrm{i}\omega}{\tilde{\kappa}(\mathrm{i}\omega)}$

Examples of attenuation models

$\kappa \in C^\infty(\mathbb{R};\overline{\mathbb{H}})$ is called a

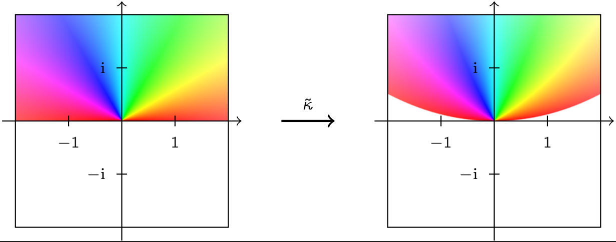
- strong attenuation coefficient if

$$\exists_{\kappa_0,\beta>0,\omega_0\geq 0} \quad \Im\kappa(\omega) \geq \kappa_0|\omega|^\beta\,, \qquad \forall \omega \in \mathbb{R}\,, |\omega| \geq \omega_0$$

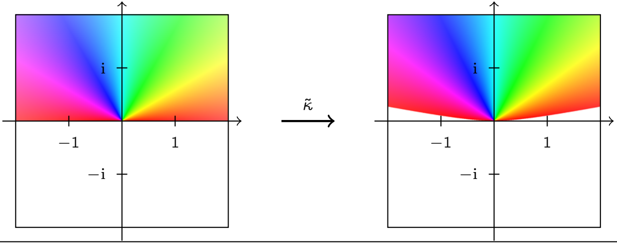
- weak attenuation coefficient if it is of the form

$$\exists_{c>0,\kappa_\infty\geq 0}\exists_{\kappa_*\in C^\infty(\mathbb{R})\cap L^2(\mathbb{R})} \quad \kappa(\omega) = \frac{\omega}{c} + \mathrm{i}\kappa_\infty + \kappa_*(\omega), \quad \forall \omega \in \mathbb{R}$$

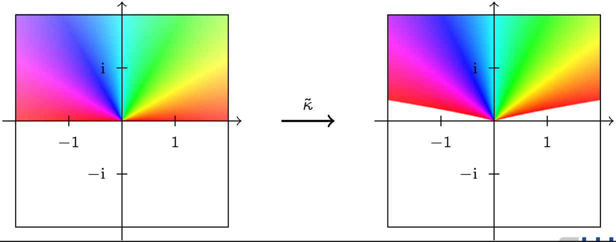
Thermo-viscous model

Name:	Thermo-viscous model
Coefficient:	$\kappa : \mathbb{R} \to \mathbb{C},\ \kappa(\omega) = \dfrac{\omega}{\sqrt{1 - \mathrm{i}\tau\omega}}, \quad \tau > 0,\ \tilde{\kappa}(z) = \kappa(z)$
Upper bound:	$\lvert\tilde{\kappa}(z)\rvert \le \lvert z\rvert$ for all $z \in \overline{\mathbb{H}}$
Speed:	$c = \lim\limits_{\omega\to\infty} \dfrac{\mathrm{i}\omega}{\tilde{\kappa}(\mathrm{i}\omega)} = \lim\limits_{\omega\to\infty} \sqrt{1 + \tau\omega} = \infty$
Type:	Strong attenuation coefficient
Range of $\tilde{\kappa}$:	

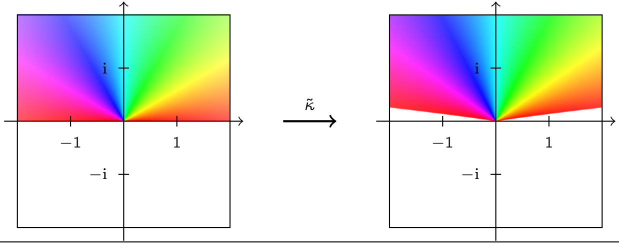
Kowar-S-Bonnefond model

Attenuation:	$\kappa : \mathbb{R} \to \mathbb{C},\ \kappa(\omega) = \omega \left(1 + \frac{\alpha}{\sqrt{1+(-\mathrm{i}\tau\omega)^\gamma}}\right)$, $\tilde\kappa(z) = \kappa(z)$
Speed:	$c = \lim_{\omega\to\infty} \frac{\mathrm{i}\omega}{\tilde\kappa(\mathrm{i}\omega)} = \lim_{\omega\to\infty} \frac{1}{1+\frac{\alpha}{\sqrt{1+(\tau\omega)^\gamma}}} = 1$
Type:	Strong attenuation coefficient
Range of $\tilde\kappa$:	

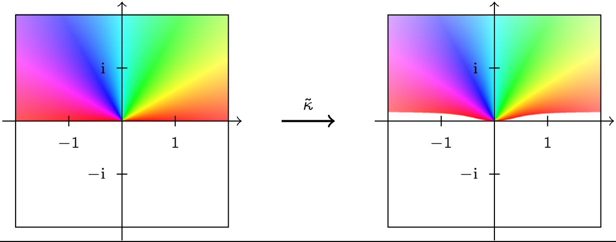
Power law $\sim$ Szabo

Model:	Power law
Coefficient:	$\kappa : \mathbb{R} \to \mathbb{C}$, $\kappa(\omega) = \omega + \mathrm{i}\alpha(-\mathrm{i}\omega)^\gamma$, $\tilde\kappa(z) = \kappa(z)$
Parameters:	$\gamma \in (0,1)$, $\alpha > 0$
Upper bound:	$\lvert\tilde\kappa(z)\rvert \leq \lvert z\rvert + \alpha\lvert z\rvert^\gamma \leq \alpha(1-\gamma) + (1+\alpha\gamma)\lvert z\rvert$
Propagation speed:	$c = \lim_{\omega\to\infty} \frac{\mathrm{i}\omega}{\tilde\kappa(\mathrm{i}\omega)} = \lim_{\omega\to\infty} \frac{1}{1+\alpha\omega^{\gamma-1}} = 1$
Attenuation type:	Strong attenuation coefficient
Range of $\tilde\kappa$:	

Modified Szabo's model

Coefficient: Parameters:	$\kappa:\mathbb{R}\to\mathbb{C},\ \kappa(\omega)=\omega\sqrt{1+\alpha(-\mathrm{i}\omega)^{\gamma-1}}$ $\gamma\in(0,1),\ \alpha>0$
Upper bound:	$\lvert\tilde\kappa(z)\rvert\le\frac12\alpha(1-\gamma)+(1+\frac{\alpha}{2}(1+\gamma))\lvert z\rvert$
Speed:	$c=\lim\limits_{\omega\to\infty}\dfrac{\mathrm{i}\omega}{\tilde\kappa(\mathrm{i}\omega)}=\lim\limits_{\omega\to\infty}\dfrac{1}{\sqrt{1+\alpha\omega^{\gamma-1}}}=1$
Type:	Strong attenuation coefficient
Range of $\tilde\kappa$:	

Nachman-Smith-Waag

Coefficient:	$\kappa : \mathbb{R} \to \mathbb{C},\ \kappa(\omega) = \frac{\omega}{c_0}\sqrt{\frac{1-\mathrm{i}\tilde\tau\omega}{1-\mathrm{i}\tau\omega}}$
Parameters:	$c_0 > 0,\ \tau > 0,\ \tilde\tau \in (0,\tau)$
Extension:	$\tilde\kappa : \overline{\mathbb{H}} \to \mathbb{C},\ \frac{z}{c_0}\sqrt{\frac{1-\mathrm{i}\tilde\tau z}{1-\mathrm{i}\tau z}},\ \lvert\tilde\kappa(z)\rvert \le \frac{1}{c_0}\lvert z\rvert$ for all $z \in \overline{\mathbb{H}}$
Speed:	$c = \lim_{\omega\to\infty}\frac{\mathrm{i}\omega}{\tilde\kappa(\mathrm{i}\omega)} = \lim_{\omega\to\infty} c_0\sqrt{\frac{1+\tau\omega}{1+\tilde\tau\omega}} = c_0\sqrt{\frac{\tau}{\tilde\tau}}$
Type:	Weak attenuation coefficient
Range of $\tilde\kappa$:	

Photoacoustic equation in $\mathcal{F}$-domain: recalled

The solution p of the attenuated wave equation in Fourier domain:

$$\check{p}(\omega, x) = \int_{\mathbb{R}^3} G_\kappa(\omega, x - y)\mathcal{H}(y)dy$$
$$= -\frac{\mathrm{i}\omega}{4\pi\sqrt{2\pi}} \int_{\mathbb{R}^3} \frac{\mathrm{e}^{\mathrm{i}\kappa(\omega)|x-y|}}{|x-y|}\mathcal{H}(y)dy, \qquad \omega \in \mathbb{R},\ x \in \mathbb{R}^3$$

Measurement data of photoacoustics

$$\check{m}(\omega, \xi) = \check{p}(\omega, \xi) \quad \forall \omega \in \mathbb{R},\ \xi \in \partial\Omega$$

Alternative equation: integrated photoacoustic operator

Inverse problem of attenuated photoacoustics: Find $\mathcal{H}$ such that

$$\boxed{\frac{1}{-\mathrm{i}\omega}\check{m}(\omega,\xi) = \frac{1}{-\mathrm{i}\omega}\check{p}(\omega,\xi), \quad \forall \omega \in \mathbb{R},\ \xi \in \partial\Omega}$$

Meaning that the data are integrated in time-domain before inversion!

Integrated photoacoustic operator

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and
$\Omega_\varepsilon = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$
κ denotes either a strong or a weak attenuation coefficient

Definition

$$\check{\mathcal{P}}_\kappa : L^2(\Omega_\varepsilon) \to L^2(\mathbb{R} \times \partial\Omega),$$
$$\mathcal{H} \to \frac{1}{4\pi\sqrt{2\pi}} \int_{\Omega_\varepsilon} \frac{\mathrm{e}^{\mathrm{i}\kappa(\omega)|\xi - y|}}{|\xi - y|} \mathcal{H}(y)dy$$

is called *integrated photoacoustic operator* of the attenuation coefficient κ in frequency domain

Properties of integrated photoacoustic operator

Let $\check{\mathcal{P}}_\kappa : L^2(\Omega_\varepsilon) \to L^2(\mathbb{R} \times \partial\Omega)$ be the integrated photoacoustic operator of a weak or a strong attenuation coefficient κ:

Theorem

Then, $\check{\mathcal{P}}_\kappa^* \check{\mathcal{P}}_\kappa : L^2(\Omega_\varepsilon) \to L^2(\Omega_\varepsilon)$ is a self-adjoint integral operator with kernel $F \in L^2(\Omega_\varepsilon \times \Omega_\varepsilon)$ given by

$$F_\kappa(x,y) = \frac{1}{32\pi^3} \int_{-\infty}^{\infty} \int_{\partial\Omega} \frac{\mathrm{e}^{\mathrm{i}\kappa(\omega)|\xi-y| - \mathrm{i}\overline{\kappa(\omega)}|\xi-x|}}{|\xi-y||\xi-x|} dS(\xi) d\omega,$$

that is

$$\check{\mathcal{P}}_\kappa^* \check{\mathcal{P}}_\kappa h(x) = \int_{\Omega_\varepsilon} F_\kappa(x,y) h(y) dy$$

In particular, $\check{\mathcal{P}}_\kappa^* \check{\mathcal{P}}_\kappa$ is a Hilbert–Schmidt operator and thus compact

Proof: lengthy

Singular values of IPO for strongly attenuating media

Kernel estimates[1]

Lemma

For strongly attenuation κ the kernel F_κ of $\check{\mathcal{P}}_\kappa^*\check{\mathcal{P}}_\kappa$ satisfies

$$\exists_{B,b>0}\frac{1}{j!}\sup_{x,y\in\Omega_\varepsilon}\sup_{v\in S^2}\left|\frac{\partial^j}{\partial s^j}\bigg|_{s=0}F_\kappa(x,y+sv)\right|\leq Bb^j j^{(\frac{N}{\beta}-1)j}\qquad\forall j\in\mathbb{N}_0$$

[1] $N\in\mathbb{N}$ denotes the exponent for $\ell=0$ (polynomial condition) and $\beta\in(0,N]$ is the exponent in the condition for the strong attenuation coefficient κ

Singular values of IPO for strongly attenuating media

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ and $\varepsilon > 0$

Corollary

Let $\check{\mathcal{P}}_\kappa : L^2(\Omega_\varepsilon) \to L^2(\mathbb{R} \times \partial\Omega)$ be the integrated photoacoustic operator of a strong attenuation coefficient κ.
Then, there exist constants $C, c > 0$ such that the eigenvalues $(\lambda_n(\check{\mathcal{P}}_\kappa^* \check{\mathcal{P}}_\kappa))_{n\in\mathbb{N}}$ (in decreasing order) satisfy

$$\lambda_n(\check{\mathcal{P}}_\kappa^* \check{\mathcal{P}}_\kappa) \leq C n \sqrt[m]{n} \exp\left(-c n^{\frac{\beta}{Nm}}\right) \quad \forall n \in \mathbb{N}$$

Singular values of IPO for weakly attenuating media

Split

$$\check{\mathcal{P}}_\kappa = \check{\mathcal{P}}_\kappa^{(0)} + \check{\mathcal{P}}_\kappa^{(1)},$$

$$\check{\mathcal{P}}_\kappa^{(0)} h(\omega,\xi) = \frac{1}{4\pi\sqrt{2\pi}} \int_{\Omega_\varepsilon} \frac{\mathrm{e}^{\mathrm{i}\frac{\omega}{c}|\xi-y|}}{|\xi-y|} \mathrm{e}^{-\kappa_\infty|\xi-y|} h(y)\,dy$$

$$\check{\mathcal{P}}_\kappa^{(1)} h(\omega,\xi) = \frac{1}{4\pi\sqrt{2\pi}} \int_{\Omega_\varepsilon} \frac{\mathrm{e}^{\mathrm{i}\frac{\omega}{c}|\xi-y|}}{|\xi-y|} \mathrm{e}^{-\kappa_\infty|\xi-y|} (\mathrm{e}^{\mathrm{i}\kappa_*(\omega)|\xi-y|} - 1) h(y)\,dy$$

$(\check{\mathcal{P}}_\kappa^{(0)}, \check{\mathcal{P}}_\kappa^{(1)})$ photoacoustic operator with constant attenuation and the perturbation

Lemma

Let κ be a weak attenuation coefficient. Then, there exist constants $C_1, C_2 > 0$ such that we have

$$C_1 n^{-\frac{2}{3}} \le \lambda_n(\check{\mathcal{P}}_\kappa^* \check{\mathcal{P}}_\kappa) \le C_2 n^{-\frac{2}{3}} \quad \forall n \in \mathbb{N}$$

Summary

Medium type	Examples	Singular values decay
No attenuation	$\kappa(\omega) = \frac{\omega}{c_0}$	$n^{-2/3}$(Palamodov)
Weak attenuation	Nachman-Smith-Waag $\kappa(\omega) = \frac{z}{c_0}\sqrt{\frac{1-\mathrm{i}\tilde{\tau}\omega}{1-\mathrm{i}\tau\omega}}$	$n^{-2/3}$
Strong attenuation	Thermo-viscous model $\kappa(\omega) = \frac{\omega}{\sqrt{1-i\tau\omega}}$	e^{-cn^C}

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p, p_a solutions of wave equation with $-\omega^2$ and $\kappa^2(\omega)$.

$$q_a(t,x) = \int_{-\infty}^{t} p_a(\tau,x)d\tau \text{ and } q(t,x) = \int_{-\infty}^{t} p(\tau,x)d\tau$$

Then

$$q^a(t,x) = \int_{\mathbb{R}} \left(\mathcal{F}f^{-1}\left(e^{i\kappa(\omega)\tau}\right)\right)(t)q(\tau,x)d\tau$$

For constant attenuation $\kappa_\infty > 0$ (independent of ω)

$$\boxed{\kappa(\omega) = \omega + \mathrm{i}\kappa_\infty}$$

we have

$$\boxed{\mathcal{H} = B_p \frac{\partial}{\partial t}(M q^a)}$$

where

- $M = e^{k_\infty t}$
- B_p backprojection without attenuation

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Thank you for your attention