Exploring a first order hydrodynamic limit of the kinetic Cucker–Smale model with singular influence function

YRW17: Current trends in kinetic theory

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Joint work with Juan Soler

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Introduction to the model

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0, & t \geq 0,\ x \in \mathbb{R}^N \\ \nabla \psi + \nu u = \phi_0 * (\rho u) - (\phi_0 * \rho) u, & t \geq 0,\ x \in \mathbb{R}^N\,. \end{cases} \tag{1}$$

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Notation:

- $t \in \mathbb{R}_0^+$ and $x \in \mathbb{R}^N$ stand for time and position.
- $\rho = \rho(t, x)$ and $u = u(t, x)$ are the density of particles and velocity field.
- $\psi = \psi(t, x)$ is the potential of an external force (maybe self-generated).
- $\nu \in \mathbb{R}_0^+$ is a friction coefficient.
- $N = 2$ or $N = 3$ are the physically meaningful cases.

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   It involves a commutator of mildly singular integrals.

   R. Coifman, R. Rochberg, G. Weiss (1976), S. Chanillo (1982).

Introduction to the model

Reminiscent of Eulerian dynamics with alignment forces and neglected inertia:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0, & t \geq 0,\ x \in \mathbb{R}^N, \\ \dfrac{\partial u}{\partial t} + (u \cdot \nabla) u + \nu u + \nabla \psi = \phi * (\rho u) - (\phi * \rho) u, & t \geq 0,\ x \in \mathbb{R}^N. \end{cases}$$

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General context:

1. Regular weights:

   S.-Y. Ha, E. Tadmor (2008). S.-Y. Ha, J.-G. Liu (2009).
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   - R. C. Fetecau, W. Sun, C. Tan (2016).
   - E. Tadmor, C. Tan (2014).
2. Singular weights:
   - J. Peszek (2014). S. M. Ahn, H. Choi, S.-Y. Ha, H. Lee (2012). P. B. Mucha, J. Peszek (2016). R. Shvydkoy, E. Tadmor (2017).

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General context:

3. Other related models: Kuramoto, aggregation, Vicsec, etc.
   - Synchronization: S.-Y. Ha et al.
   - Aggregation models: J. A. Carrillo, A. Bertozzi et al.
   - Vicsec models: P. Degond, S. Motsch, M.-J. Kang, A. Figalli, et al.
   - Soft active matter: C. Marcheti et al.

Goals of the talk

Goal 1: The hydrodynamic limit.

1. Introduce a singular hyperbolic scaling of the kinetic Cucker–Smale model with regular weights.
2. Derive the rigorous hydrodynamic limit towards (1).
3. Obtain measure-valued solutions of (1) for the range $\lambda \in \left(0, \frac{1}{2}\right]$.

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Goal 2: Well-posedness.

1. Obtain sharp estimates in Sobolev spaces of the commutator of mildly singular integrals.
2. Derive a global in time well-posedness theory for (1) in Sobolev-type spaces for the range $\lambda \in (0, N/2)$.

From micro to meso: the mean field limit

The discrete Cucker–Smale model

$$\begin{cases} \dfrac{dx_i}{dt} = v_j, \\ m\dfrac{dv_i}{dt} = \dfrac{K}{N}\displaystyle\sum_{j=1}^{N} \phi(|x_i - x_j|)(v_j - v_i) \end{cases}$$

where the influence function is

$$\phi(r) := \frac{\sigma^{2\lambda}}{(\sigma^2 + c_{\kappa,\lambda} r^2)^\lambda} \quad \text{and} \quad c_{\kappa,\lambda} := \kappa^{-1/\lambda} - 1,\ \kappa \in (0,1).$$

F. Cucker, S. Smale (2007).

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and $\xi_i(t)$ is δ-correlated white noise with

$$\langle \xi_i(t)\xi_j(t')\rangle = 2D\delta_{ij}\delta(t - t').$$

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Consider τ, the relaxation time under friction, and $\sqrt{\mu}$, the mean thermal velocity of noise and set $\nu := \frac{m}{\tau},\ D := \frac{\mu}{\tau^2}$.

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F. Cucker, S. Smale (2007).

From micro to meso: the mean field limit

The mean field limit $N \to \infty$

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \frac{1}{m}\nabla\psi \cdot \nabla_v f = \mathrm{div}_v \left(\frac{1}{\tau} v f + \frac{\mu}{\tau}\nabla_v f + \frac{K}{mM}((\phi^\sigma\, v) * f) f \right),$$

where M is the total mass of the system and $f = f(t, x, v)$ is the density of particles.

- S.-Y. Ha, E. Tadmor (2008). S.-Y. Ha, J.-G. Liu (2009).
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Adimensionalize the system

$$\bar{t} = \frac{t}{T}, \quad \bar{x} = \frac{x}{L}, \quad \bar{v} = \frac{v}{V}, \quad \bar{f}(\bar{t},\bar{x},\bar{v}) = \frac{f(t,x,v)}{f_0}, \quad \bar{\psi}(\bar{t},\bar{x}) = \frac{\psi(t,x)}{\psi_0},$$

and link characteristic units to physical parameters as follows

$$\frac{L}{T} = \frac{1}{m}\frac{\tau}{L}\psi_0, \quad M = V^N L^N f_0.$$

From micro to meso: the mean field limit

The mean field limit $N \to \infty$

$$\frac{\partial f}{\partial t}+\frac{VT}{L}v\cdot\nabla_x f-\frac{1}{m}\frac{\psi_0 T}{LV}\nabla\psi\cdot\nabla_v f=\operatorname{div}_v\left(\frac{T}{\tau}vf+\frac{T}{\tau}\frac{\mu}{V^2}\nabla_v f+\frac{TK}{m}((\phi^{\sigma/L}v)*f)f\right).$$

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Define the scaled mean free path, mean thermal velocities, mass, range of interactions and maximum strength of interactions by

$$\alpha:=\frac{\sqrt{\mu}}{L/T},\quad \beta:=\frac{\sqrt{\mu}\tau}{L},\quad \mathcal{V}:=\frac{\sqrt{\mu}}{V},\quad \mathcal{M}:=\frac{m}{M},\quad \delta:=\frac{\sigma}{L},\quad \mathcal{K}:=\tau K.$$

From micro to meso: the mean field limit

The mean field limit $N \to \infty$

$$\frac{\partial f}{\partial t} + \frac{\alpha}{\mathcal{V}} v \cdot \nabla_x f - \frac{V}{\beta} \nabla \psi \cdot \nabla_v f = \frac{\alpha}{\beta} \operatorname{div}_v \left(v f + \mathcal{V}^2 \nabla_v f + \frac{\mathcal{K}}{\mathcal{M}} ((\phi^\delta v) * f) f \right).$$

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Assume the next hyperbolic singular scaling of the dimensionless parameters

$$\alpha = 1,\ \beta = \varepsilon,\ \mathcal{V} = 1,\ \mathcal{M} = 1,\ \delta = \varepsilon,\ \mathcal{K} = \varepsilon^{-2\lambda}.$$

From micro to meso: the mean field limit

The singular hyperbolic scaling

$$\varepsilon\frac{\partial f_\varepsilon}{\partial t}+\varepsilon v\cdot\nabla_x f_\varepsilon-\nabla\psi_\varepsilon\cdot\nabla_v f_\varepsilon=\operatorname{div}_v\left(vf_\varepsilon+\nabla_v f_\varepsilon+((\phi_\varepsilon\, v)*f_\varepsilon)f_\varepsilon\right),$$

where the scaled influence function is $\phi_\varepsilon(r)=\dfrac{1}{(\varepsilon^2+c_{\kappa,\lambda}r^2)^\lambda}$.

D. P., J. Soler (2017).

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D. P., J. Soler (2017).

The limit $\varepsilon\searrow 0$ can be understood as a coupled limit where inertia vanishes and singular interactions appear.

From micro to meso: the mean field limit

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The limit $\varepsilon \searrow 0$ can be understood as a coupled limit where inertia vanishes and singular interactions appear.

Formally, let us assume that f_ε would converge to f in some weak sense. Then,

$$-\nabla\psi\cdot\nabla_v f = \operatorname{div}_v\left(v f + \nabla_v f_+ ((\phi_0\, v) * f) f\right).$$

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It then entails that

$$f(t,x,v)=\frac{\rho(t,x)}{(2\pi)^{N/2}}\exp\left(-\frac{\left|v+\nabla\psi+v(\phi_0*\rho)-\phi_0*(\rho\, u)\right|^2}{2}\right),$$

where (ρ,u) is a solution to the macroscopic singular system (1).

From micro to meso: the mean field limit

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In order to derive the rigorous limit we will assume the following **hypothesis**:

$$\left\{\begin{array}{l} f_\varepsilon^0=f_\varepsilon^0(x,v)\geq 0 \ \text{ and } \ f_\varepsilon^0\in C_c^\infty(\mathbb{R}^N\times\mathbb{R}^N),\\ \|\rho_\varepsilon^0\|_{L^1(\mathbb{R}^N)}\leq M_0 \ \text{ and } \rho_\varepsilon^0\overset{*}{\rightharpoonup}\rho^0 \text{ in } \mathcal{M}(\mathbb{R}^N),\\ \|E_\varepsilon^0\|_{L^1(\mathbb{R}^N)}\leq E_0, \end{array}\right. \tag{H_1}$$

$$\left\{\begin{array}{l} \psi_\varepsilon\in L^2(0,T;W^{1,\infty}(\mathbb{R}^N)) \text{ and } \psi\in L^2(0,T;W^{1,\infty}(\mathbb{R}^N)),\\ \nabla\psi_\varepsilon(t,\cdot)\in C_0(\mathbb{R}^N,\mathbb{R}^N) \text{ and } \nabla\psi(t,\cdot)\in C_0(\mathbb{R}^N,\mathbb{R}^N),\ \text{a.e. } t\in[0,T),\\ \|\nabla\psi_\varepsilon\|_{L^2(0,T;L^\infty(\mathbb{R}^N,\mathbb{R}^N))}\leq F_0 \ \text{ and } \ \nabla\psi_\varepsilon\to\nabla\psi \ \text{ in } L^1(0,T;C_0(\mathbb{R}^N)), \end{array}\right. \tag{H_2}$$

Hydrodynamic limit: hierarchy of velocity moments

The hierarchy of velocity moments

Density: $\rho_\varepsilon(t,x) := \int_{\mathbb{R}^N} f_\varepsilon(t,x,v)\,dv,$

Current: $j_\varepsilon(t,x) := \int_{\mathbb{R}^N} v\, f_\varepsilon(t,x,v)\,dv,$

Stress tensor: $\mathcal{S}_\varepsilon(t,x) := \int_{\mathbb{R}^N} v\otimes v\, f_\varepsilon(t,x,v)\,dv,$

Stress flux tensor: $\mathcal{T}_\varepsilon(t,x) := \int_{\mathbb{R}^N} (v\otimes v)\otimes v f_\varepsilon(t,x,v)\,dv.$

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Velocity field: $u_\varepsilon(t,x) := \dfrac{j_\varepsilon(t,x)}{\rho_\varepsilon(t,x)},$

Energy: $E_\varepsilon(t,x) := \dfrac{1}{2}\mathrm{Tr}(\mathcal{S}_\varepsilon(t,x)) = \dfrac{1}{2}\int_{\mathbb{R}^N} |v|^2 f_\varepsilon(t,x,v)\,dv,$

Energy flux: $Q_\varepsilon(t,x) := \dfrac{1}{2}\mathrm{Tr}(\mathcal{T}_\varepsilon(t,x)) = \dfrac{1}{2}\int_{\mathbb{R}^N} v|v|^2 f_\varepsilon(t,x,v)\,dv$

Hydrodynamic limit: hierarchy of velocity moments

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$$\varepsilon \frac{\partial j_\varepsilon}{\partial t} + \varepsilon \operatorname{div}_x \mathcal{S}_\varepsilon + \rho_\varepsilon \nabla_x \psi_\varepsilon + j_\varepsilon + (\phi_\varepsilon * \rho_\varepsilon)\, j_\varepsilon - (\phi_\varepsilon * j_\varepsilon)\, \rho_\varepsilon = 0.$$

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Formally, if we could pass to the limit in the nonlinear term of the current equation and we had some a priori estimate for ρ_ε, j_ε and $\mathcal{S}_\varepsilon$, then we might show that inertia vanishes when $\varepsilon \to 0$ in the moment equation and we would success in closing a limiting system in terms of ρ and j.

Hydrodynamic limit: weak form of the moment equations

The continuity and momentum equation in weak form read

$$\int_0^T \int_{\mathbb{R}^N} \left(\frac{\partial \varphi}{\partial t} \rho_\varepsilon + \nabla \varphi \cdot j_\varepsilon \right) dx\, dt = -\int_{\mathbb{R}^N} \rho_\varepsilon^0 \varphi(0,\cdot)\, dx,$$

$$\int_0^T \int_{\mathbb{R}^N} \left(-\varepsilon j_\varepsilon \frac{\partial \varphi}{\partial t} - \varepsilon \mathcal{S}_\varepsilon \nabla \varphi + \rho_\varepsilon \nabla \psi_\varepsilon \varphi + j_\varepsilon \varphi \right) dx\, dt$$

$$- \int_0^T \int_{\mathbb{R}^N} \left((\phi_\varepsilon * j_\varepsilon)\rho_\varepsilon - (\phi_\varepsilon * \rho_\varepsilon) j_\varepsilon \right) \varphi\, dx\, dt = \int_{\mathbb{R}^N} j_\varepsilon^0 \varphi(0,\cdot)\, dx,$$

for every test function $\varphi \in C_0^1([0,T) \times \mathbb{R}^N)$.

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for every test function $\varphi\in C_0^1([0,T)\times\mathbb{R}^N)$.

The nonlinear term can be restated as follows

$$\int_0^T\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\underbrace{\frac{1}{2}\phi_\varepsilon(|x-y|)\left(\varphi(t,x)-\varphi(t,y)\right)}_{H_\varphi^{\lambda,\varepsilon}(t,x,y)}\left(\rho_\varepsilon(t,x)j_\varepsilon(t,y)-\rho_\varepsilon(t,y)j_\varepsilon(t,x)\right)dx\,dy\,dt$$

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1. Passing to the limit in the **linear terms**:
   weak-* compactness of ρ_ε and j_ε along with a priori bound of E_ε.
2. Passing to the limit in the **nonlinear term**:
   strong compactness of the kernel $H_\varphi^{\lambda,\varepsilon}$ and weak-* compactness of $\rho_\varepsilon\otimes j_\varepsilon-j_\varepsilon\otimes\rho_\varepsilon$.
3. The value of λ in the **singularity** plays a role.

Hydrodynamic limit: a priori estimates

1. **Estimate for ρ_ε:**

$$\|\rho_\varepsilon\|_{L^\infty(0,T;L^1(\mathbb{R}^N))} \leq M_0.$$

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Taking traces in the equation for $\mathcal{S}_\varepsilon$ we obtain the corresponding one for the energy E_ε:

$$\varepsilon\frac{\partial E_\varepsilon}{\partial t} + \varepsilon\,\mathrm{div}_x\, Q_\varepsilon + j_\varepsilon \cdot \nabla_x \psi_\varepsilon + 2E_\varepsilon + 2(\phi_\varepsilon * \rho_\varepsilon)E_\varepsilon - 2(\phi_\varepsilon * j_\varepsilon)\cdot j_\varepsilon - N\rho_\varepsilon = 0.$$

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Then, integrating with respect to x we obtain the next bound of energy

$$\begin{aligned}&\|E_\varepsilon\|_{L^1(0,T;L^1(\mathbb{R}^N))}\\ &\qquad + \int_0^T \int_{\mathbb{R}^{4N}} \phi_\varepsilon(|x-y|)|v-w|^2 f_\varepsilon(t,x,v) f_\varepsilon(t,y,w)\,dx\,dy\,dv\,dw\,dt\\ &\qquad\qquad \leq \varepsilon E_0 + \frac{1}{2}F_0 + \frac{1}{8}M_0.\end{aligned}$$

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3. **Estimate for j_ε:**

$$\|j_\varepsilon\|_{L^2(0,T;L^1(\mathbb{R}^N))} \leq \left[2M_0\left(\varepsilon E_0 + \frac{1}{2}F_0 + \frac{1}{8}M_0\right)\right]^{1/2}.$$

Hydrodynamic limit: passing to the limit the linear terms

The above estimates for ρ_ε and j_ε yield weak-* limits $\rho \in L^\infty(0,T;\mathcal{M}(\mathbb{R}^N))$ and $j \in L^2(0,T;\mathcal{M}(\mathbb{R}^N))^N$, i.e.,

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This, along with the boundedness in $L^1(0,T;L^1(\mathbb{R}^N))$ of $\mathcal{S}_\varepsilon$ allow passing to the limit in the linear terms:

$$\int_0^T\int_{\mathbb{R}^N}\left(\frac{\partial\varphi}{\partial t}\rho_\varepsilon + \nabla\varphi\cdot j_\varepsilon\right)dx\,dt = -\int_{\mathbb{R}^N}\rho_\varepsilon^0\varphi(0,\cdot)\,dx,$$

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1. **Strong convergence of the kernel**:
   It is clear that

$$|\phi_\varepsilon(r)-\phi_0(r)| \leq C_\lambda \frac{\varepsilon^{1-2\lambda}}{r} \overset{\lambda\in(0,\frac{1}{2})}{\Longrightarrow} \lim_{\varepsilon\searrow 0} \|H_\varphi^{\lambda,\varepsilon}-H_\varphi^{\lambda,0}\|_{C([0,T],C_0(\mathbb{R}^N))} = 0.$$

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   $$\|\rho_\varepsilon\|_{C^{0,\frac{1}{2}}([0,T],W^{-1,1}(\mathbb{R}^N))} \leq T^{1/2}\|\rho_\varepsilon\|_{L^\infty(0,T;L^1(\mathbb{R}^N))} + \|j_\varepsilon\|_{L^2(0,T;L^1(\mathbb{R}^N))}.$$

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   By Ascoli-Arzelà theorem in weak-* form one can show that

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$$\Longrightarrow \rho_\varepsilon \otimes j_\varepsilon - j_\varepsilon \otimes \rho_\varepsilon \overset{*}{\rightharpoonup} \rho\otimes j - j\otimes\rho \ \text{ in } \ L^2(0,T;\mathcal{M}(\mathbb{R}^{2N})).$$

Hydrodynamic limit: the critical exponent $\lambda = \frac{1}{2}$

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   Indeed, $H_\varphi^{\frac{1}{2},0}(t,x,y)$ may exhibit jump discontinuities along the diagonal points $x = y$, but it is bounded and continuous outside such set and falls-off at infinity.

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3. **Solution: absence of concentration for all** $\lambda \in (0, N/2)$.
   Set $\Omega_R := \{(x,y) \in \mathbb{R}^{2N} \, : \, |x-y| < R\}$ and notice that

$$\|\rho_\varepsilon \otimes j_\varepsilon - j_\varepsilon \otimes \rho_\varepsilon\|_{L^2(0,T;\mathcal{M}(\Omega_R))} \leq (\varepsilon^2 + R^2)^{\lambda/2} M_0$$
$$\times \left(\int_0^T \int_{\mathbb{R}^{4N}} \phi_\varepsilon(|x-y|)|v-w|^2 f_\varepsilon(t,x,v) f_\varepsilon(t,y,w) \, dt \, dx \, dy \, dv \, dw \right)^{1/2}.$$

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Thus, there is non-concentration in the following sense

$$\liminf_{\varepsilon, R \to 0} \|\rho_\varepsilon \otimes j_\varepsilon - j_\varepsilon \otimes \rho_\varepsilon\|_{L^2(0,T;\mathcal{M}(\Omega_R))} = 0.$$

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How does the non-concentration helps ?

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$$+ \int_0^T \int_{\mathbb{R}^{2N}} H_\varphi^{\frac{1}{2},0}[(\rho_\varepsilon \otimes j_\varepsilon - j_\varepsilon \otimes \rho_\varepsilon) - (\rho \otimes j - j \otimes \rho)].$$

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1. The non-concentration result allows passing to the limit in the pink term even though $H_\varphi^{\frac{1}{2},0}$ may be discontinuous along the diagonal points.

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Hydrodynamic limit: the critical exponent $\lambda = \frac{1}{2}$

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F. Poupaud, *Diagonal defect measures, adhesion dynamics and Euler equation*, Methods Appl. Anal. **9** (2002) 533–562.

Hydrodynamic limit

Theorem

Under the hypothesis (H_1) *and* (H_2) *and for any* $\lambda \in (0, 1/2]$ *the macroscopic quantities* ρ_ε *and* j_ε *satisfy*

$$\rho_\varepsilon \to \rho, \quad \text{in } C([0,T], \mathcal{M}(\mathbb{R}^N) - \text{weak*}),$$
$$j_\varepsilon \overset{*}{\rightharpoonup} j, \quad \text{in } L^2(0,T; \mathcal{M}(\mathbb{R}^N))^N.$$

where (ρ, j) *is a local-in-time weak measure-valued solution to the Cauchy problem associated with the following Euler-type system*

$$\begin{cases} \partial_t \rho + \operatorname{div}\, j = 0, & x \in \mathbb{R}^N,\ t \in [0,T), \\ \rho\, \nabla\psi + j = (\phi_0 * j)\rho - (\phi_0 * \rho) j, & x \in \mathbb{R}^N,\ t \in [0,T) \\ \rho(0, \cdot) = \rho^0, & x \in \mathbb{R}^N. \end{cases}$$

Singular macro system: well-posedness in Sobolev spaces

In terms of ρ and u, the singular macroscopic system can be restated as

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, & x \in \mathbb{R}^N,\ t \geq 0, \\ \rho(0,x) = \rho^0(x), & x \in \mathbb{R}^N, \\ u = \phi_0 * (\rho u) - (\phi_0 * \rho) u - \nabla \psi, & x \in \mathbb{R}^N,\ t \geq 0. \end{cases}$$

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Then, we will study each problem separately and will try to find a solution via a **fixed point argument** when ρ and u are taken in spaces with **high enough Sobolev regularity**.

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Indeed, the next estimate holds

$$\|D[u]\|_{L^\infty(0,T;L^p(\mathbb{R}^N))} \leq \exp\left(\frac{1}{p'}\|u\|_{L^1(0,T;W^{1,\infty}(\mathbb{R}^N,\mathbb{R}^N))}\right)\|\rho^0\|_{L^p(\mathbb{R}^N)}.$$

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$$\mathcal{C}[\rho, u] := \phi_0 * (\rho u) - (\phi_0 * \rho)u = -\frac{1}{c_\lambda^\lambda}[u, I_{N-2\lambda}]\rho.$$

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S. Chanillo, *A note on commutator*, Indiana Univ. Math. J. **31** (1982) 7–16.

D. Cruz–Uribe, A. Fiorenza, *Endpoint estimates and weighted norm inequalities for commutator of fractional integrals*, Pub. Mat. **47** (2003), 103–131.

Singular macro system: estimates of commutators

Based on **Hardy–Littlewood–Sovolev theorem** we obtain the next results:

Theorem ($W^{1,\infty}$ regularity)

Consider $1 \leq p_1 < p_2 \leq \infty$ *with* $\frac{1}{p_2} < 1 - \frac{2\lambda}{N} < \frac{1}{p_1}$. *Then,*

$$\|([u,I_{N-2\lambda}]\rho)\|_{L^1(0,T;W^{1,\infty})} \leq C\|u\|_{L^1(0,T;W^{1,\infty})}\|\rho\|_{L^\infty(0,T;L^{p_1})}^{(1/p_2'-\frac{2\lambda}{N})/(1/p_1-1/p_2)}\|\rho\|_{L^\infty(0,T;L^{p_2})}^{(\frac{2\lambda}{N}-1/p_1')/(1/p_1-1/p_2)}.$$

Theorem (L^s regularity)

Consider $1 \leq p_1 < p_2 \leq \infty$ *and* $1 \leq s < \infty$ *with* $\frac{1}{p_2} < 1 - \frac{2\lambda}{N} < \frac{1}{p_1}$ *and* $s > \frac{N}{2\lambda}$. *Then,*

$$\|([u, I_{N-2\lambda}]\rho)\|_{L^1(0,T;L^s)} \leq C\|u\|_{L^1(0,T;L^s)}\|\rho\|_{L^\infty(0,T;L^{p_1})}^{(1/p_2'-\frac{2\lambda}{N})/(1/p_1-1/p_2)}\|\rho\|_{L^\infty(0,T;L^{p_2})}^{(\frac{2\lambda}{N}-1/p_1')/(1/p_1-1/p_2)}.$$

Singular macro system: the fixed-point argument

1. **Restating the macroscopic system**:

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$$\Phi[u]=u.$$

3. **Definition of Banach spaces**:

$$W^{k,p,q}(\mathbb{R}^N):=(W^{k-1,p}\cap W^{k-1,q}\cap W^{k,\infty})(\mathbb{R}^N).$$

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$$W^{k,p,q}(\mathbb{R}^N):=(W^{k-1,p}\cap W^{k-1,q}\cap W^{k,\infty})(\mathbb{R}^N).$$
4. **High regularity estimates of the operators**: Consider $\lambda\in(0,N/2)$ and $1\leq p_1<p_2\leq\infty$ such that $\frac{1}{p_2}<1-\frac{2\lambda}{N}<\frac{1}{p_1}$. Set any $k\in\mathbb{N}$ so that $k>\max\{\frac{k}{2N}-1,\frac{N}{2\lambda p_1}\}$. Then,
$$\|\mathcal{D}[u]\|_{L^\infty(0,T;L^{p_i})}\leq\exp\left(\frac{1}{p_i'}\|u\|_{L^1(0,T;W^{1,\infty})}\right)\|\rho^0\|_{L^{p_i}}$$
$$\|\mathcal{C}[\rho,u]\|_{L^1(0,T;W^{1,kp_1,kp_2})}\leq C\|u\|_{L^1(0,T;W^{1,kp_1,kp_2})}$$
$$\times\|\rho\|_{L^\infty(0,T;L^{p_1})}^{(1/p_2'-\frac{2\lambda}{N})/(1/p_1-1/p_2)}\|\rho\|_{L^\infty(0,T;L^{p_2})}^{(\frac{2\lambda}{N}-1/p_1')/(1/p_1-1/p_2)}.$$

5. **Domain and codomain of** Φ: Consider $\lambda \in (0, N/2)$ and $1 \leq p_1 < p_2 \leq \infty$ such that $\frac{1}{p_2} < 1 - \frac{2\lambda}{N} < \frac{1}{p_1}$. Set any $k \in \mathbb{N}$ so that $k > \max\{\frac{k}{2N} - 1, \frac{N}{2\lambda p_1}\}$. Then,

$$\Phi : L^1(0,T; W^{1,kp_1,kp_2}) \longrightarrow L^1(0,T; W^{1,kp_1,kp_2}),$$

and

$$\begin{aligned} &\|\Phi[u]\|_{L^1(0,T;W^{1,kp_1,kp_2})} \\ &\quad \leq \|\rho^0\|_{L^{p_1}}^{(1/p_2' - \frac{2\lambda}{N})/(1/p_1 - 1/p_2)} \|\rho^0\|_{L^{p_2}}^{(\frac{2\lambda}{N} - 1/p_1')/(1/p_1 - 1/p_2)} \kappa_1(\|u\|_{L^1(0,T;W^{1,kp_1,kp_2})}) \\ &\qquad + \|\nabla\psi\|_{L^1(0,T;W^{1,kp_1,kp_2})}, \end{aligned}$$

$$\begin{aligned} \|\Phi[u_1] - \Phi[u_2]\|_{L^1(0,T;W^{1,kp_1,kp_2})} &\leq \|u_1 - u_2\|_{L^1(0,T;W^{1,kp_1,kp_2})} \\ &\times \|\rho^0\|_{W^{k,p_1,p_2}} \kappa_2\left(\|u_1\|_{L^1(0,T;W^{1,kp_1,kp_2})}, \|u_2\|_{L^1(0,T;W^{1,kp_1,kp_2})}\right). \end{aligned}$$

Singular macro system: the fixed point argument

6. For any $R > \|\nabla\psi\|_{L^1(0,T;W^{1,kp_1,kp_2})}$, set

$$\mathcal{B}_R := \{u \in W^{1,kp_1,kp_2}(\mathbb{R}^N, \mathbb{R}^N) : \|u\|_{W^{1,kp_1,kp_2}} \leq R\}$$

and assume that $\rho^0 \in W^{k,p_1,p_2}(\mathbb{R}^N)$ is small enough.

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$\rightsquigarrow$ **Banach' contraction principle**.

Singular macro system: well-posedness in Sobolev spaces

Theorem

Let λ be any exponent in $(0, N/2)$ and $1 \leq p_1 < p_2 \leq \infty$ such that

$$\frac{1}{p_2} < 1 - \frac{2\lambda}{N} < \frac{1}{p_1}.$$

Consider any (large enough) positive integer k so that

$$k > \max\left\{\frac{N}{2\lambda} - 1, \frac{N}{2\lambda p_1}\right\}.$$

Then, for any positive radius R there exists some positive constant δ_R depending on R such that the macroscopic system admits one, and only one solution

$$u \in L^1(0,T; W^{1,kp_1,kp_2}(\mathbb{R}^N, \mathbb{R}^N)),\ \rho \in L^\infty(0,T; L^{p_1}(\mathbb{R}^N) \cap L^{p_2}(\mathbb{R}^N))$$

with $\|u\|_{L^1(0,T;W^{1,kp_1,kp_2}(\mathbb{R}^N,\mathbb{R}^N))} \leq R$, as long as the external force $F = -\nabla\psi$ and the initial datum ρ^0 are taken so that they fulfil the next conditions

$$\|\nabla\psi\|_{L^1(0,T;W^{1,kp_1,kp_2}(\mathbb{R}^N,\mathbb{R}^N))} < R \quad \textit{and} \quad \|\rho^0\|_{W^{k,p_1,p_2}(\mathbb{R}^N)} < \delta_R.$$

Singular macro system: applications

(Soaring)

Figure: Flock of vultures soaring in a thermal. (**Click here** *to Youtube*)

Conclusions

1. From microscopic description to mesoscopic.
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3. First order hydrodynamic and singular limit for $0 < \lambda \leq \frac{1}{2}$.
4. Regularity estimates of the commutator of mildly singular integrals.
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D. Poyato, J. Soler, *Euler-type equations and commutators in singular and hyperbolic limits of kinetic Cucker-Smale models*, Math. Mod. Meth. Appl. Sci, **27**(6) (2017), 1089-1152.

THANK YOU!!