Majority Vote Processes on Trees

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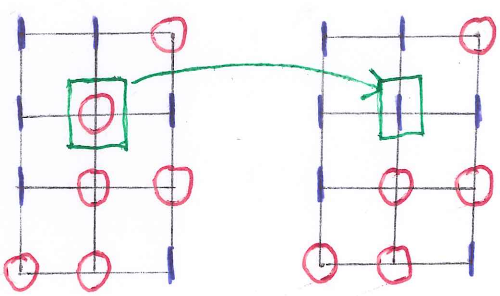
Majority Vote Process (MVP) - one of the classical interacting particle systems

others include voter model, contact process, exclusion process

The MVP is the continuous time Markov process such that: $d \geq 1$ $\varepsilon \in [0,1]$

- On $\mathbb{Z}^d$, each site has "opinion" either 0 or 1.
- Independently, at rate $1-\varepsilon$ at each site, sites align w/ the majority opinion of their immediate neighbors. ← MV point
- Independently, at rate ε at each site, the opinion "flips". ← noise point

Are primarily interested in case where ε is small, but positive.



How many equilibria does MVP have?

- Is open except in $d=1$.

Only 1 equilibrium (Gray (1992))

Compare with:

VM when $\varepsilon=0$

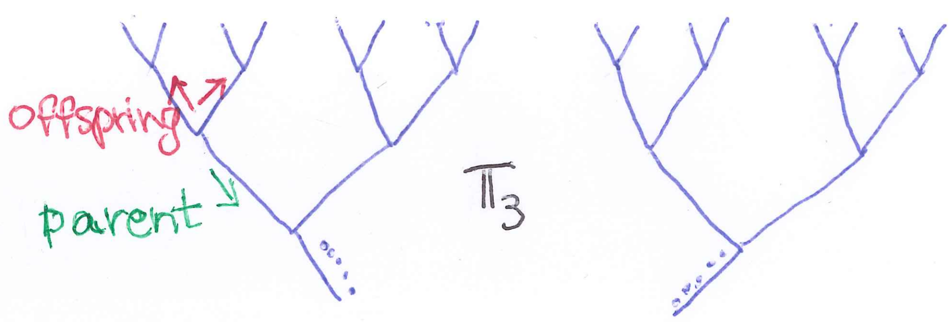
Voter Model with Noise (VMN) –

- Same as MVP, except sites randomly choose one of neighbors with which to align.

Not difficult to check: uses duality

- VMN converges exp'l quickly to its unique equilibrium.

Here, will investigate behavior of MVP on d-regular trees $\mathbb{T}_d$.



Results will discuss: (B.-Gray (2018))

<u>Theorem 1.</u> For $d \geq 5$ and $\varepsilon > 0$ small, there exist uncountably many mutually singular equilibria. will spend more time on

less time on

<u>Theorem 2.</u> For any initial state close enough to such an equilibrium, process converges exp'l quickly to this equilibrium.

Comments:

- Method does not depend on precise model. Includes, for example, stochastic Ising model on $\mathbb{T}_d$.

- Argument simplifies somewhat for oriented MVP, where above results are true for $d \geq 4$.
  parents depend only on offspring

- Gandolfo, Ruiz, Shlosman (2012) constructed uncountably many pure Gibbs states for the Ising model on trees.

Ideas behind proof of Theorem 1

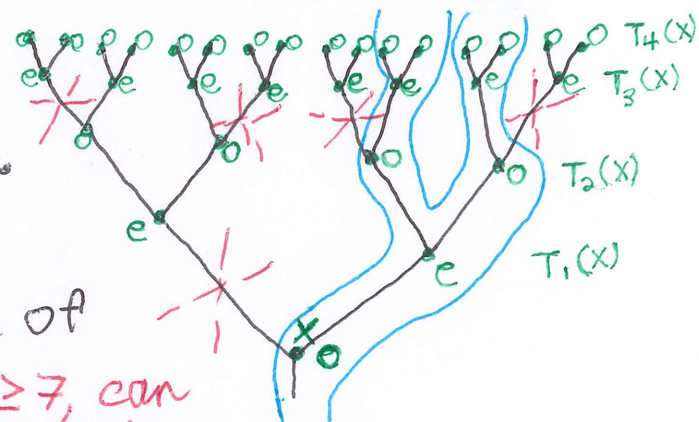
related ideas for Theorem 2, but more involved

Notation

$(\xi_t^{\mathcal{J}})_{t\geq 0}$ ← initial state

$(\xi_t^{\mathcal{J}})_{t\geq 0}$ is MVP w/ $\xi_t^{\mathcal{J}}(x) = 0$ or 1 for $x \in \mathbb{T}_d$

Will consider special $\mathcal{J}$ defined using a <u>tiling</u>:



- Partition $\mathbb{T}_d$ into even/odd sites.
- From each odd site, delete an edge to one of offspring. ← For $d \geq 7$, can instead delete an edge from all sites.
- Each connected subset is a <u>tile</u>; corresponding collection is a <u>tiling</u>. (denote by T)

  $T_n(x)$ = set of n^{th} generation offspring of x in T
- $\mathcal{J}$ is <u>compatible</u> (wrt a given tiling) if $x, y \in T$ implies $\mathcal{J}(x) = \mathcal{J}(y)$.

Note: For a given tiling,

(1) there are uncountably many compatible $\mathcal{J}$.

Reasoning for Theorem 1

For $d \geq 5$ and small noise ε, we will show that, for each tile T,

assumed compatible wrt tiling

$$(2) \quad \overline{\lim_{n\to\infty}} \frac{1}{\# T_n(x)} \sum_{y \in T_n(x)} |\xi_t^{\mathcal{J}}(y) - \mathcal{J}(y)| \leq 10\varepsilon \text{ a.s.}$$

for each $x \in T$ and t.

says ξ_t is close to $\mathcal{J}$ for all t.

Assume (2)

Let

$\mu_t^{\mathcal{J}}$ = Cesaro avg. over $[0,t]$ of measures of $\xi_t^{\mathcal{J}}$.

One has

$$\mu_{t_i}^{\mathcal{J}} \xrightarrow{i\to\infty} \mu^{\mathcal{J}} \text{ on some subsequence } t_i \to \infty,$$

where $\mu^{\mathcal{J}}$ satisfies analog of (2).

Consequently, for $\mathcal{J} \neq \mathcal{J}'$ (w/ both compatible wrt tiling),

(3) $\quad \mu^{\mathcal{J}}$ and $\mu^{\mathcal{J}'}$ are mutually singular.

Theorem 1 follows from (1) and (3).

Demonstrate (2)

To demonstrate (2), construct a process δ_t on $\{0,1\}^{\mathbb{T}_d}$ s.t.

(4) $|\xi_t^{\mathcal{J}}(y) - \mathcal{J}(y)| \leq \delta_t(y)$ for all y

with

(2') $\overline{\lim_{n\to\infty}} \frac{1}{\# T_n(x)} \sum_{y \in T_n(x)} \delta_t(y) \leq 10\varepsilon.$

Construction of δ_t:

only points where change of state occurs

- $\delta_0 \equiv 0.$
- $\delta_t(x) = 1$ at noise points (x,t).
- δ_t behaves like $\xi_t^{\mathcal{J}}$ at MV points (x,t), except that neighborhood is now $T_1(x)$, and, for $\delta_t(x) = 1$ to occur, 1 (2) fewer 1's required if x is even (odd).

These properties imply (4).

They also imply siblings evolve independently of one another. ← same parent

To demonstrate (2′), set

$$p_t = P(\delta_t(x)=1) \quad \text{for } x \in \mathbb{T}_d .$$

p_t depends on whether x is even or odd.
To simplify computations, instead delete an edge from <u>all</u> sites (not just odd).
This suffices for $d \ge 7$.

It follows that, for $d \ge 7$,

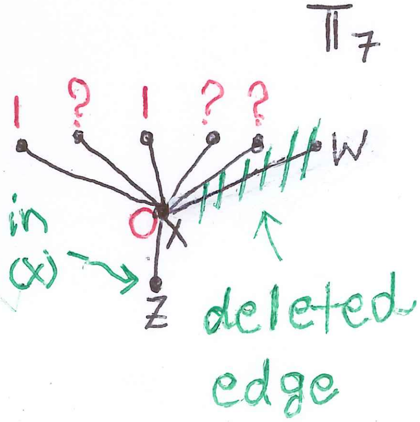
$$p_t' \le \underbrace{\varepsilon(1-p_t)}_{\text{due to noise}} + C_d\, p_t^2$$

$$\qquad - p_t + C_d\, p_t^2$$

due to rate back to 0

since siblings evolve independently; correct power improves as d increases

even if $\delta_t(z)=\delta_t(w)=1$, still require ≥ 2 sites $y \in T_1(x)$ w/ $\delta_t(y)=1$



Consequently,

(5) $p_t' \le \varepsilon - p_t(1-2C_d\, p_t)$.

Since $p_0 = 0$, for small ε,

(6) $p_t \le 2\varepsilon$ for all t.

Since all sites in $T_n(x)$ evolve independently, (2') follows from (6) and SLLN.

Basics behind proof of Theorem 2

Proof of Theorem 2 is more complicated than that of Theorem 1.

Some ideas:

The limit
$$\xi^{\mathcal{J}}_{\mathcal{S}_t} \xrightarrow{d} \xi^{\mathcal{J}}_{\mathcal{S}_\infty} \quad \text{as } t \to \infty$$
is equivalent to
$$(7) \qquad \xi^{\mathcal{J},-s}_{\mathcal{S}_0} \xrightarrow{d} \xi^{\mathcal{J}}_{\mathcal{S}_\infty} \quad \text{as } s \to \infty.$$

process starts at time $-s$ instead of 0

will show a.s. convergence in this setting

Employ δ_t^{-s}. ← analog of δ_t, but starting at time $-s$

Since δ_t^{-s} is increasing in s,

$\lim_{s\to\infty} \delta_t^{-s}$ exists. ← want to also show for $\xi^{-s}_{\mathcal{S}_0}$

Trace paths of possible influence backwards in time. ↖ refer to as search histories

Is complicated. ⇐ no duality

But can control these using δ_t^{-s}.

Corresponding to these paths are supercritical branching processes, at least one of which must eventually die out, but not die out quickly if $\xi_0^{\zeta,-s_1}(x) \neq \xi_0^{\zeta,-s_2}(x)$ for large $s_1 \le s_2$.

unlikely event

deaths are associated w/ noise pts; births w/ MV pts