Localization-delocalization transitions in random matrix models: a SPDE approach

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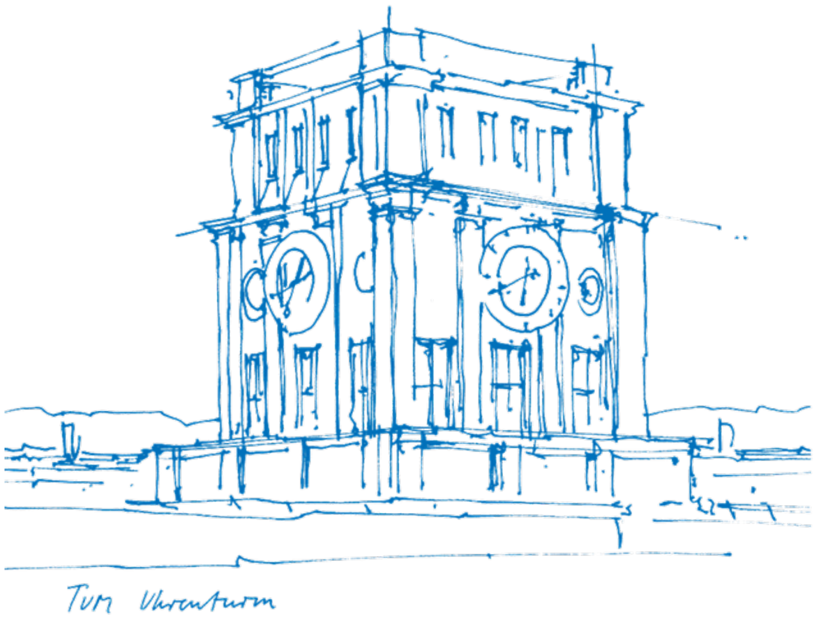


The localization-delocalization challenge

Understand the various facets of metal-insulator transitions in disordered systems, *e.g.* in the random matrix context:

Power-Law Random Band Matrices (PRBM) Mirlin/Fyodorov/Dittes/Quezada/Seligman '96

Real, symmetric $N \times N$ matrix (H_{xy}) with Gaussian rv's as entries

$$\mathbb{E}\left[H_{xy}\right] = 0$$
$$\mathbb{E}\left[H_{xy}^2\right] \sim \begin{cases} 1 & \text{if } d(x,y) \leq 1 \\ d(x,y)^{-2\alpha} & \text{else} \end{cases}$$

Simple limiting cases:

- **Random diagonal ensemble** $V = \mathrm{diag}(V_x)$ with iid rv's (V_x)
- **Gaussian orthogonal ensemble (GOE)** Real, symmetric $N \times N$ matrix $(\Phi_{x,y})$
  - $\Phi_{x,y}$ are centered Gaussian rv's with $\mathbb{E}[\Phi_{x,y}^2] = \dfrac{1+\delta_{x,y}}{N}$
  - $\Phi_{x,y}$ are independent for $x \leq y$.

Wigner, Dyson, Metha $\geq'$ 50

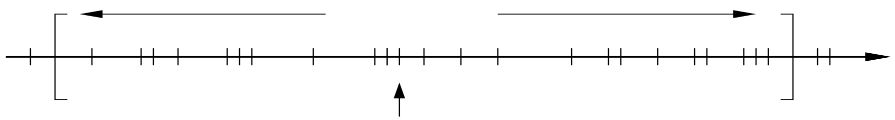
The localization-delocalization challenge

Predicted features – as a function of parameters in the model (α) and/or energy:

- **Eigenvectors** ψ undergo **localization-delocalization transition**

Inverse participation ratios for ℓ^2-normalized function: $\|\psi\|_\infty = \begin{cases} \mathcal{O}(1) & \text{localization} \\ \mathcal{O}(N^{-1/2}) & \text{delocalization} \end{cases}$

- **Eigenvalue statistics** changes from Poisson to Random Matrix (GOE)



Rescaled random process of eigenvalues close to some energy E:

$$\sum_{\lambda\in\sigma(H)} \delta_{N(\lambda-E)} \xrightarrow[N\to\infty]{} \begin{cases} \text{Poisson process} & \text{localization} \\ \text{GOE process} & \text{delocalization} \end{cases}$$

Rosenzweig-Porter Ensemble

$$H(t) = V + \sqrt{t}\,\Phi \qquad t \geq 0$$

with $N \times N$ GOE matrix Φ and initial matrix V (wlog $V = \mathrm{diag}(V_x)$)

Possible assumptions on random initial conditions:

A1 $V = \mathrm{diag}(\omega_x)$ with iid (ω_x) with density $\varrho \in L^\infty$, *or*

A2 $V = A + \mathrm{diag}(\omega_x)$ with $A = A^*$ and (ω_x) as in A1.

Rosenzweig-Porter Ensemble

$$H(t) = V + \sqrt{t}\,\Phi \qquad t \geq 0$$

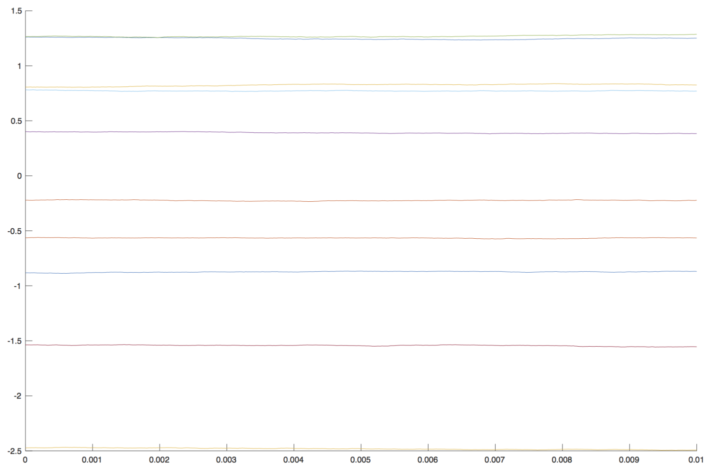
with $N \times N$ GOE matrix Φ and initial matrix V (wlog $V = \mathrm{diag}(V_x)$)

- Eigenvalues undergo **Dyson Brownian Motion (DBM)**

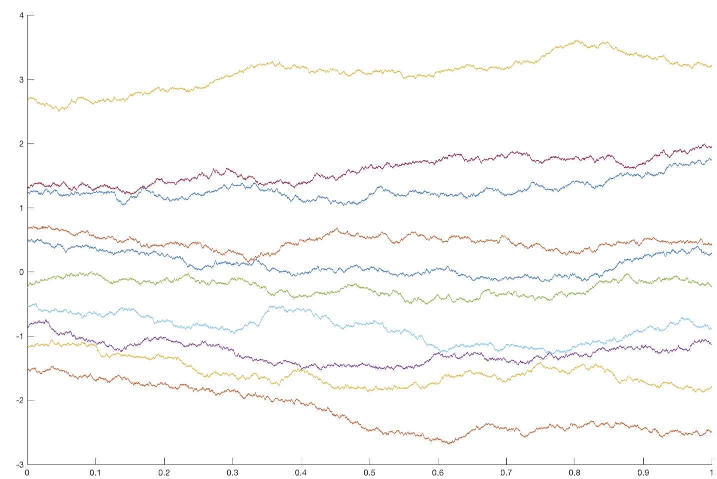
$$\sqrt{t}\,\Phi_{x,y} \equiv \sqrt{\frac{1+\delta_{xy}}{N}}\,B_{xy}(t)$$

$$d\lambda_j(t) = \sqrt{\frac{2}{N}}dB_j(t) + \frac{1}{N}\sum_{i \neq j} \frac{dt}{\lambda_j(t) - \lambda_i(t)}$$

Dyson '62



10 Trajectories until $t = N^{-1}$ of 100-particle DBM with independent initial conditions



Trajectories until $t = 1$ of 10-particle DBM with independent initial conditions

Rosenzweig-Porter Ensemble

$$H(t) = V + \sqrt{t}\,\Phi \qquad t \geq 0$$

with $N \times N$ GOE matrix Φ and initial matrix V (wlog $V = \mathrm{diag}(V_x)$)

Time scales		Results
$t \ll N^{-1}$	pertubative regime	Soosten/W. '17
$N^{-1} \ll t$	local equilibration regime	Erdős, Yau, ..., Landon , Sosoe $\geq$ '12
$1 \ll t$	global equilibration regime delocalisation of eigenvectors	Erdős, Schlein, Yau, ..., Bourgade ... Lee, Schnelli , ... $\geq$ '09
$N^{-1} \ll t \ll 1$	intermediate regime	Soosten/W. '17, Begnini '17

Further reading: *Dynamical approach to random matrix theory* by Erdős and Yau.

Physics papers: Kravtsov/Khaymovich/Cuevas/Amini '15, Facoetti/Vivo/Biroli '16, Bogomolny/Sieber '18

Spectral information

Green function: $G(x,z) = \langle \delta_x, (H-z)^{-1}\delta_x \rangle \qquad x \in \{1,\dots,N\},\ z \in \mathbb{C}^+.$

- Relation to normalized eigenfunctions ψ_j corresponding to ev λ_j:

$$G(x,z) = \sum_j \frac{|\psi_{\lambda_j}(x)|^2}{\lambda_j - z}$$

- Upper bound on normalized eigenfunction with eigenvalue in I:

$$|\psi_{\lambda_j}(x)|^2 \leq \sup_{E\in I} \sum_k \frac{\eta^2}{(\lambda_k - E)^2 + \eta^2} |\psi_{\lambda_k}(x)|^2 = \eta \sup_{E \in I} \operatorname{Im} G(x, E + i\eta)$$

  for any x and $\eta > 0$.

Stieltjes trafo of empirical eigenvalue measure: $S(z) = \frac{1}{N} \operatorname{Tr} (H-z)^{-1}$

- Local **density of states measure** μ: $\int (x-z)^{-1} \mu(dx) = \mathbb{E}\left[S(z)\right]$
- Rescaled eigenvalue processat E is captured by $S(E + z/N) = \sum_j \frac{1}{N(\lambda_j - E) - z}$.

Flow of $S_t(z)$ under DBM

Itô's lemma yields: **Viscous complex Burger's equation**

$$dS_t(z) = \left[S_t(z)\partial_z S_t(z) + \frac{1}{2N}\partial_z^2 S_t(z) \right] dt + dM_t(z)$$

with an (explicit) martingal term $dM_t(z)$.

Lemma (Soosten/W. '17)

Assuming A2 for all $t \leq N^{-(1+\varepsilon)}$ with $\varepsilon > 0$ and $z \in \mathbb{C}^+$ *('Perturbative regime')*

$$\mathbb{E}\left| S_t(z) - S_0(z) \right| \leq CN^{-\varepsilon/2} \left(1 + \frac{1}{N \operatorname{Im} z} + \frac{1}{(N \operatorname{Im} z)^3} \right)$$

Poof idea: Use regularizing effect of the random potential on the drift, diffusion and martingale term through *Wegner & Minami-type estimates*.

Thus: **Rescaled eigenvalue process** remains **Poisson** of one started with Poisson as in A1.

Flow of $S_t(z)$ under DBM

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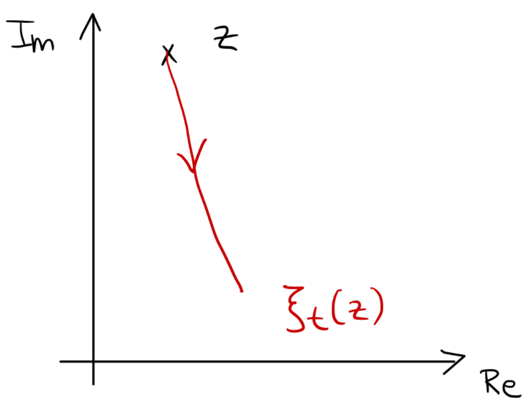
Inviscous case: $\partial_t S_t(z) = S_t(z)\partial_z S_t(z)$



Method of characteristics: Pastur flow

$$\dot{\xi}_t = -S_t(\xi_t)\,, \ \xi_0 = z \in \mathbb{C}^+$$

$$\frac{d}{dt} S_t(\xi_t(z)) = 0 \qquad \text{i.e.} \quad \xi_t(z) = z - tS_0(z)$$

Example $V = 0$: $S_0(z) = -z^{-1}$ and hence $t\, S_t(w)^2 + w\, S_t(w) + 1 = 0$, i.e. semicircular law.

Local law: $S_t(z)$ down to scale $\operatorname{Im} z \gg N^{-1}$

$$dS_t(z) = \left[S_t(z)\partial_z S_t(z) + \frac{1}{2N}\partial_z^2 S_t(z) \right] dt + dM_t(z)$$

For any $z \in \mathbb{C}$ with $\operatorname{Im} z \geq \eta > 0$, let $\xi_t(z)$ be the **random characteristics** given by

$$\dot{\xi}_s = -S_s(\xi_s)\,, \;\; \xi_0 = z \in \mathbb{C}^+$$

stopped at $\operatorname{Im} \xi_t(z) = \eta/2$.

Theorem (Soosten/W. '17)

For any $z \in \mathbb{C}$ *with* $\operatorname{Im} z \geq \eta > 0$*:*

$$\mathbb{P}\left(\sup_{\substack{|z| \leq \eta^{-1} \\ \operatorname{Im} z \geq \eta}} \sup_{s \in [0,t]} |S_s(\xi_s(z)) - S_0(z)| \geq \frac{6}{\sqrt{N\eta}} \right) \leq \frac{4N\eta}{\eta^{10}} \, e^{-N\eta/2} \,.$$

Key idea: Integration trick & large deviation estimate for BM. *(Blackboard)*

Local law: $S_t(z)$ down to scale $\operatorname{Im} z \gg N^{-1}$

Theorem (Soosten/W. '17)

For any $z \in \mathbb{C}$ *with* $\operatorname{Im} z \geq \eta > 0$*:*

$$\mathbb{P}\left(\sup_{\substack{|z| \leq \eta^{-1} \\ \operatorname{Im} z \geq \eta}} \sup_{s \in [0,t]} |S_s(\xi_s(z)) - S_0(z)| \geq \frac{6}{\sqrt{N\eta}} \right) \leq \frac{4N\eta}{\eta^{10}} e^{-N\eta/2} .$$

Remaining task: Control & invert the characteristics $\xi_s(z)$. This is done by **assuming regularity of** S_0 in some window $W \subset \mathbb{R}$, i.e. for some $K_u, K_l \in (0, \infty)$

$$K_l \leq \operatorname{Im} S_0(z) \leq |S_0(z)| \leq K_u$$

uniformly for $z \in W + [-K_u t, K_u t] + i\,[K_l t, 2]$.

Assumption is satisfied in case:

- if $t \gg N^{-1}$ under assumption A1 with $0 < \inf_{v \in W} \varrho(v)$ — Cramer's concentration bound
- if $t \gg N^{-1/2}$ under assumption A2 with $0 < \inf_{z \in W_t} \mathbb{E}\left[\operatorname{Im} S_0(z)\right]$ — *(next slide)*

Excursion: Local laws for Schrödinger matrices

Consider $N \times N$ matrix of the form

$$V = A + \operatorname{diag}(\omega_1, \ldots, \omega_N)$$

with any $A = A^*$ non-random and iid (ω_x) with density $\varrho \in L^\infty$.

Lemma (Soosten/W. '18)

Let $z = E + i\eta$. Then for any $\mu > 0$

$$\mathbb{P}\left(|S_0(z) - \mathbb{E}[S_0(z)]| > \mu\right) \leq C \exp\left(-c\mu^2 N\eta^2\right)$$

and

$$\mathbb{P}\left(\operatorname{Im} S_0(z) > e\pi \|\rho\|_\infty + \mu\right) \leq \exp\left(-\mu N \eta\right).$$

Proof idea: McDiarmid concentration inequality + rank-one perturbation theory & spectral averaging.

(Blackboard)

Delocalization of eigenvectors

Green function satisfies an SDE

Advection equation

$$dG_t(x,z) = \left(S_t(z)\partial_z G_t(x,z) + \frac{1}{2N}\partial_z^2 G_t(x,z) \right) dt + dM_t(x,z)$$

with an explicit martingal $dM_t(x,z)$.

The method of characteristics suggests:

$$G_t(x,\xi_t(z)) \approx G_0(x,z) = (V_x - z)^{-1}$$

and hence:

$$\eta \operatorname{Im} G_t(x, E+i\eta) \approx \frac{\eta \ \operatorname{Im} \xi_t^{-1}(w)}{(V_x - \operatorname{Re} \xi_t^{-1}(E+i\eta))^2 + (\operatorname{Im} \xi_t^{-1}(E+i\eta))^2}$$

$$\approx \begin{cases} C\,\dfrac{\eta}{t} & \text{if } |V_x - E| \le Ct \\ 0 & \text{else.} \end{cases}$$

Intermediate regime

Theorem (Soosten/W. '17)

Let $t = N^{-1+\delta}$ *with* $\delta \in (0,1)$ *and set* $\kappa > \delta > \theta$*,*

$$X_\lambda = \left\{ x \in \{1, ..., N\} : |\lambda - V_x| > N^{-1+\kappa} \right\},$$

and $W \Subset \operatorname{supp} \varrho$*. Then under assumption A1 there exists* $\gamma > 0$ *such that for any* $p > 0$ *and all sufficiently large N the* ℓ^2*-normalized eigenvectors in* W *carry only negligible mass inside* X_λ*:*

$$\mathbb{P}\left(\sup_{\lambda \in \sigma(H_T) \cap W} \sum_{x \in X_\lambda} |\psi_\lambda(x)|^2 > N^{-\gamma} \right) \leq N^{-p}$$

and are maximally extended outside X_λ*:*

$$\mathbb{P}\left(\sup_{\lambda \in \sigma(H_T) \cap W} \|\psi_\lambda\|_\infty > N^{-\theta/2} \right) \leq N^{-p}.$$

Extension to deformed Wigner matrices: Begnini '17

Outlook: Ultrametric ensemble

$N \times N$ random matrices $(N = 2^n)$

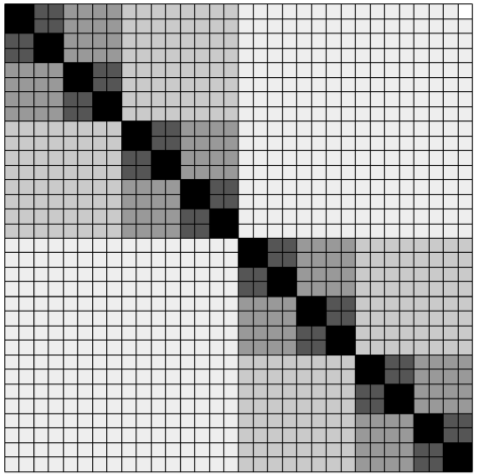
$$H_n = \frac{1}{Z_{n,c}} \sum_{r=0}^{n} 2^{-\frac{(1+c)}{2} r} \sum_{B \in \mathcal{P}_r} \Phi_B$$

with $c \in \mathbb{R}$ and (Φ_B) independent **GOE matrices.**

- Normalization $Z_{n,c}$ is chosen s.t. the variance matrix is doubly stochastic, *i.e.*

$$\sum_y \mathbb{E}\left[|\langle \delta_y, H_n \delta_x \rangle|^2 \right] = 1 \, .$$

- Hierarchical analogue of PRBM $\alpha = 1$ corresponds to $c = 0$.



Results: partially confirming predictions by Fyodorov/Ossipov/Rodriguez '09

- $c > 0$ **:** Localization regime and Poisson statistics
- $c < -1/2$**:** Delocalization regime $\|\psi\|_\infty = \mathcal{O}(N^{-1/2})$ and GOE statistics
- $c \in (-1, -1/2)$**:** Infinite-volume operator has continuous spectrum.

Thank You!

P. von Soosten, S.W

- *Non-Ergodic Delocalization in the Rosenzweig-Porter Model*, arXiv:1709.10313.
- *The Phase Transition in the Ultrametric Ensemble and Local Stability of Dyson Brownian Motion*, arXive:1705.00923.
- *Singular Spectrum and Recent Results on Hierarchical Operators*, arXive:1705.04884 (to appear in Contemp. Math.)
- *Delocalization in Ultrametric Ensembles*, in preparation.