Random Schrödinger operators with point interactions on R^d: Localization and eigenvalue statistics

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Transport and localization in random media
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2 May 2018

LES overview

Main problem: Characterize the local eigenvalue statistics for random Schrödinger operators on $L^2(\mathbb{R}^d)$ or $\ell^2(\mathbb{Z}^d)$, $d \geq 2$.

Basic Schrödinger operator:

$$H_\omega = H_0 + \lambda V_\omega$$

Hilbert space: lattice $\ell^2(\mathbb{Z}^d)$ or continuum $L^2(\mathbb{R}^d)$

- H_0 deterministic (fixed) self-adjoint operator: $H_0 = -\Delta$, Laplacian
- V_ω random potential:
  - $(V_\omega f)(k) = \omega_k f(k)$, on $\ell^2(\mathbb{Z}^d)$
  - $(V_\omega f)(x) = \sum_{k \in \mathbb{Z}^d} \omega_k u(x-k) f(x)$, $L^2(\mathbb{R}^d)$

LES overview

Randomness: The coupling constants $\{\omega_j \mid j \in \mathbb{Z}^d\}$

- family of independent, identically distributed random variables
- absolutely continuous probability measure having a density $h_0 \in L_0^\infty(\mathbb{R})$.

Deterministic spectrum: $\Sigma \subset \mathbb{R}$ (fixed) equals $\sigma(H_\omega)$ almost surely

Finite-volume operators: Λ_L cube of side-length $L > 0$
$H_\omega^\Lambda := H_\omega | \Lambda_L$ plus boundary conditions

Spectrum of H_ω^Λ is discrete: $\{E_j^\Lambda(\omega)\}_{j=1}^N$,

- $N = |\Lambda|$ for lattice $\mathbb{Z}^d$
- $N = \infty$ for continuum $\mathbb{R}^d$

LES overview

Local eigenvalue statistics Fix $E_0 \in \Sigma$ define:

$$d\xi_\omega^\Lambda(s) = \sum_{j=1}^{N} \delta(|\Lambda_L|(E_j^\Lambda(\omega) - E_0) - s)\; ds$$

Questions:

1. Does ξ_ω^Λ converge to a point process as $|\Lambda| \to \infty$?
2. How does one characterize the limiting process?

Answers:

1. Does ξ_ω^Λ converge to a point process as $|\Lambda| \to \infty$? **YES**
2. How does one characterize the limiting process? **Depends on** E_0 **and the dimension** d

LES overview

CONJECTURES: 1. If $E_0 \in \Sigma$ lies in a region for which the localization length γ_L of eigenfunctions for $H_\omega^{\Lambda_L}$ is small compared to L,

$$\frac{\gamma_L}{L} \to 0, \qquad L \to \infty,$$

then the limiting point process ξ_ω is a **Poisson point process**.
2. If $E_0 \in \Sigma$ lies in a region for which the localization length γ_L of eigenfunctions for $H_\omega^{\Lambda_L}$ is large compared to L,

$$\frac{\gamma_L}{L} > 0, \qquad L \to \infty,$$

then the limiting point process ξ_ω is the same as **random matrix theory** *GOE*.

LES overview

A Toy Model: Scaled disorder

Scaled disorder random Anderson model:

$$H_\omega^{(n)} = H_0^{(n)} + \sum_{j=-n}^{n} \frac{\sigma \omega_j}{\langle n \rangle^\alpha} \Pi_j, \quad \mathcal{H} = \ell^2([-n, n]).$$

$H_0^{(n)}$: Finite difference Laplacian on $[-n, n]$ with simple boundary conditions

$(\Pi_j f)(k) = f(j)\delta_{jk}$ and $\sigma > 0$

Localization length: $\gamma_n \sim \frac{n^{2\alpha}}{\sigma^2}$

Scaling ratio: $\frac{\gamma_n}{n} = \frac{n^{2\alpha-1}}{\sigma^2}$

LES overview

A Toy Model: Scaled disorder

Transition in LES depending on $\alpha \geq 0$.
Scaling regimes:

$$0 \leq \alpha < \frac{1}{2} \qquad \frac{\gamma_n}{n} \to 0 \qquad \textit{LES} = \textit{Poisson}$$

$$\alpha = \frac{1}{2} \qquad \frac{\gamma_n}{n} = 1 \qquad \textit{critical}$$

$$\frac{1}{2} < \alpha \quad 1 \leq \frac{\gamma_n}{n} \to \infty \quad \textit{LES} = \textit{Clock}$$

Clock is the LES of the Laplacian H_0 on $\ell^2(\mathbb{Z})$.

LES overview

Another tool: LSD, the level spacing distribution

Order eigenvalues of H_ω^Λ: $E_1^\Lambda(\omega) \leq E_2^\Lambda(\omega) \leq \cdots \leq E_N^\Lambda(\omega)$

For $E_0 \in \Sigma$, set $I_\Lambda = [E_0 - \frac{1}{|\Lambda|^{1-\epsilon}}, E_0 + \frac{1}{|\Lambda|^{1-\epsilon}}]$; $n(E_0)$ Density of states.

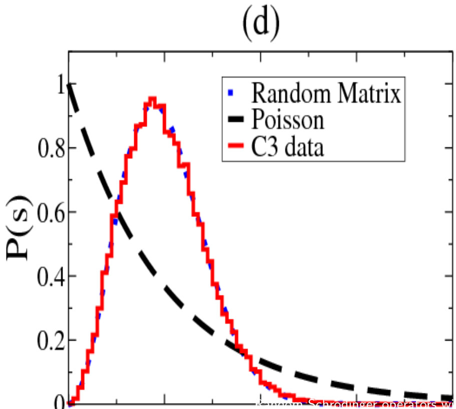
$$LSD_\omega^\Lambda(x; I_\Lambda) = \frac{\#\{j \mid E_j^\Lambda(\omega) \in I_\Lambda, \quad |\Lambda| n(E_0)(E_{j+1}^\Lambda(\omega) - E_j^\Lambda(\omega)) \geq x\}}{\#\{j \mid E_j^\Lambda(\omega) \in I_\Lambda\}}$$

$$LSD(x) = \lim_{|\Lambda| \to \infty} LSD_\omega^\Lambda(x; I_\Lambda).$$

Behavior of $LSD(x)$ depends on E_0 in *localized* or *delocalized* regime.

LES overview

Poisson: Density of the $LSD(s)$ is exponential: $P(s) = e^{-s}$.
GOE: Density of $LSD(s)$ follows the Wigner surmise: $P(s) = Ase^{-Bs^2}$, $A, B > 0$.



LES Results

Random Schrödinger operators on $\mathbb{Z}^d$

- Minami: $E_0 \in \Sigma^{\mathrm{CL}}$, LES ξ_ω is a Poisson point process.
- Germinet-Klopp: LES for unfolded eigenvalues and LSD with exponential density.

Random Schrödinger operators on $\mathbb{R}^d$

- Hislop-Krishna: LES always have limit points ξ_ω that are compound Poisson processes and LSD has exponential density.
- Hislop, Kirsch, Krishna: random Schrödinger operators with δ-interactions, LES is Poisson and exponential LSD density.
- Deitlein-Elgard: Anderson-type random Schrödinger operators has LES Poisson at the bottom of the spectrum.

Random δ-interactions: Model and results

The formal Hamiltonian for random δ-interactions:

$$H_\omega = -\Delta + \sum_{j \in \mathbb{Z}^d} \omega_j \delta(x - j),$$

on $L^2(\mathbb{R}^d)$, for $d = 1, 2, 3$.

[H1] The coupling constants $\{\omega_j \mid j \in \mathbb{Z}^d\}$ form a family of independent, identically distributed random variables with an absolutely continuous probability measure having a density $h_0 \in L_0^\infty(\mathbb{R})$.

[H2] The support of h_0 is the interval $[-b, -a]$ for some finite constants $0 < a < b < \infty$.

Rigorous description of H_ω is given via the Green's function.

Localization for the random δ-interaction model

H_ω is an *ergodic, random Schrödinger operator.*
Deterministic spectrum of H_ω: Σ.

$$\Sigma = \overline{\cup_{\lambda \in \mathrm{supp}\ h_0} H_\lambda}$$

for the periodic Schrödinger operator $H_\lambda = -\Delta + \sum_{j \in \mathbb{Z}^d} \lambda \delta(x - j)$.

$E_0(k; \lambda)$: first band function for periodic H_λ

Deterministic spectrum:

$$E_0(0; -1a) < 0, \qquad [E_0(0; -1/a), E_0(k_0; -1/b)] \cup [0, \infty) \subset \Sigma,$$

where $k_0 \in \mathcal{B}$ is the point where E_0 has its minimum.

Localization for the random δ-interaction model

Localization at negative energies:

Theorem

There exists an finite energy $\tilde{E}_0 < 0$ so that $\Sigma_{pp} \cap (-\infty, \tilde{E}_0] \subset \Sigma$ is almost surely nonempty. Furthermore, for any $\phi \in L^2_0(\mathbb{R}^d)$, any integer $q \in \mathbb{N}$, and any interval $I \subset (-\infty, \tilde{E}_0]$, we have

$$\mathbb{E}[\sup_{t>0}\{\| \|x\|^{q/2} e^{-itH_\omega} E_\omega(I)\phi\|_{\mathrm{HS}}\}] < \infty,$$

Main result: Eigenvalue statistics in the localization regime

Localization regime: Σ^{CL}: energy regime with pure point spectrum and dynamical localization.

Local Eigenvalue Statistics

$\Lambda_L = [0, L]^d$ and local Schrödinger operators $H_\omega^L := H_\omega | \Lambda_L$.

Eigenvalues: $\{E_j^L(\omega)\}$.

Rescaled local eigenvalue point process at $E_0 \in \Sigma^{\mathrm{CL}}$:

$$d\xi_\omega^L(s) := \sum_j \delta(|\Lambda_L|(E_j^L(\omega) - E_0) - s) \; ds$$

Main result: Eigenvalue statistics in the localization regime

Theorem

Consider a fixed energy $E_0 \in (-\infty, \tilde{E}_0] \subset \Sigma^{\mathrm{CL}}$ for which the density of states is nonpositive: $n(E_0) > 0$. The local eigenvalue statistics ξ_ω^L for the random point interaction model on $\mathbb{R}^d$, for $d = 1, 2, 3$, converges weakly to a Poisson point process with intensity measure $n(E_0) ds$.

This means that for $f \in C_0^+(\mathbb{R})$:

$$\lim_{L \to \infty} \mathbb{E}\{e^{-t\xi_\omega^L(f)}\} = \mathbb{E}\left\{e^{-t\xi_\omega^P(f)}\right\},$$

where

$$\mathbb{E}\{e^{-t\xi_\omega^P(f)}\} = e^{n(E_0)\int_{\mathbb{R}}\left(e^{-tf(x)}-1\right)\,dx}.$$

Green's function definition of the Hamiltonian

Remark: Consider $d = 3$ when explicit formulae are used.

The Green's function for a cube $\Lambda_L \subset \mathbb{R}^3$ with Dirichlet boundary conditions is

$$G_0^L(x, y; z) = \frac{e^{-i\sqrt{z}\|x-y\|}}{4\pi\|x - y\|} - c_{z,y}^L(x), \quad x, y \in \Lambda_L.$$

Corrector: $c_{z,y}^L(x)$ for the boundary condition

Let $\tilde{\Lambda}_L := \Lambda_L \cap \mathbb{Z}^3$ and put a δ-interaction at each point with coefficient ω_j.

The Green's function $G_\omega^L(x, y; z)$ for H_ω^L and $d = 1, 2, 3$ is related to the Green's function $G_0^L(x, y; z)$ for the unperturbed operator $H_0^L = -\Delta^L$ by

$$G_\omega^L(x, y; z) = G_0^L(x, y; z) + \sum_{j,k=1}^{|\tilde{\Lambda}_L|} G_0^L(x, j; z)[K_\omega^L(z)^{-1}]_{jk} G_0^L(k, y; z).$$

Green's function definition of the Hamiltonian

Matrix Schrödinger operator:

Let $N := |\Lambda_L| = L^d$. $K_\omega^L(z) : \mathbb{C}^N \to \mathbb{C}^N$

$$[K_\omega^L(z)]_{jk} := t^L(z) + v_\omega$$

where:

- Kinetic energy $t^L(z)$:

$$t_{jk}^L(z) := c_{z,k}^L(j)\delta_{jk} - G_0^L(j,k;z)(1-\delta_{jk}),$$

- Random diagonal potential v_ω:

$$[v_\omega]_{jk} := \frac{1}{\alpha_{d,k}}\delta_{jk}.$$

The off-diagonal part of $t^L(z)$ decays exponentially.

$$e_3(z) = \frac{i\sqrt{z}}{4\pi}, \quad \text{and} \quad \alpha_{3,j} = \omega_j.$$

Estimates for finite-volume operators: Local operators and spectral averaging

Local operators: $H_\omega^L := H_\omega | \Lambda_L$, with Dirichlet boundary conditions.

Spectral averaging of the trace.

Consider $\mathbb{E}_{\omega_j}\{\mathrm{Tr} E_{H_\omega^L}(I)\} := \mathbb{E}_{\omega_j}\{X^L_{\omega_j, \omega_j^\perp}(I)\}$, parameters $\omega_j^\perp$ fixed.

Stone's formula for the spectral projection $E_{H_\omega^L}(I)$:

$$E_{H_\omega^L}(I) = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_I \Im R_\omega^L(E + i\epsilon) \, dE.$$

$R_0^L(z)$ is analytic away from $\mathbb{R}^+$. For $E < 0$, the resolvent formula yields

$$E_{H_\omega^L}(I) = \frac{1}{\pi} \lim_{\epsilon \to 0} \sum_{\ell, m \in \tilde{\Lambda}_L} \int_I \Im [R_0^L(\cdot, \ell; E)[K^L_{\omega_j^\perp}(E + i\epsilon; \omega_j)^{-1}]_{\ell m} R_0^L(m, \cdot; E)].$$

Estimates for finite-volume operators: Spectral averaging

The trace is expressible as the integral over the diagonal of the corresponding Green's functions:

$$X^L_{\omega_j,\omega_j^\perp}(I)$$
$$= \frac{1}{\pi}\lim_{\epsilon\to 0}\int_I \sum_{\ell,m\in\tilde{\Lambda}_L}\int_{\Lambda_L} G_0^L(x,\ell;E)\Im[K^L_{\omega_j^\perp}(E+i\epsilon;\omega_j)^{-1}]_{\ell m} G_0^L(m,x;E)\,dE\,d^dx.$$

Differential inequality method: For any $\xi\in\ell^2(\tilde{\Lambda}_L)$, there is a constant $C_1>0$ so that

$$\sup_{\epsilon\to 0}\left|\int h_0(\omega_j)\langle\xi,[K^L_{\omega_j^\perp}(E+i\epsilon;\omega_j)]^{-1}\xi\rangle\ d\omega_j\right|\le C_1\|\xi\|^2.$$

Estimates for finite-volume operators: Wegner estimate

The above formulas immediately yield:

Theorem

Let $E_0 < 0$ and consider $\eta > 0$ so that

$$I_\eta := [E_0 - \eta, E_0 + \eta] \subset (-\infty, 0).$$

There exists a finite positive constant $C_W > 0$, depending on the dimension d and $|E|_0^{-1}$, so that

$$\begin{aligned} \mathbb{P}\{\mathrm{dist}(\sigma(H_\omega^L), E_0) < \eta\} &= \mathbb{P}\{\mathrm{Tr} E_{H_\omega^L}(I_\eta) \geq 1\} \\ &\leq \mathbb{E}\{\mathrm{Tr} E_{H_\omega^L}(I_\eta)\} \\ &\leq C_W |\Lambda_L| \eta. \end{aligned}$$

Estimates for finite-volume operators: Wegner estimate

Proof of the Wegner estimate: Apply spectral averaging with $\xi_m = G_0^L(m, x; E)$ and obtain:

$$\mathbb{E}_{\omega_j}\{X^{(L)}_{\omega_j,\omega_j^\perp}(I)\} \leq C_1 \sum_{m \in \tilde{\Lambda}_L} \frac{1}{\pi} \int_I \int_{\Lambda_L} G_0^L(x, m; E)^2 \; dE \; d^d x.$$

Exponential decay of $G_0^L(x, m; E)$ implies x-integral is $\mathcal{O}(1)$. After the trivial E-integration, we obtain

$$\mathbb{E}_{\omega_j}\{X^{(L)}_{\omega_j,\omega_j^\perp}(I)\} \leq C_1 |\Lambda_L| |I|.$$

Constant $C_1 > 0$ is uniform in $\omega_j^\perp$ and depends on E_0.

Estimates for finite-volume operators: Minami estimate

Theorem

Let $E_0 < 0$ and consider $\eta > 0$ so that $I_\eta := [E_0 - \eta, E_0 + \eta] \subset (-\infty, 0)$. There exists a finite positive constant $C_M > 0$, depending on the dimension d and $|E|_0^{-1}$, so that

$$\mathbb{E}\{X_\omega^L(I)(X_\omega^L(I) - 1)\} \leq C_M |\Lambda_L|^2 \eta^2.$$

Step One: One-parameter perturbation. The variation of parameter, say ω_j, results in a rank one perturbation:

$$K^L_{\omega_j^\perp}(z;\omega_j)^{-1} - K^L_{\omega_j^\perp}(z;\tau_j)^{-1}$$

is a rank-one matrix so:

$$R^L_{\omega_j}(z) - R^L_{\tau_j}(z) = \sum_{k,m \in \tilde{\Lambda}} R^L_0(\cdot, k; z)[K^L_{\omega_j^\perp}(z;\omega_j)^{-1} - K^L_{\omega_j^\perp}(z;\tau_j)^{-1}]_{km} R^L_0(m, \cdot; z)$$

Estimates for finite-volume operators: Minami estimate

Step Two: Estimate on the eigenvalue counting function.
For such $z = -E << \Sigma_0 = \inf \Sigma$, the resolvent $R_\omega^L(z)$ is a self-adjoint operator.

Let $I = (a, b)$. H_ω^L has an eigenvalue in I if and only if $R_\omega^L(z)$ has an eigenvalue in $I_z := ((b-z)^{-1}, (a-z)^{-1})$.

The eigenvalue counting function for H_ω^L and $R_\omega^L(z)$ satisfy:

$$X_\omega^L(I) := \mathrm{Tr} E_{H_\omega^L}(I) = \mathrm{Tr} E_{R_\omega^L(z)}(I_z).$$

Variation of configurations $(\omega_j, \omega_j^\perp)$ and $(\tau_j, \omega_j^\perp)$ results in:

$$X_{\omega_j}^L(I) - X_{\tau_j}^L(I) = \mathrm{Tr} E_{R_{\omega_j}^L(z)}(I_z) - \mathrm{Tr} E_{R_{\tau_j}^L(z)}(I_z).$$

Estimates for finite-volume operators: Minami estimate

The difference of the resolvents $R^L_{\omega_j}(z) - R^L_{\tau_j}(z)$ is a rank one operator.

It follows that

$$\begin{aligned} |X^L_{\omega_j}(I) - X^L_{\tau_j}(I)| &= |\mathrm{Tr} E_{R^L_{\omega_j}(z)}(I_z) - \mathrm{Tr} E_{R^L_{\tau_j}(z)}(I_z)| \\ &\leq 1. \end{aligned}$$

If, for example, $X^L_{\omega_j}(I) \geq 1$, then

$$0 \leq X^L_{\omega_j}(I) - 1 \leq X^L_{\tau_j}(I).$$

Estimates for finite-volume operators: Minami estimate

Step three: Conclusion of the proof.
Take $\tau_j \in [c, d]$, an interval disjoint from $[a, b]$ and with the same distribution as ω_j:

$$\begin{aligned} \mathbb{E}\{X_\omega^L(I)(X_\omega^L(I)-1)\} &\leq \mathbb{E}_{\tau_j}\mathbb{E}\{X^L_{\omega_j,\omega_j^\perp}(I)(X^L_{\tau_j,\omega_j^\perp}(I))\} \\ &\leq C_1|\Lambda_L||I|\left(\mathbb{E}_{\tau_j}\mathbb{E}_{\omega_j^\perp}\{X^L_{\tau_j,\omega_j^\perp}(I)\}\right) \\ &\leq C_M(|\Lambda_L||I|)^2, \end{aligned} \tag{1}$$

using the above result for $(\tau_j, \omega_j^\perp)$.

Estimates for finite-volume operators: Localization bounds

Fractional moment bound for $K_\omega^\Lambda(E)$, $E < 0$.

Proposition

For any $s \in (0,1)$, there are finite, positive constants $C_s > 0$ and $\alpha_{s,d} > 0$, uniform in $L > 0$, so that for any $E < 0$, we have

$$\mathbb{E}\{|[K_\omega^\Lambda(-E)^{-1}]_{ij}|^s\} \leq C_s e^{-s\alpha_{s,d}\|i-j\|},$$

for any $i, j \in \tilde{\Lambda}_L$.

The proof of this uses the fractional moment method of Aizenman and Molchanov.

A uniformly asymptotically negligible array: *uana*

Decomposition:

$$\Lambda_L = \cup_{p=1}^{N_L} \Lambda_{\ell,p}$$

Side length: $\ell = L^\alpha$, $0 < \alpha < 1$.

Number of subcubes

$$N_L = (L/\ell)^d$$

$H_\omega^{\ell,p}$ the local point interaction Hamiltonian restricted to $\Lambda_{\ell,p}$ with Dirichlet boundary conditions.

1. The local operators: $\sigma(H_\omega^L)$ and $\sigma(H_\omega^{\ell,p})$ discrete.
2. The spectra of the local Laplacians and lower semibounded and lie in the half-axis $[\Sigma_0, \infty)$, for $\Sigma_0 := \inf \Sigma < 0$ finite.
3. The Wegner, Miniami, and localization estimates are valid for these local random operators at negative energies.

A uniformly asymptotically negligible array: *uana*

LES for each local Hamiltonian $H_\omega^{\ell,p}$: $\eta_\omega^{\ell,p}$:

$$d\eta_\omega^{\ell,p}(s) := \sum_j \delta(|\Lambda_L|(E_j^{\ell,p}(\omega) - E_0) - s)\ ds$$

The collection $\{\eta_\omega^{\ell,p}\}_{p=1}^{N_L}$ forms a **uniformly asymptotically negligible array** (*uana*) of independent random point processes:

$$\lim_{L\to\infty} \sup_{1\leq p\leq N_L} \mathbb{P}\{\eta_\omega^{\ell,p}(I) > 0\} = 0$$

This follows from the Wegner estimate for the local Hamiltonians.
Define the point process

$$\zeta_\omega^L = \sum_{p=1}^{N_L} \eta_\omega^{\ell,p}$$

.
Properties of the process ζ_ω^Λ.

LES for a uniformly asymptotically negligible array

The density of states $n(E)$ exists, belongs to $L^1_{loc}(\mathbb{R})$.
These results follow from the Lipschitz continuity of the IDS

Condition 1: Intensity of the limiting process

Proposition

For the uana $\{\eta_\omega^{\ell,p}\}$, *and any* $E_0 \in \Sigma^{\mathrm{CL}}$ *for which* $n(E_0) \neq 0$, *we have*

$$\lim_{L\to\infty} \sum_{p=1}^{N_L} \mathbb{P}\{\eta_\omega^{\ell,p}(I) = 1\} = n(E_0)|I|.$$

LES for a uniformly asymptotically negligible array

Condition 2: Multiple eigenvalues are rare

Proposition

For the uana $\{\eta_\omega^{\ell,p}\}$*, we have*

$$\lim_{L\to\infty} \sum_{p=1}^{N_L} \mathbb{P}\{\eta_\omega^{\ell,p}(I) \geq 2\} = 0.$$

Conclusion:

Theorem

For $E_0 \in \Sigma^{\mathrm{CL}} \cap (-\infty, 0)$*, and* $n(E_0) > 0$*, the process* ζ_ω^L *constructed from the uana* $\{\eta_\omega^{\ell,p}\}$ *converges weakly to a Poisson point process with intensity measure* $n(E_0) ds$*.*

Approximation by a *uana*

Theorem

For $E_0 \in \Sigma^{\mathrm{CL}} \cap (-\infty, 0)$, the local eigenvalue point processes ξ_ω^L associated with H_ω^L, and the local point process ζ_ω^L, associated with the uana have the same weak limit point. This is the Poisson point process with intensity measure $n(E_0)ds$.

Localization estimates are used to prove

$$\xi_\omega^L(f) - \zeta_\omega^L(f) \to 0, \quad L \to \infty$$

Reduce to showing:
For $z := E_0 + \frac{\zeta}{|\Lambda_L|}$ with $\zeta = \sigma + i\tau$, with $\tau > 0$ and $\sigma \in \mathbb{R}$:

$$\lim_{|\Lambda| \to \infty} \mathbb{E} \left\{ \left| \frac{1}{|\Lambda_L|} \mathrm{Tr} \Im R_\omega^L(z) - \frac{1}{|\Lambda_L|} \sum_{p=1}^{N_L} \mathrm{Tr} \Im R_\omega^{\ell,p}(z) \right| \right\} = 0.$$

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Random delta interactions

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Appendix: Estimates for finite-volume operators: Rank-one perturbations

Let A and B be to self-adjoint operators with B rank one.
General result:

Proposition

Let A and B be self-adjoint operators with $B = \Pi_\varphi$ rank one, $\|\varphi\| = 1$. Let $I := [a, b] \subset \mathbb{R}$ be an interval so that $\sigma(A) \cap [a, b]$ is discrete.

1. $$|\mathrm{Tr}[E_A(I)] - \mathrm{Tr}[E_{A+B}(I)]| \leq 1.$$
2. *If* $\mathrm{Tr}[E_A(I)] \geq 1$, *we have* $$0 \leq \mathrm{Tr}[E_A(I)] - 1 \leq \mathrm{Tr}[E_{A+B}(I)].$$

Appendix: Estimates for finite-volume operators: Rank-one perturbations

Assumption: The vector φ is cyclic for A, and hence for $A+B$.

We define two measures

$$\mu_A^{\varphi}(\cdot) := \langle \varphi, E_A(\cdot)\varphi \rangle, \quad and \quad \mu_{A+B}^{\varphi}(\cdot) := \langle \varphi, E_{A+B}(\cdot)\varphi \rangle.$$

Lemma

Under the hypotheses of the proposition, for any $x \in \sigma(A) \cap [a,b]$, we have

$$\mu_A^{\varphi}(\{x\}) \neq 0,$$

and similarly, for any $y \in \sigma(A+B) \cap [a,b]$, we have

$$\mu_{A+B}^{\varphi}(\{y\}) \neq 0.$$

Appendix: Estimates for finite-volume operators: Rank-one perturbations

Let $\{x_1 \leq x_2 \leq \ldots \leq x_j\}$ be the eigenvalues of A in $[a, b]$. Similarly, let $\{y_1 \leq y_2 \leq \ldots \leq y_t\}$ be the eigenvalues of B in $[a, b]$.

Lemma

The map

$$F_A(E) := \langle \varphi, R_A(E)\varphi \rangle$$

restricted to each interval $F_A : (x_i, x_{i+1}) \to \mathbb{R}$ *is bijective for all* $i = 1, \ldots, k-1$. *Similarly, the map*

$$F_{A+B}(E) := \langle \varphi, R_{A+B}(E)\varphi \rangle$$

restricted to each interval $F_{A+B} : (y_j, y_{j+1}) \to \mathbb{R}$ *is bijective for all* $j = 1, \ldots, \ell - 1$.

Appendix: Estimates for finite-volume operators: Rank-one perturbations

Lemma

The poles of F_A and F_{A+B} in $[a, b]$ are intertwined. In each interval (x_i, x_{i+1}), there is exactly one pole y_j of F_{A+B} and in each interval (y_j, y_{j+1}), there is exactly one x_i.

This shows that the effect of the rank one perturbation B is to change the number of eigenvlaues in $[a, b]$ by at most one.