THREE-BODY INTERACTIONS DRIVE THE TRANSITION TO POLAR ORDER IN A SIMPLE FLOCKING MODEL

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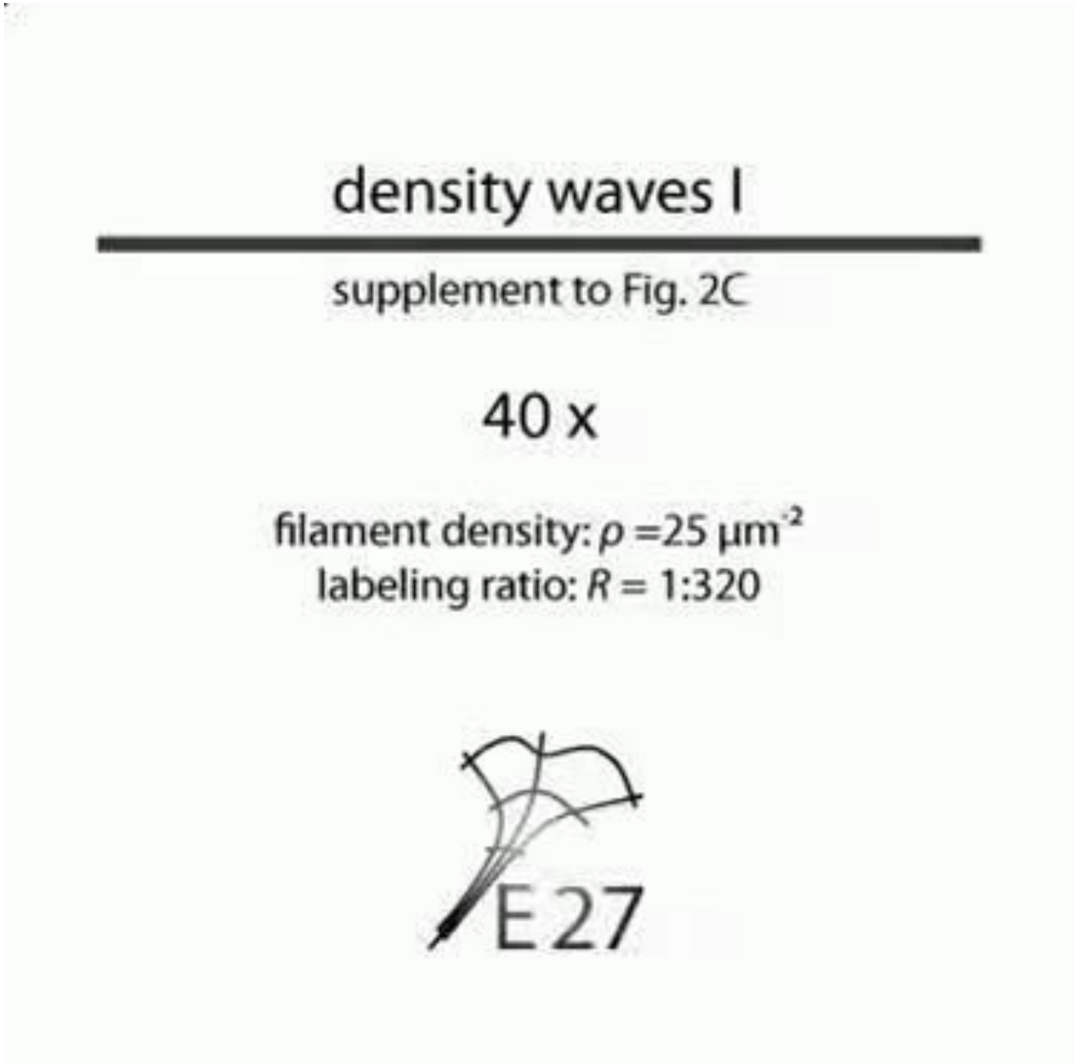
Flocking in Actomyosin motility assays



density waves I

supplement to Fig. 2C

40 x

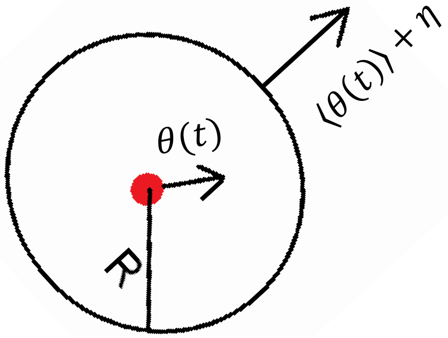
filament density: ρ =25 μm^{-2}
labeling ratio: R = 1:320

E27

V. Schallar et al, *Nature* **volume 467**, (2010)

Minimal model of flocking

- $\vec{v}(t) = v_0 e^{i\theta(t)}$
- $\theta(t+1) = \langle\theta(t)\rangle_R + \eta$



T. Vicsek et al, Phys. Rev. Lett. **75**, 1226 (1995)

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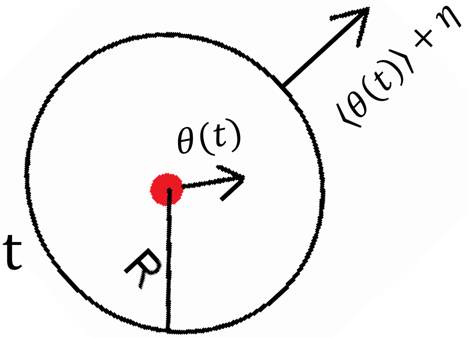

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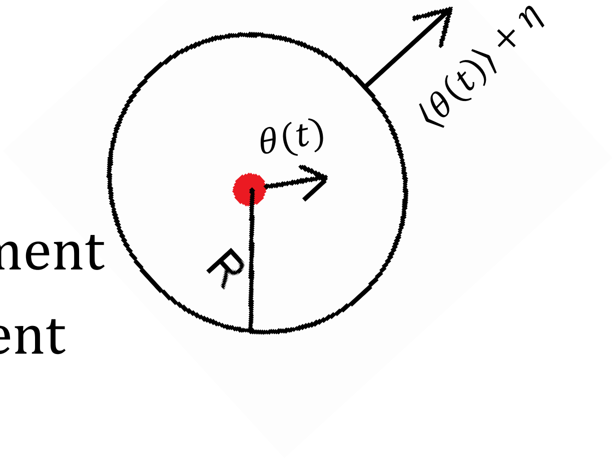

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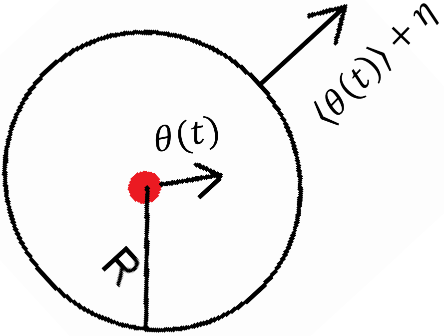
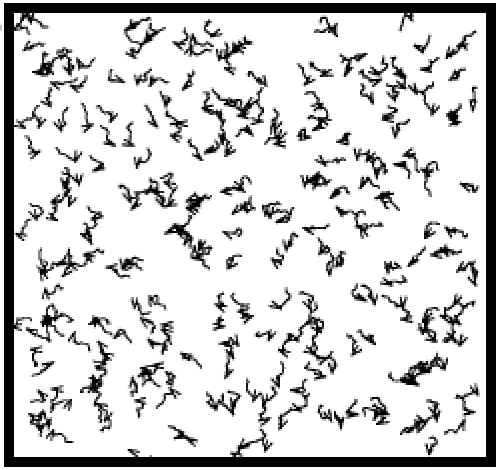
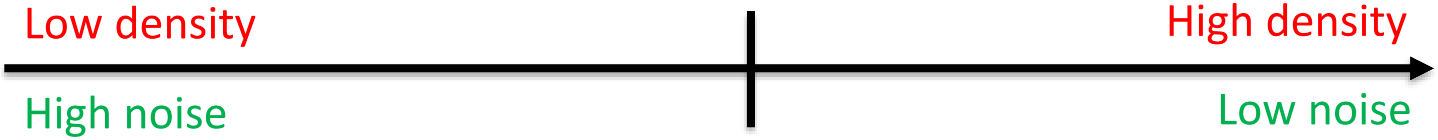




Isotropic phase

Polar ordered phase



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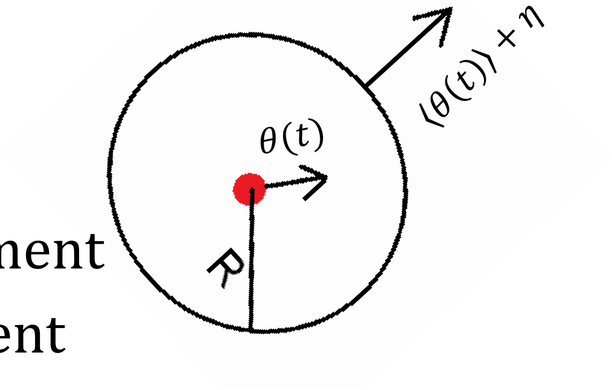
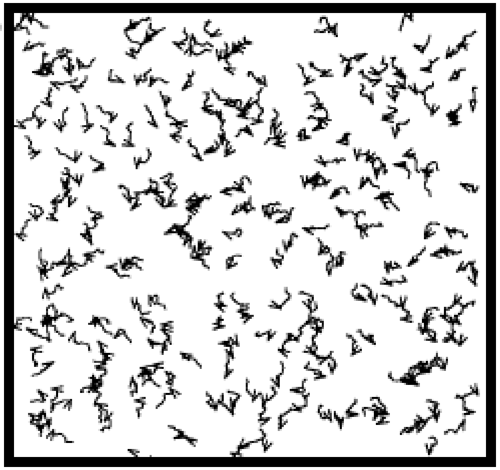
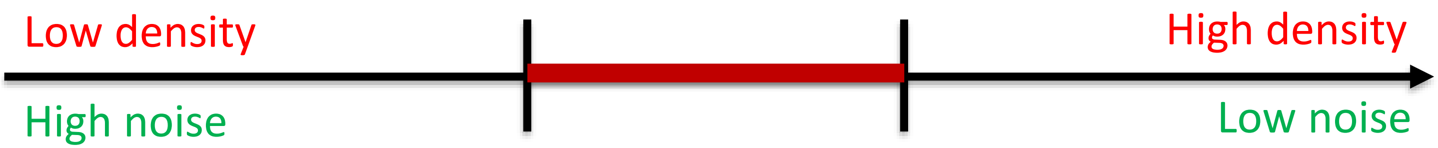




Isotropic phase

Phase coexistence

Polar ordered phase



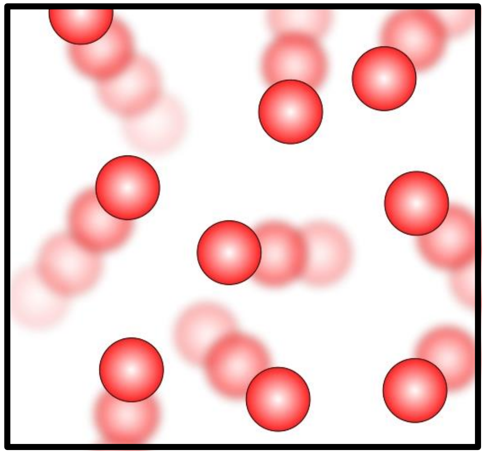
T. Vicsek et al, Phys. Rev. Lett. **75**, 1226 (1995)

Minimal model of flocking

Mapping between Active fluid and Equilibrium fluid

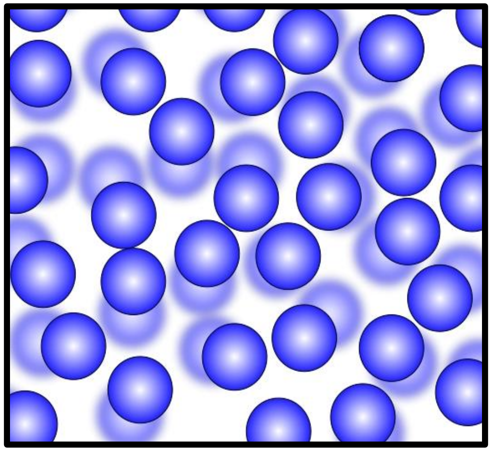
$$\eta \Leftrightarrow T$$

$$\text{density} \Leftrightarrow (\text{Volume})^{-1}$$



Gas phase

Phase coexistence



Liquid phase

Low density

High density

High noise

Low noise

T. Vicsek et al, Phys. Rev. Lett. **75**, 1226 (1995)

http://pluspng.com/png-85462.html

Minimal model of flocking

Low density

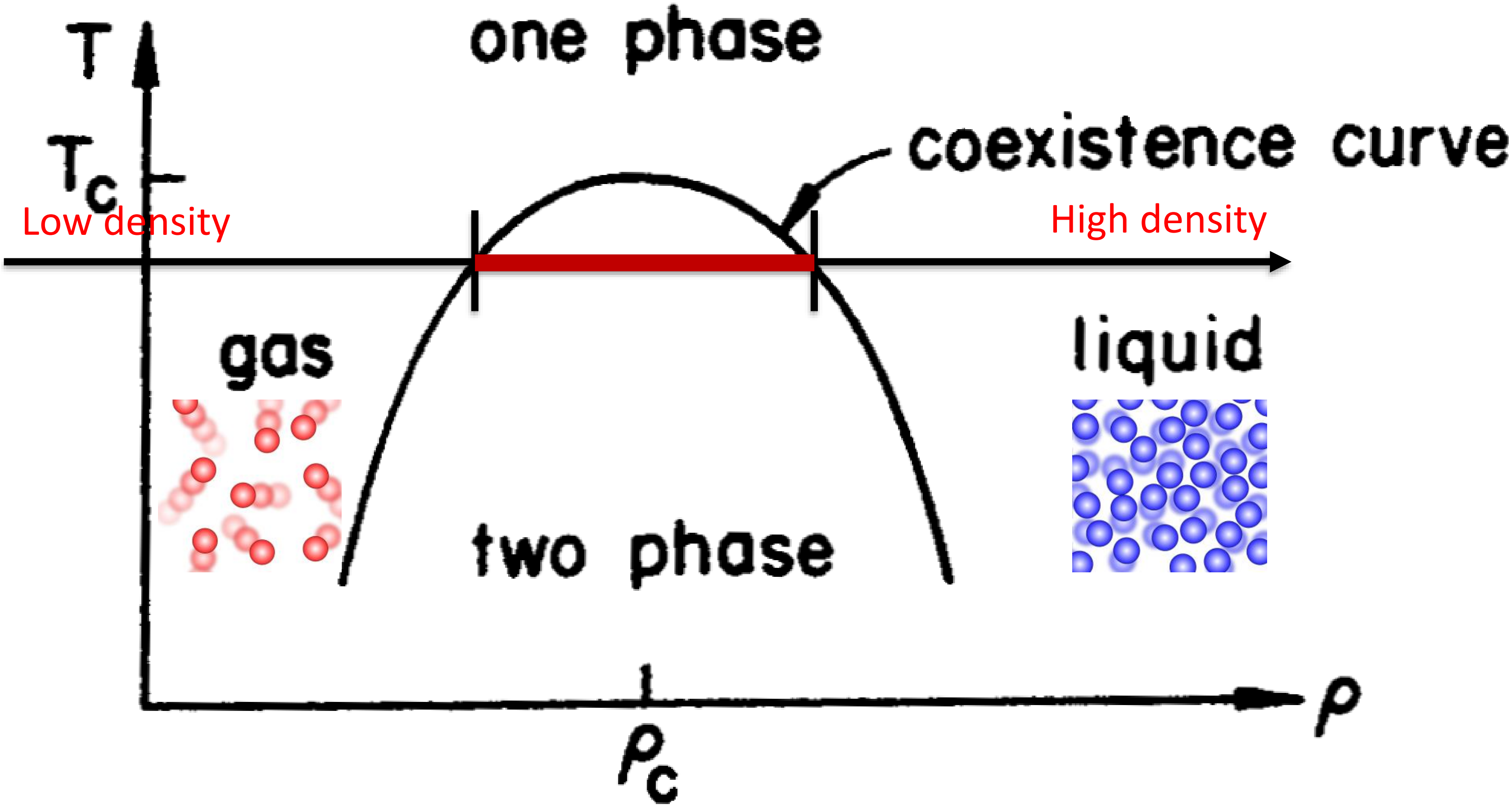
High density

High noise

Low noise

T. Vicsek et al, Phys. Rev. Lett. **75**, 1226 (1995)

Minimal model of flocking



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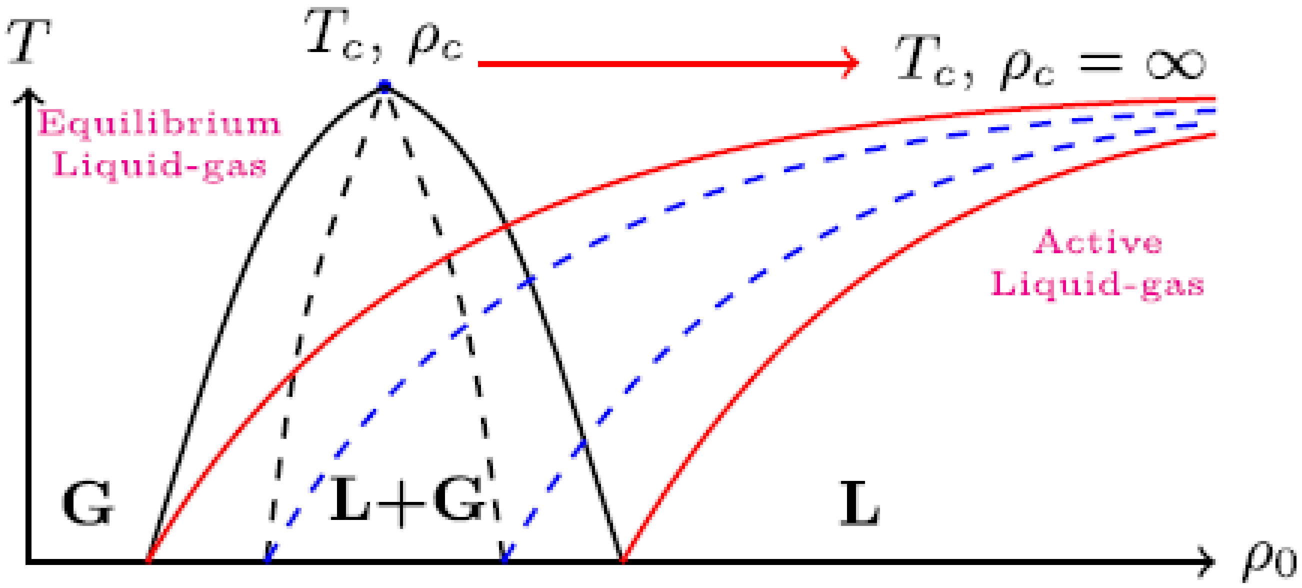
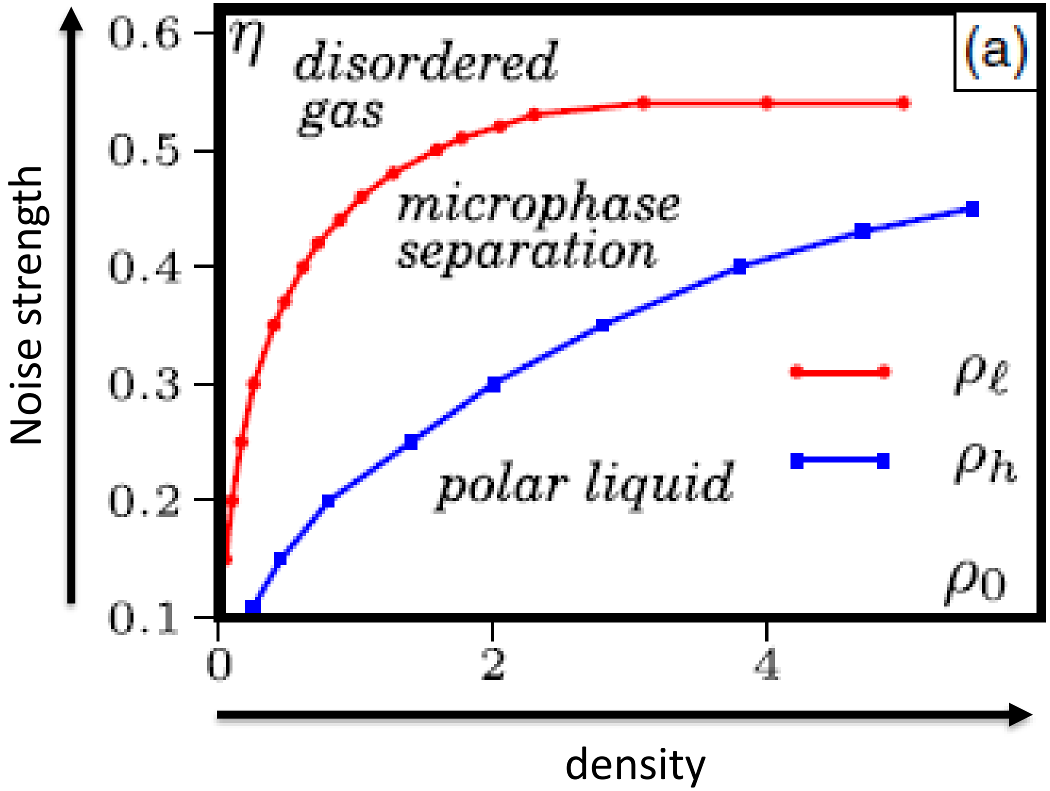
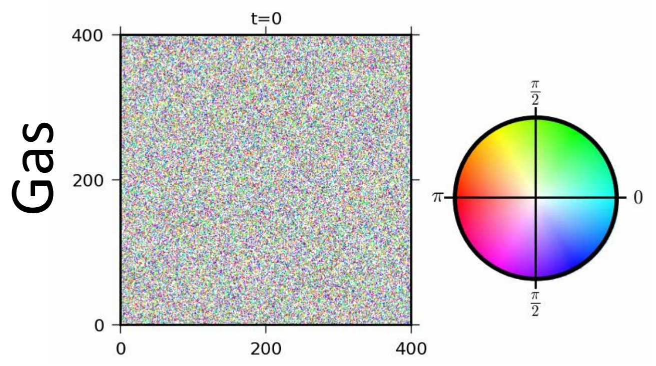
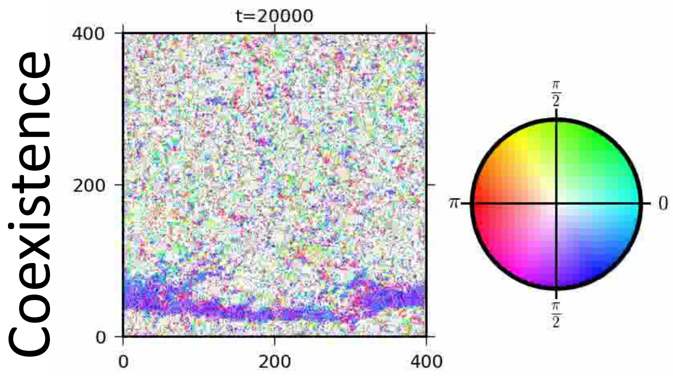


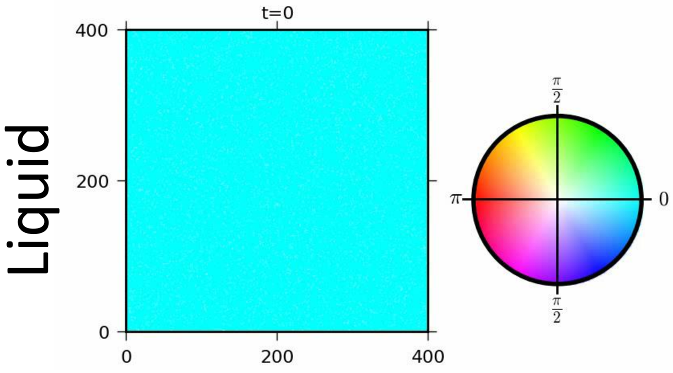
A. P. Solon and J. Tailleur, Phys. Rev. E **92**, 042119 (2015)

Minimal model of flocking



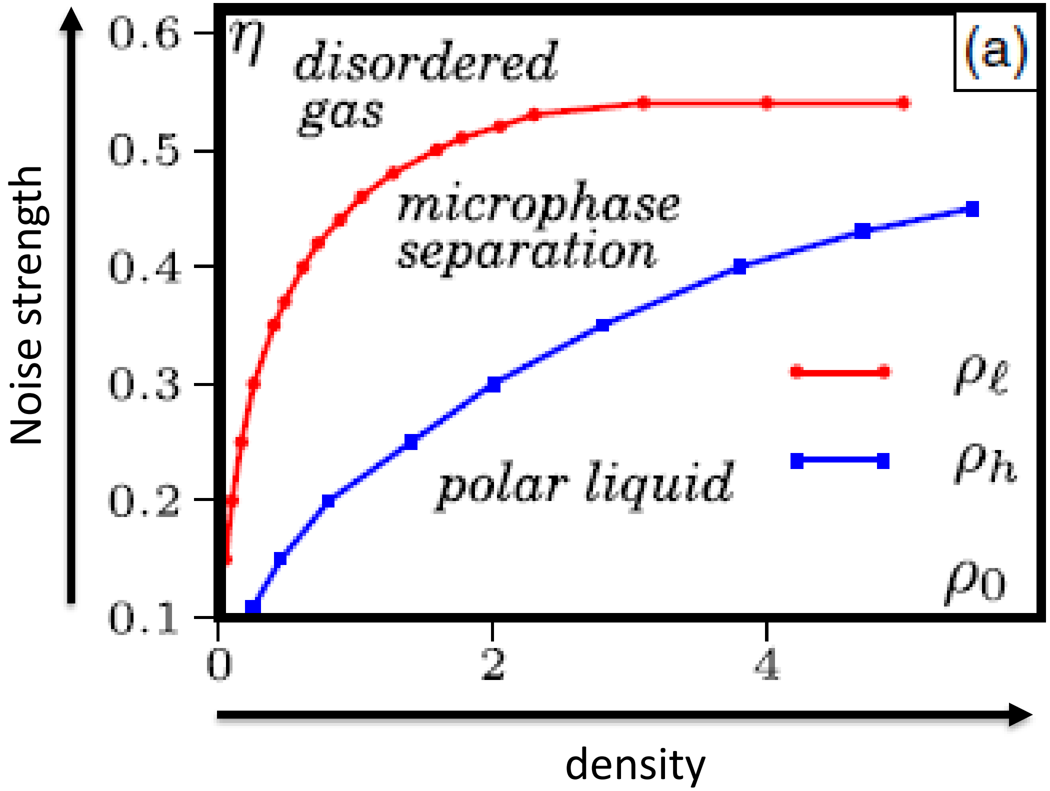






H. Chaté et al, Phys. Rev. Lett. **114**, 068101 (2015)

Questions



1) What interactions are important for the ordering transition?

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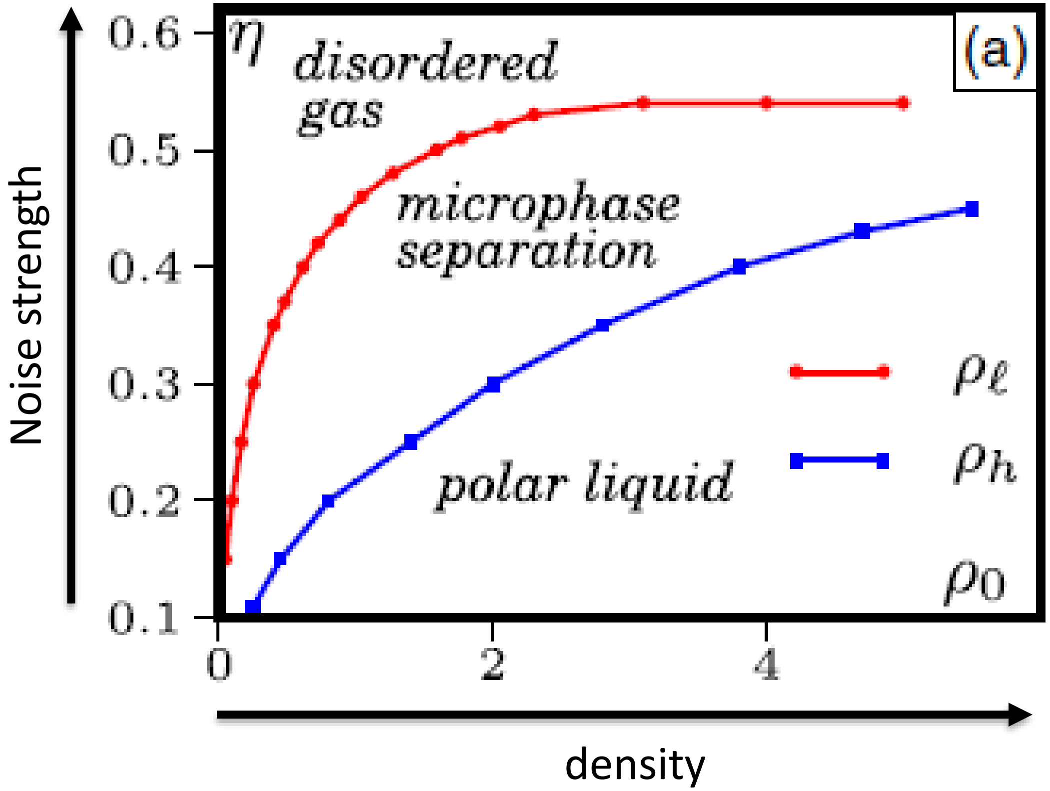

1) What interactions are important for the ordering transition?
2) What is the effect of stochasticity?

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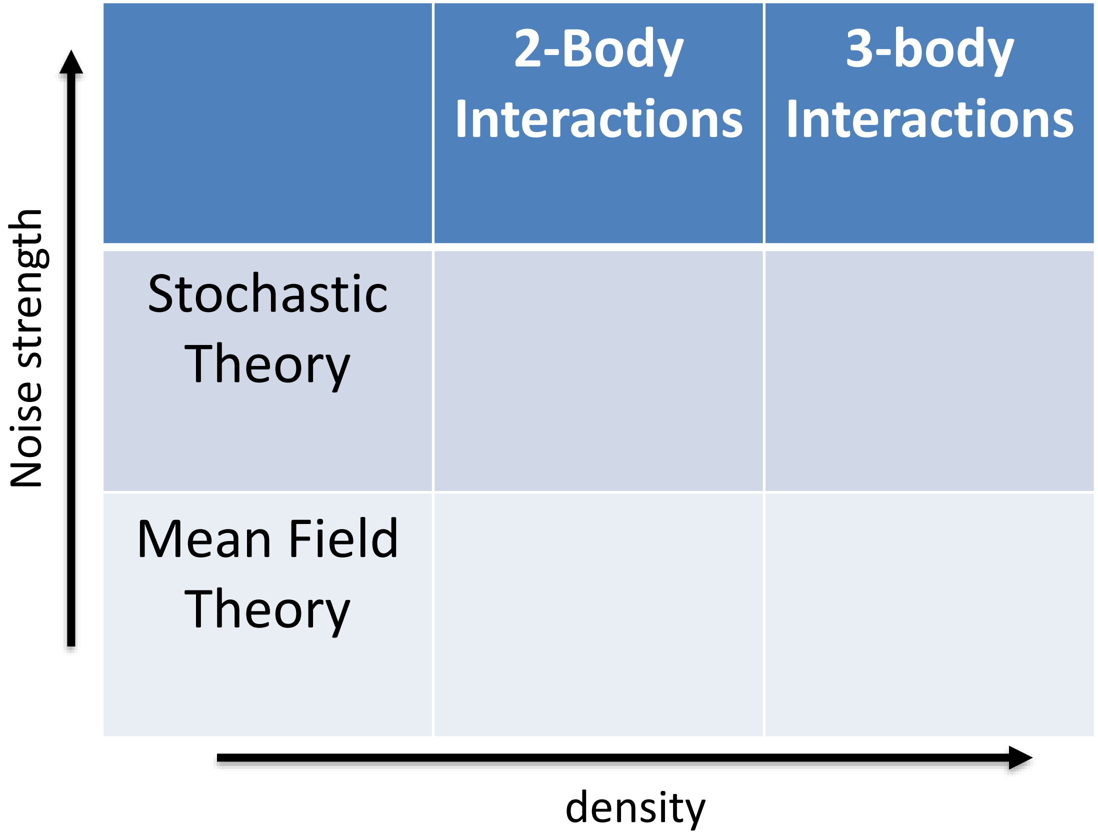


	2-Body Interactions	3-body Interactions
Stochastic Theory		
Mean Field Theory		

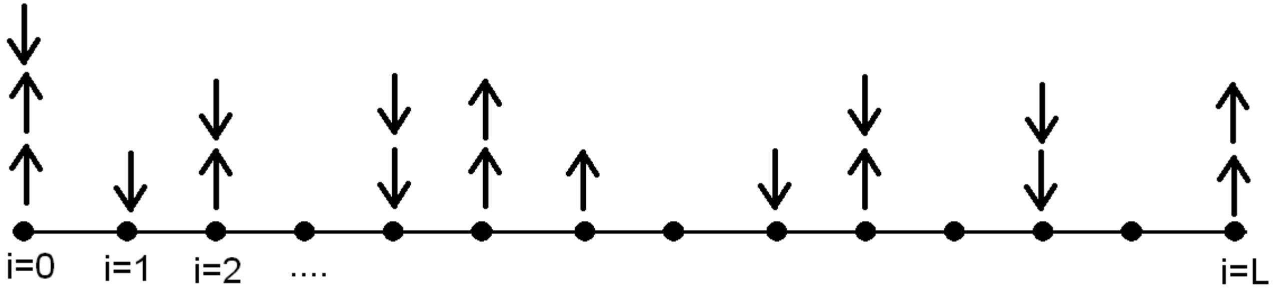
1) What interactions are important for the ordering transition?

2) What is the effect of stochasticity?

H. Chaté et al, Phys. Rev. Lett. **114**, 068101 (2015)

The Active Ising Model (AIM)

- N particles, each carrying $\pm$ spin.
- 1D lattice of size L, sites labelled by 'i'.
- $n_i^{\pm}$ $\equiv$ number of '+' and '-'spins on site i.
- $\rho_i = n_i^+ + n_i^- \equiv$ local density at site i.
- $m_i = n_i^+ - n_i^- \equiv$ local magnetization at site i.
- No exclusion: arbitrary number of $\pm$ spins on each site.



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Interactions

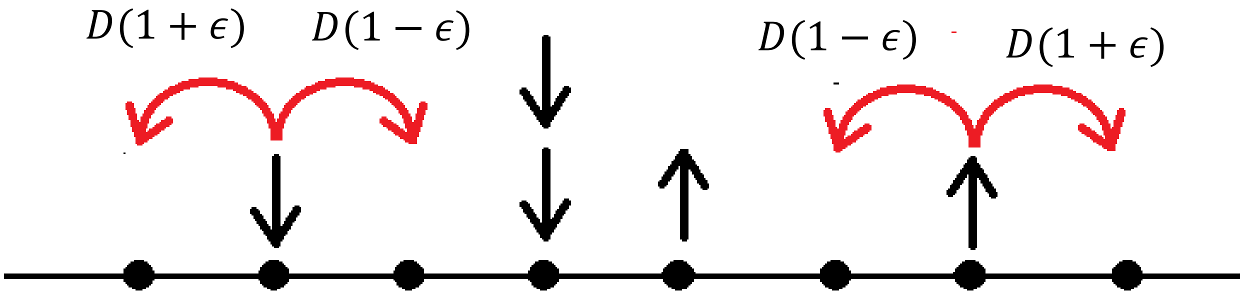
Interactions

- Self-Propulsion:

$$N_i^+ \xrightarrow{D(1+\epsilon)} N_{i+1}^+ \qquad N_i^+ \xrightarrow{D(1-\epsilon)} N_{i-1}^+$$

$$N_i^- \xrightarrow{D(1-\epsilon)} N_{i+1}^- \qquad N_i^- \xrightarrow{D(1+\epsilon)} N_{i-1}^-$$

where $\epsilon \in (0,1)$



Interactions (contd.)

- Alignment:
  - Random spin flip:

$$N_i^+ \xrightarrow{T} N_i^-$$

$$N_i^- \xrightarrow{T} N_i^+$$

  - Two-body interaction

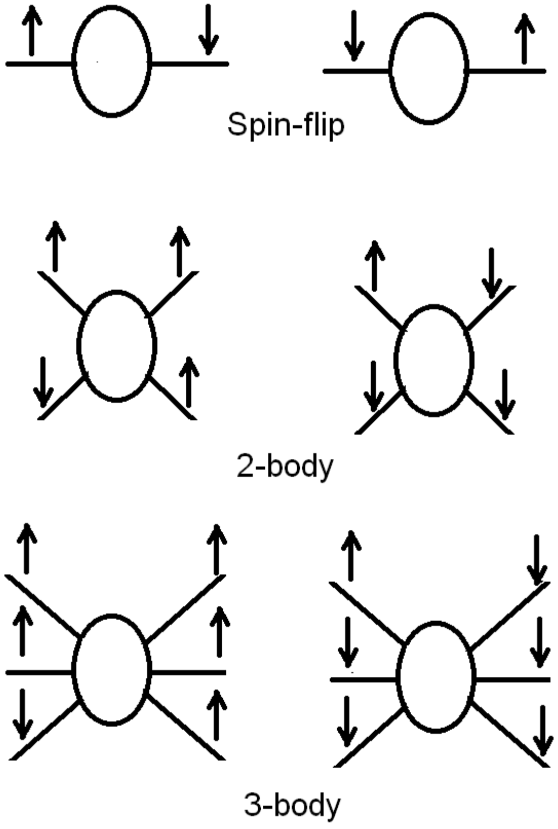
$$N_i^+ + N_i^- \xrightarrow{r_2/\rho_i} 2N_i^+$$

$$N_i^+ + N_i^- \xrightarrow{r_2/\rho_i} 2N_i^-$$

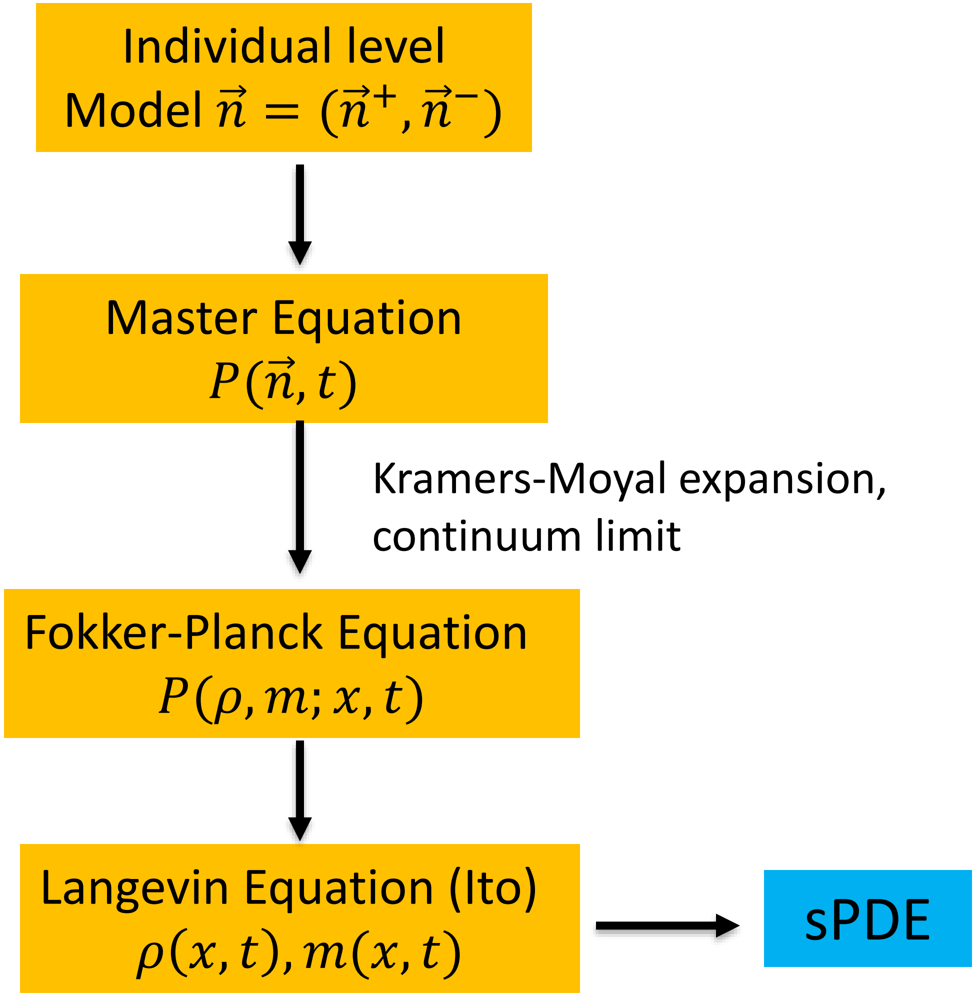
  - Three body interaction

$$2N_i^+ + N_i^- \xrightarrow{r_3/\rho_i^2} 3N_i^+$$

$$N_i^+ + 2N_i^- \xrightarrow{r_3/\rho_i^2} 3N_i^-$$



Stochastic Hydrodynamics



Individual level
Model $\vec{n} = (\vec{n}^{+}, \vec{n}^{-})$

Master Equation
$P(\vec{n}, t)$

Kramers-Moyal expansion,
continuum limit

Fokker-Planck Equation
$P(\rho, m; x, t)$

Langevin Equation (Ito)
$\rho(x, t), m(x, t)$

sPDE

Stochastic Hydrodynamics

$$\partial_t \rho = \partial_{xx}\rho - v\partial_x m,$$

$$\partial_t m = \partial_{xx} m - v\partial_x \rho - m\left[2\left(T - \frac{r_3}{4}\right) + \frac{r_3}{2}\frac{m^2}{\rho^2}\right] + \ 2\sqrt{\frac{\beta}{\rho}\left(\frac{T+\beta}{\beta}\rho^2 - m^2\right)}\,\eta$$

$$v = 2D\epsilon, \qquad \beta = \frac{r_2}{2} + \frac{r_3}{4}$$

$$\langle \eta(x,t)\eta(x',t')\rangle = \delta(x-x')\delta(t-t')$$

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- No noise in ρ because the noise comes from the alignment interactions, and the local ρ is only affected by the hopping processes, not alignment.

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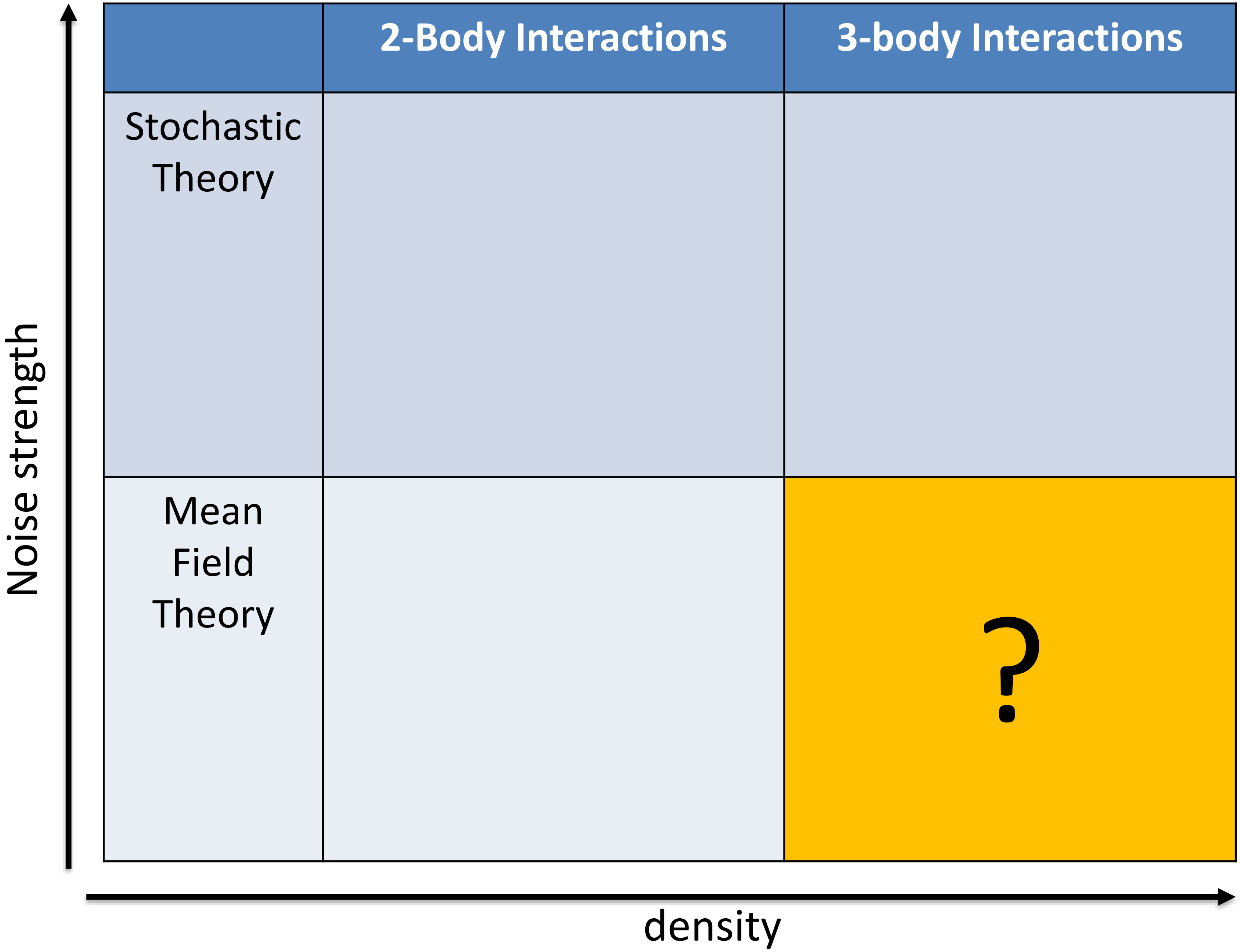
- The gradient of m changes ρ, and m is noisy, and this is why ρ has fluctuations as well



	2-Body Interactions	3-body Interactions
Stochastic Theory		
Mean Field Theory		?

H. Chaté et al, Phys. Rev. Lett. **114**, 068101 (2015)

Mean Field Theory (MFT)

$$\partial_t \rho = \partial_{xx}\rho - v\partial_x m,$$

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Homogeneous steady states:

- For $T > \frac{r_3}{4}$, homogeneous isotropic
  - $\rho = \rho_0, \qquad m = m_0 = 0$

Mean Field Theory (MFT)

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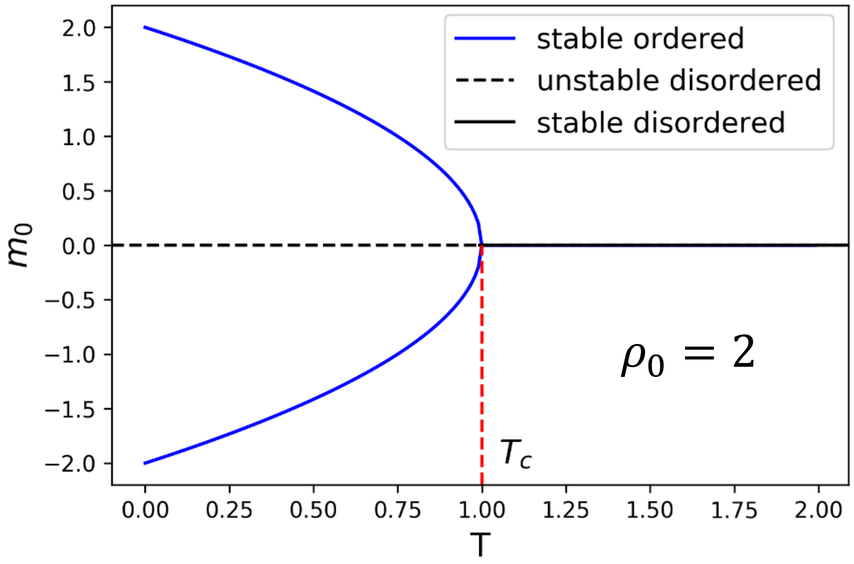
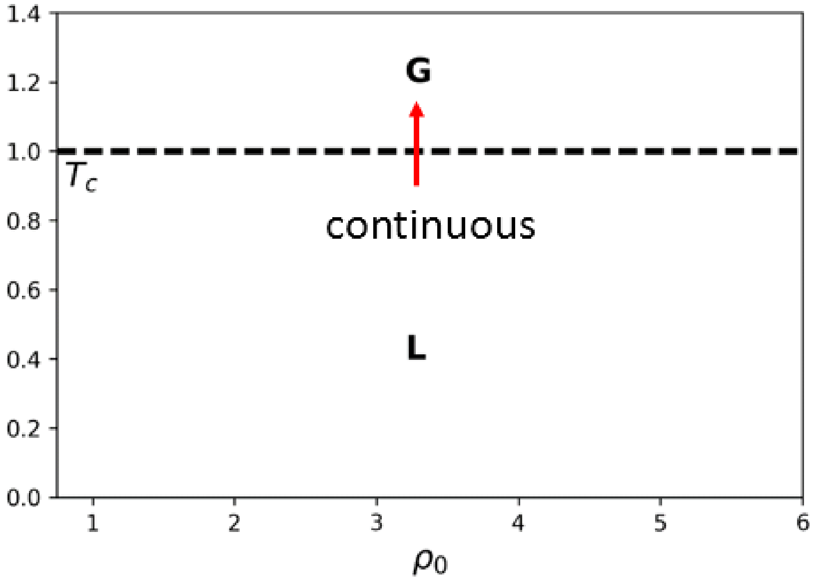
Homogeneous steady states:

- For $T > \frac{r_3}{4}$, homogeneous isotropic
  - $\rho = \rho_0, \qquad m = m_0 = 0$
- For $T < \frac{r_3}{4}$, homogeneous polar order
  - $\rho = \rho_0, \qquad m = m_0 = \pm\rho_0\sqrt{\frac{r_3-4T}{r_3}}$

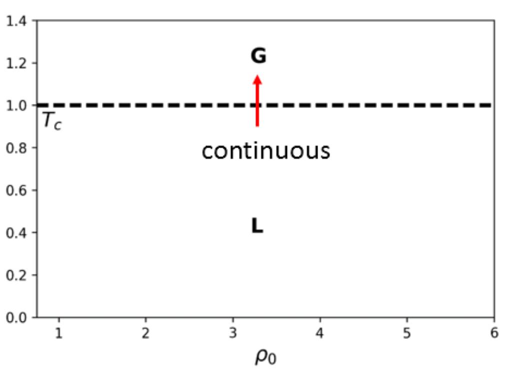
MFT Phase Diagrams





Phase diagram in mean field approximation exhibits no phase coexistence.

Noise strength

	2-Body Interactions	3-body Interactions
Stochastic Theory		
Mean Field Theory		Gas, Liquid, no coexistence. 

density

H. Chaté et al, Phys. Rev. Lett. **114**, 068101 (2015)

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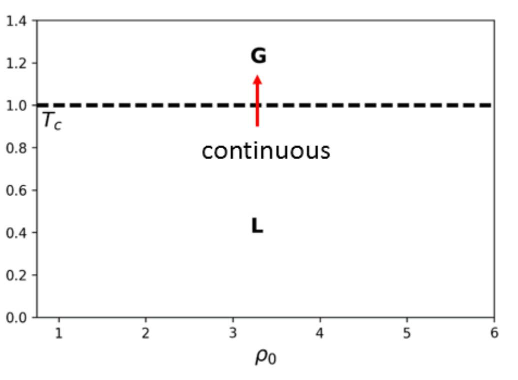
Mean Field Theory	?	Gas, Liquid, no coexistence. 

density

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Mean Field Theory (MFT)

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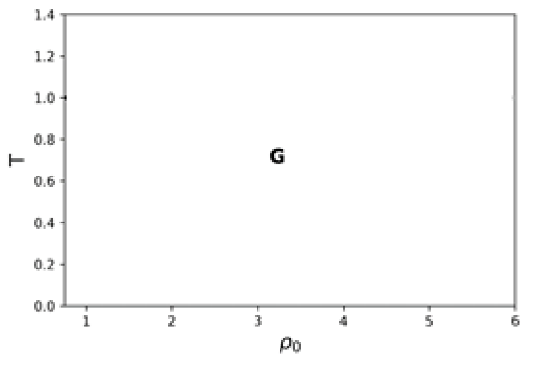
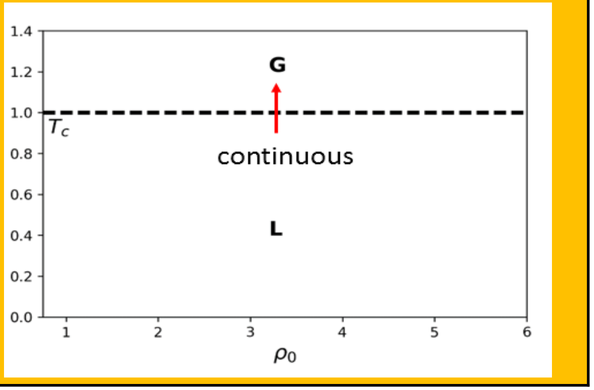
For $r_3 = 0$, only stable homogeneous state:

$$\rho = \rho_0, \qquad m = m_0 = 0$$

No ordered state with just 2-body interactions.

Noise strength

	2-Body Interactions	3-body Interactions
Stochastic Theory		
Mean Field Theory	Only Gas phase  	Gas, Liquid, no coexistence. 

density

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Beyond Mean Field Theory

- MFT :

$$P(\rho, m; x, t) = \delta(\rho - \bar{\rho})\delta(m - \bar{m})$$

Beyond Mean Field Theory

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$$P(\rho, m; x, t) = \delta(\rho - \bar{\rho})\delta(m - \bar{m})$$

- Weak Fluctuation Theory (WFT) :

$$P(\rho, m; x, t) = \mathcal{N}\left(\rho - \bar{\rho}, a_\rho \, \bar{\rho}\right)\mathcal{N}(m - \bar{m}, a_m \, \bar{\rho})$$

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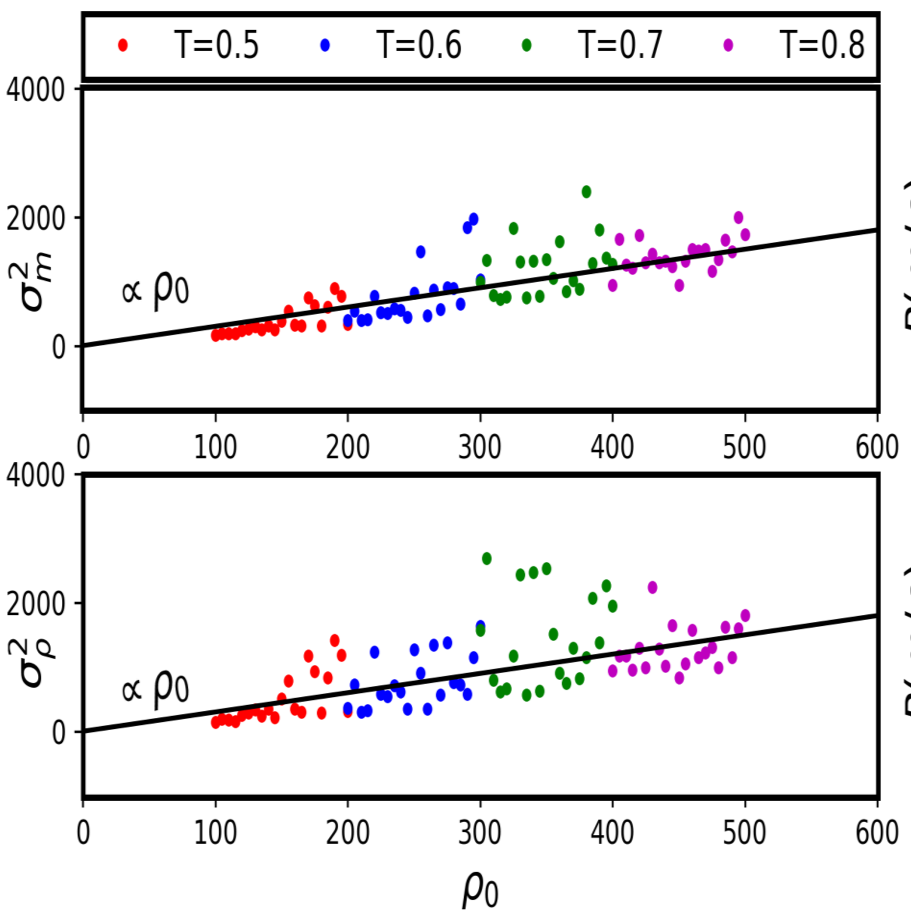
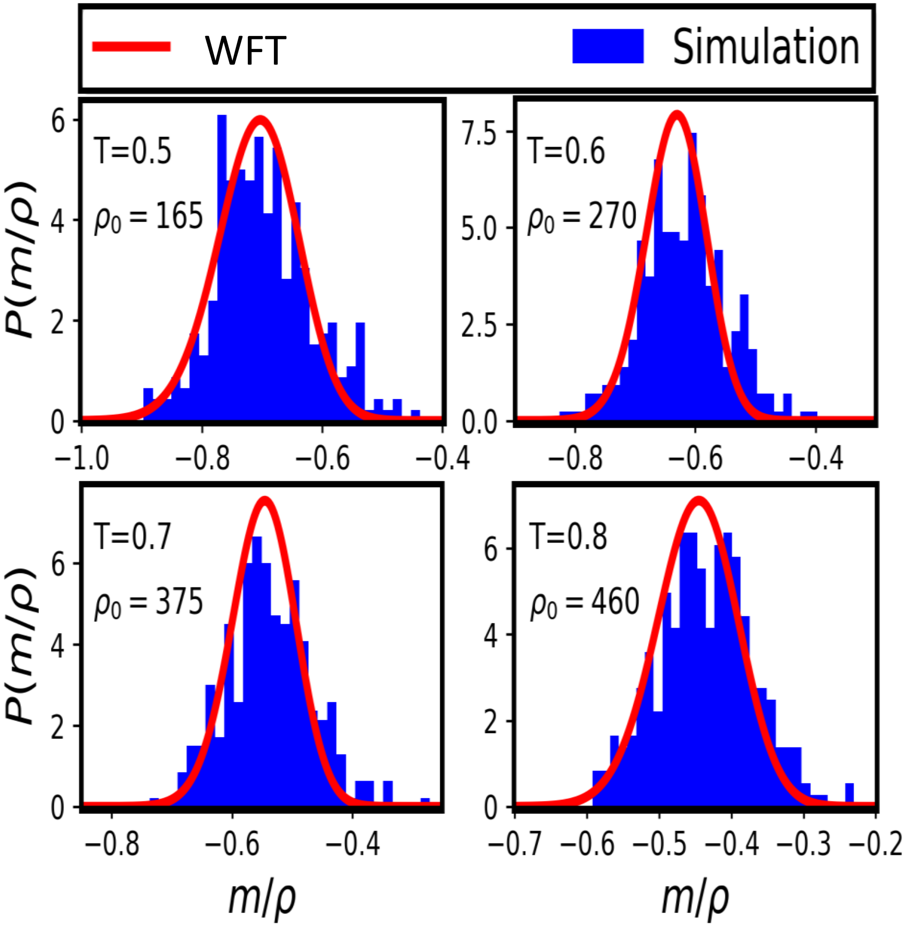
Assumption 1: Fluctuations are Gaussian

Assumption 2: The variances of these Gaussian distributions are proportional to the average density $\bar{\rho}$, i.e. locally the number of fluctuating degrees of freedom is proportional to $\bar{\rho}$

Testing the WFT approximation





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$$\left\langle m\left[2\left(T - \frac{r_3}{4}\right) + \frac{r_3}{2}\frac{m^2}{\rho^2}\right]\right\rangle \approx m\left[2\left(T - \frac{r_3}{4} + \frac{r}{\rho}\right) + \frac{r_3}{2}\frac{m^2}{\rho^2}\right]$$

Where $r = 3r_3 a_m/4$

$$T_c = \frac{r_3}{4} - \frac{r}{\rho} = T_c^{MFT} - r/\rho$$

MFT, WFT and sPDE

- MFT:

$$\partial_t \rho = \partial_{xx}\rho - v\partial_x m,$$

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- sPDE

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MFT, WFT and sPDE

No fluctuations or correlations.

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$$\partial_t m = \partial_{xx} m - v\partial_x \rho - m\left[2\left(T - \frac{r_3}{4}\right) + \frac{r_3}{2}\frac{m^2}{\rho^2}\right]$$

- WFT:

$$\partial_t \rho = \partial_{xx}\rho - v\partial_x m,$$

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- sPDE

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$$\partial_t m = \partial_{xx}m - v\partial_x \rho - m\left[2\left(T - \frac{r_3}{4}\right) + \frac{r_3}{2}\frac{m^2}{\rho^2}\right]$$

No fluctuations or correlations.

- WFT:

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Weak fluctuations-renormalized MFT

- sPDE

$$\partial_t \rho = \partial_{xx}\rho - v\partial_x m,$$

$$\partial_t m = \partial_{xx}m - v\partial_x \rho - m\left[2\left(T - \frac{r_3}{4}\right) + \frac{r_3}{2}\frac{m^2}{\rho^2}\right] + \ 2\sqrt{\frac{\beta}{\rho}\left(\frac{T+\beta}{\beta}\rho^2 - m^2\ \right)}\eta$$

MFT, WFT and sPDE

- MFT:

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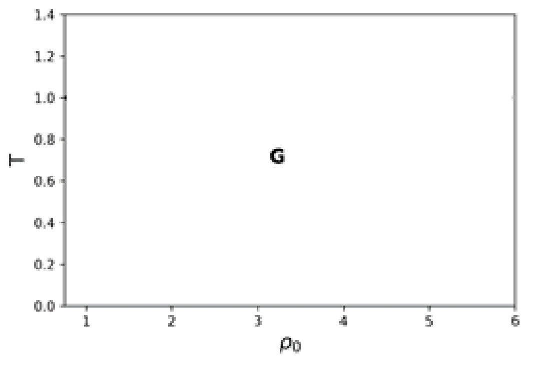
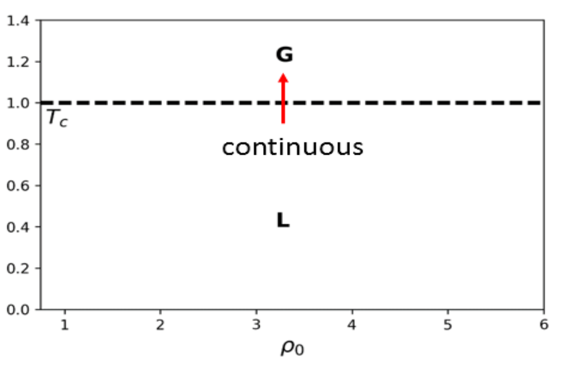
Full stochastic theory

Noise strength

	2-Body Interactions	3-body Interactions
Stochastic Theory		?
Mean Field Theory	Only Gas phase  	Gas, Liquid, no coexistence. 

density

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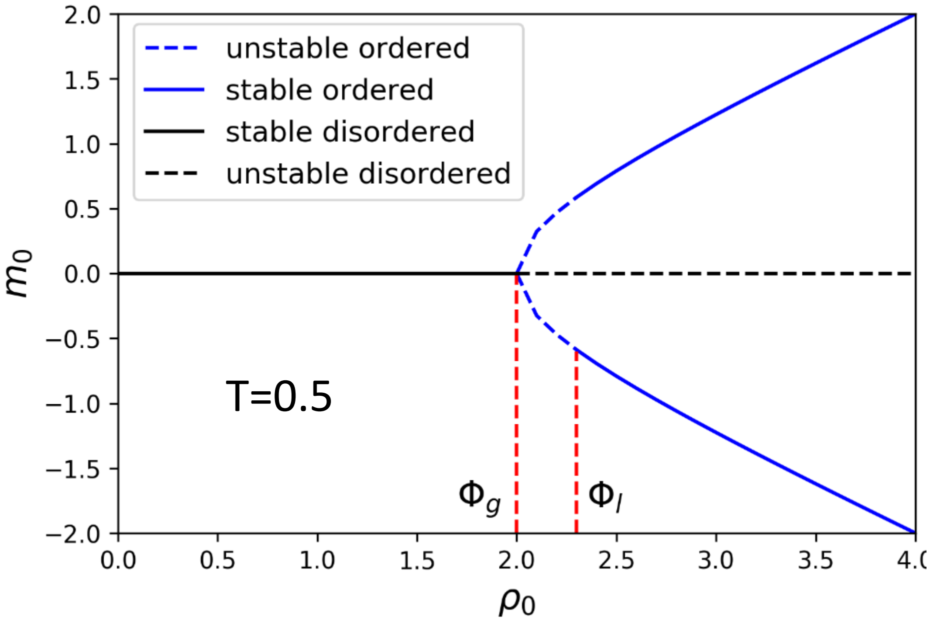
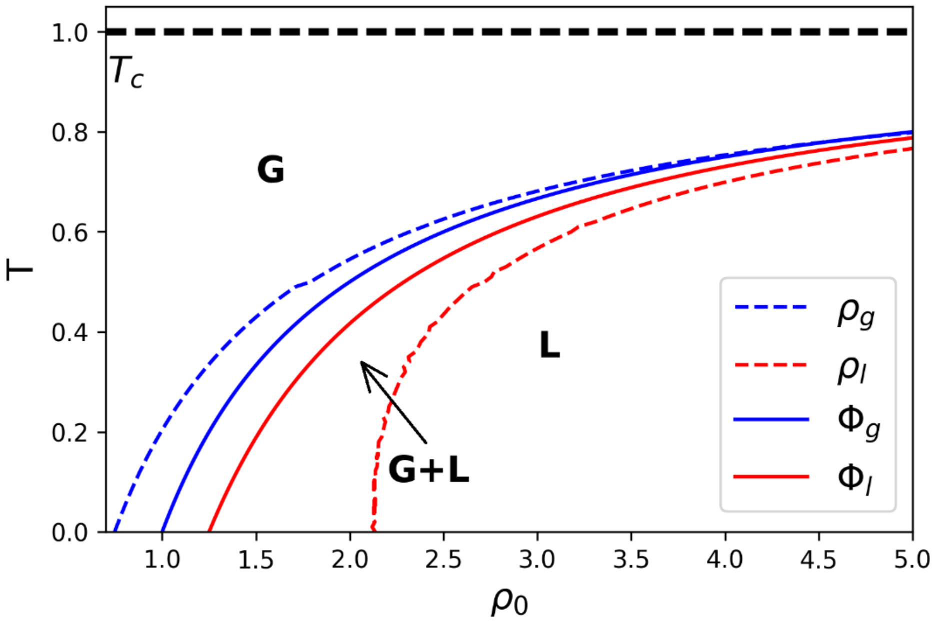
Weak Fluctuation Theory
Phase Diagrams



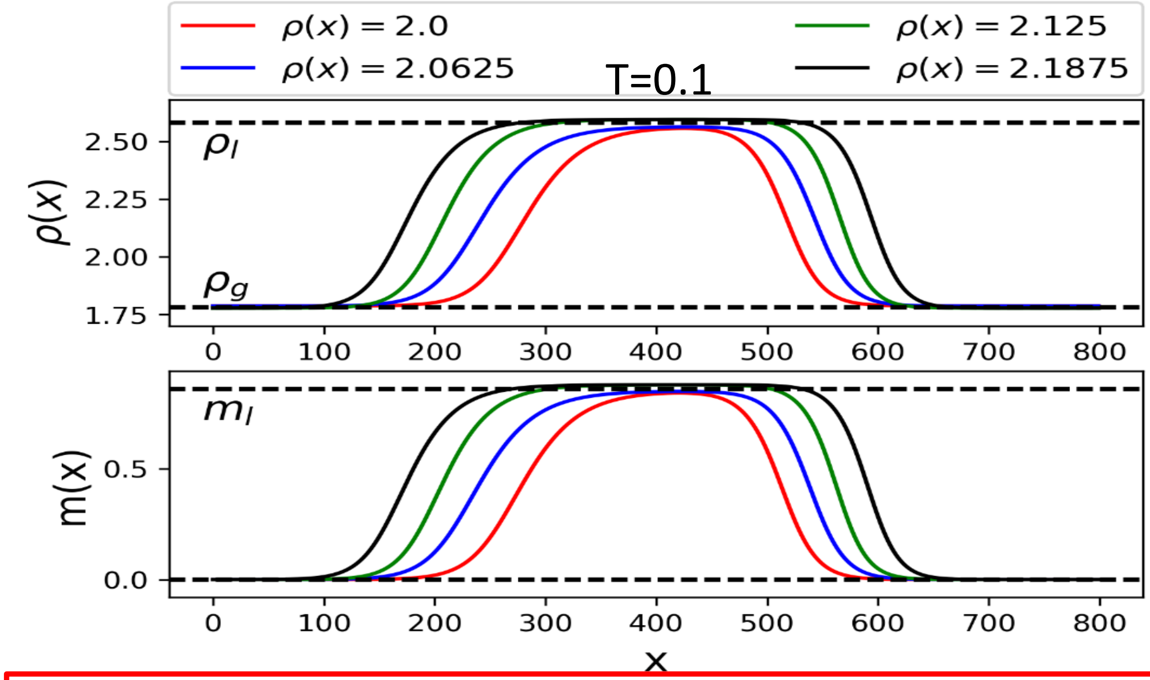


The Weak Fluctuation Theory recovers the phase coexistence regime.

WFT Simulation: Coexistence Phase

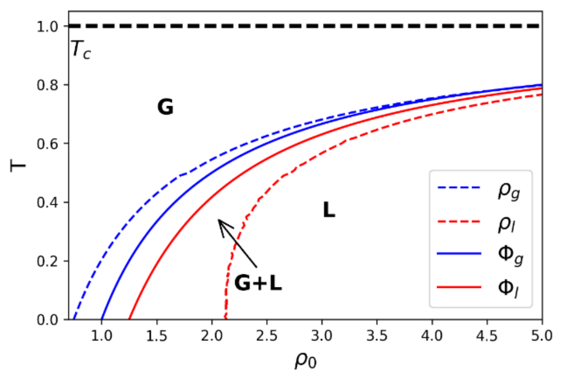
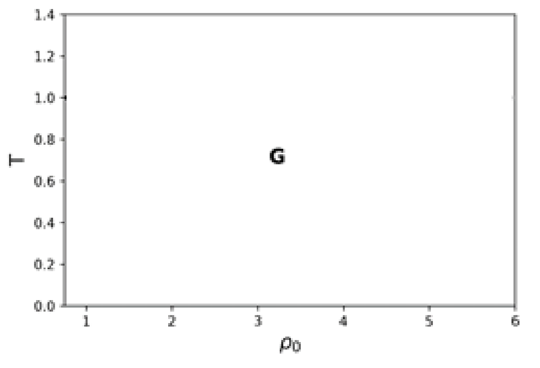
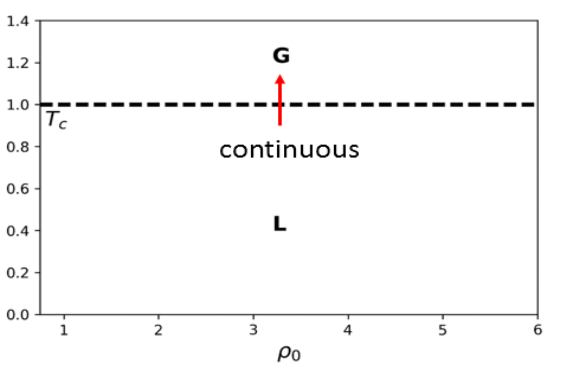
- In the coexistence phase, liquid fraction:

$$\phi = \frac{\rho_0 - \rho_g}{\rho_l - \rho_g}$$



Increasing density only increases the liquid fraction.

Noise strength

	2-Body Interactions	3-body Interactions
Stochastic Theory		Gas, liquid AND coexistence phase. 
Mean Field Theory	Only Gas phase  	Gas, liquid, no coexistence. 

density

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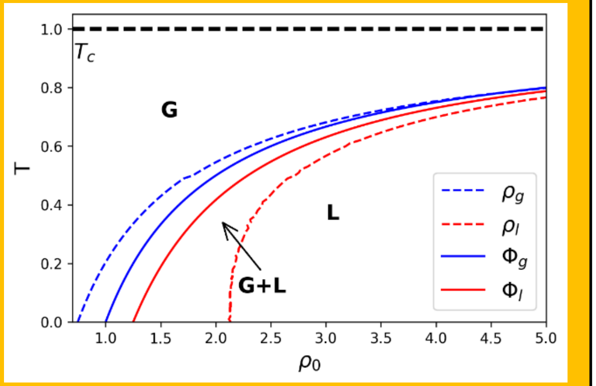
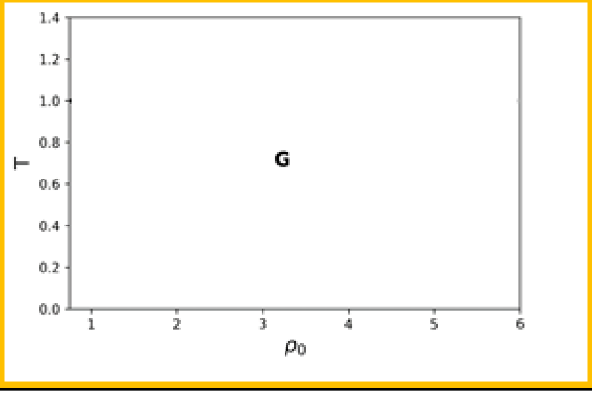
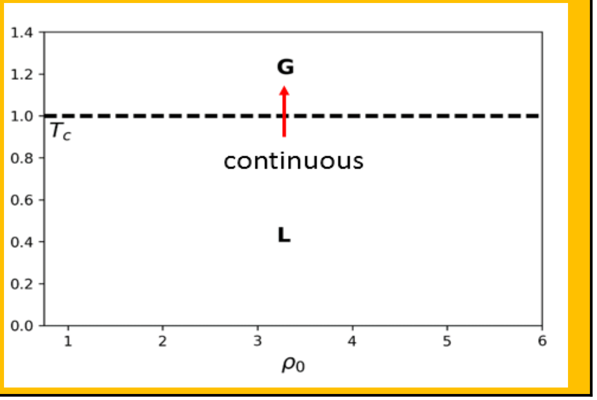
Noise strength

	2-Body Interactions	3-body Interactions
Stochastic Theory	?	Gas, liquid AND coexistence phase. 
Mean Field Theory	Only Gas phase 	Gas, liquid, no coexistence. 

density

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Absence of Three-body Interactions

- WFT:

$$\partial_t \rho = \partial_{xx}\rho - v\partial_x m,$$

$$\partial_t m = \partial_{xx} m - v\partial_x \rho - m\left[2\left(T - \frac{r_3}{4} + \frac{r}{\rho}\right) + \frac{r_3}{2}\frac{m^2}{\rho^2}\right]$$

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Only Gas phase when $r_3 = 0$

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$$\partial_t \rho = \partial_{xx}\rho - v\partial_x m,$$

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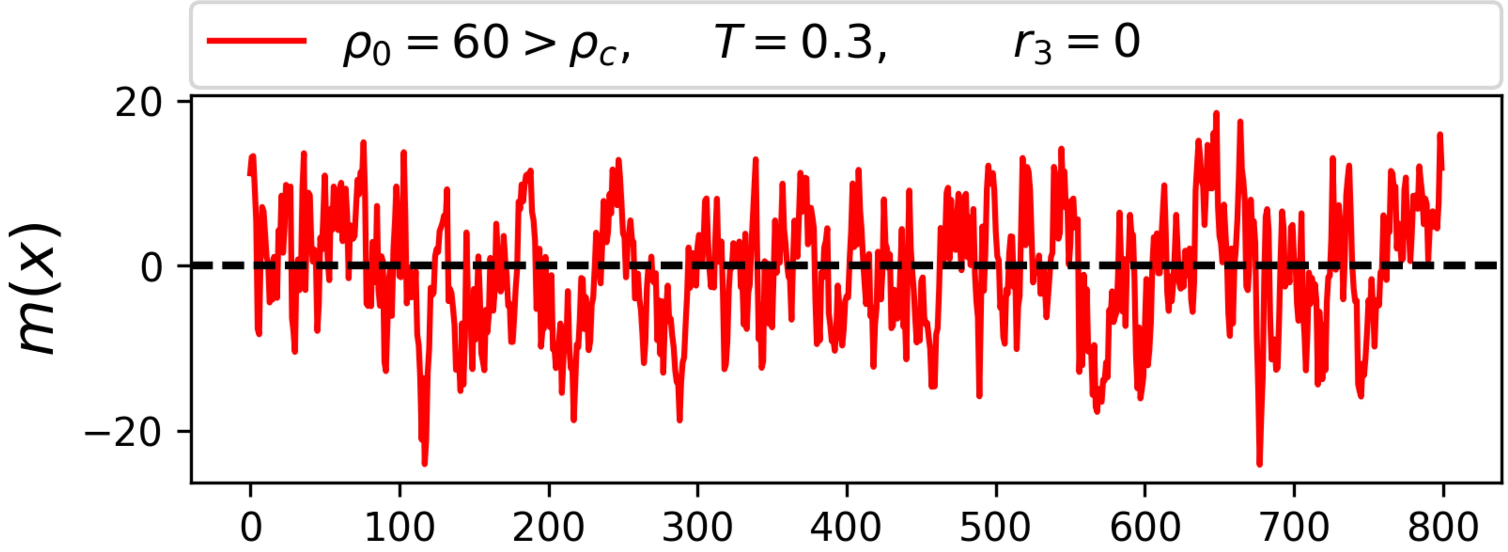
sPDE exhibits Gas phase and Switching phases.

sPDE Simulation: Absence of three-body interactions

- With $r_3 = 0$, no homogeneous ordered state
- For $\rho > \rho_c(T) = r_2/2T^2$: only Gas phase, $m_0 = 0$.



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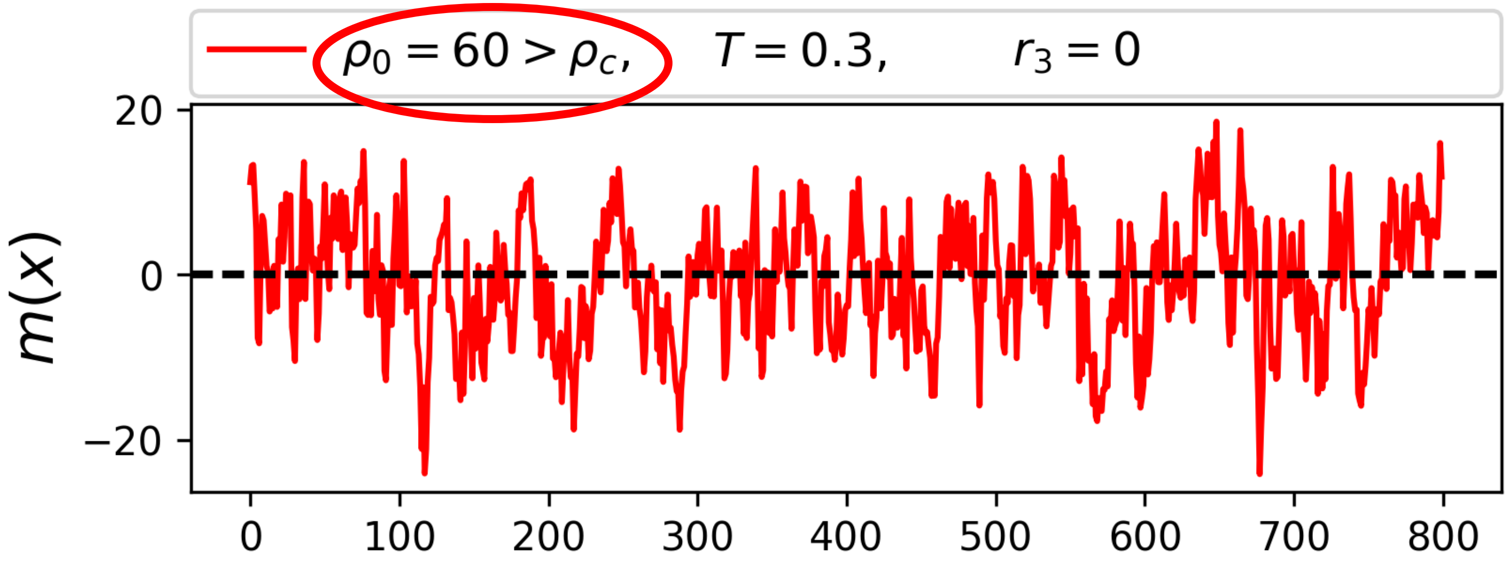

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- For $\rho < \rho_c = r_2/2T^2$, local switching of magnetization between $\pm\, m_{max}(x) = \rho(x)\sqrt{\frac{T+r_2/2}{r_2/2}}$

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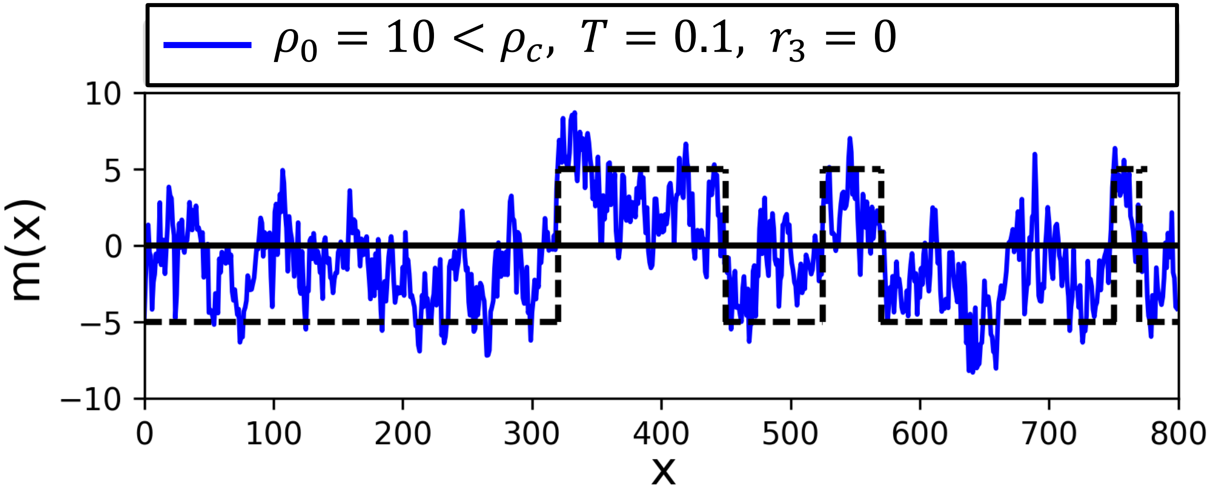

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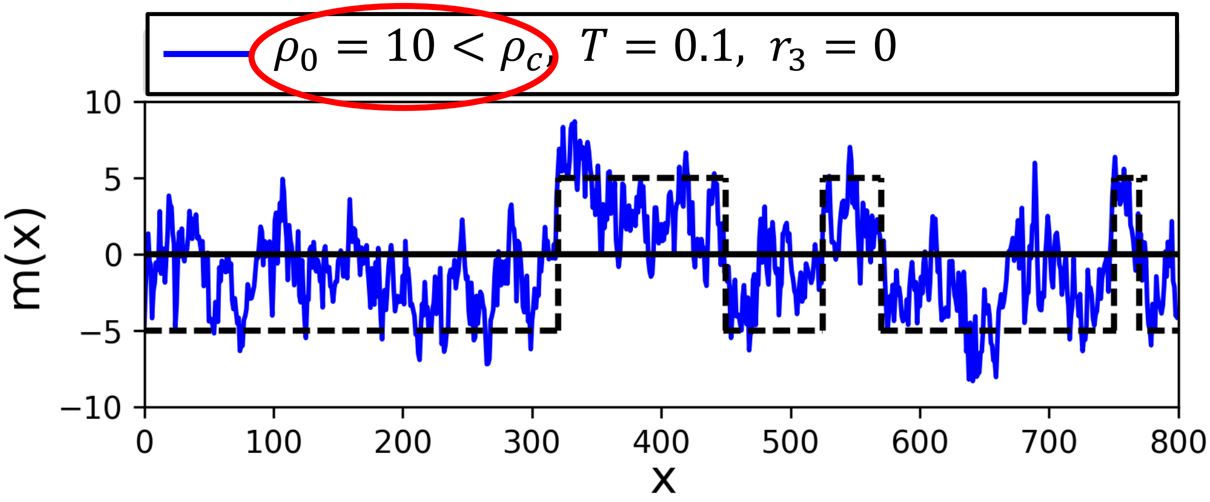


sPDE Simulation:
Absence of three-body interactions

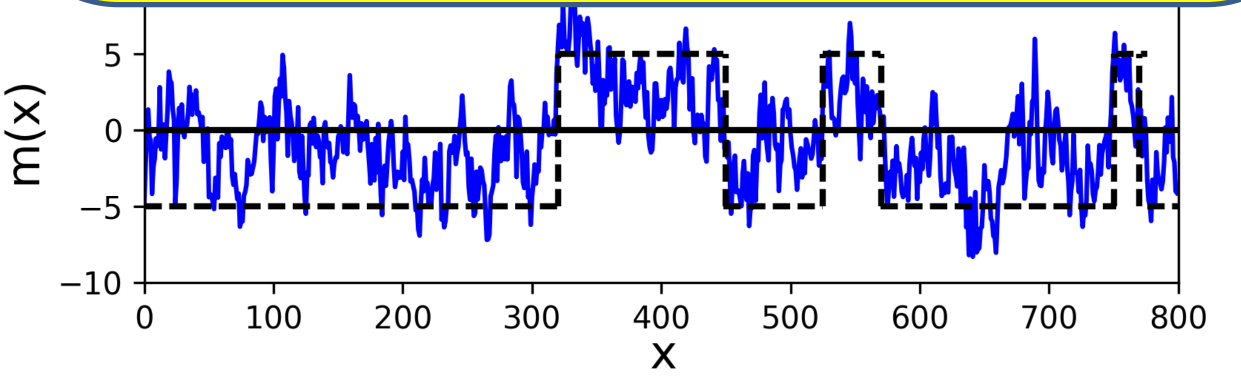
- With $r_3 =$
- sPDE:

$$\partial_t m = \partial_{xx} m \quad \cdots \quad {}^2 \Big) \eta$$

- For $\rho <$

$\pm\, m_{max}($

- No global order.
- Switching due to number fluctuations, which is a collective effect at low density.



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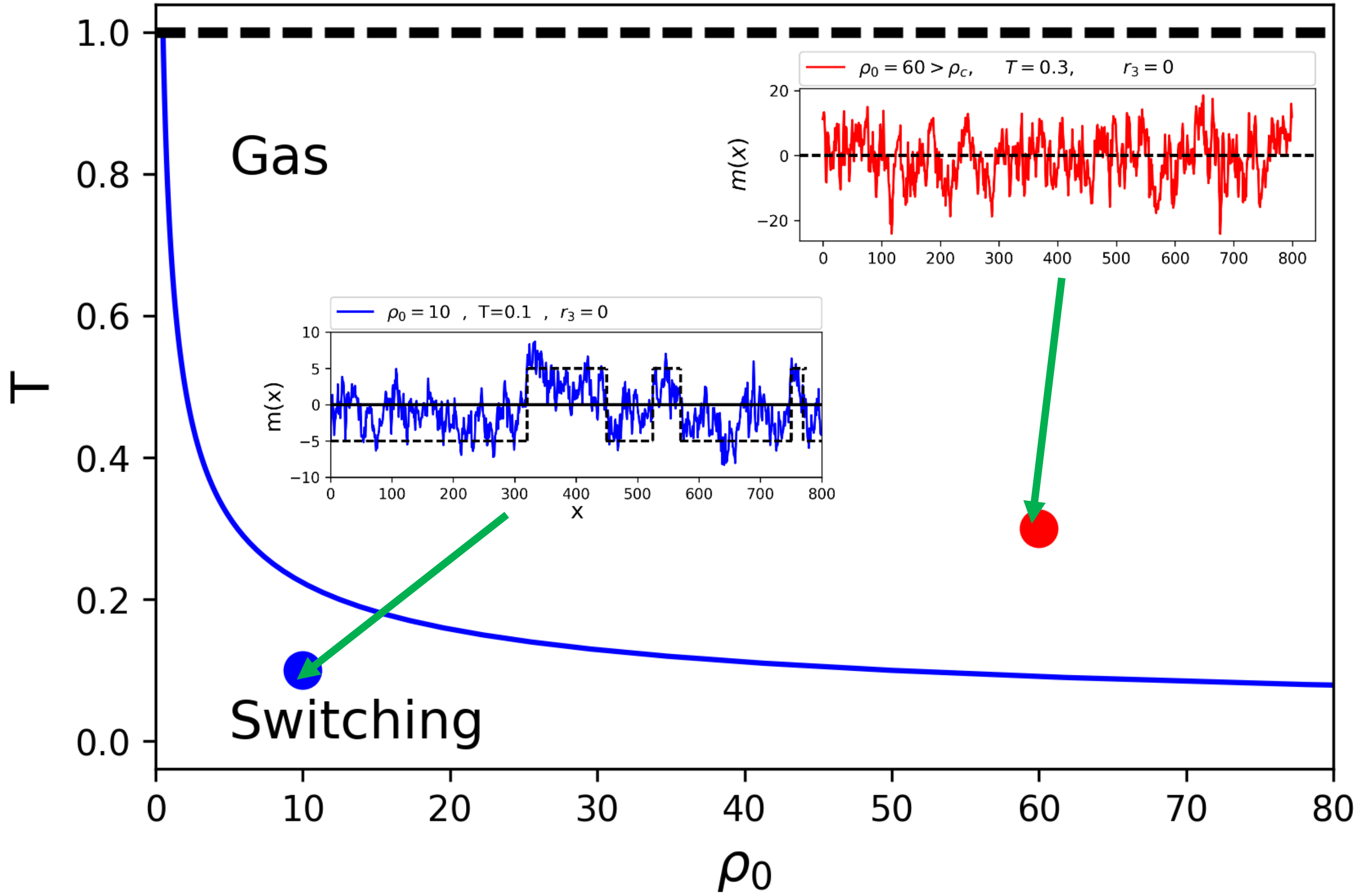
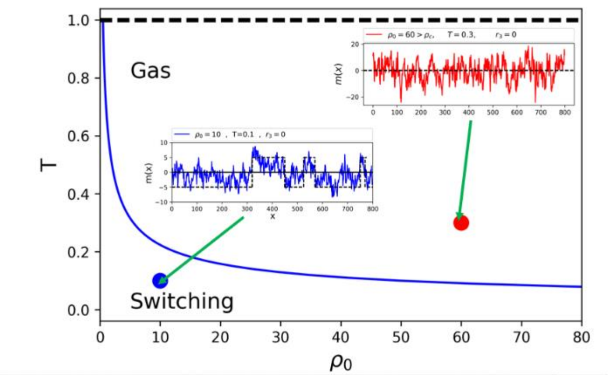
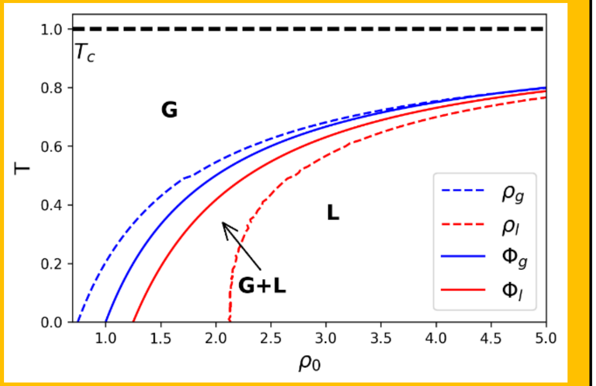
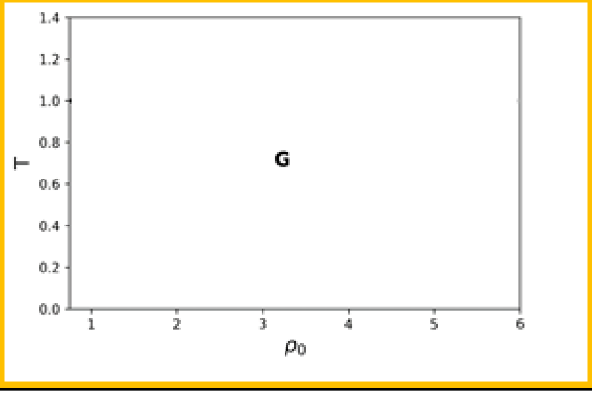
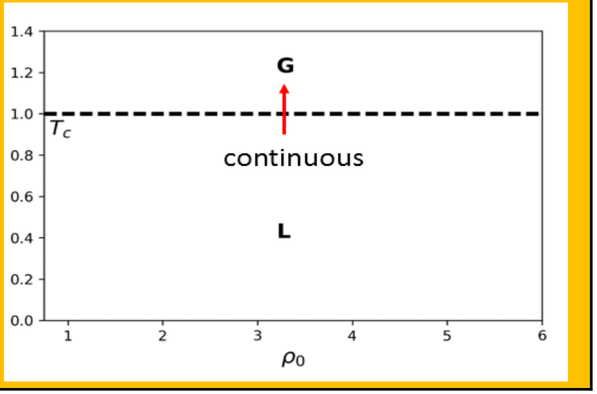


Noise strength

	2-Body Interactions	3-body Interactions
Stochastic Theory	Gas, switching phase at low densities-NO GLOBAL ORDER  	Gas, liquid AND coexistence phase. 
Mean Field Theory	Only Gas phase  	Gas, liquid, no coexistence. 

density

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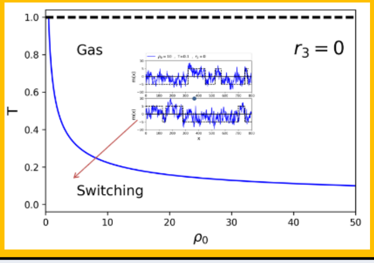
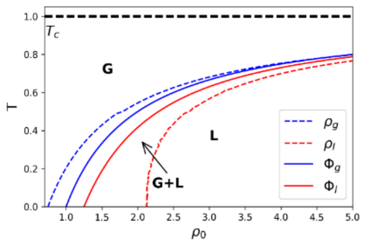
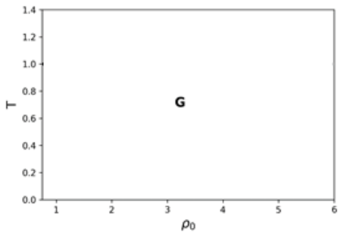
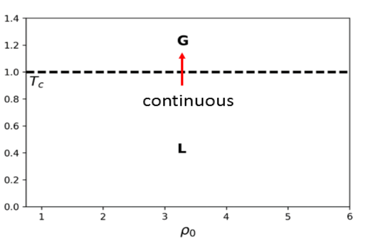
Noise strength

	2-Body Interactions	3-body Interactions
Stochastic Theory	Gas, switching phase at low densities-NO GLOBAL ORDER  	Gas, liquid AND coexistence phase. 
Gillespie Simulations of Individual Level Model (ILM)		?
Mean Field Theory	Only Gas phase  	Gas, liquid, no coexistence. 

density

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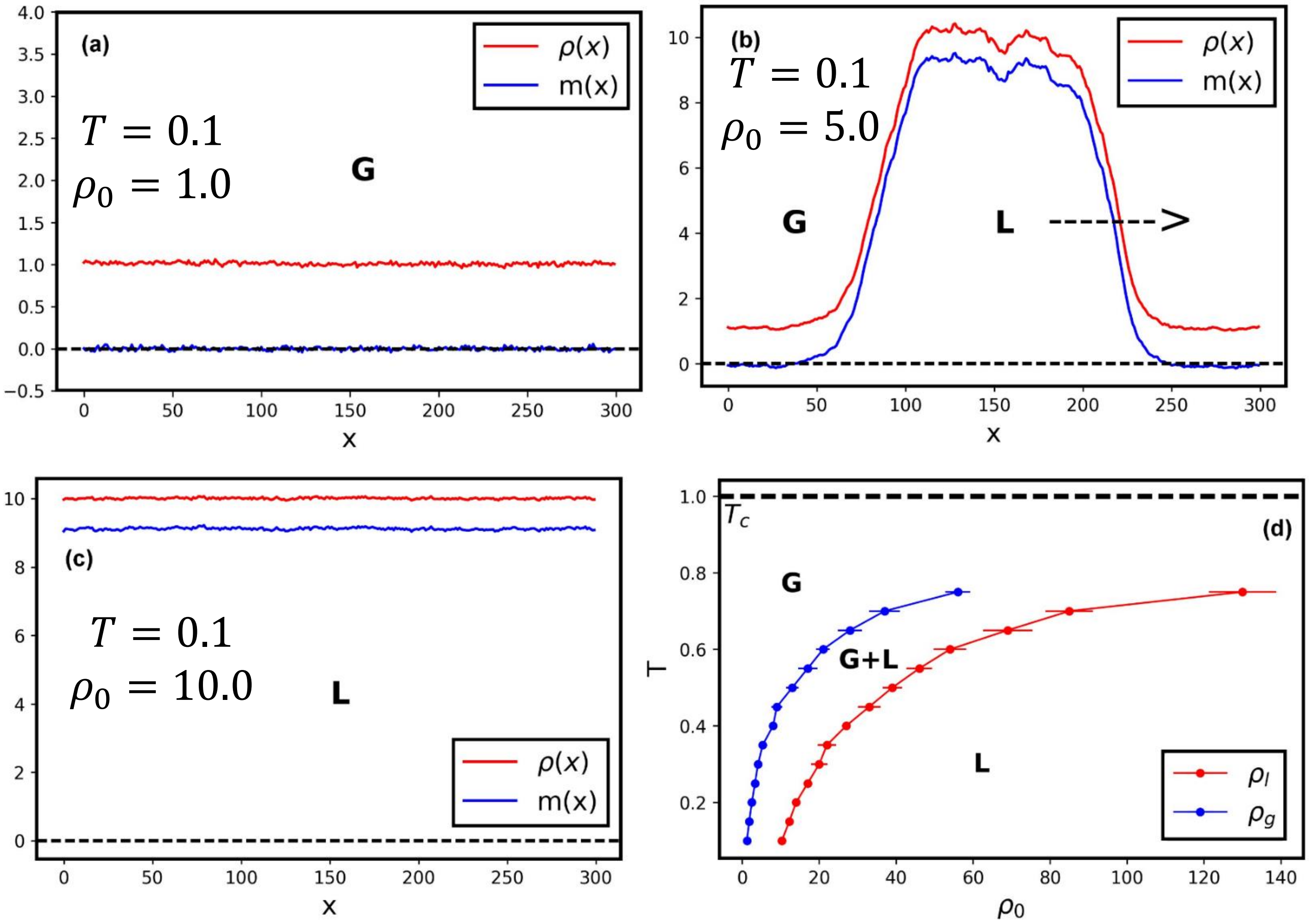
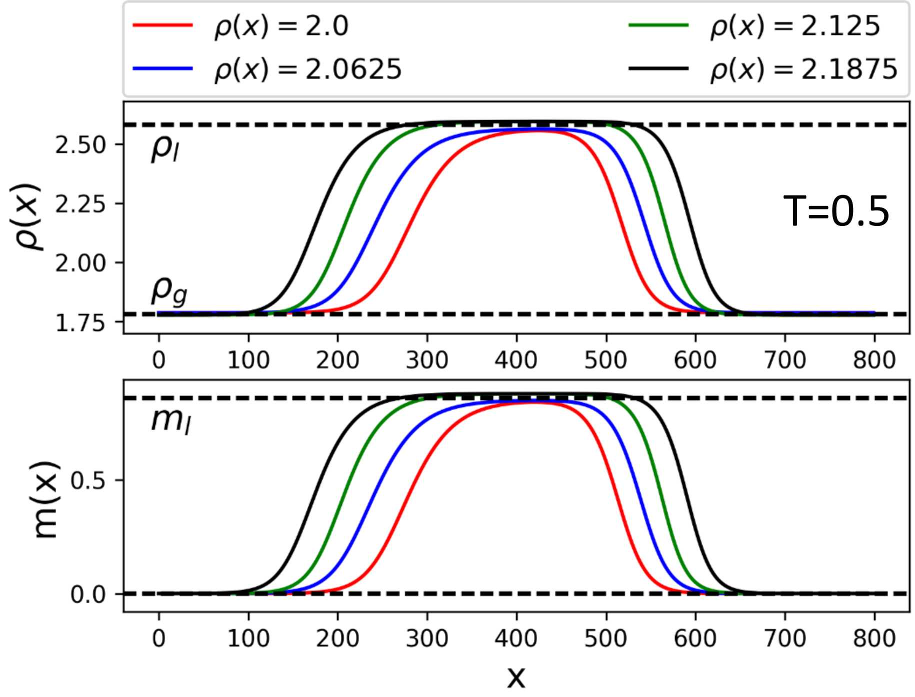
Results from Gillespie Simulations



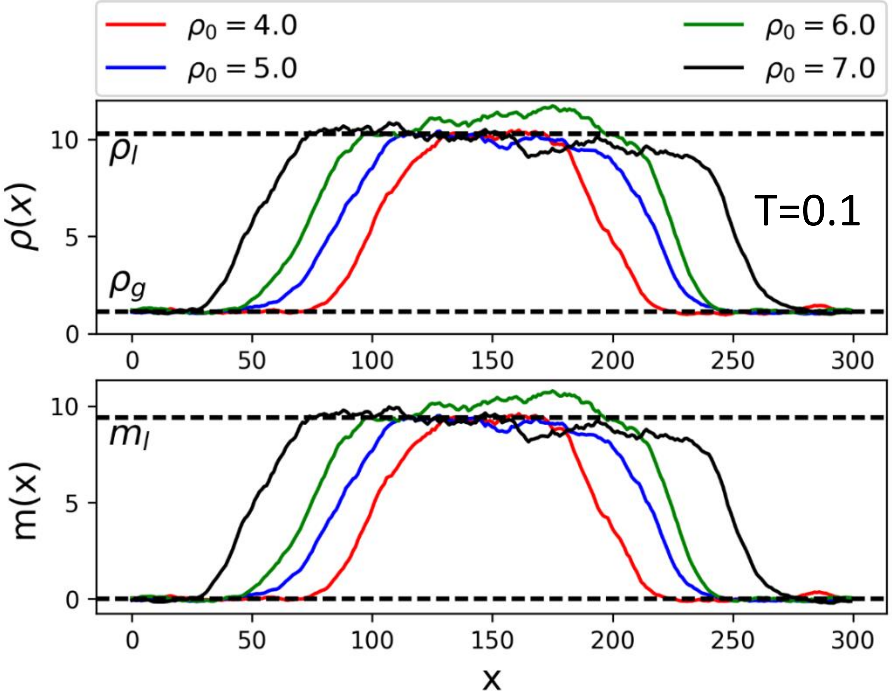
$$D = 1, \epsilon = 0.9, r_2 = 1, r_3 = 4, L = 300$$

Weak Fluctuation Theory vs Gillespie Simulation results

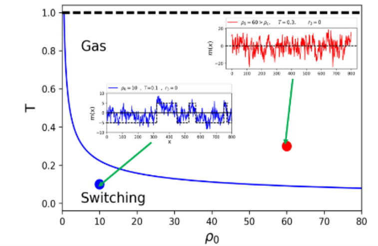
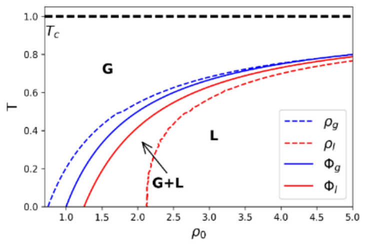
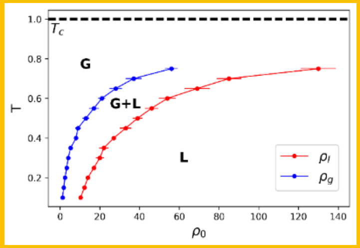
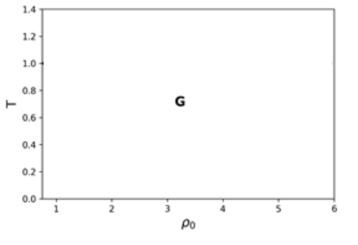
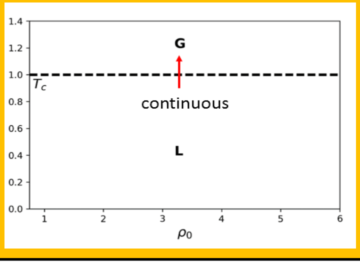
WFT



Simulation



- WFT and Simulation results agree qualitatively.
- Density is slaved to the magnetization and its fluctuations

	2-Body Interactions	3-body Interactions
Stochastic Theory	Gas, switching phase at low densities-NO GLOBAL ORDER  	Gas, liquid AND coexistence phase. 
Gillespie Simulations of Individual Level Model (ILM)		Gas, liquid AND coexistence phase 
Mean Field Theory	Only Gas phase  	Gas, liquid, no coexistence. 

Noise strength

density

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Noise strength

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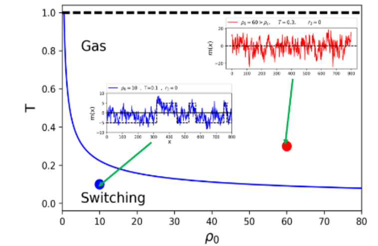
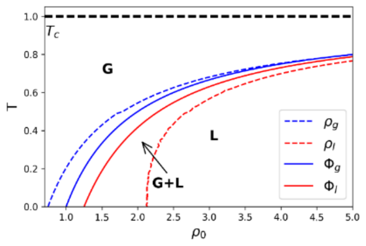
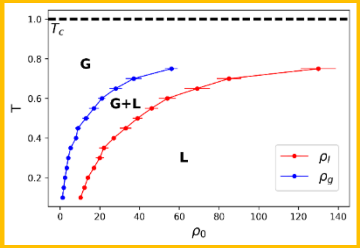
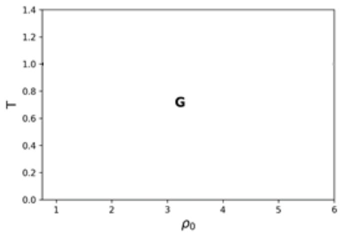
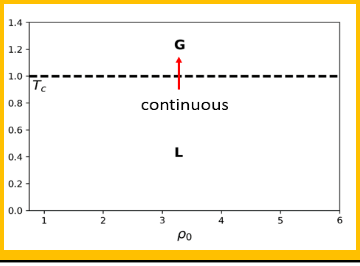
Stochastic Theory	Gas, switching phase at low densities-NO GLOBAL ORDER  	Gas, liquid AND coexistence phase. 
Gillespie Simulations of Individual Level Model (ILM)	?	Gas, liquid AND coexistence phase 
Mean Field Theory	Only Gas phase  	Gas, liquid, no coexistence. 

density

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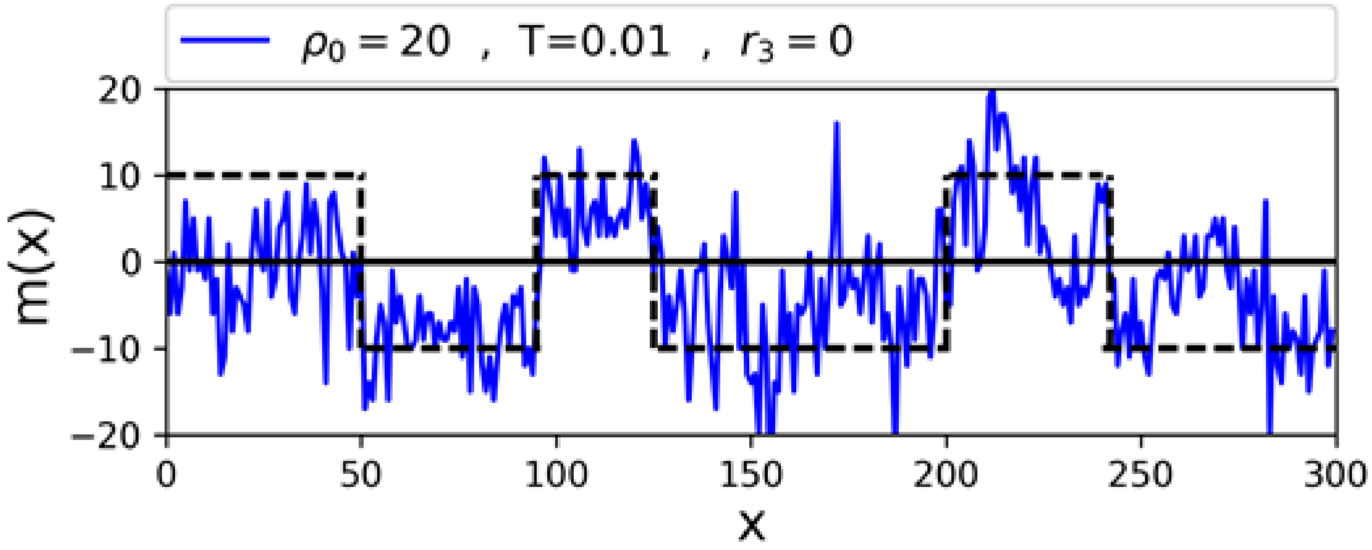
Gillespie Simulations:
Absence of Three-body Interactions

- In the absence of three-body interactions (r_3=0):

  no homogeneous ordered phase.

- Switching states for low densities.



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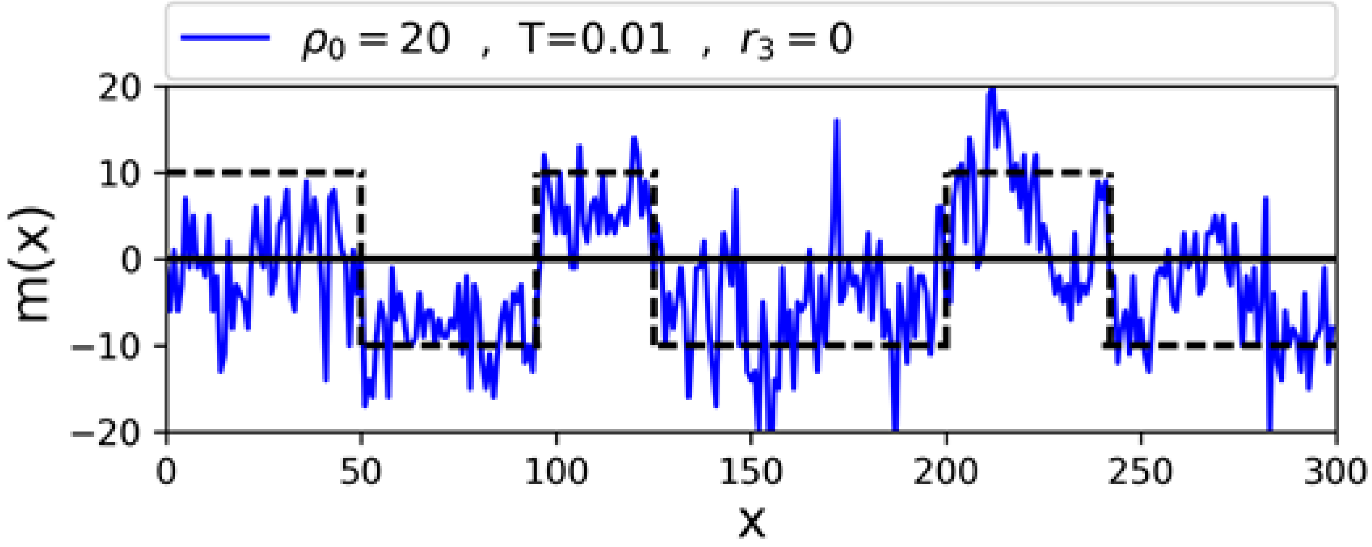
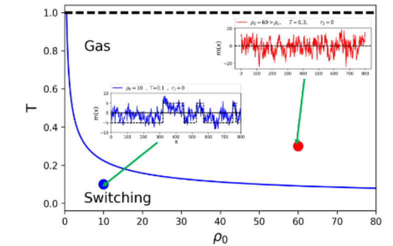
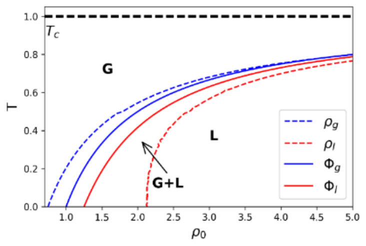
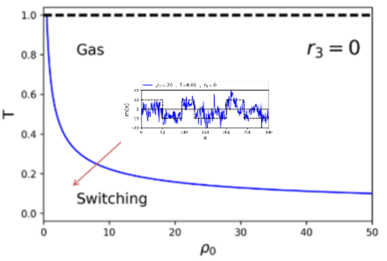
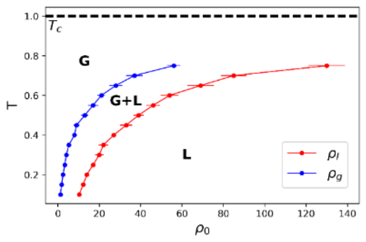
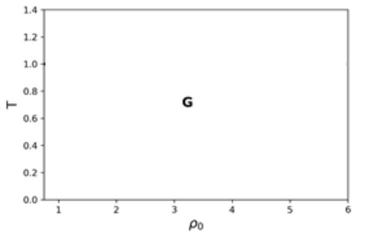
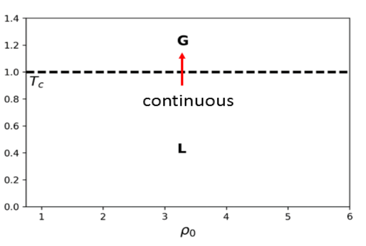


Gillespie simulations of the Individual Level Model exhibit switching states, like in the full sPDE

	2-Body Interactions	3-body Interactions
Stochastic Theory	Gas, switching phase at low densities-NO GLOBAL ORDER 	Gas, liquid AND coexistence phase. 
Gillespie Simulations of Individual Level Model (ILM)	Gas, switching phase at low densities-NO GLOBAL ORDER 	Gas, liquid AND coexistence phase 
Mean Field Theory	Only Gas phase 	Gas, liquid, no coexistence. 

Noise strength

density

H. Chaté et al, Phys. Rev. Lett. **114**, 068101 (2015)

Conclusion

- In the absence of three-body interactions there is no ordering transition.
- With three-body interactions, correct phase diagram for flocking is recovered.
- Local switching of magnetization at low densities, due to large relative number fluctuations.

Future Directions

- Generalising results to higher dimensions.
- Generalising to the case of continuous orientational parameter.

Reserve slides

Gillespie

$\bar{X} = \{X_1, X_2, \dots X_N\}$, Reactions R_μ, μ=1,2,…,M

$P(\tau, \mu)d\tau = a_\mu \exp(-a_0\tau)$

≡probability that the next reaction takes place between time t $+ \tau$ and time $t + \tau + d\tau$, and is a μ reaction, given state $\bar{X}$ at time t.

a_μ ≡probability per unit time that a μ reaction takes place.

$$a_0 = \sum_{\mu}^{M} a_\mu$$

$$\boxed{\begin{array}{lll} R_1: & X_1 \overset{c_1}{\rightarrow} X_2, & a_1 \;\; = c_1 X_1 \\ R_2: & X_1 + X_2 \overset{c_2}{\rightarrow} 2X_1, & a_2 = c_2 X_1 X_2 \\ R_3: & 2X_1 + X_2 \overset{c_3}{\rightarrow} 3X_1, & a_3 = \dfrac{c_3 X_1 (X_1 - 1) X_2}{2} \end{array}}$$

Algorithm:

1. Generate (τ, μ) according to $P(\tau, \mu)$ at each time t.
2. Update $\bar{X}$ to reflect R_μ having taken place.
3. Advance time by τ.
4. Repeat from step 1.

Generating (τ, μ):

1. Generate two uniformly distributed random numbers r_1, r_2.
2. Choose $\tau = \frac{1}{a_0} \ln\left(\frac{1}{r_1}\right)$
3. Choose μ: $\sum_{\nu=1}^{\mu-1} a_\nu < a_0 r_2 \leq \sum_{\nu=1}^{\mu} a_\nu$

WFT Steady States

$$\partial_t \rho = \partial_{xx}\rho - v\partial_x m,$$

$$\partial_t m = \partial_{xx} m - v\partial_x \rho - m\left[2\left(T - \frac{r_3}{4} + \frac{r}{\rho}\right) + \frac{r_3}{2}\frac{m^2}{\rho^2}\right]$$

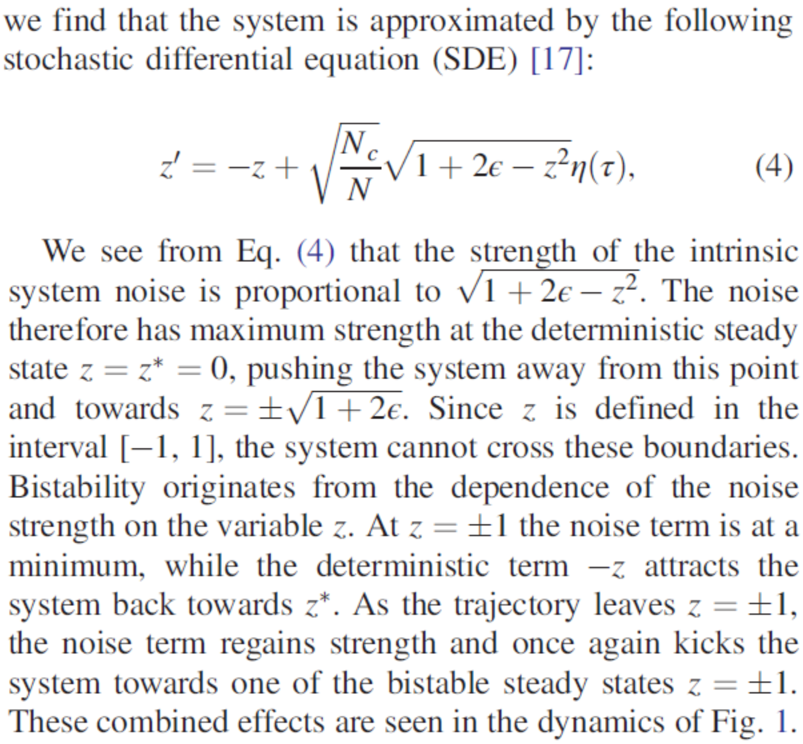
Homogeneous steady states:

- For $T > \frac{r_3}{4}$, homogeneous isotropic
  - $\rho = \rho_0, \qquad m = m_0 = 0$
- For $T < \frac{r_3}{4}$
  - For $\rho < \phi_g = 4r/(r_3 - 4T)$, homogeneous isotropic, $m_0 = 0$
  - For $\rho \in (\phi_g, \phi_l)$, coexistence : homogeneous ordered state exists but unstable ---->travelling fronts.

$$\phi_l = \phi_g \frac{v\sqrt{r\_3[v^2T + \left(\frac{D}{4}\right)(r_3 - 4T)^2]} + 2v^2T + Dr_3(r_3 - 4T)}{4v^{2T} + Dr\_3(r_3 - 4T)}$$

  - For $\rho > \phi_l$, homogeneous polar order, $m_0 = \pm\rho_0\sqrt{\frac{r_3 - 4T}{r_3}}$

Switching Phase



we find that the system is approximated by the following stochastic differential equation (SDE) [17]:

$$z' = -z + \sqrt{\frac{N_c}{N}}\sqrt{1 + 2\epsilon - z^2}\eta(\tau), \qquad (4)$$

We see from Eq. (4) that the strength of the intrinsic system noise is proportional to $\sqrt{1 + 2\epsilon - z^2}$. The noise therefore has maximum strength at the deterministic steady state $z = z^* = 0$, pushing the system away from this point and towards $z = \pm\sqrt{1 + 2\epsilon}$. Since z is defined in the interval $[-1, 1]$, the system cannot cross these boundaries. Bistability originates from the dependence of the noise strength on the variable z. At $z = \pm 1$ the noise term is at a minimum, while the deterministic term $-z$ attracts the system back towards z^*. As the trajectory leaves $z = \pm 1$, the noise term regains strength and once again kicks the system towards one of the bistable steady states $z = \pm 1$. These combined effects are seen in the dynamics of Fig. 1.

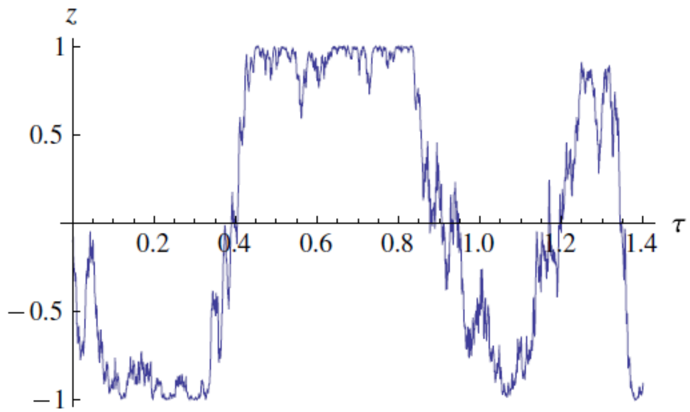
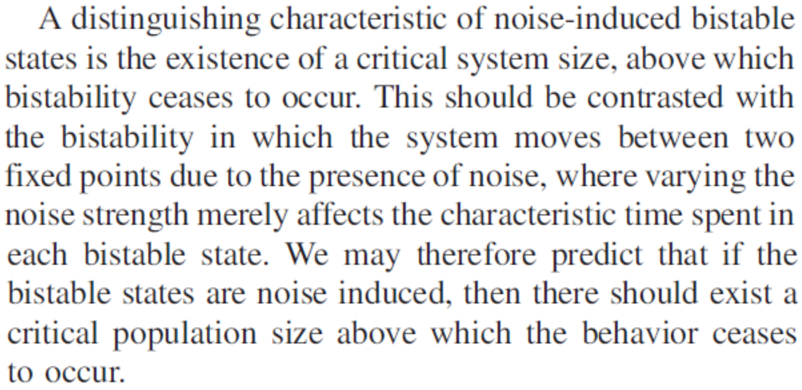


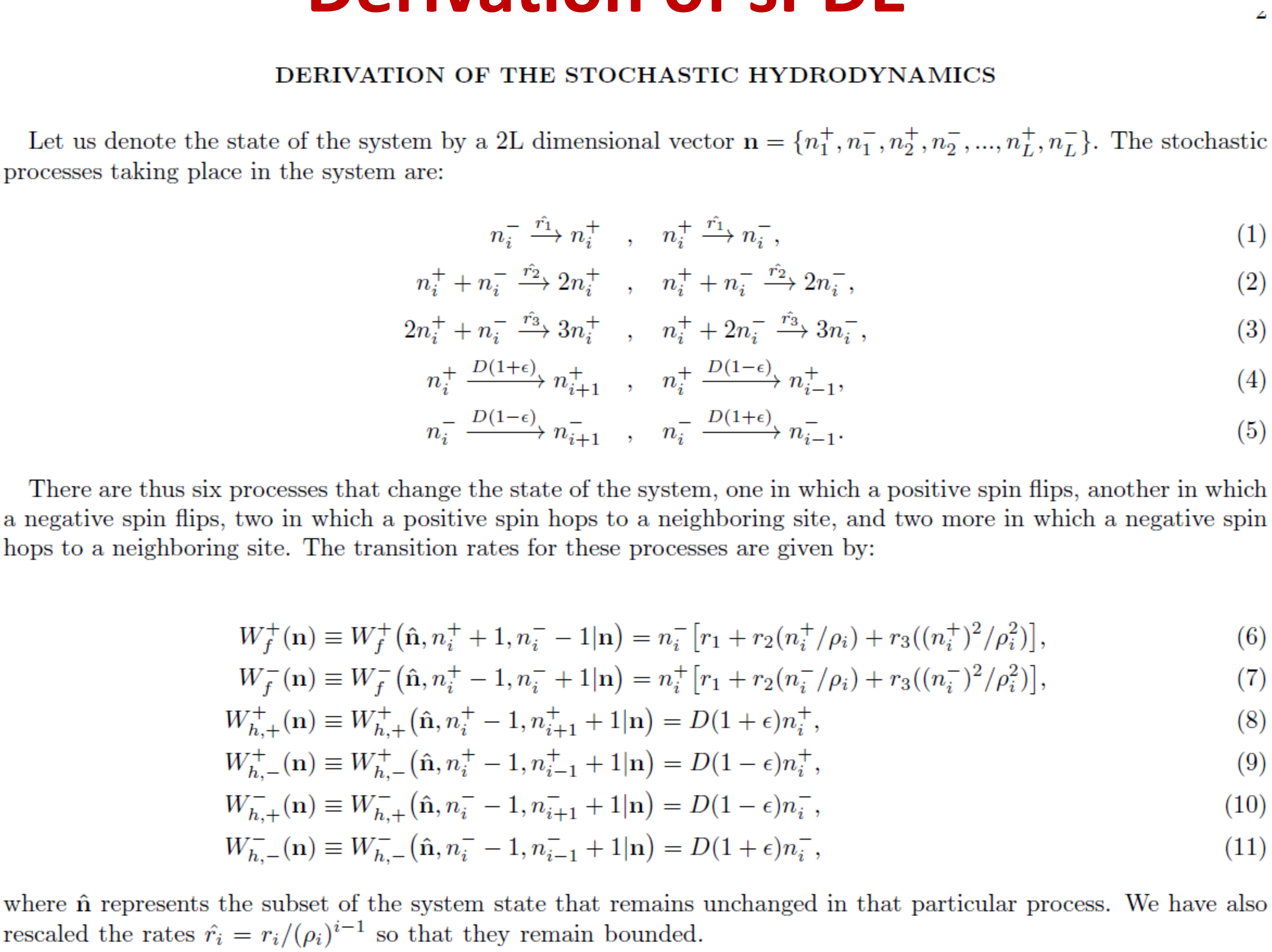
FIG. 1 (color online). Snapshot of the time series for z, obtained with stochastic simulations of the scheme of reactions (1). Parameter values are $\epsilon = 1/500$ and $N = 250$. Time is expressed in units of $\tau = 2\epsilon t/N$.



A distinguishing characteristic of noise-induced bistable states is the existence of a critical system size, above which bistability ceases to occur. This should be contrasted with the bistability in which the system moves between two fixed points due to the presence of noise, where varying the noise strength merely affects the characteristic time spent in each bistable state. We may therefore predict that if the bistable states are noise induced, then there should exist a critical population size above which the behavior ceases to occur.

Biancalani et al, Phys. Rev. Lett. **112**, 038101 (2014)

Derivation of sPDE



DERIVATION OF THE STOCHASTIC HYDRODYNAMICS

Let us denote the state of the system by a 2L dimensional vector $\mathbf{n} = \{n_1^+, n_1^-, n_2^+, n_2^-, ..., n_L^+, n_L^-\}$. The stochastic processes taking place in the system are:

$$n_i^- \xrightarrow{\hat{r_1}} n_i^+ \quad , \quad n_i^+ \xrightarrow{\hat{r_1}} n_i^- , \tag{1}$$

$$n_i^+ + n_i^- \xrightarrow{\hat{r_2}} 2n_i^+ \quad , \quad n_i^+ + n_i^- \xrightarrow{\hat{r_2}} 2n_i^- , \tag{2}$$

$$2n_i^+ + n_i^- \xrightarrow{\hat{r_3}} 3n_i^+ \quad , \quad n_i^+ + 2n_i^- \xrightarrow{\hat{r_3}} 3n_i^- , \tag{3}$$

$$n_i^+ \xrightarrow{D(1+\epsilon)} n_{i+1}^+ \quad , \quad n_i^+ \xrightarrow{D(1-\epsilon)} n_{i-1}^+ , \tag{4}$$

$$n_i^- \xrightarrow{D(1-\epsilon)} n_{i+1}^- \quad , \quad n_i^- \xrightarrow{D(1+\epsilon)} n_{i-1}^- . \tag{5}$$

There are thus six processes that change the state of the system, one in which a positive spin flips, another in which a negative spin flips, two in which a positive spin hops to a neighboring site, and two more in which a negative spin hops to a neighboring site. The transition rates for these processes are given by:

$$W_f^+(\mathbf{n}) \equiv W_f^+\left(\hat{\mathbf{n}}, n_i^+ + 1, n_i^- - 1 | \mathbf{n}\right) = n_i^- \left[r_1 + r_2(n_i^+/\rho_i) + r_3((n_i^+)^2/\rho_i^2)\right], \tag{6}$$

$$W_f^-(\mathbf{n}) \equiv W_f^-\left(\hat{\mathbf{n}}, n_i^+ - 1, n_i^- + 1 | \mathbf{n}\right) = n_i^+ \left[r_1 + r_2(n_i^-/\rho_i) + r_3((n_i^-)^2/\rho_i^2)\right], \tag{7}$$

$$W_{h,+}^+(\mathbf{n}) \equiv W_{h,+}^+\left(\hat{\mathbf{n}}, n_i^+ - 1, n_{i+1}^+ + 1 | \mathbf{n}\right) = D(1+\epsilon) n_i^+ , \tag{8}$$

$$W_{h,-}^+(\mathbf{n}) \equiv W_{h,-}^+\left(\hat{\mathbf{n}}, n_i^+ - 1, n_{i-1}^+ + 1 | \mathbf{n}\right) = D(1-\epsilon) n_i^+ , \tag{9}$$

$$W_{h,+}^-(\mathbf{n}) \equiv W_{h,+}^-\left(\hat{\mathbf{n}}, n_i^- - 1, n_{i+1}^- + 1 | \mathbf{n}\right) = D(1-\epsilon) n_i^- , \tag{10}$$

$$W_{h,-}^-(\mathbf{n}) \equiv W_{h,-}^-\left(\hat{\mathbf{n}}, n_i^- - 1, n_{i-1}^- + 1 | \mathbf{n}\right) = D(1+\epsilon) n_i^- , \tag{11}$$

where $\hat{\mathbf{n}}$ represents the subset of the system state that remains unchanged in that particular process. We have also rescaled the rates $\hat{r_i} = r_i/(\rho_i)^{i-1}$ so that they remain bounded.

Derivation of sPDE (contd.)

The master equation for the probability $P(\mathbf{n},t)$ is

$$\partial_t P(\mathbf{n},t) = \sum_{\mathbf{n}'} \Big\{ W(\mathbf{n}|\mathbf{n}')P(\mathbf{n}',t) - W(\mathbf{n}'|\mathbf{n})P(\mathbf{n},t) \Big\}. \tag{12}$$

Defining local creation and destruction operators:

$$a_i^{\pm} f(n_i^+) \equiv f(n_i^+ \pm 1) \quad , \quad b_i^{\pm} f(n_i^-) \equiv f(n_i^+ \pm 1), \tag{13}$$

we can then write down the master equation as

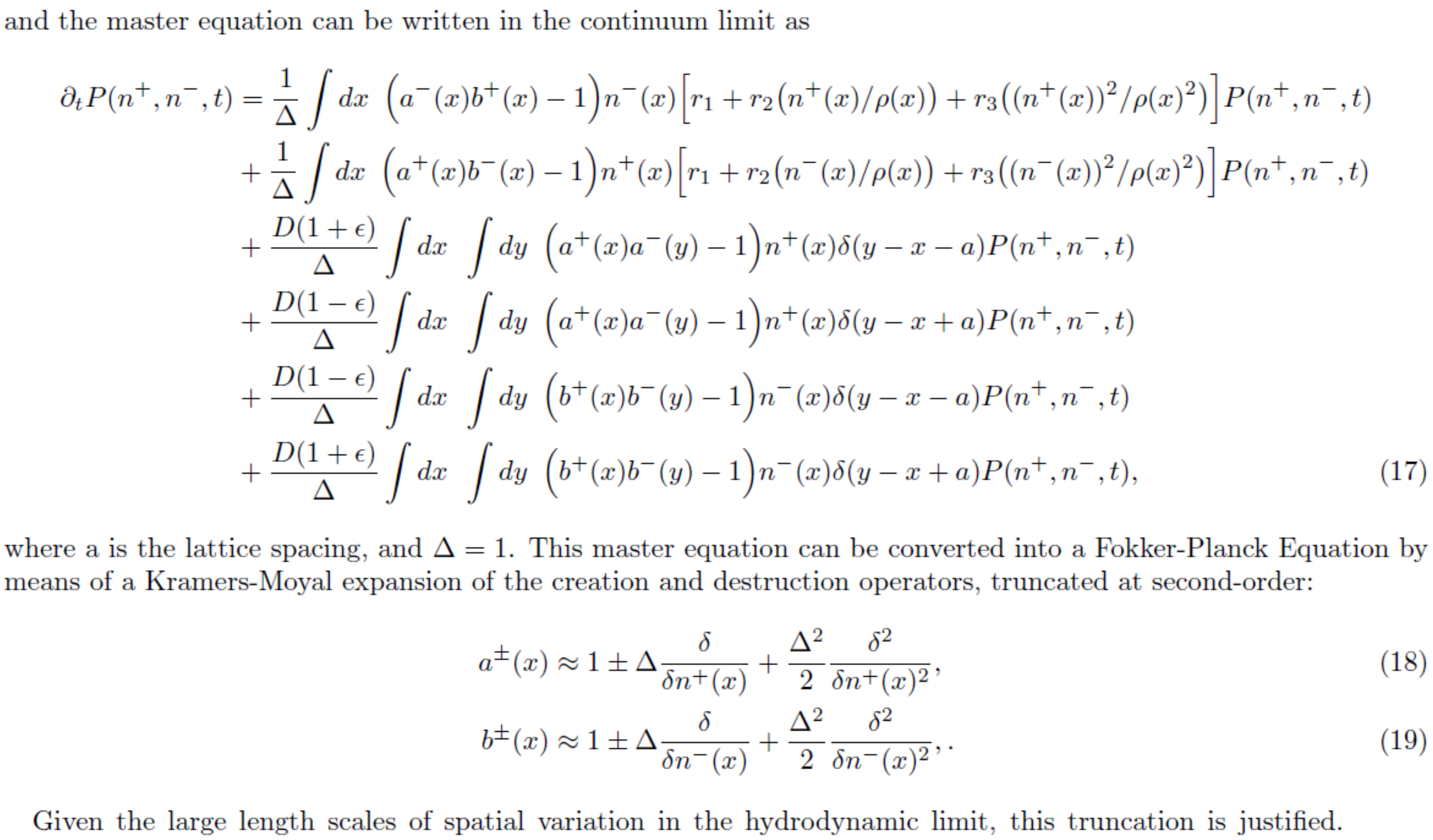
$$\begin{aligned}\partial_t P(\mathbf{n},t) = \sum_i \Big\{ &(a_i^- b_i^+ - 1) W_f^+(\mathbf{n}) + (a_i^+ b_i^- - 1) W_f^-(\mathbf{n}) \\ &+ (a_i^+ a_{i+1}^- - 1) W_{h,+}^+(\mathbf{n}) + (a_i^+ a_{i-1}^- - 1) W_{h,-}^+(\mathbf{n}) + (b_i^+ b_{i+1}^- - 1) W_{h,+}^-(\mathbf{n}) + (b_i^+ b_{i-1}^- - 1) W_{h,-}^-(\mathbf{n}) \Big\} P(\mathbf{n},t),\end{aligned} \tag{14}$$

$$\begin{aligned}= \sum_i \Big\{ &(a_i^- b_i^+ - 1) n_i^- \left[r_1 + r_2 (n_i^+/\rho_i) + r_3 ((n_i^+)^2/\rho_i^2) \right] \\ &+ (a_i^+ b_i^- - 1) n_i^+ \left[r_1 + r_2 (n_i^-/\rho_i) + r_3 ((n_i^-)^2/\rho_i^2) \right] \\ &+ (a_i^+ a_{i+1}^- - 1) n_i^+ D(1+\epsilon) + (a_i^+ a_{i-1}^- - 1) n_i^+ D(1-\epsilon) \\ &+ (b_i^+ b_{i+1}^- - 1) n_i^- D(1-\epsilon) + (b_i^+ b_{i-1}^- - 1) n_i^- D(1+\epsilon) \Big\} P(\mathbf{n},t).\end{aligned} \tag{15}$$

In the continuum limit, $i \to x$ and $f_i \to f(x)$. The creation and destruction operators then become

$$a^{\pm}(y) f[n^+(x)] \equiv f[n^+(x) \pm \Delta\delta(y-x)] \quad , \quad b^{\pm}(y) f[n^-(x)] \equiv f[n^-(x) \pm \Delta\delta(y-x)], \tag{16}$$

Derivation of sPDE (contd.)



and the master equation can be written in the continuum limit as

$$\begin{aligned}
\partial_t P(n^+,n^-,t) = {} & \frac{1}{\Delta}\int dx\ \Big(a^-(x)b^+(x)-1\Big)n^-(x)\Big[r_1 + r_2\big(n^+(x)/\rho(x)\big) + r_3\big((n^+(x))^2/\rho(x)^2\big)\Big]P(n^+,n^-,t) \\
& + \frac{1}{\Delta}\int dx\ \Big(a^+(x)b^-(x)-1\Big)n^+(x)\Big[r_1 + r_2\big(n^-(x)/\rho(x)\big) + r_3\big((n^-(x))^2/\rho(x)^2\big)\Big]P(n^+,n^-,t) \\
& + \frac{D(1+\epsilon)}{\Delta}\int dx\ \int dy\ \Big(a^+(x)a^-(y)-1\Big)n^+(x)\delta(y-x-a)P(n^+,n^-,t) \\
& + \frac{D(1-\epsilon)}{\Delta}\int dx\ \int dy\ \Big(a^+(x)a^-(y)-1\Big)n^+(x)\delta(y-x+a)P(n^+,n^-,t) \\
& + \frac{D(1-\epsilon)}{\Delta}\int dx\ \int dy\ \Big(b^+(x)b^-(y)-1\Big)n^-(x)\delta(y-x-a)P(n^+,n^-,t) \\
& + \frac{D(1+\epsilon)}{\Delta}\int dx\ \int dy\ \Big(b^+(x)b^-(y)-1\Big)n^-(x)\delta(y-x+a)P(n^+,n^-,t),
\end{aligned} \tag{17}$$

where a is the lattice spacing, and $\Delta = 1$. This master equation can be converted into a Fokker-Planck Equation by means of a Kramers-Moyal expansion of the creation and destruction operators, truncated at second-order:

$$a^\pm(x) \approx 1 \pm \Delta\frac{\delta}{\delta n^+(x)} + \frac{\Delta^2}{2}\frac{\delta^2}{\delta n^+(x)^2}, \tag{18}$$

$$b^\pm(x) \approx 1 \pm \Delta\frac{\delta}{\delta n^-(x)} + \frac{\Delta^2}{2}\frac{\delta^2}{\delta n^-(x)^2},. \tag{19}$$

Given the large length scales of spatial variation in the hydrodynamic limit, this truncation is justified.

Derivation of sPDE (contd.)

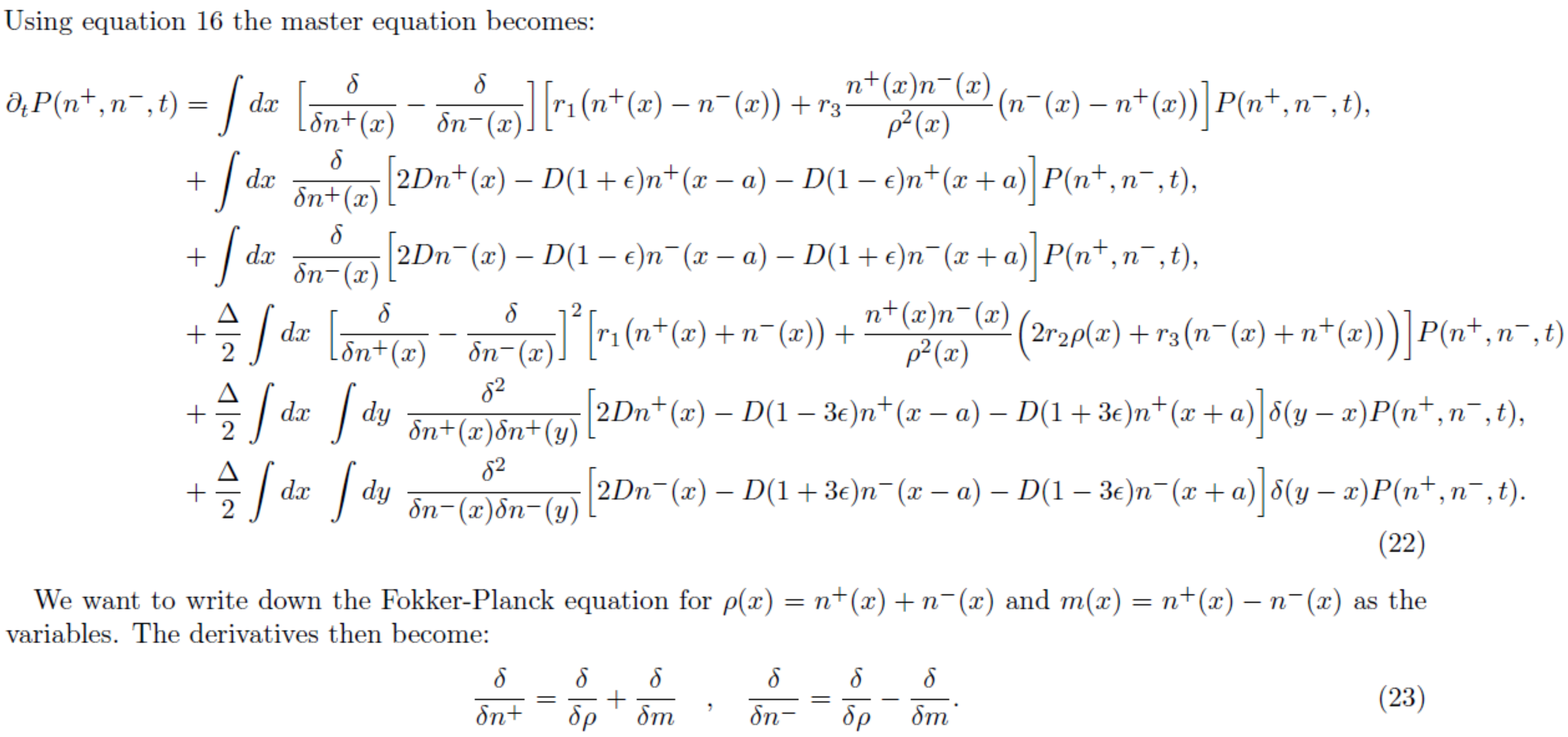
the master equation can be written as:

$$
\begin{aligned}
\partial_t P(n^+,n^-,t) = \int dx\ & \left[\frac{\delta}{\delta n^+(x)} - \frac{\delta}{\delta n^-(x)}\right]\left[r_1\left(n^+(x) - n^-(x)\right) + r_3\frac{n^+(x)n^-(x)}{\rho^2(x)}\left(n^-(x) - n^+(x)\right)\right]P(n^+,n^-,t),\\
+ \int dx\ & \frac{\delta}{\delta n^+(x)}\Big[2Dn^+(x) - D(1+\epsilon)n^+(x-a) - D(1-\epsilon)n^+(x+a)\Big]P(n^+,n^-,t),\\
+ \int dx\ & \frac{\delta}{\delta n^-(x)}\Big[2Dn^-(x) - D(1-\epsilon)n^-(x-a) - D(1+\epsilon)n^-(x+a)\Big]P(n^+,n^-,t),\\
+ \frac{\Delta}{2}\int dx\ & \left[\frac{\delta}{\delta n^+(x)} - \frac{\delta}{\delta n^-(x)}\right]^2\left[r_1\left(n^+(x) + n^-(x)\right) + \frac{n^+(x)n^-(x)}{\rho^2(x)}\Big(2r_2\rho(x) + r_3\left(n^-(x) + n^+(x)\right)\Big)\right]P(n^+,n^-,t)\\
+ \frac{\Delta}{2}\int dx\ & \int dy\ \frac{\delta^2}{\delta n^+(x)\delta n^+(y)}\Big[2Dn^+(x) + D(1+\epsilon)n^+(x-a) + D(1-\epsilon)n^+(x+a)\Big]\delta(y-x)P(n^+,n^-,t),\\
+ \frac{\Delta}{2}\int dx\ & \int dy\ \frac{\delta^2}{\delta n^-(x)\delta n^-(y)}\Big[2Dn^-(x) + D(1-\epsilon)n^-(x-a) + D(1+\epsilon)n^-(x+a)\Big]\delta(y-x)P(n^+,n^-,t),\\
- \frac{\Delta}{2}\int dx\ & \int dy\ \frac{\delta^2}{\delta n^+(x)\delta n^+(y)}\Big[2D(1+\epsilon)n^+(x)\delta(y-x-a) + 2D(1-\epsilon)n^+(x)\delta(y-x+a)\Big]P(n^+,n^-,t),\\
- \frac{\Delta}{2}\int dx\ & \int dy\ \frac{\delta^2}{\delta n^-(x)\delta n^-(y)}\Big[2D(1-\epsilon)n^-(x)\delta(y-x-a) + 2D(1+\epsilon)n^-(x)\delta(y-x+a)\Big]P(n^+,n^-,t).
\end{aligned}
\tag{20}
$$

In the last two integrals, we can expand the $\delta(y - x \pm a)$ under the integral sign, to get:

$$
\begin{aligned}
n^\pm(x)\delta(y-x\pm a) &\approx n^\pm(x)\Big\{\delta(y-x) \pm a\delta^{'}(y-x) + \frac{a^2}{2}\delta^{''}(y-x)\Big\},\\
&= \delta(y-x)\Big\{n^\pm(x) \mp a\partial_x n^\pm(x) + \frac{a^2}{2}\partial_{xx}n^\pm(x)\Big\},\\
&= n^\pm(x\mp a)\delta(y-x).
\end{aligned}
\tag{21}
$$

Derivation of sPDE (contd.)



Using equation 16 the master equation becomes:

$$
\begin{aligned}
\partial_t P(n^+,n^-,t) = &\int dx\ \left[\frac{\delta}{\delta n^+(x)} - \frac{\delta}{\delta n^-(x)}\right]\Big[r_1\big(n^+(x)-n^-(x)\big) + r_3\frac{n^+(x)n^-(x)}{\rho^2(x)}\big(n^-(x)-n^+(x)\big)\Big]P(n^+,n^-,t),\\
&+\int dx\ \frac{\delta}{\delta n^+(x)}\Big[2Dn^+(x) - D(1+\epsilon)n^+(x-a) - D(1-\epsilon)n^+(x+a)\Big]P(n^+,n^-,t),\\
&+\int dx\ \frac{\delta}{\delta n^-(x)}\Big[2Dn^-(x) - D(1-\epsilon)n^-(x-a) - D(1+\epsilon)n^-(x+a)\Big]P(n^+,n^-,t),\\
&+\frac{\Delta}{2}\int dx\ \left[\frac{\delta}{\delta n^+(x)} - \frac{\delta}{\delta n^-(x)}\right]^2\Big[r_1\big(n^+(x)+n^-(x)\big) + \frac{n^+(x)n^-(x)}{\rho^2(x)}\Big(2r_2\rho(x) + r_3\big(n^-(x)+n^+(x)\big)\Big)\Big]P(n^+,n^-,t)\\
&+\frac{\Delta}{2}\int dx\int dy\ \frac{\delta^2}{\delta n^+(x)\delta n^+(y)}\Big[2Dn^+(x) - D(1-3\epsilon)n^+(x-a) - D(1+3\epsilon)n^+(x+a)\Big]\delta(y-x)P(n^+,n^-,t),\\
&+\frac{\Delta}{2}\int dx\int dy\ \frac{\delta^2}{\delta n^-(x)\delta n^-(y)}\Big[2Dn^-(x) - D(1+3\epsilon)n^-(x-a) - D(1-3\epsilon)n^-(x+a)\Big]\delta(y-x)P(n^+,n^-,t).
\end{aligned}
\tag{22}
$$

We want to write down the Fokker-Planck equation for $\rho(x) = n^+(x) + n^-(x)$ and $m(x) = n^+(x) - n^-(x)$ as the variables. The derivatives then become:

$$
\frac{\delta}{\delta n^+} = \frac{\delta}{\delta \rho} + \frac{\delta}{\delta m}\quad,\quad \frac{\delta}{\delta n^-} = \frac{\delta}{\delta \rho} - \frac{\delta}{\delta m}. \tag{23}
$$

Derivation of sPDE (contd.)

Finally, after expanding $n^{\pm}(x \pm a)$ around x,

$$n^{\pm}(x \pm a) \approx n^{\pm}(x) \pm a\partial_x n^{\pm}(x) + \frac{a^2}{2}\partial_{xx} n^{\pm}(x), \tag{24}$$

we obtain the FPE for the probability distribution $P(\rho, m, t)$:

$$\begin{aligned} \partial_t P(\rho, m, t) = &- \int dx \; \left\{ \frac{\delta}{\delta\rho(x)}\Big[\mathcal{A}_\rho(\rho, m, x)P\Big] + \frac{\delta}{\delta m(x)}\Big[\mathcal{A}_m(\rho, m, x)P\Big]\right\} \\ &+ \frac{\Delta}{2}\int dx \; \int dy \; \left\{ \frac{\delta^2}{\delta\rho(x)\delta\rho(y)}\Big[\mathcal{B}_{\rho,\rho}(\rho, m, x, y)P\Big] \right. \\ &+ \left. \frac{\delta^2}{\delta m(x)\delta m(y)}\Big[\mathcal{B}_{m,m}(\rho, m, x, y)P\Big] + 2\frac{\delta^2}{\delta\rho(x)\delta m(y)}\Big[\mathcal{B}_{\rho,m}(\rho, m, x, y)P\Big]\right\}, \end{aligned} \tag{25}$$

with

$$\mathcal{A}_\rho(\rho, m, x) = D\partial_{xx}\rho - v\partial_x m, \tag{26}$$

$$\mathcal{A}_m(\rho, m, x) = D\partial_{xx} m - v\partial_x\rho - m\left[2\Big(r_1 - \frac{r_3}{4}\Big) + \frac{r_3}{2}\frac{m^3}{\rho^2}\right], \tag{27}$$

$$\mathcal{B}_{\rho,\rho}(\rho, m, x, y) = \Big(-D\partial_{xx}\rho - 6v\partial_x m\Big)\delta(y - x), \tag{28}$$

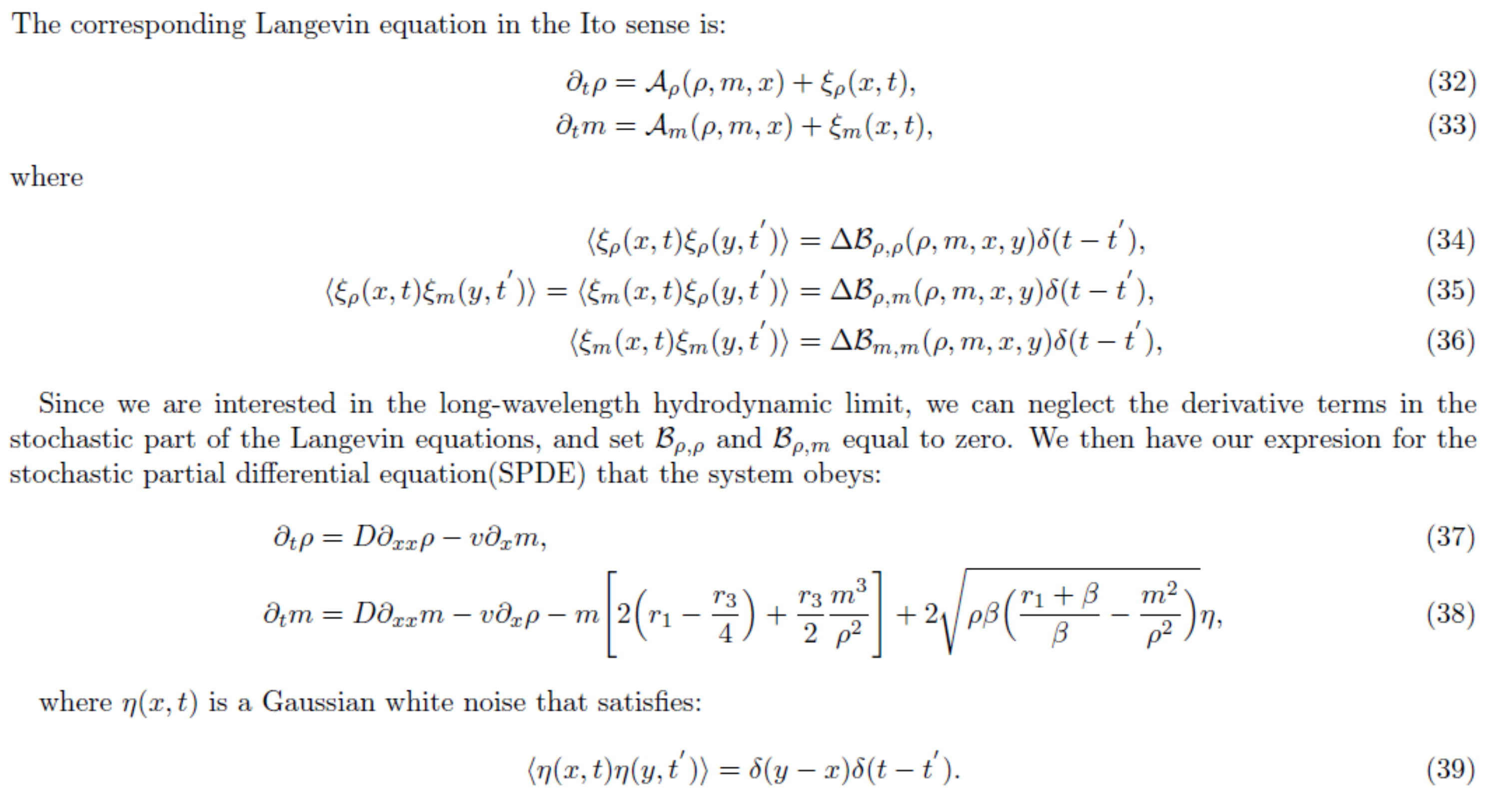
$$\mathcal{B}_{m,m}(\rho, m, x, y) = \left[-D\partial_{xx}\rho - 6v\partial_x m + 4\rho\beta\Big(\frac{r_1 + \beta}{\beta} - \frac{m^2}{\rho^2}\Big)\right]\delta(y - x), \tag{29}$$

$$\mathcal{B}_{\rho,m}(\rho, m, x, y) = \Big(-D\partial_{xx} m - 6v\partial_x\rho\Big)\delta(y - x), \tag{30}$$

and

$$\beta = \frac{r_2}{2} + \frac{r_3}{4}. \tag{31}$$

Derivation of sPDE (contd.)



The corresponding Langevin equation in the Ito sense is:

$$\partial_t \rho = \mathcal{A}_\rho(\rho, m, x) + \xi_\rho(x,t), \tag{32}$$

$$\partial_t m = \mathcal{A}_m(\rho, m, x) + \xi_m(x,t), \tag{33}$$

where

$$\langle \xi_\rho(x,t)\xi_\rho(y,t^{'})\rangle = \Delta\mathcal{B}_{\rho,\rho}(\rho,m,x,y)\delta(t-t^{'}), \tag{34}$$

$$\langle \xi_\rho(x,t)\xi_m(y,t^{'})\rangle = \langle \xi_m(x,t)\xi_\rho(y,t^{'})\rangle = \Delta\mathcal{B}_{\rho,m}(\rho,m,x,y)\delta(t-t^{'}), \tag{35}$$

$$\langle \xi_m(x,t)\xi_m(y,t^{'})\rangle = \Delta\mathcal{B}_{m,m}(\rho,m,x,y)\delta(t-t^{'}), \tag{36}$$

Since we are interested in the long-wavelength hydrodynamic limit, we can neglect the derivative terms in the stochastic part of the Langevin equations, and set $\mathcal{B}_{\rho,\rho}$ and $\mathcal{B}_{\rho,m}$ equal to zero. We then have our expresion for the stochastic partial differential equation(SPDE) that the system obeys:

$$\partial_t \rho = D\partial_{xx}\rho - v\partial_x m, \tag{37}$$

$$\partial_t m = D\partial_{xx}m - v\partial_x \rho - m\left[2\Big(r_1 - \frac{r_3}{4}\Big) + \frac{r_3}{2}\frac{m^3}{\rho^2}\right] + 2\sqrt{\rho\beta\Big(\frac{r_1+\beta}{\beta} - \frac{m^2}{\rho^2}\Big)}\eta, \tag{38}$$

where $\eta(x,t)$ is a Gaussian white noise that satisfies:

$$\langle \eta(x,t)\eta(y,t^{'})\rangle = \delta(y-x)\delta(t-t^{'}). \tag{39}$$

Results from Exact Stochastic Simulations (contd.)

- Giant number fluctuations in the coexistence phase.