Well posedness for the Hughes model

P-E Jabin
In collaboration with : D. Gomes, P.A. Markowich

The so-called Hughes model

It is a macroscopic model for pedestrian flow with, as only unknown the density of pedestrian $n(t,x)$, which reads

$$\partial_t n(t,x) + \mathrm{div}\,(a(t,x)\, n(t,x)\, f^2(n(t,x))) = 0,$$

where $f(.) \to 0$ as n approaches a critical value n_c, for instance

$$f(n) = (n_c - n)_+^k.$$

On the other hand, $a = -\nabla\phi$ where ϕ solves an eikonal equation

$$f(n(t,x))\,|\nabla\phi| = 1.$$

The so-called Hughes model

It is a macroscopic model for pedestrian flow with, as only unknown the density of pedestrian $n(t, x)$, which reads

$$\partial_t n(t,x) + \operatorname{div}\left(a(t,x)\, n(t,x)\, f^2(n(t,x))\right) - \Delta n = 0,$$

where $f(.) \to 0$ as n approaches a critical value n_c, for instance

$$f(n) = (n_c - n)_+^k.$$

On the other hand, $a = -\nabla\phi$ where ϕ solves an eikonal equation

$$f(n(t,x))\,|\nabla\phi| = 1.$$

Diffusion may be added in the transport.

The so-called Hughes model

It is a macroscopic model for pedestrian flow with, as only unknown the density of pedestrian $n(t,x)$, which reads

$$\partial_t n(t,x) + \text{div}\,(a(t,x)\,n(t,x)\,f^2(n(t,x))) = 0,$$

where $f(.) \to 0$ as n approaches a critical value n_c, for instance

$$f(n) = (n_c - n)_+^k.$$

On the other hand, $a = -\nabla\phi$ where ϕ solves an eikonal equation

$$-\Delta\phi + f(n(t,x))\,|\nabla\phi| = 1.$$

Diffusion may be added in the transport, in the eikonal Eq.

The so-called Hughes model

It is a macroscopic model for pedestrian flow with, as only unknown the density of pedestrian $n(t,x)$, which reads

$$\partial_t n(t,x) + \mathrm{div}\,(a(t,x)\,n(t,x)\,f^2(n(t,x))) - \Delta n = 0,$$

where $f(.) \to 0$ as n approaches a critical value n_c, for instance

$$f(n) = (n_c - n)_+^k.$$

On the other hand, $a = -\nabla\phi$ where ϕ solves an eikonal equation

$$-\Delta\phi + f(n(t,x))\,|\nabla\phi| = 1.$$

Diffusion may be added in the transport, in the eikonal Eq., or in both.

Non linear continuity equations

The Classical, linear continuity equation reads

$$\partial_t n(t,x) + \mathrm{div}\,(a(t,x)\,n(t,x)) = 0,$$

where the velocity field a is either given or is related to n through another equation.
Recently new models were introduced in various settings (traffic flow for cars or pedestrian, movement of bacteria/cells...) taking local non linear effects into account

$$\partial_t n(t,x) + \mathrm{div}\,(a(t,x)\,F(n(t,x))) = 0, \quad t \in \mathbb{R}_+,\ x \in \mathbb{R}^d$$

The function F is given and typically decreases as the density increases. F models complicated, localized interactions between individuals leading to a local reduction of the velocity when the density is too large.

Non linear continuity equations

The Classical, linear continuity equation reads

$$\partial_t n(t,x) + \mathrm{div}\,(a(t,x)\,n(t,x)) = 0,$$

Recently new models were introduced in various settings (traffic flow for cars or pedestrian, movement of bacteria/cells...) taking local non linear effects into account

$$\partial_t n(t,x) + \mathrm{div}\,(a(t,x)\,F(n(t,x))) = 0, \quad t \in \mathbb{R}_+,\ x \in \mathbb{R}^d$$

As before the field a is given or related to n.

The eikonal equation

The natural interpretation of the eikonal equation

$$f(n(t,x))\,|\nabla\phi| = 1, \quad x \in \Omega,$$
$$\phi = \bar{\phi}, \quad x \in \partial\Omega,$$

is that the individual at position x solves an optimization problem to find their optimal trajectory $X(s,x)$ and an exit time T given the density of all other individuals at a given time:

$$X(s=t,x) = x, \quad X(s=T,x) \in \partial\Omega,$$

while X minimizes

$$\int_t^T \left(\frac{|\partial_s X(s,x)|^2}{2} + \frac{1}{2\,f^2(n(t,X(s,x)))} \right) ds + \bar{\phi}(X(T,x)).$$

Many "false" assumptions in the model

It is easy to criticize the model

- Assumes that individuals have perfect information on the density
- Assumes that individuals only consider the position of other individuals and not the direction they are going
- ...

However the big advantage of the model is that it is a relatively simple macroscopic system on n, ϕ, which still takes interesting and complex behaviors into account. Any more accurate model would likely be much more complicated.

Existence theory?

The key difficulty to obtain existence is to pass to the limit in the terms $\nabla\phi\, n\, f(n)$ and $|\nabla\phi|^2$. This usually requires compactness of both $\nabla\phi$ and $n(t,x)$.
The $1-d$ case is special with many additional estimates, see Amadori, Di Francesco, Markowich, Pietschmann, Wolfram...
In the more realistic $2-d$ case, some viscosity seems needed, leading to

$$\partial_t n(t,x) - \mathrm{div}\left(\nabla\phi\, n(t,x)\, f^2(n(t,x))\right) = 0.$$

$$-\Delta\phi + f(n(t,x))\,|\nabla\phi| = 1, \quad x\in\Omega.$$

See for instance Ben Belgacem-J., or Colombo-Garavello-Mercier.

Existence theory?

The key difficulty to obtain existence is to pass to the limit in the terms $\nabla\phi\, n\, f(n)$ and $|\nabla\phi|^2$. This usually requires compactness of both $\nabla\phi$ and $n(t,x)$.
The $1-d$ case is special with many additional estimates, see Amadori, Di Francesco, Markowich, Pietschmann, Wolfram...
In the more realistic $2-d$ case, some viscosity seems needed, leading to

$$\partial_t n(t,x) - \mathrm{div}\left(\nabla\phi\, n(t,x)\, f^2(n(t,x))\right) = \Delta n.$$
$$f(n(t,x))\,|\nabla\phi| = 1, \quad x\in\Omega,$$

which we focus on here.

The main result

Consider an initial data $n(t,x)$ uniformly bounded in $L^1 \cap L^\infty$; smooth boundary conditions $\bar{\phi}$, $\bar{n}$. Assume that there exists $n_c > 0$ s.t. $f(n) = 0$ if $n \geq n_c$ and that for some $k \geq 1$,

$$C^{-1}\,(n_c - n)^k \leq f(n) \leq C\,(n_c - n).$$

Theorem

Under the previous assumptions, there exists $n \leq n_c$ s.t. $\nabla n \in L^2_{t,x}$, $(n - n_c)^{-1} \in L^p_{t,x}$ for all $p < \infty$; there exists ϕ s.t. $\nabla \phi \in L^p_{t,x} \cap L^2_t H^{2/5-0}_x$ for any $p < \infty$, solution to

$$\partial_t n(t,x) - div(\nabla\phi\, n(t,x)\, f^2(n(t,x))) = \Delta n.$$
$$f(n(t,x))\,|\nabla\phi| = 1, \quad x \in \Omega,$$

with boundary conditions $\bar{n}$ and $\bar{\phi}$.

Note that the notion of solution to the eikonal Eq. is not clear as the r.h.s. is not continuous.

Estimates on n, Part I

Consider any non linear convex function $\chi(n)$ and calculate

$$\frac{d}{dt}\int \chi(n(t,x))\,dx = -\int \chi''(n)\,|\nabla n|^2 + \int \chi''\,\nabla n \cdot \nabla\phi\, n\, f^2(n) + \textit{boundary conditions}.$$

Recall that $|\nabla \phi| = 1/f(n)$ so that

$$\frac{d}{dt}\int \chi(n(t,x))\,dx \le C - \frac{1}{2}\int \chi''(n)\,|\nabla n|^2 + 2\int_{n\le n_c} \chi''(n)\, f^2(n)\, n^2\,dx.$$

Estimates on n, conclusion

- First take $\chi = (n - n_c)_+^{2+0}$ and note that $\chi''(n)\, n^2 = 0$ if $n \leq n_c$. Conclude that $n \leq n_c$.
- Take $\chi = n^2$ and observe that $\chi''(n)\, f^2\, n$ is now uniformly bounded. Conclude that $\nabla n \in L^2_{t,x}$.
- Take $\chi = (n_c - n)^{-p}$ for which

$$\chi''(n)\, f^2(n)\, n^2 \leq \frac{C_p}{(n_c - n)^p},$$

since $f(n) \leq C\,(n_c - n)_+$.
This proves that $(n_c - n)^{-1} \in L^p$ for all $p > 1$.

- Observing that $\partial_t n \in L^2_t H^{-1}_x$ lets us obtain compactness on n.

The problem

Now focus on

$$\frac{1}{2}|\nabla\phi(t,x)|^2 = R(t,x), \quad x \in \Omega,$$
$$\phi = \bar{\phi}, \quad x \in \partial\Omega,$$

for a given right hand side $R(t,x) \geq c > 0$ with $R \in L^p_{t,x}$ for all $p < \infty$ and $\nabla_x R \in L^q_{t,x}$ for all $q < 2$.
Since R is not continuous, in x or in t, the classical theory of viscosity solutions does not apply.
Even obtaining the equation pointwise, requires some compactness of $\nabla\phi$ in x and in t...

Compactness in x by kinetic formulation

Follow an idea introduced in J.-Perthame and define

$$\chi(t,x,v) = \mathbb{I}_{v\cdot\nabla^\perp\phi\leq 0}, \quad v\in S^1.$$

Calculate, formally, using the equation

$$\begin{aligned} v\cdot\nabla_x\chi =& v\cdot\nabla^2\phi\cdot v^\perp\,\delta_{v=\pm\nabla\phi/|\nabla\phi|} \\ =& \pm\frac{\nabla\phi}{\sqrt{R(t,x)}}\cdot\nabla^2\phi\cdot v^\perp\,\delta_{v=\pm\nabla\phi/|\nabla\phi|} \\ =& -\pm\frac{\nabla R}{\sqrt{R(t,x)}}\cdot\frac{\nabla^\perp\phi}{|\nabla\phi|}\,\delta_{v=\pm\nabla\phi/|\nabla\phi|}. \end{aligned}$$

Compactness in x: Conclusion

Therefore one obtains the kinetic equation

$$v \cdot \nabla_x \chi = \partial_v m,$$

where m is bounded in $L^p_{t,x}$ for any $p < 2$ (and even in fact in M^1) and $\chi \in L^2_{t,x}\, H^s_v$ for any $s < 1/2$.
By velocity averaging, one may deduce that the average of χ

$$\int_{S^1} v\, \chi(t, x, v)\, dv = c\, \nabla \phi \in L^2_t\, H^s_x, \quad s < 2/5.$$

Hence using the regularizing properties of the eikonal equation, we obtain explicit compactness in x.

Compactness in t, the problem of uniqueness

As time is only a parameter, the compactness in time is equivalent to the uniqueness problem: For two solutions ϕ_1, ϕ_2 to

$$\frac{1}{2}|\nabla\phi_i(x)|^2 = R_i(x), \quad x \in \Omega,$$
$$\phi_i = \bar{\phi}, \quad x \in \partial\Omega,$$

estimate $\phi_1 - \phi_2$ in terms of $\|R_1 - R_2\|_{L^p}$ for some $p < \infty$ provided that the R_i are also in $W^{1,q}$ for all $q < 2$.
For that we cannot use viscosity solutions but have to go back to the optimal control formulation:

$$\phi_i(x) = \inf_{X, X(s=t,x)=x} \int_t^T \left(\frac{|\partial_s X(s,x)|^2}{2} + R_i(X(s,x)) \right) ds + \bar{\phi}(X(T,x)).$$

Compactness in time, the regularity of the trajectory

Following Figalli-Mandorino, it is possible to show that for $R(x) \in W^{1,q}$ with $q > 1$

- For *a.e.* x, there exists an optimal trajectory X.
- For *a.e.* x, $X \in W^{2,q}(\mathbb{R}_+,\ \Omega)$ and one has $\partial_s^2 X = \nabla R(t, X)$.
- The exit time can be estimated $T \leq C\,(\phi(t,x) - \bar{\phi}(X(T,x)))$.
- For *a.e.* x, the total length of the trajectory is finite and in average of length $T \leq C\,(\phi(t,x) - \bar{\phi}(X(T,x)))$.

The stability argument

Now consider again our two solutions ϕ_1 and ϕ_2. Take a point x which is "typical" for ϕ_1 and introduce the optimal trajectory X_1 for ϕ_1 at x. Then

$$\phi_2(x) - \phi_1(x) \leq C \int_t^T (R_2(X_1(s,x)) - R_1(X_1(s,x)))\, ds.$$

By the previous argument, the trajectory X_1 is rectifiable and $R \in W^{s,r}(\mathbb{R}^2)$ has an L^1_{loc} trace if $s > 1/r$. Thus

$$\begin{aligned}\phi_2(x) - \phi_1(x) &\leq C_{X_1} \|R_2 - R_1\|_{H^{1/2+0}(\Omega)} \\ &\leq C_{X_1} \left(\|R_2\|_{W^{1,2-0}} + \|R_1\|_{W^{1,2-0}}\right)^{1/2+0} \|R_2 - R_1\|_{L^2(\Omega)}^{1/2-0}.\end{aligned}$$

Compactness in time, conclusion

By integrating over x, one finally obtains

$$\int_\Omega |\nabla\phi(t,x) - \nabla\phi(t',x)|\, dx \le C_n \|n(t,.) - n(t',.)\|_{L^2(\Omega)}^{1/2-0},$$

where the constant C_n depends in particular on the H^1 norm of n and the L^p norm of $(n_c - n)^{-1}$ for p large enough.

From the compactness of n, one then deduces the compactness in time of $\nabla\phi$.

Note finally that this theory also provides a proper notion of solution to the eikonal equation with uniqueness.

Thank you!