Propagation of Monokinetic Measures with Rough Momentum Profiles II

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STRUCTURE OF $\mu(t)$ ON CAUSTIC FIBER

Theorems and examples.

All the examples are constructed in 1 d with the free flow:

$$(y,\xi) \mapsto \Phi_t(y,\xi) = (y + t\xi, \xi)$$

$$(y, U^{in}(y)) \mapsto F_t(y) = y + tU^{in}(y)$$

Structure of $\mu(t)$ and $\rho(t)$ outside caustic fiber, recall from previous talk

Thm A: Assume Hamiltonian H satisfies condition (H) and that momentum profile U^{in} satisfies (SL+DU). Then

(a) for a.e. $x \in \mathbf{R}^N$ and all $t \in \mathbf{R}$, the set $F_t^{-1}(\{x\})$ is finite

(b) the following conditions are equivalent

$$\rho(t)(C_t) = 0 \Leftrightarrow \rho(t)(\mathbf{R}^N \setminus C_t) = 1 \Leftrightarrow \rho^{in} = 0 \text{ a.e. on } Z_t$$

(c) under the equivalent conditions in (b), $\rho(t) \ll \mathscr{L}^N$ and

$$\rho(t,x) := \frac{d\rho(t)}{d\mathscr{L}^N}(x) = \sum_{F_t(y)=x} \frac{\rho^{in}(y)}{J_t(y)} \quad \text{for a.e. } x \in \mathbf{R}^N$$

(d) under the equivalent conditions in (b)

$$\mu(t,x,\cdot) = \sum_{F_t(y)=x} \frac{\rho^{in}(y)}{J_t(y)} \delta_{\Xi_t(y,U^{in}(y))} \quad \text{for a.e. } x \in \mathbf{R}^N$$

Lebesgue decomposition of $\rho(t)$

Thm C: Assume Hamiltonian H satisfies condition (H) and that momentum profile U^{in} satisfies (SL+DU). Then

(a) for each $t \in \mathbf{R}$, one has

$$\mathrm{supp}(\mu(t)) \subset \Lambda_t$$

(b) writing the Lebesgue decomposition of $\rho(t)$ w.r.t. $\mathscr{L}^N$ as

$$\rho(t) = \rho_a(t) + \rho_s(t) \text{ with } \rho_a(t) \ll \mathscr{L}^N \text{ and } \rho_s(t) \perp \mathscr{L}^N$$

then

$$\rho_a(t) = F_t\#(\rho^{in}\mathbf{1}_{P_t}\mathscr{L}^N) \quad \text{and } \rho_s(t) = F_t\#(\rho^{in}\mathbf{1}_{Z_t}\mathscr{L}^N)$$

Structure of $\mu(t)$, general case

Let μ_a^{in} and μ_s^{in} be the monokinetic measures with densities $\rho^{in}\mathbf{1}_{P_t}$ and $\rho^{in}\mathbf{1}_{Z_t}$ respectively and momentum profile U^{in}:

$$\mu_a^{in}(x,\cdot) := \rho^{in}(x)\mathbf{1}_{P_t}(x)\delta_{U^{in}(x)}\,, \quad \mu_s^{in}(x,\cdot) := \rho^{in}(x)\mathbf{1}_{Z_t}(x)\delta_{U^{in}(x)}$$

Propagate these measures by Hamiltonian flow

$$\mu(t) = \mu_a(t) + \mu_s(t) \text{ with } \mu_a(t) = \Phi_t\#\mu_a^{in} \text{ and } \mu_s(t) = \Phi_t\#\mu_s^{in}$$

•Structure of $\mu_a(t)$ and of $\rho_a(t) = \Pi\#\mu_a(t)$ described by **Thm A**

•Structure of $\mu_s(t)$? of $\rho_s(t)$?

Thm D: Assume Hamiltonian H satisfies condition (H) and that momentum profile U^{in} satisfies (SL+DU). For each $t \in \mathbf{R}$, let

$$A_t := \{x \text{ s.t. } \mathscr{L}^N(F_t^{-1}(\{x\}) \cap Z_t) > 0\}$$

(a) For each $t \in \mathbf{R}$, one has $A_t \subset C_t$

(b) For each $t > 0$ the set A_t is at most countable

(c) Let $\rho^{in} \in L^1(\mathbf{R}^N)$ s.t. $\rho^{in} > 0$ a.e. on Z_t; then

$$\rho(t)(\{x\}) > 0 \Leftrightarrow x \in A_t$$

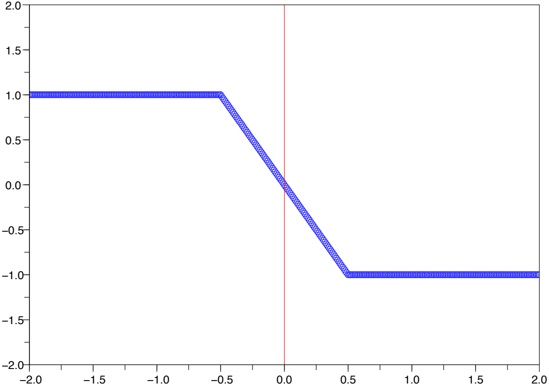
Remark

•If $N = 1$, if $H(x,\xi) = \frac{1}{2}\xi^2$ and if U^{in} is real analytic+sublinear at infinity, then $F_t = \mathrm{id} + tU^{in}$ is real analytic+proper. Therefore $F_t^{-1}(\{x\})$ is finite for all $x \in \mathbf{R}$ — even if $x \in C_t$.
•In particular, $\mathscr{L}^N(F_t^{-1}(\{x\}) \cap Z_t) = 0$ and therefore $\rho(t)(\{x\}) = 0$ for all $t \in \mathbf{R}$ and all $x \in \mathbf{R}$

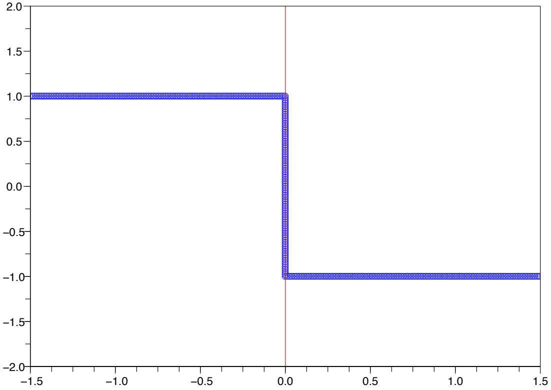
In space dimension 1, and
for analytic flow+momentum profile
$\rho_s(t)$ does not have atoms

•However, $\rho_s(t)$ may have atoms even if the flow and the initial momentum profiles are C^∞.

Example A: *Lip* and C^∞ case

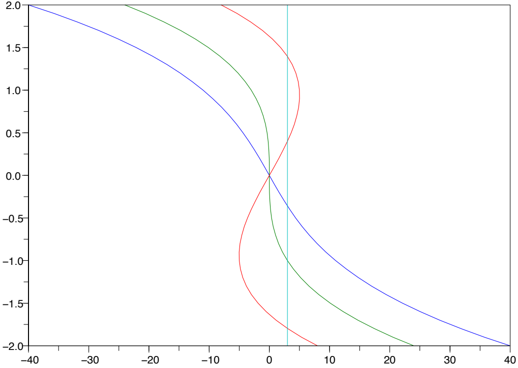


Lagrangian Λ_t at $t = 0$ for $U^{in}(y) = \begin{cases} -y/|y| & \text{if } |y| > 1 \\ -y & \text{if } |y| \leq 1 \end{cases}$



Lagrangian Λ_t at $t = 1$. Here, $A_1 = \{0\}$ and $F_1^{-1}(\{0\}) \cap Z_1 = (-1, 1)$. Analogous picture if U^{in} is regularized near $y = \pm 1$.

Example B: analytic case



Lagrangian at times $t = 0, 8, 16$ for U^{in} =inverse of $y \mapsto -8y - 3y^3$. Here $\#F_t^{-1}(\{x\}) \leq 3$ for all t and all x. Therefore $A_t = \varnothing$ for all t.

APPLICATIONS TO THE CLASSICAL LIMIT

WKB method for Schrödinger's equation

Classical limit of Schrödinger's equation for $x \in \mathbf{R}^N$:

$$i\epsilon\partial_t\psi_\epsilon + \tfrac{1}{2}\epsilon^2\Delta_x\psi_\epsilon = V(x)\psi_\epsilon\,, \quad \psi_\epsilon(0,x) = a^{in}(x)e^{iS^{in}(x)/\epsilon}$$

WKB ansatz for wave function ψ_ϵ

$$\psi_\epsilon(t,x) \simeq \sum_{n\geq 0}\epsilon^n a_n(t,x)e^{iS(t,x)/\epsilon}$$

Explicit solution of Cauchy pbm for Schrödinger's eqn when $V \equiv 0$

$$\psi_\epsilon(t,x) = \frac{1}{\sqrt{2\pi i\epsilon}^N}\int_{\mathbf{R}^N} e^{\frac{i}{\epsilon}\left(\frac{|x-y|^2}{2t}+S^{in}(y)\right)}a^{in}(y)dy$$

If $V \not\equiv 0$, replace explicit solution with FIO parametrix (Laptev-Sigal)

Caustic fiber (case $S^{in} \in C^2$): set $F_t(y) := y + t\nabla S^{in}(y)$ and $J_t(y) := |\det DF_t(y)|$; since $F_t \in C^1(\mathbf{R}^N, \mathbf{R}^N)$, one has $E = \varnothing$ and

$$C_t := \{\text{critical values of } F_t\}$$

Thm (Maslov) for $x \notin C_t$, $a^{in} \in C_c^\infty(\mathbf{R}^N)$ and $S^{in} \in C^\infty(\mathbf{R}^N)$

$$\psi_\epsilon(t,x) = \sum_{F_t(y)=x} \frac{a^{in}(y)}{\sqrt{J_t(y)}} e^{i\left(\frac{S^{in}(y)}{\epsilon} - \#(\sigma(D^2F_t(y)) \cap \mathbf{R}_+^*)\frac{\pi}{2}\right)} + O(t\epsilon)$$

Thus $\psi_\epsilon \simeq$ locally finite sum of WKB ansatz away from caustic fibers

Proof: apply stationary phase

Classical limit for non C^2 phase functions

Assume initial phase function $S^{in} \in C^1(\mathbf{R}^N)$ with

$$\nabla^2 S^{in} \in L^{N,1}_{loc}(\mathbf{R}^N) \quad \text{and } \nabla S^{in}(x) = o(|y|) \quad \text{as } |y| \to \infty$$

Let

$$H(x,\xi) = \tfrac{1}{2}|\xi|^2 + V(x)$$

with $V \in C^\infty_b(\mathbf{R}^N)$ such that, for some $\alpha > N/2$,

$$V(x) = o(|x|) \quad \text{and } V^-(x) = o(|x|^{-\alpha}) \qquad \text{as } |x| \to \infty$$

Thm E: Let $a^{in} \in L^2(\mathbf{R}^N)$, $\theta, \chi \in C_b(\mathbf{R}^N)$ with $\|a^{in}\|_{L^2} = 1$, and

$$\psi_\epsilon(t,\cdot) := e^{i\frac{t}{\epsilon}(\frac{1}{2}\epsilon^2\Delta_x - V(x))}(a^{in}e^{iS^{in}/\epsilon}) \quad t \in \mathbf{R}\,,\ \epsilon > 0$$

(a) If $\theta = 0$ on C_t then

$$\lim_{\epsilon\to 0}\int_{\mathbf{R}^N}\theta(x)|\psi_\epsilon(t,x)|^2dx = \int_{\mathbf{R}^N}\theta(x)\sum_{F_t(y)=x}\frac{|a^{in}|^2\mathbf{1}_{P_t}}{J_t}(y)dx$$

(b) If $y \in Z_t \Rightarrow \tilde{\chi}_t(y) := \chi(\Xi_t(y, \nabla S^{in}(y))) = 0$, then

$$\lim_{\epsilon\to 0}\int_{\mathbf{R}^N}\chi(-\epsilon\xi)|\hat{\psi}_\epsilon(t,\xi)|^2\frac{d\xi}{(2\pi)^N} = \int_{\mathbf{R}^N}\sum_{F_t(y)=x}\tilde{\chi}(y)\frac{|a^{in}|^2\mathbf{1}_{P_t}}{J_t}(y)dx$$

Sketch of the proof of Thm E

1. That

$$|\psi_\epsilon(t,\cdot)|^2 \to \rho(t) \text{ while } (2\pi\epsilon)^{-N}|\hat{\psi}_\epsilon(t,\cdot/\epsilon)|^2 \to \int \mu(t,dx,\cdot)$$

weakly in the sense of probability measures as $\epsilon \to 0$ follows from [Lions-Paul, Rev. Mat. Iberoam., 1993], especially Theorem III.1.3 and Theorem IV.1.2. Notice that there is no mass loss at infinity because $\mu(t) = \Phi_t \# \mu^{in}$ is a probability measure for all $t \in \mathbf{R}$

2. The formulas for the limits follow from our theorem (Theorem A, previous talk) on the structure of $\mu(t)$.

MORE EXAMPLES AND COUNTER-EXAMPLES

Example 1: set $N = 1$ with $H(x,\xi) := \frac{1}{2}\xi^2$ and

$$U^{in}(y) = y\sin(\ln|y|) \text{ for } y \neq 0\,, \quad U^{in}(0) = 0$$

so that $F_t = \mathrm{id}_{\mathbf{R}^N} + tU^{in} \in \mathrm{Lip}(\mathbf{R}) \setminus C^1(\mathbf{R})$ with $E = \{0\}$
For $t < -1$ one has

$$F_t^{-1}(\{0\}) \cap (-e^{\pi}, e^{\pi}) = \{0\} \cup \{\pm y_n(t) \,|\, n \geq 0\} \cup \{\pm z_n(t) \,|\, n \geq 0\}\,,$$

where

$$y_n(t) := e^{\arcsin(-1/t) - 2\pi n}\,, \quad z_n(t) := e^{\pi - \arcsin(-1/t) - 2\pi n}$$

On the other hand

$$F_t'(y) = 1 + t \sin \ln |y| + t \cos \ln |y|$$

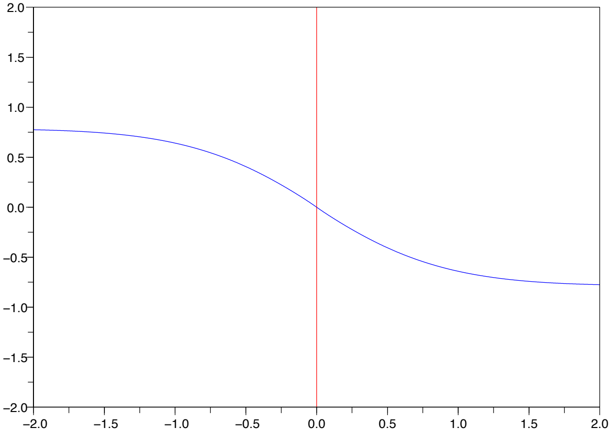
so that

$$|F_t'(y_n(t))| = |F_t'(z_n(t))| = \sqrt{t^2 - 1} \neq 0\,.$$

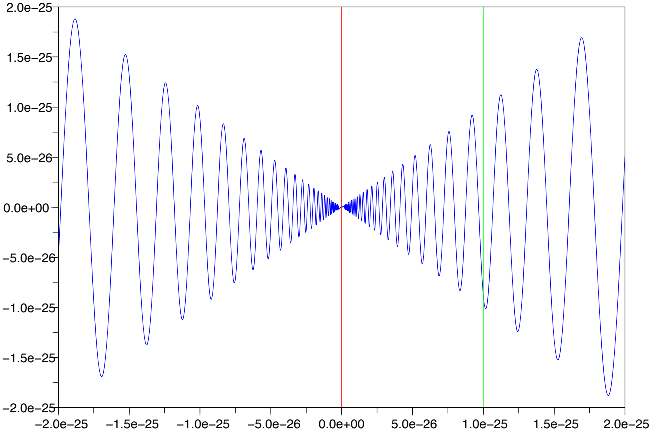
Hence 0 is not a critical value of the restriction of F_t to $(-e^\pi, e^\pi)$, and yet $F_t^{-1}(\{0\}) \cap (-e^\pi, e^\pi)$ is infinite

Conclusion: if U^{in} is not C^1, one cannot keep **both** the usual definition of the caustic fiber$= \{$critical values of $F_t\}$ and the fact that $F_t^{-1}(\{x\})$ is finite for all $x \notin C_t$

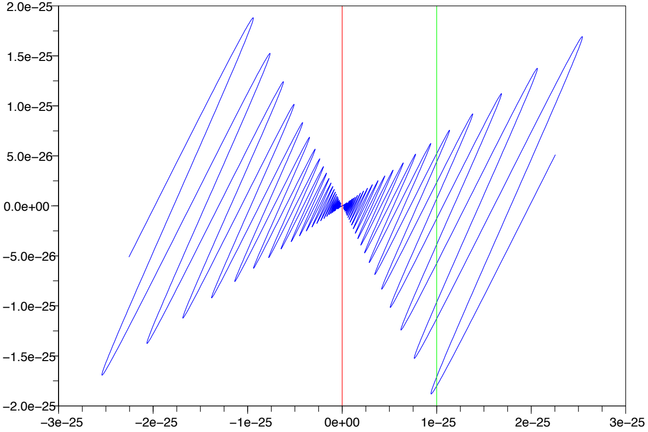
This is one reason for including the nondifferentiability set E in the definition of the caustic fiber C_t



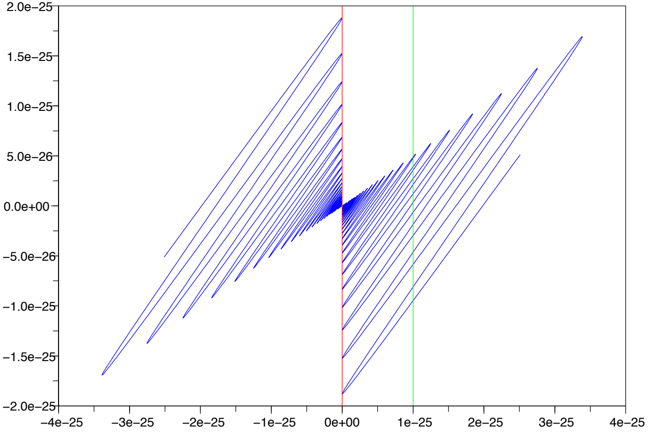
Initial profile $U^{in}(y) = -\tanh(y)\sin(|y|^{10^{-n}-1})$



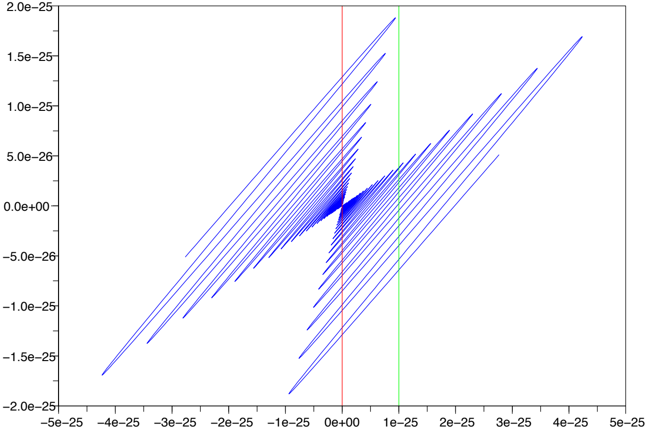
Initial profile $U^{in}(y) = -\tanh(y)\sin(|y|^{10^{-n}-1})$, zoom near origin



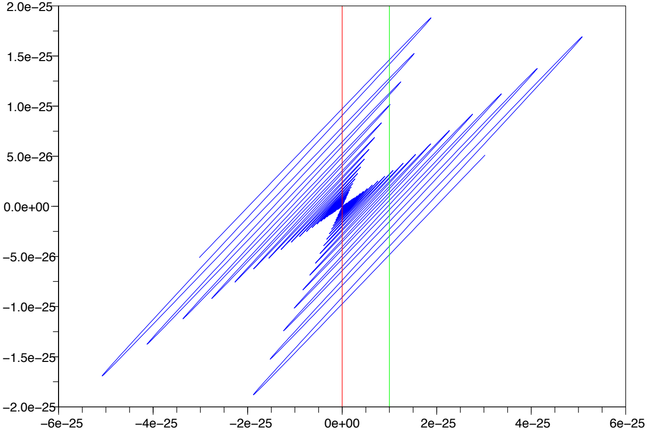
Lagrangian at time $t = 0.5$ for $U^{in}(y) = -\tanh(y)\sin(|y|^{10^{-n}-1})$
zoom near origin



Lagrangian at time $t = 1$ for $U^{in}(y) = -\tanh(y)\sin(|y|^{10^{-n}-1})$
zoom near origin



Lagrangian at time $t = 1.5$ for $U^{in}(y) = -\tanh(y)\sin(|y|^{10^{-n}-1})$
zoom near origin



Lagrangian at time $t = 2$ for $U^{in}(y) = -\tanh(y)\sin(|y|^{10^{-n}-1})$
zoom near origin

The singular component of $\rho(t)$ can be diffuse

Example 2: set $N = 1$ and $H(x, \xi) := \frac{1}{2}\xi^2$
Let $K \subset (0, 1) \setminus \mathbf{Q}$ s.t. $\mathscr{L}^1(K) \in (\frac{1}{2}, 1)$ and $\Omega := (0, 1) \setminus K$

$$U^{in}(y) := \begin{cases} 0 & \text{if } y < 0 \\ \mathscr{L}^1(\Omega \cap [0, y]) - y & \text{if } y \in [0, 1] \\ \mathscr{L}^1(\Omega) - 1 & \text{if } y > 1 \end{cases}$$

One has $\Phi_t(x, \xi) := (x + t\xi, \xi)$ so that $F_t : y \mapsto y + tU^{in}(y)$. Then

(a) the map $F_1 : \mathbf{R} \ni y \mapsto y + U^{in}(y) \in \mathbf{R}$ is increasing and onto

(b) for each $y \in (-\infty, 0) \cup \Omega \cup (1, \infty)$, one has $F_1'(y) = 1$, while $F_1'(y) = 0$ for a.e. $y \in K$

(c) for $\rho^{in} := \mathbf{1}_K / (\mathscr{L}^1(K))$, the measure $\rho(1) := F_1\#(\rho^{in}\mathscr{L}^1) \perp \mathscr{L}^1$ and

$$\rho(1)(\{x\}) = 0 \text{ for all } x \in \mathbf{R}$$

Example 3: set $N = 1$ and $H(x,\xi) := \frac{1}{2}\xi^2$
Let $K \subset [0,1]$ =ternary Cantor set with Hausdorff dimension $s = \frac{\ln 2}{\ln 3}$

$$U^{in}(z) := \mathbf{1}_{0\leq z\leq 1}(\mathscr{H}^s([0,z]\cap K) - z)$$

(a) Momentum profile $U^{in} \in C_c(\mathbf{R})\cap BV(\mathbf{R})$ but $(U^{in})' \notin L^{1,1}(\mathbf{R})$

One has $\Phi_t(x,\xi) := (x + t\xi, \xi)$ so that $F_t : y \mapsto y + tU^{in}(y)$. Then

(b) the map $F_1 \in C(\mathbf{R})$ and is increasing $\Rightarrow F_1 \in BV_{loc}(\mathbf{R})$
(c) the map F_1 is not differentiable on K and differentiable on $\mathbf{R} \setminus K$

$$F_1'(y) = 0 \Leftrightarrow y \in [0,1] \setminus K$$

(d) the caustic fiber is $C_1 = [0,1]$ and $\mathscr{L}^1(C_1) > 0$

If U^{in} is less regular than in assumption (DU) — i.e. if $DU^{in} \notin L^{N,1}_{loc}$ — it may happen that the caustic fiber is not Lebesgue negligeable

In this case, the propagated monokinetic measure may fail to be a.e. equal to a finite sum of monokinetic measures

In fact, if (DU) is not satisfied, it can happen that F_t doesn't map Lebesgue-negligeable sets on Lebesgue-negligeable sets; then including E (or any other Lebesgue-negligeable set) in the definition of the caustic fiber C_t may result in $\mathscr{L}^N(C_t) > 0$

However this choice does not have any effect on the propagated measure $\mu(t)$ since

$$\mu^{in}(E \times \mathbf{R}^N) = \int_E \rho^{in}(x)dx = 0$$

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$$\mu^{in}(E \times \mathbf{R}^N) = \int_E \rho^{in}(x)dx = 0$$

Example 4:

Set $N = 1$ and $H(x,\xi) := \frac{1}{2}\xi^2$ so that $\Phi_t(x,\xi) := (x + t\xi, \xi)$. Then

for each $s \in (0,1)$, there exists

•a compact $K(s) \subset [0,1]$ s.t. $\mathscr{H}^s(K(s)) = 1$

•a momentum profile $U^{in} \in \text{Lip}(\mathbf{R}^N)$ & a probability density ρ^{in} s.t.

$$C_1 = \text{supp}(F_1 \# \rho^{in}) = K(s) \quad \text{where } F_t(y) := y + tU^{in}(y)$$

On the Hausdorff dimension of supp($\rho(t)$) (end)

Construction for $s = \frac{\ln 2}{\ln 3}$, set K :=ternary Cantor set and

$$\begin{cases} \mathcal{O} := [0,1] \setminus K =: \bigcup\limits_{\substack{1 \le k \le 2^{m-1} \\ m \ge 1}} (a_{m,k} - \frac{1}{2}3^{-m}, a_{m,k} + \frac{1}{2}3^{-m}) \\ \Omega := \bigcup\limits_{\substack{1 \le k \le 2^{m-1} \\ m \ge 1}} (a_{m,k} - \frac{1}{6}3^{-m}, a_{m,k} + \frac{1}{6}3^{-m}) =: [0,1] \setminus \tilde{K} \end{cases}$$

Define

$$\rho^{in} = \frac{3}{2}\mathbf{1}_{\tilde{K}} \quad \text{and } U^{in}(y) = \mathbf{1}_{0 \le y \le 1}(3\mathscr{L}^1(\Omega \cap [0,y]) - y)$$

$$\theta = \frac{1}{3}, \quad r_m = \frac{3^{-m}}{6},$$

$$\mu(1) = \frac{1}{1-\theta} \sum_{m \geq 1} \sum_{k=1}^{2^{m-1}} (\delta_{a_{m,k}-r_m} \otimes \mathbf{1}_{(-(1-\theta) r_m, 0} + \delta_{a_{m,k}+r_m} \otimes \mathbf{1}_{(0,(1-\theta) r_m)})$$

$$\rho(1) = \frac{1}{2} \frac{1}{1-2\theta} \sum_{m \geq 1} \theta^{m-1} \sum_{k=1}^{2^{m-1}} (\delta_{a_{m,k}-r_m} + \delta_{a_{m,k}+r_m})$$

$\rho(1) =$ denumerable convex combination of Dirac masses at 3-adic rationals

Conclusions

Our results on the Hamiltonian propagation of monokinetic measures provide information on the classical limit of the Schrödinger equation for WKB initial wave functions with L^2 amplitudes and rough phase functions ($S^{in} \in C^1(\mathbf{R}^N)$ but $\nabla^2 S^{in} \in L^{N,1}_{loc} \setminus C(\mathbf{R}^N)$)

Specifically, we obtain formulas for the position and momentum densities in the classical limit, that are consistent with Maslov's theory in the case of smooth amplitudes and phase functions

Various examples show that our results are sharp — especially regarding the regularity assumptions on the momentum profile, the "size" of the caustic fiber, and the structure of the propagated measure on the caustic fiber