REGULARITY OF THE BOLTZMANN EQUATION IN CONVEX DOMAINS

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Abstract. A basic question about regularity of Boltzmann solutions in the presence of physical boundary conditions has been open due to characteristic nature of the boundary as well as the non-local mixing of the collision operator. Consider the Boltzmann equation in a strictly convex domain with the specular, bounce-back and diffuse boundary condition. With the aid of a distance function toward the grazing set, we construct weighted classical C^1 solutions away from the grazing set for all boundary conditions. For the diffuse boundary condition, we construct $W^{1,p}$ solutions for $1 < p < 2$ and weighted $W^{1,p}$ solutions for $2 \leq p \leq \infty$ as well. On the other hand, we show second derivatives do not exist up to the boundary in general by constructing counterexamples for all boundary conditions.

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Introduction

Boundary effects play an important role in the dynamics of Boltzmann solutions of

$$\partial_t F + v \cdot \nabla_x F = Q(F, F), \tag{1}$$

where $F(t, x, v)$ denotes the particle distribution at time t, position $x \in \Omega$ and velocity $v \in \mathbb{R}^3$. Throughout this paper, the collision operator takes the form

$$\begin{aligned} Q(F_1, F_2) &\equiv Q_{\text{gain}}(F_1, F_2) - Q_{\text{loss}}(F_1, F_2) \\ &= \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |v-u|^\kappa q_0(\theta) \Big[F_1(u')F_2(v') - F_1(u)F_2(v) \Big] \mathrm{d}\omega \mathrm{d}u, \end{aligned}$$

where $u' = u + [(v-u)\cdot\omega]\omega$, $v' = v - [(v-u)\cdot\omega]\omega$ and $0 \leq \kappa \leq 1$ (hard potential) and $0 \leq q_0(\theta) \leq C|\cos\theta|$ (angular cutoff) with $\cos\theta = \frac{v-u}{|v-u|}\cdot\omega$.

Despite extensive developments in the study of the Boltzmann equation, many basic questions regarding solutions in a physical bounded domain, such as their regularity, have remained largely open. This is partly due to the characteristic nature of boundary conditions in the kinetic theory. In [8], it is shown that in convex domains, Boltzmann solutions are continuous away from the grazing set. On the other hand, in [11], it is shown that singularity (discontinuity) does occur for

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Boltzmann solutions in a non-convex domain, and such singularity propagates precisely along the characteristics emanating from the grazing set. The boundary of the phase space is

$$\gamma := \{(x,v) \in \partial\Omega \times \mathbb{R}^3\},$$

where $n = n(x)$ the outward normal direction at $x \in \partial\Omega$. We decompose γ as

$$\begin{aligned}
\gamma_- &= \{(x,v) \in \partial\Omega \times \mathbb{R}^3 : n(x)\cdot v < 0\}, && (\textit{the incoming set}),\\
\gamma_+ &= \{(x,v) \in \partial\Omega \times \mathbb{R}^3 : n(x)\cdot v > 0\}, && (\textit{the outcoming set}),\\
\gamma_0 &= \{(x,v) \in \partial\Omega \times \mathbb{R}^3 : n(x)\cdot v = 0\}, && (\textit{the grazing set}).
\end{aligned}$$

In general the boundary condition is imposed only for the incoming set γ_- for general kinetic PDEs [1, 3, 6, 8].

Throughout this paper we assume that Ω is a bounded open subset of $\mathbb{R}^3$ and there exists $\xi : \mathbb{R}^3 \to \mathbb{R}$ such that $\Omega = \{x \in \mathbb{R}^3 : \xi(x) < 0\}$, and $\partial\Omega = \{x \in \mathbb{R}^3 : \xi(x) = 0\}$. Moreover for all $\xi(x) \leq 0$(therefore $x \in \bar{\Omega} = \Omega \cup \partial\Omega$) we assume the domain is *strictly convex*

$$\sum_{i,j} \partial_{ij}\xi(x)\zeta_i\zeta_j \geq C_\xi |\zeta|^2. \tag{2}$$

We assume that $\nabla\xi(x) \neq 0$ when $|\xi(x)| \ll 1$ and we define the *outward normal* as $n(x) \equiv \frac{\nabla\xi(x)}{|\nabla\xi(x)|}$.

In this paper, we consider the following basic boundary conditions on $(x,v) \in \gamma_-$

(1) *Diffuse boundary condition:*

$$F(t,x,v) = c_\mu \mu(v) \int_{n(x)\cdot u>0} F(t,x,u)\{n(x)\cdot u\}\mathrm{d}u,$$

where $c_\mu \int_{n(x)\cdot u>0} \mu(u)\{n(x)\cdot u\}\mathrm{d}u = 1$.

(2) *Specular reflection boundary condition:*

$$F(t,x,v) = F(t,x,R_x v),$$

where $R_x v := v - 2n(x)(n(x)\cdot v)$.

(3) *Bounce-back reflection boundary condition:*

$$F(t,x,v) = F(t,x,-v).$$

For $(x,v) \in \bar{\Omega} \times \mathbb{R}^3$ we define $t_{\mathbf{b}}(x,v)$ be *the backward exit time* as

$$t_{\mathbf{b}}(x,v) = \inf\{\tau > 0 : x - sv \notin \Omega\},$$

and $x_{\mathbf{b}}(v) = x - t_{\mathbf{b}} v$.

The characteristics ODE of the Boltzmann equation (1) is

$$\frac{\mathrm{d}X(s)}{\mathrm{d}s} = V(s), \quad \frac{\mathrm{d}V(s)}{\mathrm{d}s} = 0.$$

Before the trajectory hits the boundary, $t - s < t_{\mathbf{b}}(x,v)$, we have $[X(s;t,x,v), V(s;t,x,v)] = [x - (t-s)v, v]$ with the initial condition $[X(t;t,x,v), V(t;t,x,v)] = [x,v]$. On the other hand, when the trajectory hits the boundary we define the generalized characteristics as follows:

Definition 1 ([8])**.** *Let* $(x,v) \notin \gamma_0$.

(1) Let $(t^0, x^0, v^0) = (t,x,v)$ *and define the stochastic(diffuse) cycles,* $\ell \geq 1$,

$$(t^{\ell+1}, x^{\ell+1}, v^{\ell+1}) = (t^\ell - t_{\mathbf{b}}(x^\ell, v^\ell), x_{\mathbf{b}}(x^\ell, v^\ell), v^{\ell+1}).$$

(2) Let $(t^0, x^0, v^0) = (t,x,v)$ *and define the specular cycles,* $\ell \geq 1$,

$$(t^{\ell+1}, x^{\ell+1}, v^{\ell+1}) = (t^\ell - t_{\mathbf{b}}(x^\ell, v^\ell), x_{\mathbf{b}}(x^\ell, v^\ell), v^\ell - 2n(x^\ell)(v^\ell \cdot n(x^\ell))).$$

(3) Let $(t^0, x^0, v^0) = (t,x,v)$ *and define the bounce-back cycles,* $\ell \geq 1$,

$$(t^{\ell+1}, x^{\ell+1}, v^{\ell+1}) = (t^\ell - t_{\mathbf{b}}(x^\ell, v^\ell), x_{\mathbf{b}}(x^\ell, v^\ell), -v^\ell).$$

Then for $\ell \geq 1$

$$t^\ell = t^1 - (\ell-1)t_{\mathbf{b}}(x^1,v^1), \quad x^\ell = \frac{1-(-1)^\ell}{2}x^1 + \frac{1+(-1)^\ell}{2}x^2, \quad v^{\ell+1} = (-1)^{\ell+1}v.$$

(4) We define the backward trajectory as

$$X_{\mathbf{cl}}(s;t,x,v) \;=\; \sum_\ell \mathbf{1}_{[t^{\ell+1},t^\ell)}(s)\{x^\ell - (t^\ell - s)v^\ell\}, \quad V_{\mathbf{cl}}(s;t,x,v) \;=\; \sum_\ell \mathbf{1}_{[t^{\ell+1},t^\ell)}(s)v^\ell.$$

Note that if $G(t,x,v)$ solves $\partial_t G + v\cdot\nabla_x G = 0$ with boundary conditions then

$$G(t,x,v) = G(s, X_{\mathbf{cl}}(s;t,x,v), V_{\mathbf{cl}}(s;t,x,v)),$$

where $[X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s)]$ is defined respectively([8]).

In this paper we establish the first Sobolev regularity away from the grazing set γ_0 for Boltzmann solutions in convex domains. One of the crucial ingredient is the construction of a distance function towards the grazing set γ_0 to achieve this goal.

Definition 2 (Kinetic Distance)**.** *For $(x,v)\in\bar{\Omega}\times\mathbb{R}^3$,*

$$\alpha(x,v) := |V_{\mathbf{cl}}(s)\cdot\nabla\xi(X_{\mathbf{cl}}(s))|^2 - 2\{V_{\mathbf{cl}}(s)\cdot\nabla^2\xi(X_{\mathbf{cl}}(s))\cdot V_{\mathbf{cl}}(s)\}\xi(X_{\mathbf{cl}}(s)).$$

Due to (2), $\alpha(x,v)$ is zero if and only if (x,v) is at the grazing set γ_0. We observe that

$$\begin{aligned} v\cdot\nabla_x\alpha &= \{2v\cdot\nabla\xi(x)[v\cdot\nabla^2\xi\cdot v] - 2v\cdot\nabla\xi(x)[v\cdot\nabla^2\xi\cdot v] - 2v\{v\cdot\nabla^3\xi(x)\cdot v\}\xi(x)\} \\ &= -2v\{v\cdot\nabla^3\xi(x)\cdot v\}\xi(x), \end{aligned}$$

which is bounded by $|v|\alpha(x,v)$ since $\{v\cdot\nabla^2\xi(x)\cdot v\}\backsim|v|^2$ from (2). This crucial invariant property of α under operator $v\cdot\nabla_x$ is the key for our approach. On the other hand, unless $\nabla^3\xi\equiv 0$ (for example the domain is a ball or an ellipsoid), a growth factor $|v|$ creates a geometric effect which is out of control for our analysis. We introduce a strong decay factor $e^{-\varpi\langle v\rangle t}$ with sufficiently large $\varpi>0$ to overcome such a geometric effect :

$$e^{-\varpi\langle v\rangle t}\alpha(x,v). \tag{3}$$

A direct computation yields

$$\begin{aligned} \{\partial_t + v\cdot\nabla_x\}[e^{-\varpi\langle v\rangle t}\alpha(x,v)] &= -\varpi\langle v\rangle e^{-\varpi\langle v\rangle t}\alpha(x,v) - e^{-\varpi\langle v\rangle t}2v\{v\cdot\nabla^3\xi(x)\cdot v\} \\ &\lesssim (-\varpi + O_\xi(1))\langle v\rangle e^{-\varpi\langle v\rangle t}\alpha(x,v), \end{aligned}$$

where we used the convexity of ξ in (2). Here $O_\xi(1) = \frac{2v\{v\cdot\nabla^3\xi(x)\cdot v\}\xi}{\alpha\langle v\rangle}$ represents the geometric effect. Throughout this paper we assume

$$\varpi > \max\frac{2v\{v\cdot\nabla^3\xi(x)\cdot v\}\xi}{\alpha\langle v\rangle}. \tag{4}$$

Remark that if ξ is quadratic (for example the domain is a ball or an ellipsoid) then we are able to set $\varpi=0$ and $\{\partial_t + v\cdot\nabla_x\}\alpha\equiv 0$.

The important technique to treat α along the trajectory is based on the geometric Velocity Lemma :

Lemma 1 (Lemma 1, [8])**.** *Let Ω be convex (2). Along the backward trajectory we define*

$$\alpha(s;t,x,v) := \alpha(X_{\mathbf{cl}}(s;t,x,v), V_{\mathbf{cl}}(s;t,x,v)).$$

Then there exists $C=C_\xi>0$ such that, for all $0\leq s_1,s_2\leq t$,

$$e^{-C|v||s_1-s_2|}\alpha(s_1;t,x,v) \;\leq\; \alpha(s_2;t,x,v) \;\leq\; e^{C|v||s_1-s_2|}\alpha(s_1;t,x,v).$$

We denote $F=\mu+\sqrt{\mu}f$(f could be large) where $\mu = e^{-\frac{|v|^2}{2}}$ is a global normalized Maxwellian. The perturbation f satisfies

$$\partial_t f + v\cdot\nabla_x f + \nu(F)f - Kf = \Gamma_{\text{gain}}\,(f,f).$$

Here

$$\begin{aligned}\nu(F)(v) &:= \frac{1}{\sqrt{\mu(v)}}Q_{\text{loss}}(F,\sqrt{\mu}f)(v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\omega)F(u)\mathrm{d}\omega\mathrm{d}u \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\omega)\{\mu(u)+\sqrt{\mu(u)}f(u)\}\mathrm{d}\omega\mathrm{d}u,\end{aligned} \tag{5}$$

and the (non-local) linear Boltzmann operator is given by

$$\begin{aligned}Kf(v) &:= K_2f(v) - K_1f(v) \\ &:= \frac{1}{\sqrt{\mu}}\Big[Q_{\text{gain}}(\mu,\sqrt{\mu}f) + Q_{\text{gain}}(\sqrt{\mu}f,\mu)\Big](v) - \frac{1}{\sqrt{\mu}}Q_{\text{loss}}(\sqrt{\mu}f,\mu)(v),\end{aligned}$$

and the gain part of the nonlinear Boltzmann operator is given by

$$\Gamma_{\text{gain}}(f_1,f_2)(v) := \frac{1}{\sqrt{\mu}}Q_{\text{gain}}(\sqrt{\mu}f_1,\sqrt{\mu}f_2)(v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u)\sqrt{\mu(u)}f_1(u')f_2(v')\mathrm{d}\omega\mathrm{d}u.$$

The boundary conditions for f are

(1) *Diffuse boundary condition:*

$$f(t,x,v) = c_\mu\sqrt{\mu(v)}\int_{n(x)\cdot u>0} f(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u, \tag{6}$$

(2) *Specular reflection boundary condition:*

$$f(t,x,v) = f(t,x,R_xv), \tag{7}$$

(3) *Bounce-back reflection boundary condition:*

$$f(t,x,v) = f(t,x,-v). \tag{8}$$

1. Diffuse Reflection

We denote $||\cdot||_p$ the $L^p(\Omega\times\mathbb{R}^3)$ norm, while $|\cdot|_{\gamma,p}$ is the $L^p(\partial\Omega\times\mathbb{R}^3;\mathrm{d}\gamma)$ norm and $|\cdot|_{\gamma_\pm,p} = |\cdot\mathbf{1}_{\gamma_\pm}|_{\gamma,p}$ where $\mathrm{d}\gamma = |n(x)\cdot v|\mathrm{d}S_x\mathrm{d}v$ with the surface measure $\mathrm{d}S_x$ on $\partial\Omega$. Denote $\langle v\rangle = \sqrt{1+|v|^2}$. We define

$$\partial_t f(0) = \partial_t f_0 \equiv -v\cdot\nabla_x f_0 - \nu(F_0)f_0 + Kf_0 + \Gamma_{\text{gain}}(f_0,f_0). \tag{9}$$

Theorem 1. *Assume that $f_0\in W^{1,p}(\Omega\times\mathbb{R}^3)$ and $||\partial_t f_0||_p + ||\nabla_x f_0||_\infty + ||\nabla_v f_0||_\infty + ||\langle v\rangle^\zeta f_0||_\infty < +\infty$ for $\zeta\geq 4$ and any fixed $1<p<2$, and the compatibility condition on $(x,v)\in\gamma_-$,*

$$f_0(x,v) = c_\mu\sqrt{\mu(v)}\int_{n(x)\cdot u>0} f_0(x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u, \tag{10}$$

then there exists $T = T(||\langle v\rangle^\zeta f_0||_\infty) > 0$ such that $f\in L^\infty_{loc}([0,T];W^{1,p}(\Omega\times\mathbb{R}^3))$ such that for all $0\leq t\leq T$

$$\begin{aligned}&||\partial_t f(t)||_p^p + ||\nabla_x f(t)||_p^p + ||\nabla_v f(t)||_p^p + \int_0^t \Big[\,|\partial_t f(s)|_{\gamma,p}^p + |\nabla_x f(s)|_{\gamma,p}^p + |\nabla_v f(s)|_{\gamma,p}^p\Big]\mathrm{d}s \\ &\lesssim_T \;||\partial_t f_0||_p^p + ||\nabla_x f_0||_p^p + ||\nabla_v f_0||_p^p.\end{aligned} \tag{11}$$

Furthermore, if $||\langle v\rangle^\beta f_0||_\infty \ll 1$, then T can be arbitrarily large, $T=+\infty$.

There can be no size restriction on initial data. We also remark that, from [8, 2], the assumption $||\langle v\rangle^\zeta f_0||_\infty \ll 1$ without a mass constraint $\iint_{\Omega\times\mathbb{R}^3} f_0\sqrt{\mu}\mathrm{d}v\mathrm{d}x = 0$ ensures a uniform-in-time bound as $\sup_{0\leq t\leq\infty}||\langle v\rangle^\beta f(t)||_\infty \lesssim ||\langle v\rangle^\beta f_0||_\infty$ (not a decay). We also remark that this estimate is a global-in-x estimate which includes the grazing set γ_0 and the constant grows exponentially with time. Moreover, in Lemma 10, the estimate of (11) in Theorem 1 for $p<2$ is indeed optimal even for the free transport equation $\partial_t f + v\cdot\nabla_x f = 0$ with the diffuse boundary condition. In fact, the boundary integral blows up at $p=2$. We therefore conjecture that $f\notin H^1$ in the bulk.

We now illustrate main ideas of the proof of Theorem 1. Clearly, both t and v derivatives behave nicely for the diffuse boundary condition as for $(x,v)\in\gamma_-$,

$$\partial_t f(t,x,v) = c_\mu\sqrt{\mu(v)}\int_{n(x)\cdot u>0}\partial_t f(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u, \tag{12}$$

$$\nabla_v f(t,x,v) = c_\mu\nabla_v\sqrt{\mu(v)}\int_{n(x)\cdot u>0} f(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u. \tag{13}$$

Let $\tau_1(x)$ and $\tau_2(x)$ be unit vectors satisfying $\tau_1(x)\cdot n(x)=0=\tau_2(x)\cdot n(x)$ and $\tau_1(x)\times\tau_2(x)=n(x)$. Define the orthonormal transformation from $\{n,\tau_1,\tau_2\}$ to the standard bases $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$, i.e. $\mathcal{T}(x)n(x)=\mathbf{e}_1,\ \mathcal{T}(x)\tau_1(x)=\mathbf{e}_2,\ \mathcal{T}(x)\tau_2(x)=\mathbf{e}_3$, and $\mathcal{T}^{-1}=\mathcal{T}^t$. Upon a change of variable: $u'=\mathcal{T}(x)u$, we have

$$n(x)\cdot u = n(x)\cdot\mathcal{T}^t(x)u' = n(x)^t\mathcal{T}^t(x)u' = [\mathcal{T}(x)n(x)]^t u' = \mathbf{e}_1\cdot u' = u_1',$$

then

$$c_\mu\sqrt{\mu(v)}\int_{n(x)\cdot u>0} f(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u = c_\mu\sqrt{\mu(v)}\int_{u_1'>0} f(t,x,\mathcal{T}^t(x)u')\sqrt{\mu(u')}\{u_1'\}\mathrm{d}u',$$

so that we can further take tangential derivatives ∂_{τ_i} as, for $(x,v)\in\gamma_-$,

$$\begin{aligned}
&\partial_{\tau_i} f(t,x,v)\\
&= c_\mu\sqrt{\mu(v)}\int_{u_1'>0}\Big\{\partial_{\tau_i} f(t,x,\mathcal{T}^t(x)u') + \nabla_v f(t,x,\mathcal{T}^t(x)u')\frac{\partial\mathcal{T}^t(x)}{\partial\tau_i}u'\Big\}\sqrt{\mu(u')}\{u_1'\}\mathrm{d}u'\\
&= c_\mu\sqrt{\mu(v)}\int_{n(x)\cdot u>0}\partial_{\tau_i} f(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u\\
&\quad + c_\mu\sqrt{\mu(v)}\int_{n(x)\cdot u>0}\nabla_v f(t,x,u)\frac{\partial\mathcal{T}^t(x)}{\partial\tau_i}\mathcal{T}(x)u\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u.
\end{aligned} \tag{14}$$

The difficulty is always the control of the normal spatial derivative of ∂_n. From the general method of proving regularity in PDE with boundary conditions, it is natural to use the Boltzmann equation to solve the normal derivative $\partial_n f$ inside the region, in terms of $\partial_t f$, $\nabla_v f$, and $\partial_\tau f$ as:

$$\partial_n f(t,x,v) = -\frac{1}{n(x)\cdot v}\left\{\partial_t f + \sum_{i=1}^{2}(v\cdot\tau_i)\partial_{\tau_i} f + \nu(F)f - Kf - \Gamma_{\mathrm{gain}}(f,f)\right\}, \tag{15}$$

at least near $\partial\Omega$. Unfortunately, this standard approach encounters a severe difficulty: $\frac{1}{n(x)\cdot v}\notin L^1_{loc}$ in the velocity space (a L^∞ bound is desirable for any $W^{1,p}$ estimate).

The first new ingredient of our approach is to use (15) *not* inside the domain, but at the boundary $\partial\Omega$. Using special feature of the diffuse boundary condition and (12), (13) and (14), we can express $\partial_n f$ at $(x,v)\in\gamma_-$ as

$$\begin{aligned}
&\partial_n f(t,x,v)\\
&= -\frac{1}{n(x)\cdot v}\Big\{\sqrt{\mu(v)}\int_{n(x)\cdot u>0}\partial_t f(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u\\
&\qquad + \sum_{i=1}^{2}(v\cdot\tau_i)\sqrt{\mu(v)}\int_{n(x)\cdot u>0}\partial_{\tau_i} f(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u\\
&\qquad + \sum_{i=1}^{2}(v\cdot\tau_i)\sqrt{\mu(v)}\int_{n(x)\cdot u>0}\nabla_v f(t,x,u)\frac{\partial\mathcal{T}^t(x)}{\partial\tau_i}\mathcal{T}(x)u\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u\\
&\qquad + \nu(F)f - Kf - \Gamma_{\mathrm{gain}}(f,f)\quad\Big\},
\end{aligned} \tag{16}$$

Due to the additional u integral in (16) and the crucial factor $|n(x)\cdot u|$ in the measure $d\gamma$ on the boundary γ, it is clear that the singularity $|\partial_n f|^p|n\cdot v|$ in (16) is roughly of the order

$$\frac{1}{\{n\cdot v\}^{p-1}},$$

so that its v integration is precisely finite if $1\leq p<2$, and indeed its v integration is *uniformly* bounded with respect to x.

However, in order to control $\partial_t f, \nabla_v f$ and $\partial_\tau f$ for $p<2$, a new difficulty arises. It is well-known from [8, 2] that a crucial boundary estimate for diffuse boundary takes the form of a L^2-contraction:

$$\int_{\gamma_-} h^2 \mathrm{d}\gamma \leq \int_{\gamma_+} h^2 \mathrm{d}\gamma.$$

Unfortunately, this is not expected to be valid for $p\neq 2$, so it is impossible to absorb the incoming part γ_- solely by the outgoing part γ_+ part.

Our second new ingredient is to split the γ_+ integral into near grazing set γ_+^ε and the rest for $p\neq 2$ for our boundary representation for derivatives (12), (13), (14), and (16). For small $\varepsilon>0$ we define γ_+^ε, the set of almost grazing velocities or large velocities

$$\gamma_+^\varepsilon=\{(x,v)\in\gamma_+ : v\cdot n(x)<\varepsilon \text{ or } |v|>1/\varepsilon\}. \tag{17}$$

Denote $\partial=[\partial_t,\nabla_x,\nabla_v]$. We can roughly obtain

$$\begin{aligned}\int_{\gamma_-}|\partial f|^p &\lesssim \int_{\partial\Omega}\left(\int_{n\cdot v>0}|\partial f|\mu^{1/4}\{n\cdot v\}\mathrm{d}v\right)^p + \text{good terms},\\ &\lesssim \int_{\partial\Omega}\left(\int_{\{v:(x,v)\in\gamma_+^\varepsilon\}}|\partial f|\mu^{1/4}\{n\cdot v\}\right)^p+\int_{\partial\Omega}\left(\int_{\{v:(x,v)\in\gamma_+\setminus\gamma_+^\varepsilon\}}|\partial f|\mu^{1/4}\{n\cdot v\}\right)^p + \text{good terms},\\ &\lesssim \sup_x\left(\int_{\{v:(x,v)\in\gamma_+^\varepsilon\}}\mu^{q/4}\{n\cdot v\}\mathrm{d}v\right)^{p/q}\int_{\gamma_+^\varepsilon}|\partial f|^p\mathrm{d}\gamma+\int_{\gamma_+\setminus\gamma_+^\varepsilon}|\partial f|^p\mathrm{d}\gamma+\text{good terms}.\end{aligned}$$

It is important to realize that $\sup_x\left(\int_{\{v:(x,v)\in\gamma_+^\varepsilon\}}\mu^{q/4}\{n\cdot v\}\mathrm{d}v\right)^{p/q}$ has a small measure of order ε, for $p>1$, so that it can be absorbed by the outgoing part $\int_{\gamma_+}$. Fortunately, the outgoing boundary integral $\int_{\gamma_+\setminus\gamma_+^\varepsilon}$ can be further bounded by the integration in the bulk and initial data by Lemma 7 with a crucial time integration. On the other hand, such a process produces a large constant in the Gronwall estimates and leads to a growth in time. Of course, such approach breaks down at $p=1$.

Theorem 2. *Assume the compatibility condition (10) and recall (9) and $0<\kappa\leq 1$.*

For any fixed $2\leq p<\infty$ and $\frac{p-2}{2p}<\beta<\frac{p-1}{2p}$, if $||\alpha^\beta\nabla_{t,x,v}f_0||_p+||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty<\infty$ for some $0<\theta<\frac14, 0<\zeta$, then there exists $T=T(||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty)>0$ such that $e^{-\varpi\langle v\rangle t}\alpha^\beta\partial_t f, e^{-\varpi\langle v\rangle t}\alpha^\beta\nabla_x f, e^{-\varpi\langle v\rangle t}\alpha^\beta\nabla_v f\in L^\infty_{loc}([0,T];L^p(\Omega\times\mathbb{R}^3))$ such that for all $0\leq t\leq T$,

$$\begin{aligned}&||e^{-\varpi\langle v\rangle t}\alpha^\beta\nabla_{t,x,v}f(t)||_p^p+\int_0^t|e^{-\varpi\langle v\rangle t}\alpha^\beta\nabla_{t,x,v}f(s)|_{\gamma,p}^p\mathrm{d}s\\ &\lesssim_T\ ||\alpha^\beta\nabla_{t,x,v}f_0||_p^p+P(||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty^p),\end{aligned}$$

where P is some polynomial.

If $||\alpha^{1/2}\nabla_{t,x,v}f_0||_\infty+||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty<+\infty$ for some $0<\theta<\frac14,\zeta>0$, then $e^{-\varpi\langle v\rangle t}\alpha^{1/2}\nabla_{t,x,v}f\in L^\infty([0,T];L^\infty(\Omega\times\mathbb{R}^3))$ such that for all $0\leq t\leq T$,

$$||e^{-\varpi\langle v\rangle t}\alpha^{1/2}\nabla_{t,x,v}f(t)||_\infty\ \lesssim_t\ ||\alpha^{1/2}\nabla_{t,x,v}f_0||_\infty+P(||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty).$$

If $\alpha^{1/2}\nabla f_0\in C^0(\bar\Omega\times\mathbb{R}^3)$ and

$$v\cdot\nabla_x f_0+\nu(F_0)f_0-\Gamma(f_0,f_0)=c_\mu\sqrt{\mu}\int_{n\cdot u>0}\{u\cdot\nabla_x f_0+\nu(F_0)f_0-\Gamma(f_0,f_0)\}\sqrt{\mu}\{n\cdot u\}\mathrm{d}u, \tag{18}$$

is valid for $\gamma_- \cup \gamma_0$, then $f \in C^1$ away from the grazing set γ_0.

Furthermore, if $||\langle v\rangle^{\zeta} e^{\theta|v|^2} f_0||_\infty \ll 1$ then T can be arbitrarily large.

There can be no size restriction on initial data. Remark that $||\langle v\rangle^{\zeta} e^{\theta|v|^2} f_0||_\infty \ll 1$ ensures a uniform-in-time bound $\sup_{0\leq t\leq\infty} ||\langle v\rangle^{\zeta} e^{\theta|v|^2} f(t)||_\infty \lesssim ||\langle v\rangle^{\zeta} e^{\theta|v|^2} f_0||_\infty$ due to [8, 2]. We remark for $\varpi \neq 0$, $\partial f(t) \sim e^{\varpi\langle v\rangle t}$ so that in terms of solution $f(t)$, such an estimate not only creates an exponential growth in time, but also creates less integrability in velocity. Furthermore, when $\varpi \neq 0$, we crucially need a strong weight function $e^{\zeta|v|^2}$ to balance such a factor $e^{-\varpi\langle v\rangle t}$, which produces a super exponential growth e^{t^2} in time in controls of the non-local collision operator. We suspect that it is impossible to obtain a uniform in time estimate especially when $\varpi \neq 0$. Distance functions α play an important role in the study of regularity in convex domains for Vlasov equations ([6, 10]), which can be controlled along the characteristics via so-called the geometric Velocity Lemma(Lemma 1). However, such an approach has not been successful in the study of Boltzmann equation due to the non-local nature of the Boltzmann collision operator, which mixes up different velocities so that their distance towards γ_0 can not be controlled. In addition to the key boundary representation, we establish a delicate estimate for interaction of $e^{-\varpi\langle v\rangle t}\alpha(x,v)$ and the collision kernel $e^{-\varpi\langle v\rangle t}\alpha(x,v)^{\beta} K(\frac{f}{e^{-\varpi\langle v\rangle t}\alpha^{\beta}})$ in (95) for $\beta < \frac{p-1}{2p}$. An additional requirement $\beta > \frac{p-2}{2p}$ is needed to control the boundary singularity in (99). These estimates are sufficient to treat the case for $\beta < 1/2$, but unfortunately these fail for the case $\beta = 1/2$, which accounts for the important C^1 estimate. In order to establish the C^1 estimate, we employ the Lagrangian view point, estimating along the stochastic cycles [8, 2] or Definition 1.

Our fourth new ingredient is the dynamical non-local to local estimates(Lemma 2). Even though $e^{-\varpi\langle v\rangle t}\alpha(x,v)^{1/2} K(\frac{f}{e^{-\varpi\langle v\rangle t}\alpha(x,v)^{1/2}})$ is impossible to estimate directly due to severe singularity of $\frac{1}{e^{-\varpi\langle v\rangle t}\alpha(x,v)^{1/2}}$ in the velocity space, along the characteristics, $\frac{1}{e^{-\varpi\langle v\rangle s}\alpha(x-sv,v)^{1/2}}$ is integrable in time for a convex domain. Therefore, the integral

$$\int_0^t \int_{\mathbb{R}^3} e^{-\varpi\langle v\rangle(t-s)}\alpha(x,v)^{1/2} K(\frac{f}{e^{-\varpi\langle v\rangle(t-s)}\alpha(x-(t-s)v,v)^{1/2}})$$

can be controlled by first integrating over time, and we can close the estimate.

2. Dynamical non-local to local estimates

Lemma 2. *Let $(t,x,v) \in [0,\infty)\times\bar{\Omega}\times\mathbb{R}^3$.*

(1) For $\frac{1}{2} < \beta < \frac{3}{2}$ and $0 < \kappa \leq 1$ and $r \in \mathbb{R}$ and $\varepsilon > 0$, we have

$$\begin{aligned}
&\int_0^{t_{\mathbf{b}}(x,v)} \int_{\mathbb{R}^3} e^{-l\langle v\rangle(t-s)} \frac{e^{-\theta|v-u|^2}}{|v-u|^{2-\kappa}[\alpha(x-(t_{\mathbf{b}}(x,v)-s)v,u)]^{\beta}} \frac{\langle u\rangle^r}{\langle v\rangle^r} Z(s,x,v)\mathrm{d}u\mathrm{d}s \\
&\lesssim_{\theta,r} \min\left\{\frac{\varepsilon^{\frac{3}{2}-\beta}}{|v|^2\{\alpha(x,v)\}^{\beta-1}}, \frac{\{\alpha(x,v)\}^{\frac{3}{4}-\frac{\beta}{2}}|t_Z|^{\frac{3}{2}-\beta}}{|v|^{2\beta-1}}\right\} \sup_{s\in[0,t_{\mathbf{b}}(x,v)]}\{e^{-l\langle v\rangle(t-s)}Z(s,x,v)\} \qquad (19)\\
&\quad + \frac{1}{\varepsilon^2\{\alpha(x,v)\}^{\beta-1/2}} \int_0^{t_{\mathbf{b}}(x,v)} e^{-Cl\langle v\rangle(t-s)} Z(s,x,v)\mathrm{d}s,
\end{aligned}$$

where $t_Z = \sup\{s : Z(s,x,v) \neq 0\}$.

(2) Let $[X_{\mathbf{cl}}(s;t,x,v), V_{\mathbf{cl}}(s;t,x,v)]$ be the specular backward trajectory or the bounce-back trajectory in Definition 1.

For $\frac{1}{2} < \beta < \frac{3}{2}$ and $0 < \kappa \leq 1$ and $\delta > 0$ and $1 \gg \tilde{\delta} > 0$ and $r \in \mathbb{R}$, there exists $l \gg_{\xi} 1$ and $C_{l,\beta,\xi,r} > 0, C_{\tilde{\delta},\delta,\beta} > 0$ such that

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^3}e^{-l\langle v\rangle(t-s)}\frac{e^{-\theta|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}}\frac{\langle u\rangle^r}{\langle v\rangle^r}\frac{Z(s,x,v)}{\big[\alpha(X_{\mathbf{cl}}(s;t,x,v),u)\big]^\beta}\mathrm{d}u\mathrm{d}s\\&\lesssim_{\xi,r}\frac{C_\delta O(\tilde{\delta})+C_{\tilde{\delta},\delta}O(l^{-1})}{\langle v\rangle\big[\alpha(x,v)\big]^{\beta-1/2}}\sup_{0\le s\le t}\big\{e^{-C_{l,\beta,\xi,r}\langle v\rangle(t-s)}Z(s,x,v)\big\}.\end{aligned}\tag{20}$$

The control of $K(\frac{\cdot}{\alpha^\beta})$ is addressed throughout such so-called non-local to local estimates. We discover that the non-local u integration does not destroy the local property, upon a crucial time integration along the characteristics. The proof of such non-local to local estimates are a combination of analytical and geometrical arguments. The first part is a precise estimate of u integration which is bounded via $\frac{1}{|v|^{2\beta-1}|\xi(x-vt)|^{\beta-1/2}}$. In this part of the proof we make use of a series of change of variables to obtain the precise power. The second part is to relate $\frac{1}{|\xi(x-vt)|^{\beta-1/2}}$ back to $\frac{1}{\alpha}$. Clearly,

$$\frac{1}{|\xi(x-vt)|^2}\backsim\frac{1}{\alpha}\backsim\frac{1}{v\cdot\nabla\xi(x-vt\}^2+\xi(x-vt)|v|^2}.$$

for $|\xi||v|^2$ is larger than $|v\cdot\nabla\xi|$. On the other hand, when $|v\cdot\nabla\xi|$ dominates, this can only be achieved through a crucial use of time integration and geometric Velocity Lemma(Lemma 1), by connecting

$$\mathrm{d}t\backsim\frac{\mathrm{d}\xi}{|v\cdot\nabla\xi|}$$

and recover α as in the bound of ξ integration through the geometric Velocity Lemma(Lemma 1).

The more striking feature is that not only our estimates retain the local structure for α, but they *gain* $\sqrt{\alpha}$ order of regularity. Such a precise gain of regularity is exactly enough to balance out the singularity in α appeared in $\partial X_{\mathbf{cl}}(s;t,x,v)$ and $\partial V_{\mathbf{cl}}(s;t,x,v)$ in both the specular and bounce-back cycles. In order to squeeze out a small constant for $|v|\gg 1$, we need to use the decay of $e^{-l\langle v\rangle(t-s)}$. This requires a precise regrouping of the cycles according to the time scale of

$$t|v|\sim 1.$$

Within such an important time scale, $V_{\mathbf{cl}}(s;t,x,v)$ stays almost *invariant* due to the Velocity Lemma(Lemma 1). We then are able to obtain precise estimate for the number of bounces within $t|v|\backsim 1$ and extract smallness from $e^{-l\langle v\rangle(t-s)}$ for $t-s\ge\frac{1}{|v|}$. On the other hand, for $t-s\le\frac{1}{|v|}$, the smallness comes from Lemma 2.

3. Specular Reflection

Recall the specular reflection (7) and the specular cycles in Definition 1. Our main theorem is as follows.

Theorem 3. *Assume $f_0\in W^{1,\infty}(\Omega\times\mathbb{R}^3)$ and $0<\kappa\le 1$ for $1<\beta<\frac{3}{2},\ 0<\zeta,\ 0\le\theta,$ and $b\in\mathbb{R}$,*

$$\Big|\Big|\frac{\alpha^{\beta-\frac12}}{\langle v\rangle^b}\partial_x f_0\Big|\Big|_\infty+\Big|\Big|\frac{|v|^2\alpha^{\beta-1}}{\langle v\rangle^b}\partial_v f_0\Big|\Big|_\infty+||\partial_t f_0||_\infty+||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty<\infty,$$

and the compatibility condition on $(x,v)\in\gamma_-$

$$f_0(x,v)=f_0(x,R_x(v)).\tag{21}$$

Then for $T=T(||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty)>0$ we have for all $0\le t\le T$

$$\begin{aligned}&||e^{-\varpi\langle v\rangle t}\frac{\alpha^\beta}{\langle v\rangle^{b+1}}\partial_x f(t)||_\infty+||e^{-\varpi\langle v\rangle t}\frac{|v|\alpha^{\beta-1/2}}{\langle v\rangle^b}\partial_v f(t)||_\infty+||\partial_t f(t)||_\infty\\&\lesssim_{\xi,t}\Big|\Big|\frac{\alpha^{\beta-\frac12}}{\langle v\rangle^b}\partial_x f_0\Big|\Big|_\infty+\Big|\Big|\frac{|v|^2\alpha^{\beta-1}}{\langle v\rangle^b}\partial_v f_0\Big|\Big|_\infty+||\partial_t f_0||_\infty+P(||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty).\end{aligned}\tag{22}$$

Furthermore, if $f_0 \in C^1$ and satisfies

$$[v\cdot\nabla_x f_0+\nu(F_0)f_0-Kf_0-\Gamma_{\text{gain}}(f_0,f_0)](x,v)=[v\cdot\nabla_x f_0+\nu(F_0)f_0-Kf_0-\Gamma_{\text{gain}}(f_0,f_0)](x,R_xv) \tag{23}$$

is valid for $\gamma_-\cup\gamma_0$ then $f\in C^1$ away from the grazing set γ_0.

Moreover, of Ω is real analytic(ξ is real analytic) and $||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty \ll 1$ then this theorem holds for the arbitrarily large time.

There can be no size restriction on initial data. We remark from the local existence theorem, $T>0$. The analyticity is a crucial assumption to ensure global stability in [8]. We also remark that the specular theorem is drastically different from the diffusive theorem: in addition to the loss of moments, there is a loss of regularity of α with respect to the initial data. This makes it impossible to use the continuity argument to choose small time interval to close the estimates. We need to use large ϖ in $e^{-\varpi\langle v\rangle t}$ to extract a small constant to close, which requires extra precise estimates. We note that in 3D case, $\beta>1/2$, due to the failure of the proof of the non-local to local estimates for the critical $\beta=1/2$(Lemma 2). On the other hand, in 2D, due to boundedness of $\partial_{v_3}f$, which is rather direct from the symmetry, we are able to estimate $\partial_v\Gamma_{\text{gain}}$ for the critical case $\beta=1/2$ by Lemma 15.

In additional to the dynamical non-local to local estimate, the second important ingredient for the specular reflection BC is the following crucial estimate for the derivatives of specular cycles $[X_{\mathbf{cl}}(s;t,x,v)$,$V_{\mathbf{cl}}(s;t,x,v)]$.

Theorem 4. *Assume Ω is convex (2). Then there exists $C=C(\Omega)>0$ such that for all $(s;t,x,v)\in\mathbb{R}\times\mathbb{R}\times\bar{\Omega}\times\mathbb{R}^3$,*

$$\begin{aligned}
|\partial_x X_{\mathbf{cl}}(s;t,x,v)| &\lesssim e^{C|v|(t-s)}\frac{|v|}{\sqrt{\alpha(x,v)}},\\
|\partial_v X_{\mathbf{cl}}(s;t,x,v)| &\lesssim e^{C|v|(t-s)}\frac{1}{|v|},\\
|\partial_x V_{\mathbf{cl}}(s;t,x,v)| &\lesssim e^{C|v|(t-s)}\frac{|v|^3}{\alpha(x,v)},\\
|\partial_v V_{\mathbf{cl}}(s;t,x,v)| &\lesssim e^{C|v|(t-s)}\frac{|v|}{\sqrt{\alpha(x,v)}}.
\end{aligned} \tag{24}$$

Our estimates are optimal in terms of the order of $\frac{1}{\alpha}$, and $e^{C|v|(t-s)}$ relates to the $|v|$ growth in the Velocity Lemma(Lemma 1). We remark that these precise orders of singularity, play a critical role for our design of the anisotropic norms. In fact, if $|\partial_x X_{\mathbf{cl}}(s;t,x,v)|\backsim\frac{1}{\alpha}$, it would have been too singular for the half power gain of α from the dynamical non-local to local estimate, and our method should fail. Moreover, it is also crucial to have precise $|v|$ growth in both $|\partial_x X_{\mathbf{cl}}(s;t,x,v)|,|\partial_v V_{\mathbf{cl}}(s;t,x,v)|$ to be controlled by $e^{-\varpi\langle v\rangle t}$.

We remark that $|\partial_x X_{\mathbf{cl}}(s;t,x,v)|\backsim\frac{1}{\sqrt{\alpha}}$ is unexpected. Even after one bounce, $\partial_x X_{\mathbf{cl}}(s;t,x,v)\backsim\frac{1}{\sqrt{\alpha}}$ and it is natural to expect $\partial_x X_{\mathbf{cl}}(s;t,x,v)$ picks up additional power of $\frac{1}{\sqrt{\alpha}}$ in the accumulation of $\frac{1}{\sqrt{\alpha}}$ number of bounces. However, via direct computations in 2D disk, we discover that even though

$$\partial_x t^\ell\backsim\frac{1}{\alpha},\ \text{and}\ \partial_x x^\ell\backsim\frac{1}{\alpha},$$

but surprisingly

$$\partial_x X_{\mathbf{cl}}(s;t,x,v)=\partial_x[x^\ell-(t^\ell-s)v^\ell]\backsim\frac{1}{\sqrt{\alpha}}\ !$$

Clearly, certain cancellations take place in the disk, which is difficult to even expect for general domains.

The proof of our theorem is split into nine steps, and it is the most delicate proof throughout this paper. We first remark that, due to the 'discontinuous behaviors' of the normal component of $v\cdot n$ at each specular reflection, it is impossible to apply the standard techniques for ODE to estimate $|\partial X_{\mathbf{cl}}(s;t,x,v)|$ and $|\partial V_{\mathbf{cl}}(s;t,x,v)|$. We have to develop different strategies to overcome several analytical difficulties to finally complete the proof.

Topological obstruction and moving frames. It turns out that we only need to consider the most delicate case in which all the bounces are almost grazing and staying near the boundary for $\mathbf{r}^\ell = \frac{|v^\ell \cdot n|}{|v^\ell|} \ll 1$. It is important for us to introduce the spherical co-ordinate system to cover the whole cycle and transform it into the ODE (114). Unfortunately, due to 'hair-ball' theorem in Topology, such a change of coordinate system (or any change of coordinates) can not be smooth everywhere in the 2D surface $\partial\Omega$. In the case of a ball, all the trajectories are confined in a plane, so that one may choose a single chart to cover the whole trajectories. However, in other convex domains except the ball case, with large t for the specular reflection case, trajectories are extremely complicated, which can reach almost every point so that choosing a single chart is all but impossible. On the other hand, a 'sudden' change of a chart may create new order of singularity of α from the matrix $\mathcal{P}$ as in (142). which will ruin the estimates. It is therefore important to design a 'continuous' changes of charts associated with the almost grazing bounces. Given $n(x)$, we need to construct another globally defined, orthogonal, and continuous vector field. This would have been impossible if we were to seek it only in the physical space, in light of the 'hair-ball' theorem. The key observation is that, we need continuity not from just $\partial\Omega$, but from the phase space $\partial\Omega \times \mathbb{R}^3$. In fact, for almost grazing bounces, the velocity field v is almost perpendicular to $n(x)$, which provides a natural choice for construction of the desired moving frames. These continuous moving frames cost manageable errors for each bounce, which are controlled by the next method.

Matrix Method for normal parts of $\partial X_{\mathbf{cl}}(s)$ *and* $\partial V_{\mathbf{cl}}(s)$. With such a well-defined moving charts, via the chain rule, one can represent $\partial X_{\mathbf{cl}}(s;t,x,v)$ and $\partial V_{\mathbf{cl}}(s;t,x,v)$ via a multiplication of Jacobian matrices $(t^\ell, x^\ell, v^\ell) \to (t^{\ell-1}, x^{\ell-1}, v^{\ell-1})$ in the spherical coordinate system. The 'matrix method' refers to the study of each discrete Jacobian matrix and precise estimate of their multiplication ($\frac{1}{\sqrt{\alpha}}$ of them!). One important step is to bound such a matrix by $J(\mathbf{r}^\ell)$ in (138) which can be diagonalized as $J(\mathbf{r}^\ell) = \mathcal{P}^{-1}\Lambda\mathcal{P}$, with a diagonal matrix Λ. Based on the crucial cancellation property (**141**) we extract crucial second order of $\mathbf{r}^\ell \ll 1$ appeared in $J(\mathbf{r}^\ell)$. Therefore, over the interval $t|v| \backsim 1$, we are able to estimate $\Pi_{\ell=1}^{\frac{1}{\sqrt{\alpha}}} J(\mathbf{r}^\ell) \sim \frac{1}{\sqrt{\alpha}}$. Together with $\frac{1}{\sqrt{\alpha}}$ from the initial bounce, we expect a $\frac{1}{\alpha}$ for $\partial X_{\mathbf{cl}}(s;t,x,v)$ and $\partial V_{\mathbf{cl}}(s;t,x,v)$ as in (151). Even though such estimate is too singular for our purpose, upon a closed inspection, we can improve that for the normal component of $X_{\mathbf{cl}}(s)$,

$$|\partial_x \mathbf{X}_\perp(s;t,x,v)| \lesssim \frac{1}{\sqrt{\alpha}},$$

which is based on the fact $v_\perp^\ell \sim \sqrt{\alpha}$ via the Velocity Lemma(Lemma 1, [8]). Unfortunately, the tangential part $\partial_x \mathbf{X}_{||}(s;t,x,v) \backsim \frac{1}{\alpha}$ is still too singular.

ODE Method for tangential parts of $\partial X_{\mathbf{cl}}(s)$ *and* $\partial V_{\mathbf{cl}}(s)$. To improve such estimate, we observe that given the estimates for the normal parts $[\mathbf{X}_\perp(s;t,x,v), \mathbf{V}_\perp(s;t,x,v)]$, the sub-system of ODE for $[\mathbf{X}_{||}(s;t,x,v), \mathbf{V}_{||}(s;t,x,v)]$, enjoys much better property. In fact, at each specular reflection, $[\mathbf{X}_{||}(s;t,x,v), \mathbf{V}_{||}(s;t,x,v)]$ are continuous, unlike the the normal $\mathbf{V}_\perp(s;t,x,v)$. Upon integrating over time as $\mathbf{V}_\perp(s;t,x,v) = \dot{\mathbf{X}}_\perp(s;t,x,v)$ (position $\mathbf{X}_\perp(s;t,x,v)$ is still continuous at specular reflection), we are able to derive an integral equations of $[\mathbf{X}_{||}(s;t,x,v), \mathbf{V}_{||}(s;t,x,v)]$ without broken into small discontinuous pieces (159) at each specular reflection. In other words, we can use the standard ODE theory to estimate these tangential parts. Our ODE method refers such ODE (Gronwall) estimates (155) which lead to the final conclusion of the theorem.

With such crucial estimates, we are able to design anisotropic norms in terms of singularity of $\frac{1}{\alpha}$ so that along a specular cycle of $[X_{\mathbf{cl}}(s),\, V_{\mathbf{cl}}(s)]$, very formally, around $|v| \backsim 1$,

$$\begin{aligned}
|\alpha^\beta \nabla_x f_\infty| &\lesssim \text{good} + \int_0^t \alpha^\beta |\partial_x X_{\mathbf{cl}} K(|\nabla_x f|)| + \int_0^t \alpha^\beta |\partial_x V_{\mathbf{cl}} K(\nabla_v f)| \\
&\lesssim \text{good} + \int_0^t \alpha^{\beta-1/2} |K(\frac{1}{\alpha^\beta})| \left\|\alpha^\beta \nabla_x f\right\|_\infty + \int_0^t \alpha^{\beta-1} |K(\frac{1}{\alpha^{\beta-1/2}})| \left\|\alpha^{\beta-1/2}\nabla_v f\right\|_\infty, \\
\left|\alpha^{\beta-1/2}\nabla_v f\right| &\lesssim \text{good} + \int_0^t \left|\alpha^{\beta-1/2}\partial_v X_{\mathbf{cl}} K(\nabla_x f)\right| + \int_0^t |\alpha^{\beta-1/2}\partial_v V_{\mathbf{cl}} K(\nabla_v f)| \\
&\lesssim \text{good} + \int_0^t \alpha^{\beta-1/2} |K(\frac{1}{\alpha^\beta})| \left\|\alpha^\beta \nabla_x f\right\|_\infty + \int_0^t \alpha^{\beta-1} |K(\frac{1}{\alpha^{\beta-1/2}})| \left\|\alpha^{\beta-1/2}\nabla_v f\right\|_\infty.
\end{aligned}$$

Thanks to $\int_0^t |K(\frac{1}{\alpha^\beta})| \lesssim \alpha^{1/2-\beta}$ and $\int_0^t |K(\frac{1}{\alpha^{\beta-1/2}})| \lesssim \alpha^{1-\beta}$ from the dynamical non-local to local estimates for $\beta > 1$, we have exact cancellations of the power of α in the coefficients on the right hand side, and we are able to close. For $|v|$ either small or large, more careful analysis is needed. In particular, it is important to use the weight function of $e^{-\varpi\langle v\rangle t}$ in (3) to control both the growth in Theorem 4 as well as $|v|$ in front of $\partial_x X_{\mathbf{cl}}$ and $\partial_v V_{\mathbf{cl}}$.

4. Bounce-back reflection

We recall the bounce-back reflection boundary condition (8) and the bounce-back cycles in Definition 1. Our main theorem is

Theorem 5. *Assume $f_0 \in W^{1,\infty}(\Omega\times\mathbb{R}^3)$ and $0 < \kappa \le 1$ for $\theta > 0, \zeta > 0$,*

$$||\langle v\rangle \partial_x f_0||_\infty + ||\partial_v f_0||_\infty + ||\langle v\rangle^\zeta e^{\theta|v|^2}\partial_t f_0||_\infty + ||\langle v\rangle^\zeta e^{\theta|v|^2} f_0||_\infty < +\infty,$$

and the compatibility condition on $(x,v)\in\gamma_-$

$$f_0(x,v) = f_0(x,-v). \tag{25}$$

Then for $T = T(||\langle v\rangle^\zeta e^{\theta|v|^2} f_0||_\infty) > 0$ we have for all $0 \le t \le T$

$$\begin{aligned}
&||e^{-\varpi\langle v\rangle t}\frac{\alpha}{\langle v\rangle^2}\partial_x f(t)||_\infty + ||e^{-\varpi\langle v\rangle t}\frac{|v|\alpha^{1/2}}{\langle v\rangle^2}\partial_v f(t)||_\infty + ||\langle v\rangle^\zeta e^{\theta|v|^2}\partial_t f(t)||_\infty \\
&\lesssim_{\xi,t} ||\langle v\rangle \partial_x f_0||_\infty + ||\partial_v f_0||_\infty + P(||\langle v\rangle^\zeta e^{\theta|v|^2}\partial_t f_0||_\infty) + P(||\langle v\rangle^\zeta e^{\theta|v|^2} f_0||_\infty).
\end{aligned} \tag{26}$$

Furthermore, if $f_0 \in C^1$ and satisfies

$$[v\cdot\nabla_x f_0 + \nu(F_0)f_0 - Kf_0 - \Gamma_{\text{gain}}(f_0,f_0)](x,v) = [v\cdot\nabla_x f_0 + \nu(F_0)f_0 - Kf_0 - \Gamma_{\text{gain}}(f_0,f_0)](x,-v) \tag{27}$$

is valid for $\gamma_-\cup\gamma_0$ then $f\in C^1$ away from the grazing set γ_0.

Moreover, if $||\langle v\rangle^\zeta e^{\theta|v|^2} f_0||_\infty \ll 1$ then this theorem holds for the arbitrarily large time.

There can be no size restriction on initial data. We remark that the bounce-back case enjoys the advantage of explicit expression of $\partial X_{\mathbf{cl}}(s;t,x,v)$ and $\partial V_{\mathbf{cl}}(s;t,x,v)$. Since $\partial_x t^\ell \sim \frac{1}{\alpha}$ and $\partial_x x^\ell \sim \frac{1}{\sqrt{\alpha}}$, a new difficulty arises in the estimate

$$\partial_x X_{\mathbf{cl}}(s;t,x,v) \sim \frac{1}{\alpha},$$

which is too singular to control by the non-local to local estimates. Roughly speaking, the new difficulty is exactly the opposite to the specular case, ∂x^ℓ and ∂v^ℓ are in desired form but not $\partial_x X_{\mathbf{cl}}(s;t,x,v)$. The crucial observation is the following:

Lemma 3. *In the sense of distribution,*

$$
\begin{aligned}
&\partial_{\mathbf{e}}\Big[\int_{t^{j+1}}^{t^j} f(\tau, x^j-(t^j-\tau)v^j, v^j)\mathrm{d}\tau\Big]\\
&=\int_{t^{j+1}}^{t^j}\big[\partial_{\mathbf{e}}t^j, \partial_{\mathbf{e}}x^j+\tau\partial_{\mathbf{e}}v^j, \partial_{\mathbf{e}}v^j\big]\cdot\nabla_{t,x,v}f(\tau, x^j-(t^j-\tau)v^j, v^j)\mathrm{d}\tau\\
&+\lim_{\tau\uparrow t^j}\big[\partial_{\mathbf{e}}t^j f(\tau, x^j-(t^j-\tau)v^j, v^j)\big]+\lim_{\tau\downarrow t^{j+1}}\big[\partial_{\mathbf{e}}t^{j+1} f(\tau, x^j-(t^j-\tau)v^j, v^j)\big].
\end{aligned}
$$

The key idea is to make a change of variable to transform

$$\partial_x X_{\mathbf{cl}}(s;t,x,v)\backsim v^\ell\partial_x t^\ell+\partial_x x^\ell,$$

while $\partial_x t^\ell$ captures the worst singulairty of $\frac{1}{\alpha}$. Fortunately, $\partial_x t^\ell$ is paired with $\partial_t f$, which is bounded, rather direct from the time-invariance of the problem and we are able to close the estimate.

5. Non-existence of $\nabla^2 f$ up to the boundary

In the appendix, we demonstrate that, our estimates can not be valid for higher order derivatives. Otherwise, if $\partial^2 f$ exists up to the boundary, we observe that from taking second derivatives of the Boltzmann equation:

$$v_n\partial_n^2 f=-\partial_{tn}f-(\partial_n v_n)\partial_n f-\sum_{i=1}^2\partial_n(v_{\tau_i})\partial_{\tau_i}f-\sum_{i=1}^2 v_{\tau_i\partial_{n\tau_i}}f-\nu(F)\partial_n f+K(\partial_n f)+\partial_n\Gamma_{\text{gain}}(f,f).$$

If $|\partial_n f|\geq\frac{1}{\sqrt{\alpha}}$, then at the boundary we have

$$|\partial_n f|\geq\frac{1}{|v_n|}\notin L^1_{loc}(\mathbb{R}^1)$$

so that $K(\partial_n f)$ is not defined. Since $|\partial_n f|$ is expected to behave at least as bad as $\frac{1}{\sqrt{\alpha}}$ for all diffusive, specular and bounce-back cases, we are able to identify initial conditions such that $|\partial_n f|\geq\frac{1}{|v_n|}$ for some future time.

Remark that by the trace theorem(Lemma 7) our negative results imply that second derivatives can not exist up to the boundary, so that they can only exist in the interior of Ω. Due to the transport property of the Boltzmann equation this is highly unlikely with non-flat boundaries.

1. Preliminary

For the hard potential $0<\kappa\leq 1$ and the global Maxwellian $\mu(v)=e^{-\frac{|v|^2}{2}}$,

$$Kg:=\int_{\mathbb{R}^3}\mathbf{k}_2(v,u)g(u)\mathrm{d}u-\int_{\mathbb{R}^3}\mathbf{k}_1(v,u)g(u)\mathrm{d}u,$$

where

$$
\begin{aligned}
\mathbf{k}_1(u,v)&=|u-v|^\kappa e^{-\frac{|v|^2+|u|^2}{2}}\int_{\mathbb{S}^2}q_0(\frac{v-u}{|v-u|}\cdot\omega)\mathrm{d}\omega,\\
\mathbf{k}_2(u,v)&=\frac{2}{|u-v|^2}e^{-\frac{1}{8}|u-v|^2-\frac{1}{8}\frac{(|u|^2-|v|^2)^2}{|u-v|^2}}\\
&\quad\times\int_{w\cdot(u-v)=0}q_0\Big(\frac{u-v}{\sqrt{|u-v|^2+|w|^2}}\cdot\frac{u-v}{|u-v|}\Big)e^{-|w+\varsigma|^2}(|w|^2+|u-v|^2)^{\kappa/2}\mathrm{d}w,
\end{aligned}\tag{28}
$$

where $\varsigma:=\big(\frac{v+u}{2}\cdot\frac{w}{|w|}\big)\frac{w}{|w|}$. See page 315 of [7] for details.

The gain term of the nonlinear Boltzmann operator equals

$$
\begin{aligned}
&\Gamma_{\mathrm{gain}}(g_1,g_2)(v)\\
=&C\int_{\mathbb{R}^3}\mathrm{d}u_\parallel\int_{u_\parallel\cdot u_\perp=0}\mathrm{d}u_\perp\ g_1(v+u_\perp)g_2(v+u_\parallel)q_0^*\big(\frac{|u_\parallel|}{|u_\parallel+u_\perp|}\big)\frac{|u_\parallel+u_\perp|^{\kappa-1}}{|u_\parallel|}e^{-\frac{|u_\parallel+v+u_\perp|^2}{4}},\\
=&C\int_{\mathbb{R}^3}\mathrm{d}u_\parallel\int_{u_\parallel\cdot u_\perp=0}\mathrm{d}u_\perp\ g_2(v+u_\perp)g_1(v+u_\parallel)q_0^*\big(\frac{|u_\parallel|}{|u_\parallel+u_\perp|}\big)\frac{|u_\parallel+u_\perp|^{\kappa-1}}{|u_\parallel|}e^{-\frac{|u_\parallel+v+u_\perp|^2}{4}},\\
=&C\int_{\mathbb{R}^3}\mathrm{d}u_\parallel\int_{(u_\parallel-v)\cdot u_\perp=0}\mathrm{d}u_\perp\ g_1(v+u_\perp)g_2(u_\parallel)q_0^*\big(\frac{|u_\parallel-v|}{|u_\parallel-v+u_\perp|}\big)\frac{\big|u_\parallel-v+u_\perp\big|^{\kappa-1}}{|u_\parallel-v|}e^{-\frac{|u_\parallel+u_\perp|^2}{4}},\\
=&C\int_{\mathbb{R}^3}\mathrm{d}u_\parallel\int_{(u_\parallel-v)\cdot u_\perp=0}\mathrm{d}u_\perp\ g_2(v+u_\perp)g_1(u_\parallel)q_0^*\big(\frac{|u_\parallel-v|}{|u_\parallel-v+u_\perp|}\big)\frac{\big|u_\parallel-v+u_\perp\big|^{\kappa-1}}{|u_\parallel-v|}e^{-\frac{|u_\parallel+u_\perp|^2}{4}},
\end{aligned}
\tag{29}
$$

where $q_0^*(\cos\theta)=\frac{q_0(\cos\theta)}{|\cos\theta|}$. This is due to two change of variables (37),(38) and page 316 of [7].

We define the convenient notation

$$
\mathbf{k}_{\kappa,\varrho}(v,u):=\frac{1}{|v-u|^{2-\kappa}}e^{-\frac{\varrho}{8}|v-u|^2}. \tag{30}
$$

Lemma 4. *Recall (28) and the Grad estimate* [9] *for hard potential,* $0\le\kappa\le 1$*,*

$$
|\mathbf{k}(u,v)|\ \lesssim\ \{|v-u|^\kappa+|v-u|^{-2+\kappa}\}e^{-\frac{1}{8}|v-u|^2-\frac{1}{8}\frac{(|v|^2-|u|^2)^2}{|v-u|^2}}\lesssim\frac{e^{-\frac{1}{10}|v-u|^2-\frac{1}{10}\frac{(|v|^2-|u|^2)^2}{|v-u|^2}}}{|v-u|^{2-\kappa}}.
$$

For $\varrho>0$ *and* $-2\varrho<\theta<2\varrho$ *and* $\zeta\in\mathbb{R}$*, we have for* $0<\kappa\le 1$*,*

$$
\int_{\mathbb{R}^3}\mathbf{k}_{\kappa,\varrho}(v,u)\frac{\langle v\rangle^\zeta e^{\theta|v|^2}}{\langle u\rangle^\zeta e^{\theta|u|^2}}\mathrm{d}u\ \lesssim\ \langle v\rangle^{-1}.
$$

Proof. The proof is based on [8]. Note that

$$
\frac{\langle v\rangle^\zeta e^{\theta|v|^2}}{\langle u\rangle^\zeta e^{\theta|u|^2}}\ \lesssim\ [1+|v-u|^2]^{\frac{\zeta}{2}}e^{-\theta(|u|^2-|v|^2)}.
$$

Set $v-u=\eta$ and $u=v-\eta$ in the integration of (31). Now we compute the total exponent of the integrand of (31) and (**??**) as

$$
\begin{aligned}
&-\varrho|\eta|^2-\varrho\frac{||\eta|^2-2v\cdot\eta|^2}{|\eta|^2}-\theta\{|v-\eta|^2-|v|^2\}=-2\varrho|\eta|^2+4\varrho\{v\cdot\eta\}-4\varrho\frac{|v\cdot\eta|^2}{|\eta|^2}-\theta\{|\eta|^2-2v\cdot\eta\}\\
&=(-\theta-2\varrho)\,|\eta|^2+(4\varrho+2\theta)\,v\cdot\eta-4\varrho\frac{|v\cdot\eta|^2}{|\eta|^2}.
\end{aligned}
$$

Since $-2\varrho<\theta<2\varrho$, the discriminant of the above quadratic form of $|\eta|$ and $\frac{v\cdot\eta}{|\eta|}$ is negative : $(4\varrho+2\theta)^2+16\varrho(-\theta-2\varrho)=4\theta^2-16\varrho^2<0$. We thus have

$$
-\varrho|\eta|^2-\varrho\frac{||\eta|^2-2v\cdot\eta|^2}{|\eta|^2}-\theta\{|v-\eta|^2-|v|^2\}\ \lesssim_{\varrho,\theta}\ \frac{|\eta|^2}{2}+|v\cdot\eta|.
$$

Therefore, for $0<\kappa\le 1$

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\Big\{|v-u|^\kappa+|v-u|^{-2+\kappa}\Big\}e^{-\varrho|v-u|^2-\varrho\frac{(|v|^2-|u|^2)^2}{|v-u|^2}}\frac{\langle v\rangle^\zeta e^{\theta|v|^2}}{\langle u\rangle^\zeta e^{\theta|u|^2}}\mathrm{d}u\\
&\lesssim\int_{\mathbb{R}^3}\Big\{|\eta|^\kappa+|\eta|^{-2+\kappa}\Big\}\langle\eta\rangle^\zeta e^{-C_{\varrho,\theta}|\eta|^2}\ \lesssim_{\varrho,\theta,\kappa}\ 1.
\end{aligned}
$$

Therefore, in order to show (31) it suffices to consider the case $|v|\ge 1$. We make another change of variables $\eta_\parallel=\Big\{\eta\cdot\frac{v}{|v|}\Big\}\frac{v}{|v|}$ and $\eta_\perp=\eta-\eta_\parallel$, so that $|v\cdot\eta|=|v||\eta_\parallel|$ and $|v-u|\ge|\eta_\perp|$. We can absorb $\langle\eta\rangle^{\zeta}$, $|\eta|\langle\eta\rangle^{\zeta}$ by $e^{-C_{\varrho,\theta}|\eta|^2}$, and bound the integral of (31) by

$$\int_{\mathbb{R}^3}\{1+|\eta|^{-2+\kappa}\}e^{-C_{\varrho,\theta}\left\{\frac{|\eta|^2}{2}+|v\cdot\eta|\right\}}\mathrm{d}\eta \;\le\; \int_{\mathbb{R}^3}\{1+|\eta|^{-2+\kappa}\}e^{-\frac{C_{\varrho,\theta}}{2}|\eta|^2}e^{-C_{\varrho,\theta}|v\cdot\eta|}\mathrm{d}\eta$$
$$\le \int_{\mathbb{R}^2}\{1+|\eta_\perp|^{-2+\kappa}\}e^{-\frac{C_{\varrho,\theta}}{2}|\eta_\perp|^2}\left\{\int_{\mathbb{R}}e^{-C_{\varrho,\theta}|v|\times|\eta_\parallel|}\mathrm{d}|\eta_\parallel|\right\}\mathrm{d}\eta_\perp$$
$$\lesssim \langle v\rangle^{-1}\int_{\mathbb{R}^2}\{1+|\eta_\perp|^{-2+\kappa}\}e^{-\frac{C_{\varrho,\theta}}{2}|\eta_\perp|^2}\left\{\int_0^\infty e^{-C_{\varrho,\theta}y}\mathrm{d}y\right\}\mathrm{d}\eta_\perp,\quad (y=|v||\eta_\parallel|).$$

□

Lemma 5. *(1) For $p\in[1,\infty)$,*

$$\|Kh\|_p \;\lesssim_p\; \|h\|_p,\qquad \|\nabla_v Kh\|_p \;\lesssim_p\; \|h\|_p+\|\nabla_v h\|_p,$$

and

$$|\partial_v\mathbf{k}(v,u)|+|\partial_u\mathbf{k}(v,u)|\lesssim \mathbf{k}_{\kappa-1,\frac12}(v,u)=\frac{1}{|v-u|^{3-\kappa}}e^{-\frac{1}{16}|v-u|^2}.$$

(2) For $p\in[1,\infty)$ and $(i,j)=(1,2)$ and for $(i,j)=(2,1)$

$$\left|\iint_{\Omega\times\mathbb{R}^3}\Gamma_{\text{gain}}(g_1,g_2)g_3\mathrm{d}v\mathrm{d}x\right| \;\lesssim\; ||\langle v\rangle^{\zeta}g_i||_\infty\|g_j\|_p\|g_3\|_q,\quad \textit{for}\ \ \zeta>2,$$
$$|\Gamma_{\text{gain}}(g_1,g_2)| \;\lesssim\; ||\langle v\rangle^{\zeta}e^{\theta|v|^2}g_i||_\infty\int_{\mathbb{R}^3}\mathbf{k}_{\kappa,\frac{\theta}{2}}(v,u)|g_j|\mathrm{d}u,\quad \textit{for}\ \ \theta>0,\ \zeta\ge 0.$$

For $\zeta>2,\ \theta>0$,

$$\left|\Gamma_{\text{gain}}(g_1,g_2)(v)\right| \;\lesssim\; \langle v\rangle^{-\zeta}||\langle v\rangle^{\zeta}g_1||_\infty||\langle v\rangle^{\zeta}g_2||_\infty,$$
$$\left|\Gamma_{\text{gain}}(g_1,g_2)(v)\right| \;\lesssim\; \langle v\rangle^{-\zeta}e^{-\theta|v|^2}||\langle v\rangle^{\zeta}e^{\theta|v|^2}g_1||_\infty||\langle v\rangle^{\zeta}e^{\theta|v|^2}g_2||_\infty,$$

and for all $\zeta,\theta\ge 0$,

$$|\nu(\mu+\sqrt{\mu}g_1)g_2| \;\lesssim_{\zeta,\theta}\; (1+||\langle v\rangle^{\zeta}e^{\theta|v|^2}g_1||_\infty)|\langle v\rangle^{\kappa}g_2|.$$

(3) For $\zeta>2$,

$$\left|\iint_{\Omega\times\mathbb{R}^3}\nabla_v\Gamma_{\text{gain}}(g_1,g_2)g_3\mathrm{d}v\mathrm{d}x\right|\lesssim ||\langle v\rangle^{\zeta}g_1||_\infty||\nabla_v g_2||_p||g_3||_q+||\langle v\rangle^{\zeta}g_2||_\infty||\nabla_v g_1||_p||g_3||_q.$$

Define

$$\Gamma_{\text{gain},v}(g_1,g_2)(v):=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(v-u,\omega)\nabla_v(\sqrt{\mu})(u)g_1(u')g_2(v')\mathrm{d}\omega\mathrm{d}u,$$

where $u_\parallel=(u\cdot\omega)\omega$ and $u_\perp=u-(u\cdot\omega)\omega$.

Then, for $\zeta>2,\theta>0$,

$$|\Gamma_{\text{gain},v}(g_1,g_2)(v)| \;\lesssim\; \langle v\rangle^{-\zeta}||\langle v\rangle^{\zeta}g_1||_\infty||\langle v\rangle^{\zeta}g_2||_\infty,$$
$$|\Gamma_{\text{gain},v}(g_1,g_2)(v)| \;\lesssim\; \langle v\rangle^{-\zeta}e^{-\theta|v|^2}||\langle v\rangle^{\zeta}e^{\theta|v|^2}g_1||_\infty||\langle v\rangle^{\zeta}e^{\theta|v|^2}g_2||_\infty.$$

(4) For $\theta,\zeta\ge 0$,

$$\begin{aligned}&|\partial Kg(s,X,V)|\\&=\left|\partial\int_{\mathbb{R}^3}\mathbf{k}(V,u)g(s,X,u)\mathrm{d}u\right|\\&\lesssim\int_{\mathbb{R}^3}\Big\{|\partial V||\partial_v\mathbf{k}(V,u)|\langle u\rangle^{-\zeta}e^{-\theta|u|^2}||\langle u\rangle^{\zeta}e^{\theta|u|^2}g_0||_\infty+\mathbf{k}(V,u)|\partial X||\partial_x g(s,X,u)|\Big\}\mathrm{d}u\\&\lesssim\int_{\mathbb{R}^3}\ \{\mathbf{k}_{\kappa-1,\frac12}(V,u)e^{-\theta|u|^2}||\langle v\rangle^{\zeta}e^{\theta|v|^2}g||_\infty|\partial V|+\mathbf{k}_{\kappa,\frac12}(V,u)|\partial_x g(s,X_{\mathbf{cl}}(s),u)||\partial X|\}\mathrm{d}u.\end{aligned}\tag{31}$$

For $\zeta \geq 0$ and $\theta > 0$,

$$
\begin{aligned}
&|\partial\Gamma_{\text{gain}}(g,g)(X,V)| \\
&= \Big|\partial \int_{\mathbb{S}^2}\int_{\mathbb{R}^3} |u|^\kappa q_0\Big(\frac{u}{|u|}\cdot\omega\Big) e^{-\frac{|u+V|^2}{4}} g(X,V+[u\cdot\omega]\omega) g(X,V+u-(u\cdot\omega)\omega)\mathrm{d}\omega\mathrm{d}u\Big| \\
&= |\Gamma_{\text{gain}}(\partial X\cdot\nabla_x g,g)(X,V)| + |\Gamma_{\text{gain}}(g,\partial X\cdot\nabla_x g)(X,V)| \\
&\quad + |\Gamma_{\text{gain}}(\partial V\cdot\nabla_v g,g)(X,V)| + |\Gamma_{\text{gain}}(g,\partial V\cdot\nabla_v g)(X,V)| \\
&\quad + \Big|\int_{\mathbb{S}^2}\int_{\mathbb{R}^3} |u|^\kappa q_0\Big(\frac{u}{|u|}\cdot\omega\Big)(\frac{-1}{2})(u+V)\cdot\partial V\sqrt{\mu(u+V)} \\
&\qquad\qquad \times g(X,V+[u\cdot\omega]\omega) g(X,V+u-(u\cdot\omega)\omega)\mathrm{d}\omega\mathrm{d}u\Big| \\
&\lesssim |\partial X| ||\langle v\rangle^\zeta e^{\theta|v|^2} g||_\infty \int_{\mathbb{R}^3} \mathbf{k}_{\kappa,\frac{\theta}{2}}(u,V)|\partial_x g(X,u)|\mathrm{d}u \\
&\quad + |\partial V| ||\langle v\rangle^\zeta e^{\theta|v|^2} g||_\infty \int_{\mathbb{R}^3} \mathbf{k}_{\kappa,\frac{\theta}{2}}(u,V)|\partial_v g(X,u)|\mathrm{d}u + |\partial V|\langle v\rangle^{-\zeta} e^{-\theta|v|^2} ||\langle v\rangle^\zeta e^{\theta|v|^2} g||_\infty^2,
\end{aligned}
\tag{32}
$$

where we have used $u - V \mapsto u$.

Proof. (1) Since $|Kh(v)| \leq \left(\int |\mathbf{k}(u,v)|\mathrm{d}u\right)^{1/q} \times \left(\int |\mathbf{k}(u,v)||h(u)|^p\mathrm{d}u\right)^{1/p}$,

$$
\|Kh\|_p \;\leq\; \sup_v \left(\int |\mathbf{k}(u,v)|\mathrm{d}u\right)^{1/q} \sup_u \left(\int |\mathbf{k}(u,v)|\mathrm{d}v\right)^{1/p} \|h\|_p \;\lesssim\; ||h||_p,
$$

where we have used (31).

For the second estimate we use a change of variable $u = v - u$ in the integral (28) : $Kf(v) = \int \mathbf{k}(v-u,v)f(v-u)\mathrm{d}u$, so that

$$
\begin{aligned}
\nabla_v(Kh) &= \int_{\mathbb{R}^3} \{\nabla_u \mathbf{k}(v-u,v) + \nabla_v \mathbf{k}(v-u,v)\} h(v-u)\mathrm{d}u + \int_{\mathbb{R}^3} \mathbf{k}(v-u,v)\nabla_v h(v-u)\mathrm{d}u \\
&= \int_{\mathbb{R}^3} \{\nabla_u \mathbf{k}(u,v) + \nabla_v \mathbf{k}(u,v)\} h(u)\mathrm{d}u + Kh_v \equiv K_v h + Kh_v.
\end{aligned}
$$

From (28) we have

$$
|\nabla_u \mathbf{k}_1| + |\nabla_v \mathbf{k}_1| \leq C\sqrt{\mu(u)}\sqrt{\mu(v)}(|u|^{\kappa+1} + |v|^{\kappa+1}),
$$

so that $\sup_v \left(\int |\nabla_u \mathbf{k}_1| + |\nabla_v \mathbf{k}_1|\mathrm{d}u\right)$ and $\sup_u \left(\int |\nabla_u \mathbf{k}_1| + |\nabla_v \mathbf{k}_1|\mathrm{d}v\right)$ are finite. For $\mathbf{k}_2$, we absorb the terms in $|w+\varsigma|$, in $|u-v|$ and in $\frac{||u|^2-|v|^2|}{|u-v|}$ in the associated exponential terms to have

$$
\begin{aligned}
|\nabla_u \mathbf{k}_2| + |\nabla_v \mathbf{k}_2| &\lesssim \frac{2}{|u-v|^3} e^{-\frac{1}{10}|u-v|^2 - \frac{1}{10}\frac{(|u|^2-|v|^2)^2}{|u-v|^2}} \\
&\qquad \times \int_{w\cdot(u-v)=0} q_0\Big(\frac{u-v}{\sqrt{|u-v|^2+|w|^2}}\cdot\frac{u-v}{|u-v|}\Big) e^{-|w+\varsigma|^2}(|w|^2+|u-v|^2)^{\kappa/2}\mathrm{d}w.
\end{aligned}
$$

This expression is of the same shape as the expression of $\mathbf{k}_2$ up to some constants in the exponential terms. The proof of the Grad estimate (31) is also valid for this expression, so that

$$
|\nabla_u \mathbf{k}_2(v,u)| + |\nabla_v \mathbf{k}_2(v,u)| \;\lesssim\; \{|v-u|^{\kappa-1} + |v-u|^{-3+\kappa}\} e^{-\frac{1}{10}|v-u|^2 - \frac{1}{10}\frac{(|v|^2-|u|^2)^2}{|v-u|^2}},
$$

and we deduce that $\sup_v \left(\int |\nabla_u \mathbf{k}_2| + |\nabla_v \mathbf{k}_2|\mathrm{d}u\right)$ and $\sup_u \left(\int |\nabla_u \mathbf{k}_2| + |\nabla_v \mathbf{k}_2|\mathrm{d}v\right)$ are finite for $0 < \kappa \leq 1$.

(2)(i) First we use the first equality of (29),

$$\begin{aligned}
&\int_{\mathbb{R}^3}\Gamma_{\text{gain}}(g_1,g_2)(v)g_3(v)\mathrm{d}v\\
&=\int_{\mathbb{R}^3}\mathrm{d}v\ |g_3(v)|\int_{\mathbb{R}^2}\mathrm{d}u_2\ |g_1(v+u_2)|\int_{\mathbb{R}^3}\mathrm{d}u_1\ \langle u_1\rangle^\zeta|g_2(u_1)|\frac{\langle u_2\rangle^\zeta e^{-\frac{|u_1+u_2|^2}{4}}}{\langle u_1\rangle^\zeta\langle u_2\rangle^\zeta|u_1-v|^{2-\kappa}}\\
&\lesssim\int_{\mathbb{R}^3}\mathrm{d}v\ |g_3(v)|\int_{\mathbb{R}^2}\mathrm{d}u_2\ |g_1(v+u_2)|\int_{\mathbb{R}^3}\mathrm{d}u_1\ \langle u_1\rangle^\zeta|g_2(u_1)|\frac{e^{-\frac{|u_1+u_2|^2}{8}}}{\langle u_2\rangle^\zeta|u_1-v|^{2-\kappa}}\\
&\lesssim\int_{\mathbb{R}^3}|g_3(v)|\int_{\mathbb{R}^2}\langle u_2\rangle^{-\zeta}|g_1(v+u_2)|||\langle v\rangle^\zeta g_2||_{L^\infty_{x,v}}\Big|\Big|\frac{e^{-\frac{|u_1|^2}{8}}}{|u_1-(v+u_2)|^{2-\kappa}}\Big|\Big|_{L^1_{u_1}}\mathrm{d}u_2\mathrm{d}v\\
&\lesssim||\langle v\rangle^\zeta g_2||_\infty\int_{\mathbb{R}^2}||g_3||_{L^p_v}\Big[\int|g_1(v+u_2)|^q\mathrm{d}v\Big]^{1/q}\langle u_2\rangle^{-\zeta}\mathrm{d}u_2\\
&\lesssim_\zeta||\langle v\rangle^\zeta g_2||_\infty||g_1||_{L^q_v}||g_3||_{L^p_v}
\end{aligned}$$

where we have used $\langle u_2\rangle^{-\zeta}L^1(\mathbb{R}^2)$ for $\zeta>2$, and $\frac{\langle u_2\rangle^\zeta}{\langle u_1\rangle^\zeta}\lesssim\langle u_2+u_1\rangle^\zeta\lesssim e^{\frac{|u_1+u_2|^2}{8}}$, and $\Big|\Big|\frac{e^{-\frac{|u_1|^2}{8}}}{|u_1-(v+u_2)|^{2-\kappa}}\Big|\Big|_{L^q(\{u_1\in\mathbb{R}^3\})}\lesssim\langle v+u_2\rangle^{-2+\kappa}\lesssim 1$. Then we integrate over the domain Ω and use Hölder's inequality in space to conclude one side of estimates. For the another side of estimates we use the second equality of (29) and follow the same proof.

(ii) From the definition, $0<\zeta<\frac{1}{2}$(therefore $\sqrt{\mu(v)}\le e^{-\zeta|v|^2}$)

$$\begin{aligned}
\Gamma_{\text{gain}}(g_1,g_2)&\le\frac{||\langle v\rangle^\zeta e^{\theta|v|^2}g_1||_\infty}{\sqrt{\mu}}Q_{\text{gain}}(\langle v\rangle^{-\zeta}e^{-\theta|v|^2}\sqrt{\mu},\sqrt{\mu}g_2)\\
&\lesssim\frac{||\langle v\rangle^\zeta e^{\theta|v|^2}g_1||_\infty}{\sqrt{\mu}}Q_{\text{gain}}(e^{-\frac{3\theta}{4}|v|^2}\sqrt{\mu},\sqrt{\mu}|g_2|)\\
&\lesssim||\langle v\rangle^\zeta e^{\theta|v|^2}g_1||_\infty\iint B(v-u,\omega)\sqrt{\mu(u)}e^{-\frac{3\theta}{4}|u'|^2}|g_2(v')|\mathrm{d}\omega\mathrm{d}u\\
&\lesssim||\langle v\rangle^\zeta e^{\theta|v|^2}g_1||_\infty\iint B(v-u,\omega)e^{-\frac{3\theta}{4}|u|^2}e^{-\frac{3\theta}{4}|u'|^2}|g_2(v')|\mathrm{d}\omega\mathrm{d}u.
\end{aligned}$$

Then we follow the Grad estimate([7, 9]) to have

$$\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(v-u,\omega)e^{-\frac{3\theta}{4}|u|^2}e^{-\frac{3\theta}{4}|u'|^2}|g_2(v')|\mathrm{d}\omega\mathrm{d}u\lesssim\int_{\mathbb{R}^3}\mathbf{k}_{\kappa,\frac{\theta}{2}}(v,u)|g_2(u)|\mathrm{d}u.$$

The proof of another side is same but we extract $||e^{\zeta|v|^2}g_2||_\infty$ first and then follow Grad estimate.

(iii) From (29)

$$\langle v\rangle^\zeta|\Gamma_{\text{gain}}(g_1,g_2)(v)|\le\ ||\langle v\rangle^\zeta g_1||_\infty||\langle v\rangle^\zeta g_2||_\infty\iint\frac{\langle v\rangle^\zeta e^{-\frac{|u_1+u_2|^2}{4}}}{\langle v+u_2\rangle^\zeta\langle u_1\rangle^\zeta}\frac{\big[|v-u_1|^2+|u_2|^2\big]^{\frac{\kappa-1}{2}}}{|v-u_1|}\mathrm{d}u_2\mathrm{d}u_1,$$

where

$$\begin{aligned}
&\iint\frac{\langle v\rangle^\zeta e^{-\frac{|u_1+u_2|^2}{4}}}{\langle v+u_2\rangle^\zeta\langle u_1\rangle^\zeta}\frac{\big[|v-u_1|^2+|u_2|^2\big]^{\frac{\kappa-1}{2}}}{|v-u_1|}\le\int_{\mathbb{R}^2}\frac{\langle v\rangle^\zeta\mathrm{d}u_2}{\langle v+u_2\rangle^\zeta}\int_{\mathbb{R}^3}\frac{e^{-\frac{|u_1+u_2|^2}{4}}\mathrm{d}u_1}{\langle u_1\rangle^\zeta|v-u_1|^{2-\kappa}}\\
&\lesssim\int_{\mathbb{R}^2}\frac{\langle v\rangle^\zeta\mathrm{d}u_2}{\langle v+u_2\rangle^\zeta}\int_{\mathbb{R}^3}\frac{e^{-\frac{|u_1+u_2|^2}{4}}\mathrm{d}u_1}{\langle u_2\rangle^\zeta|v-u_1|^{2-\kappa}}=\int_{\mathbb{R}^2}\frac{\langle v\rangle^\zeta\mathrm{d}u_2}{\langle v+u_2\rangle^\zeta\langle u_2\rangle^\zeta}\int_{\mathbb{R}^3}\frac{e^{-\frac{|u_1|^2}{4}}\mathrm{d}u_1}{|v+u_2-u_1|^{2-\kappa}}\\
&\lesssim\int_{\mathbb{R}^2}\frac{\langle v\rangle^\zeta\mathrm{d}u_2}{\langle v+u_2\rangle^{\zeta+2-\kappa}\langle u_2\rangle^\zeta}=\int_{|u_2|\le\frac{|v|}{2}}+\int_{|u_2|\ge\frac{|v|}{2}}\\
&\lesssim\frac{\langle v\rangle^\zeta}{\langle v\rangle^{\zeta+2-\kappa}}\int_{\mathbb{R}^2}\frac{\mathrm{d}u_2}{\langle u_2\rangle^\zeta}+\frac{\langle v\rangle^\zeta}{\langle v\rangle^\zeta}\int_{\mathbb{R}^2}\frac{\mathrm{d}u_2}{\langle v+u_2\rangle^{\zeta+2-\kappa}}\lesssim 1.
\end{aligned}$$

where we have used $\frac{\langle u_2\rangle^\zeta}{\langle u_1\rangle^\zeta}=\frac{\langle -u_2\rangle^\zeta}{\langle u_1\rangle^\zeta}\lesssim (1+|u_1+u_2|^2)^{\zeta/2}\lesssim e^{\frac{|u_1+u_2|^2}{4}}$.

If the weight function is $\langle v\rangle^\zeta e^{\theta|v|^2}$ then

$$\begin{aligned}\Gamma_{\text{gain}}(\langle v\rangle^{-\zeta}e^{-\theta|v|^2},\langle v\rangle^{-\zeta}e^{-\theta|v|^2}) &= \iint B(v-u,\omega)\mu(u)^{\frac{1}{2}}\langle u'\rangle^\zeta e^{-\theta|u'|^2}\langle v'\rangle^\zeta e^{-\theta|v'|^2}\mathrm{d}\omega\mathrm{d}u\\ &= \mu(v)^{-2\theta}\iint B(v-u,\omega)\mu(u)^{\frac{1}{2}+2\theta}\langle u'\rangle^\zeta\langle v'\rangle^\zeta\mathrm{d}\omega\mathrm{d}u.\end{aligned}$$

Then we follow page 315-316 in [7] to have

$$\langle v\rangle^\zeta e^{\theta|v|^2}\Gamma_{\text{gain}}(\langle v\rangle^{-\zeta}e^{-\theta|v|^2},\langle v\rangle^{-\zeta}e^{-\theta|v|^2})=\iint\frac{\langle v\rangle^\zeta e^{-(\frac{1}{4}-\theta)|u_1+u_2|^2}}{\langle v+u_2\rangle^\zeta\langle u_1\rangle^\zeta}\frac{\left[|v-u_1|^2+|u_2|^2\right]^{\frac{\kappa-1}{2}}}{|v-u_1|}\mathrm{d}u_2\mathrm{d}u_1.$$

Note that $0<\theta<\frac{1}{4}$. Then we follow the same proof of $\langle v\rangle^\zeta-$weight case(only changing the constant in the exponent from $-\frac{1}{4}$ to $-(\frac{1}{4}-\theta)$) to have

$$\langle v\rangle^\zeta e^{\theta|v|^2}\Gamma_{\text{gain}}(\langle v\rangle^{-\zeta}e^{-\theta|v|^2},\langle v\rangle^{-\zeta}e^{-\theta|v|^2})\lesssim 1.$$

(iv) Directly

$$\begin{aligned}\nu(\mu+\sqrt{\mu}g_1)g_2 &\leq \iint B(v-u,\omega)\big\{\mu(u)+\sqrt{\mu(u)}\langle u\rangle^{-\zeta}e^{-\theta|u|^2}||\langle v\rangle^\zeta e^{\theta|v|^2}g_1(u)||_\infty\big\}\mathrm{d}\omega\mathrm{d}u g_2(v)\\ &\leq (1+||\langle v\rangle^\zeta e^{\theta|v|^2}g_1(u)||_\infty)\langle v\rangle^\kappa|g_2(v)|.\end{aligned}$$

(3) We compute the velocity derivative of Γ_{gain} after the change of variable $u:=v-u$:

$$\begin{aligned}\nabla_v\Gamma_{\text{gain}}(g_1,g_2) &= \Gamma_{\text{gain}}(g_1,\nabla_v g_2)+\Gamma_{\text{gain}}(\nabla_v g_1,g_2)+\Gamma_{\text{gain},v}(g_1,g_2)\\ &:= \Gamma_{\text{gain}}(g_1,\nabla_v g_2)+\Gamma_{\text{gain}}(\nabla_v g_1,g_2)\\ &\quad+\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\nabla_v(\sqrt{\mu})(v-u)g_1(v-u_\perp)g_2(v-u_\parallel)|u|^\kappa q_0(\theta)\mathrm{d}\omega\mathrm{d}u,\end{aligned}$$

where $u_\parallel=(u\cdot\omega)\omega$ and $u_\perp=u-(u\cdot\omega)\omega$. The two first terms are estimated directly with *(2)*. Since $|\nabla_v(\sqrt{\mu})(v-u)|\leq C\sqrt{\mu}^{1/2}(v-u)$, the last remaining term is estimated as *(2)*.

(4) The proof is due to direct computation and *(1)*, *(2)*, *(3)* of this lemma. □

Lemma 6 (Local Existence)**.** *For $\zeta>2+\kappa$ and $0\leq\theta<1/4$, if $||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty<+\infty$ then there exists $T>0$ depending on $||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty$ such that there exists unique $F=\mu+\sqrt{\mu}f$ solves the Boltzmann equation (1) in $[0,T]$ and satisfies the initial condition and boundary conditions (6), (7), (8) respectively, and f satisfies*

$$\sup_{0\leq t\leq T}||\langle v\rangle^\zeta e^{\theta|v|^2}f(t)||_\infty\lesssim||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty, \tag{33}$$

and $F(t,x,v)\geq 0$ on $[0,T]\times\bar{\Omega}\times\mathbb{R}^3$. Moreover if f_0 is continuous and satisfies the compatibility conditions (10), (21), (25) respectively then f is continuous away from the grazing set γ_0.

Moreover, for $\zeta',\theta'\geq 0$, if $||\langle v\rangle^{\zeta'}e^{\theta'|v|^2}\partial_t f_0||_\infty\equiv||\langle v\rangle^{\zeta'}e^{\theta'|v|^2}\frac{-v\cdot\nabla_x F_0+Q(F_0,F_0)-\mu}{\sqrt{\mu}}||_\infty<+\infty$ then

$$\sup_{0\leq t\leq T^*}||\langle v\rangle^{\zeta'}e^{\theta'|v|^2}\partial_t f(t)||_\infty\lesssim||\langle v\rangle^{\zeta'}e^{\theta'|v|^2}\partial_t f_0||_\infty+\mathcal{P}(||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty).$$

Furthermore for the diffuse and bounce-back boundary conditions if $||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty\ll 1$ then the results hold with arbitrarily large $T=+\infty$. For the specular reflection boundary condition, if ξ is real analytic(ξ is real analytic), and if $||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty\ll 1$ then the results hold with arbitrarily large $T=+\infty$.

Proof. We use the positive preserving iteration([8, 11])

$$\partial_t F^{m+1}+v\cdot\nabla_x F^{m+1}+\nu(F^m)F^{m+1}=Q_{\text{gain}}(F^m,F^m),\quad F^{m+1}|_{t=0}=F_0\geq 0, \tag{34}$$

which is equivalent to, with $F^m:=\mu+\sqrt{\mu}f^m$,

$$\partial_t f^{m+1}+v\cdot\nabla_x f^{m+1}+\nu(F^m)f^{m+1}-Kf^m=\Gamma_{\text{gain}}(f^m,f^m),\quad f^{m+1}|_{t=0}=f_0. \tag{35}$$

The starting of this iteration is $F^0 \equiv F_0 \geq 0$, $f^0 \equiv f_0$ and let $F^{-m} \equiv F^0$, $f^{-m} \equiv f^0$ for all $m \in \mathbb{N}$. This iteration scheme is evolved with boundary conditions accordingly:

(1) Diffuse reflection boundary condition, on $(x,v) \in \gamma_-$,

$$f^{m+1}(t,x,v)|_{\gamma_-} = c_\mu \sqrt{\mu(v)} \int_{n(x)\cdot u>0} f^m(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u. \quad (36)$$

(2) Specular reflection boundary condition, on $(x,v) \in \gamma_-$,

$$f^{m+1}(t,x,v)|_{\gamma_-} = f^m(t,x,R_x v), \quad (37)$$

where $R_x v = v - 2n(x)(n(x)\cdot v)$.

(3) Bounce-back reflection boundary condition, on $(x,v) \in \gamma_-$,

$$f^{m+1}(t,x,v)|_{\gamma_-} = f^m(t,x,-v). \quad (38)$$

For details see [8, 2]. □

2. Traces and the In-flow Problems

Recall the almost grazing set γ_+^ε defined in (17). We first estimate the outgoing trace on $\gamma_+ \setminus \gamma_+^\varepsilon$. We remark that for the outgoing part, our estimate is global in time without cut-off, in contrast to the general trace theorem.

Lemma 7. *Assume that $\varphi = \varphi(v)$ is $L^\infty_{loc}(\mathbb{R}^3)$. For any small parameter $\varepsilon > 0$, there exists a constant $C_{\varepsilon,T,\Omega} > 0$ such that for any h in $L^1([0,T], L^1(\Omega\times\mathbb{R}^3))$ with $\partial_t h + v\cdot\nabla_x h + \varphi h$ is in $L^1([0,T], L^1(\Omega\times\mathbb{R}^3))$, we have for all $0 \le t \le T$,*

$$\int_0^t \int_{\gamma_+\setminus\gamma_+^\varepsilon} |h|\mathrm{d}\gamma \mathrm{d}s \le C_{\varepsilon,T,\Omega}\left[\ ||h_0||_1 + \int_0^t \left\{\|h(s)\|_1 + \left\|[\partial_t + v\cdot\nabla_x + \varphi]h(s)\right\|_1\right\}\mathrm{d}s\ \right].$$

Furthermore, for any (s,x,v) in $[0,T]\times\Omega\times\mathbb{R}^3$ the function $h(s+s', x+s'v, v)$ is absolutely continuous in s' in the interval $[-\min\{t_{\mathbf{b}}(x,v),s\}, \min\{t_{\mathbf{b}}(x,-v), T-s\}]$.

Proof. With a proper change of variables (e.g. Page 247 in [1]) we have

$$\begin{aligned}
&\int_0^T \iint_{\Omega\times\mathbb{R}^3} h(t,x,v\mathrm{d})\mathrm{d}v\mathrm{d}x\mathrm{d}t \quad (39)\\
&= \int_{-\min\{T,t_{\mathbf{b}}(x,v)\}}^0 \iint_{\Omega\times\mathbb{R}^3} h(T+s,x+sv,v)\mathrm{d}v\mathrm{d}x\mathrm{d}s + \int_0^{\min\{T,t_{\mathbf{b}}(x,-v)\}} \iint_{\Omega\times\mathbb{R}^3} h(0+s,x+sv,v)\mathrm{d}v\mathrm{d}x\mathrm{d}s\\
&\quad + \int_0^T\int_{\gamma_+}\int_{-\min\{t,t_{\mathbf{b}}(x,v)\}}^0 h(t+s,x+sv,v)\mathrm{d}s\mathrm{d}\gamma\mathrm{d}t + \int_0^T\int_{\gamma_-}\int_0^{\min\{T-t,t_{\mathbf{b}}(x,-v)\}} h(t+s,x+sv,v)\mathrm{d}s\mathrm{d}\gamma\mathrm{d}t.
\end{aligned}$$

For $(t,x,v)\in[0,T]\times\gamma_+$ and $0\le s\le \min\{t,t_{\mathbf{b}}(x,v)\}$,

$$h(t,x,v) = h(t-s,x-sv,v)e^{-\varphi(v)s} + \int_{-s}^0 e^{\varphi(v)\tau}[\partial_t h + v\cdot\nabla_x h + \varphi(v)h](t+\tau,x+\tau v,v)\mathrm{d}\tau.$$

Now for $(t,x,v)\in[\varepsilon_1,T]\times\gamma_+\setminus\gamma_+^\varepsilon$, we integrate over $\int_{\varepsilon_1}^T\int_{\gamma_+\setminus\gamma_+^\varepsilon}\int^0_{\min\{t,t_{\mathbf{b}}(x,v)\}}$ to get

$$\begin{aligned}
&\min\{\varepsilon_1,\varepsilon^3\}\times\int_{\varepsilon_1}^T\int_{\gamma_+\setminus\gamma_+^\varepsilon}|h(t,x,v)|\mathrm{d}\gamma\mathrm{d}t \lesssim \min_{[\varepsilon_1,T]\times[\gamma_+\setminus\gamma_+^\varepsilon]}\{t,t_{\mathbf{b}}(x,v)\}\times\int_{\varepsilon_1}^T\int_{\gamma_+\setminus\gamma_+^\varepsilon}|h(t,x,v)|\mathrm{d}\gamma\mathrm{d}t\\
&\lesssim \int_0^T\int_{\gamma_+}\int_{-\min\{t,t_{\mathbf{b}}(x,v)\}}^0 |h(t+s,x+sv,v)|\mathrm{d}s\mathrm{d}\gamma\mathrm{d}t\\
&\quad + T\int_0^T\int_{\gamma_+}\int_{-\min\{t,t_{\mathbf{b}}(x,v)\}}^0 |\partial_t h + v\cdot\nabla_x h + \varphi h|(t+\tau,x+\tau v,v)\mathrm{d}\tau\mathrm{d}\gamma\mathrm{d}t\\
&\lesssim \int_0^T ||h(t)||_1\mathrm{d}t + \int_0^T ||[\partial_t + v\cdot\nabla_x + \varphi]h(t)||_1\mathrm{d}t,
\end{aligned}$$

where we have used the integration identity (39), and (40) of [8] to obtain $t_{\mathbf{b}}(x,v) \geq C_\Omega |n(x)\cdot v|/|v|^2 \geq C_\Omega \varepsilon^3$ for $(x,v)\in\gamma_+\setminus\gamma_+^\varepsilon$. Now we choose $\varepsilon_1=\varepsilon_1(\Omega,\varepsilon)$ as

$$\varepsilon_1 \leq C_\Omega\varepsilon^3 \leq \inf_{(x,v)\in\gamma_+\setminus\gamma_+^\varepsilon} t_{\mathbf{b}}(x,v).$$

We only need to show, for $\varepsilon_1\leq C_\Omega\varepsilon^3$,

$$\int_0^{\varepsilon_1}\int_{\gamma_+\setminus\gamma_+^\varepsilon}|h(t,x,v)|\mathrm{d}\gamma\mathrm{d}t \lesssim_{\Omega,\varepsilon,\varepsilon_1} ||h_0||_1+\int_0^{\varepsilon_1}||[\partial_t+v\cdot\nabla_x+\varphi]h(t)||_1\mathrm{d}t.$$

Because of our choice ε and ε_1, $t_{\mathbf{b}}(x,v)>t$ for all $(t,x,v)\in[0,\varepsilon_1]\times\gamma_+\setminus\gamma_+^\varepsilon$. Then

$$|h(t,x,v)|\lesssim |h_0(x-tv,v)|+\int_0^t \Big|[\partial_t+v\cdot\nabla_x+\varphi(v)]h(s,x-(t-s)v,v)\Big|\mathrm{d}s,$$

where the second contribution is bounded, from (39), by

$$\int_0^{\varepsilon_1}\int_{\gamma_+\setminus\gamma_+^\varepsilon}\int_0^t \Big|[\partial_t+v\cdot\nabla_x+\varphi(v)]h(s,x-(t-s)v,v)\Big|\mathrm{d}s\mathrm{d}\gamma\mathrm{d}t \lesssim \int_0^{\varepsilon_1}||[\partial_t+v\cdot\nabla_x+\varphi(v)]h(t)||_1\mathrm{d}t.$$

Consider the initial datum contribution of $|h_0(x-tv,v)|$: Assume $\partial_{x_3}\xi(x_0)\neq 0$. By the implicit function theorem $\partial\Omega$ can be represented locally by the graph $\eta=\eta(x_1,x_2)$ satisfying $\xi(x_1,x_2,\eta(x_1,x_2))=0$ and $(\partial_{x_1}\eta(x_1,x_2),\partial_{x_2}\eta(x_1,x_2))=(-\partial_{x_1}\xi/\partial_{x_3}\xi,-\partial_{x_2}\xi/\partial_{x_3}\xi)$ at $(x_1,x_2,\eta(x_1,x_2))$. We define the change of variables

$$(x,t)\in\partial\Omega\cap\{x\sim x_0\}\times[0,\varepsilon_1]\mapsto y=x-tv\in\bar{\Omega},$$

where $\left|\frac{\partial y}{\partial(x,t)}\right| = -v_1\frac{\partial_{x_1}\xi}{\partial_{x_3}\xi}-v_2\frac{\partial_{x_2}\xi}{\partial_{x_3}\xi}-v_3$. Therefore

$$|n(x)\cdot v|\mathrm{d}S_x\mathrm{d}t=(n(x)\cdot v)\Big[1+(\frac{\partial_{x_1}\xi}{\partial_{x_3}\xi})^2+(\frac{\partial_{x_2}\xi}{\partial_{x_3}\xi})^2\Big]^{1/2}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}t=\Big[-v_1\frac{\partial_{x_1}\xi}{\partial_{x_3}\xi}-v_2\frac{\partial_{x_2}\xi}{\partial_{x_3}\xi}-v_3\Big]\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}t=\mathrm{d}y,$$

and $\int_0^{\varepsilon_1}\int_{\gamma_+\setminus\gamma_+^\varepsilon\cap\{x\sim x_0\}}|h_0(x-tv,v)|\mathrm{d}\gamma\mathrm{d}t \lesssim_{\varepsilon,\varepsilon_1,x_0} \iint_{\Omega\times\mathbb{R}^3}|h_0(y,v)|\mathrm{d}y\mathrm{d}v$. Since $\partial\Omega$ is compact we can choose a finite covers of $\partial\Omega$ and repeat the same argument for each piece to conclude

$$\int_0^{\varepsilon_1}\int_{\gamma_+\setminus\gamma_+^\varepsilon}|h_0(x-tv,v)|\mathrm{d}\gamma\mathrm{d}t \lesssim_{\Omega,\varepsilon,\varepsilon_1} \iint_{\Omega\times\mathbb{R}^3}|h_0(y,v)|\mathrm{d}y\mathrm{d}v.$$

□

Lemma 8 (Green's Identity)**.** *For $p\in[1,\infty)$ assume that $f,\partial_t f+v\cdot\nabla_x f\in L^p([0,T];L^p(\Omega\times\mathbb{R}^3))$ and $f_{\gamma_-}\in L^p([0,T];L^p(\gamma))$. Then $f\in C^0([0,T];L^p(\Omega\times\mathbb{R}^3))$ and $f_{\gamma_+}\in L^p([0,T];L^p(\gamma))$ and for almost every $t\in[0,T]$:*

$$||f(t)||_p^p+\int_0^t|f|_{\gamma_+,p}^p=||f(0)||_p^p+\int_0^t|f|_{\gamma_-,p}^p+\int_0^t\iint_{\Omega\times\mathbb{R}^3}\{\partial_t f+v\cdot\nabla_x f\}|f|^{p-2}f.$$

See [8] for the proof. Now we state and prove following propositions for the in-flow problems:

$$\{\partial_t+v\cdot\nabla_x+\nu\}f=H,\quad f(0,x,v)=f_0(x,v),\quad f(t,x,v)|_{\gamma_-}=g(t,x,v), \tag{40}$$

where $\nu(t,x,v)\geq 0$. For notational simplicity, we define

$$\partial_t f_0 \;\equiv\; -v\cdot\nabla_x f_0-\nu f_0+H(0,x,v), \tag{41}$$

$$\nabla_x g \;\equiv\; \frac{n}{n\cdot v}\Big\{-\partial_t g-\sum_{i=1}^2(v\cdot\tau_i)\partial_{\tau_i}g-\nu g+H\Big\}+\sum_{i=1}^2\tau_i\partial_{\tau_i}g. \tag{42}$$

We remark that $\partial_t f_0$ is obtained from formally solving (40), and (42) leads to the usual tangential derivatives of $\partial_{\tau_i}g$, while defines new 'normal derivative' $\partial_n g$ from the equation (40).

Proposition 1. *Assume a compatibility condition*

$$f_0(x,v) = g(0,x,v) \quad \text{for} \quad (x,v) \in \gamma_-. \tag{43}$$

For any fixed $p \in [1,\infty)$*, assume*

$$\begin{array}{rcl} \nabla_x f_0, \nabla_v f_0, -v\cdot\nabla_x f_0 - \nu f_0 + H(0,x,v) & \in & L^p(\Omega\times\mathbb{R}^3), \\ \langle v\rangle g, \partial_t g, \nabla_v g, \partial_{\tau_i} g, \dfrac{1}{n(x)\cdot v}\{-\partial_t g - \sum_i (v\cdot\tau_i)\partial_{\tau_i} g - \nu(v) g + H\} & \in & L^p([0,T]\times\gamma_-), \end{array}$$

and, assume $1/p + 1/q = 1$ *there exist* $TC_T \sim O(T)$ *and* $\varepsilon \ll 1$ *such that for all* $t \in [0,T]$

$$\left| \iint_{\Omega\times\mathbb{R}^3} \partial H(t) h(t) \mathrm{d}x\mathrm{d}v \right| \le C_T ||h(t)||_q.$$

Then for sufficiently small $T > 0$ *there exists a unique solution* f *to (40) such that* $f, \partial_t f, \nabla_x f, \nabla_v f \in C^0([0,T]; L^p(\Omega\times\mathbb{R}^3))$ *and the traces satisfy*

$$\begin{array}{c} \partial_t f|_{\gamma_-} = \partial_t g,\ \nabla_v f|_{\gamma_-} = \nabla_v g,\ \nabla_x f|_{\gamma_-} = \nabla_x g, \quad \text{on} \quad \gamma_-, \\ \nabla_x f(0,x,v) = \nabla_x f_0,\ \nabla_v f(0,x,v) = \nabla_v f_0,\ \partial_t f(0,x,v) = \partial_t f_0, \quad \text{in} \quad \Omega\times\mathbb{R}^3, \end{array} \tag{44}$$

where $\partial_t f_0$ *and* $\nabla_x g$ *are given by (41) and (42). Moreover*

$$||\partial_t f(t)||_p^p + \int_0^t |\partial_t f|_{\gamma_+,p}^p \le ||\partial_t f_0||_p^p + \int_0^t |\partial_t g|_{\gamma_-,p}^p + p\int_0^t \iint_{\Omega\times\mathbb{R}^3} |\partial_t H||\partial_t f|^{p-2}, \tag{45}$$

$$||\nabla_x f(t)||_p^p + \int_0^t |\nabla_x f|_{\gamma_+,p}^p \le ||\nabla_x f_0||_p^p + \int_0^t |\nabla_x g|_{\gamma_-,p}^p + p\int_0^t \iint_{\Omega\times\mathbb{R}^3} |\nabla_x H||\nabla_x f|^{p-1}, \tag{46}$$

$$\begin{aligned} ||\nabla_v f(t)||_p^p + \int_0^t |\nabla_v f|_{\gamma_+,p}^p \le{} & ||\nabla_v f_0||_p^p + \int_0^t |\nabla_v g|_{\gamma_-,p}^p \\ & + p\int_0^t \iint_{\Omega\times\mathbb{R}^3} |\{\nabla_v H - \nabla_x f - \nabla_v \nu f\}||\nabla_v f|^{p-1}. \end{aligned} \tag{47}$$

Proof. We apply the trace theorem to the derivatives of f by explicit computations. Denote $\nu(s) = \nu(s, x-(t-s)v, v)$. First we assume f_0, g and H have compact support in $v \in \mathbb{R}^3$. We integrate the equation (40) along the backward trajectories. If the initial condition is reached before hitting the boundary (case $t < t_{\mathbf{b}}$), we have

$$f(t,x,v) = e^{-\int_0^t \nu} f_0(x-tv,v) + \int_0^t e^{-\int_0^s \nu} H(t-s, x-vs, v)\mathrm{d}s.$$

If the boundary is first reached (case $t > t_{\mathbf{b}}$), we have

$$f(t,x,v) = e^{-\int_0^{t_{\mathbf{b}}} \nu} g(t-t_{\mathbf{b}}, x_{\mathbf{b}}, v) + \int_0^{t_{\mathbf{b}}} e^{-\int_0^s \nu} H(t-s, x-vs, v)\mathrm{d}s.$$

Let us rewrite it

$$\begin{aligned} f(t,x,v) ={} & \mathbf{1}_{\{t\le t_{\mathbf{b}}\}} e^{-\int_0^t \nu} f_0(x-tv,v) + \mathbf{1}_{\{t> t_{\mathbf{b}}\}} e^{-\int_0^{t_{\mathbf{b}}} \nu} g(t-t_{\mathbf{b}}, x_{\mathbf{b}}, v) \\ & + \int_0^{\min(t,t_{\mathbf{b}})} e^{-\int_0^s \nu} H(t-s, x-vs, v)\mathrm{d}s. \end{aligned} \tag{48}$$

We take derivative of f with respect to time, space and velocity for $t \neq t_{\mathbf{b}}$. Recall the following derivatives of $x_{\mathbf{b}}$ and $t_{\mathbf{b}}$ (see lemma 2 in [8]) :

$$\nabla_x t_{\mathbf{b}} = \frac{n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})}, \quad \nabla_v t_{\mathbf{b}} = -\frac{t_{\mathbf{b}} n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})}, \quad \nabla_x x_{\mathbf{b}} = I - \frac{n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})} \otimes v, \quad \nabla_v x_{\mathbf{b}} = -t_{\mathbf{b}} I + \frac{t_{\mathbf{b}} n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})} \otimes v. \tag{49}$$

Since g is defined on a surface, we cannot define its space gradient. We then use directly the gradient in space of $g(x_{\mathbf{b}})$. Regarding $g(t-t_{\mathbf{b}}, x_{\mathbf{b}}(x,v), v)$ as function on $[0,T]\times\bar{\Omega}\times\mathbb{R}^3$ we obtain from (49)

$$
\begin{aligned}
\nabla_x[g(t-t_{\mathbf{b}},x_{\mathbf{b}},v)] &= -\nabla_x t_{\mathbf{b}}\partial_t g + \nabla_x x_{\mathbf{b}}\nabla_\tau g = -\frac{n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})}\partial_t g + \left(I - \frac{n\otimes v}{n\cdot v}\right)\nabla_\tau g \\
&= \tau_1\partial_{\tau_1} g + \tau_2\partial_{\tau_2} g - \frac{n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})}\left\{\partial_t g + v\cdot\tau_1\partial_{\tau_1} g + v\cdot \tau_2\partial_{\tau_2} g\right\}, \\
\nabla_v[g(t-t_{\mathbf{b}},x_{\mathbf{b}},v)] &= -t_{\mathbf{b}}\nabla_x[g(t-t_{\mathbf{b}},x_{\mathbf{b}},v)] + \nabla_v g,
\end{aligned}
$$

where $\tau_1(x)$ and $\tau_2(x)$ are unit vectors satisfying $\tau_1(x)\cdot n(x) = 0 = \tau_2(x)\cdot n(x)$ and $\tau_1(x)\times\tau_2(x) = n(x)$.

Therefore by direct computation for $t\neq t_{\mathbf{b}}$, we deduce

$$
\begin{aligned}
\partial_t f(t,x,v)\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}} &= -\mathbf{1}_{\{t<t_{\mathbf{b}}\}} e^{-\int_0^t \nu}[\nu f_0 + v\cdot\nabla_x f_0 - H_{|t=0}](x-tv,v) + \mathbf{1}_{\{t>t_{\mathbf{b}}\}}e^{-\int_0^{t_{\mathbf{b}}}\nu}\partial_t g(t-t_{\mathbf{b}},x_{\mathbf{b}},v) \\
&+ \int_0^{\min(t,t_{\mathbf{b}})} e^{-\int_0^s\nu}\partial_t H(t-s,x-vs,v)\mathrm{d}s, \\
\nabla_x f(t,x,v)\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}} &= \mathbf{1}_{\{t<t_{\mathbf{b}}\}} e^{-\int_0^t \nu}\nabla_x f_0(x-tv,v) \\
&+ \mathbf{1}_{\{t>t_{\mathbf{b}}\}}e^{-\int_0^{t_{\mathbf{b}}}\nu}\Big\{\sum_{i=1}^2 \tau_i\partial_{\tau_i}g - \frac{n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})}\big\{\partial_t g + \sum_{i=1}^2 (v\cdot\tau_i)\partial_{\tau_i}g + \nu g - H\big\}\Big\}(t-t_{\mathbf{b}},x_{\mathbf{b}},v) \\
&+ \int_0^{\min(t,t_{\mathbf{b}})} e^{-\int_0^s\nu}\nabla_x H(t-s,x-vs,v)\mathrm{d}s, \\
\nabla_v f(t,x,v)\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}} &= \mathbf{1}_{\{t<t_{\mathbf{b}}\}} e^{-\int_0^t \nu}[-t\nabla_x f_0 + \nabla_v f_0 - t\nabla_v\nu(v) f_0](x-tv,v) \\
&- \mathbf{1}_{\{t>t_{\mathbf{b}}\}}t_{\mathbf{b}}e^{-\int_0^{t_{\mathbf{b}}}\nu}\Big\{\sum_{i=1}^2 \tau_i\partial_{\tau_i}g - \frac{n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})}\big\{\partial_t g + \sum_{i=1}^2 (v\cdot\tau_i)\partial_{\tau_i}g + \nu g - H\big\}\Big\}(t-t_{\mathbf{b}},x_{\mathbf{b}},v) \\
&+ \mathbf{1}_{\{t>t_{\mathbf{b}}\}}e^{-\int_0^{t_{\mathbf{b}}}\nu}\Big\{\nabla_v g(t-t_{\mathbf{b}},x_{\mathbf{b}},v) - t_{\mathbf{b}}\nabla_v\nu(v) g(t-t_{\mathbf{b}},x_{\mathbf{b}},v)\Big\} \\
&+ \int_0^{\min(t,t_{\mathbf{b}})} e^{-\int_0^s\nu}\{\nabla_v H - s\nabla_x H - s\nabla\nu H\}(t-s,x-vs,v)\mathrm{d}s.
\end{aligned}
$$

We, first, show that $\partial f\mathbf{1}_{\{t>t_{\mathbf{b}}\}}\in L^p$ and $\partial f\mathbf{1}_{\{t<t_{\mathbf{b}}\}}$ separately. Now we take L^p norms above with the changes of variables in Lemma 2.1 of [6] and using Jensen's inequality in $[0,t]$. More precisely, for $\phi\in L^1$ with $\phi\geq 0$,

$$
\begin{aligned}
&\iint_{\Omega\times\mathbb{R}^3}\mathbf{1}_{\{x-tv\in\Omega\}}\phi(x-tv,v) = \int_{\mathbb{R}^3}\left[\int_\Omega \mathbf{1}_{\{x-tv\in\Omega\}}\phi(x-tv,v)\mathrm{d}x\right]\mathrm{d}v \leq \iint_{\Omega\times\mathbb{R}^3}\phi(x,v), \\
&\iint_{\{\Omega\times\mathbb{R}^3\}\cap B((x_0,v_0);\delta)}\mathbf{1}_{\{t\geq t_{\mathbf{b}}\}}\phi(t-t_{\mathbf{b}}(x,v),x_{\mathbf{b}}(x,v),v) \leq \int_0^t\int_{\partial\Omega\times\mathbb{R}^3}\phi(s,x,v)|n(x)\cdot v|\mathrm{d}S_x\mathrm{d}v\mathrm{d}s,
\end{aligned}
\tag{50}
$$

where for the second inequality we have used the change of variables for fixed t,v,

$$
x\mapsto (t-t_{\mathbf{b}}(x,v),x_{\mathbf{b}}(x,v)). \tag{51}
$$

In fact, without the loss of generality we may assume $\partial_{x_3}\xi(x_{\mathbf{b}}(x,v))\neq 0$ for $(x,v)\in B((x_0,v_0);\delta)$ so that $x_{\mathbf{b}}(x,v) = (x_{\mathbf{b},1},x_{\mathbf{b},2},\eta(x_{\mathbf{b},1},x_{\mathbf{b},2}))$. Using (49), we compute the Jacobian

$$
\det\begin{pmatrix} -\nabla_x t_{\mathbf{b}} \\ -\nabla_x x_{\mathbf{b},1} \\ -\nabla_x x_{\mathbf{b},2}\end{pmatrix} = \det\begin{pmatrix} -(v\cdot n)^{-1}n \\ -\nabla_x x_{\mathbf{b},1} \\ -\nabla_x x_{\mathbf{b},2}\end{pmatrix} = \left| -v_1\frac{\partial_{x_1}\xi}{\partial_{x_3}\xi} - v_2\frac{\partial_{x_2}\xi}{\partial_{x_3}\xi} + v_3\right|^{-1}.
$$

Therefore $\mathrm{d}x\mathrm{d}v = \left|-v_1\frac{\partial_{x_1}\xi}{\partial_{x_3}\xi} - v_2\frac{\partial_{x_2}\xi}{\partial_{x_3}\xi} + v_3\right| \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}v\mathrm{d}t = |n\cdot v|\mathrm{d}S_x\mathrm{d}v\mathrm{d}t = \mathrm{d}\gamma\mathrm{d}t$. Using these changes of variables, we obtain

$$\|f(t)\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}}\|_p \le \|f_0\|_p + \left[\int_0^t\int_{\gamma_-}|g|^p\mathrm{d}\gamma\mathrm{d}s\right]^{1/p} + t^{(p-1)/p}\left[\int_0^t\|H\|_p^p\mathrm{d}s\right]^{1/p},$$

$$\begin{aligned}\|\partial_t f(t)\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}}\|_p &\le \|v\cdot\nabla_x f_0 + \nu f_0 - H(0,\cdot,\cdot)\|_p\\ &\quad + \left[\int_0^t\int_{\gamma_-}|\partial_t g|^p\mathrm{d}\gamma\mathrm{d}s\right]^{1/p} + t^{(p-1)/p}\left[\int_0^t\|\partial_t H\|_p^p\mathrm{d}s\right]^{1/p},\end{aligned}$$

and

$$\begin{aligned}&\|\nabla_x f(t)\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}}\|_p\\ &\le \|\nabla_x f_0\|_p + t^{(p-1)/p}\left[\int_0^t\|\nabla_x H(s)\|_p^p\mathrm{d}s\right]^{1/p}\\ &+\left[\int_0^t\int_{\gamma_-}\left|\Big\{\sum_{i=1}^2\tau_i\partial_{\tau_i}g - \frac{n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})}\big\{\partial_t g + \sum_{i=1}^2(v\cdot\tau_i)\partial_{\tau_i}g + \nu g - H(0,x,v)\big\}\Big\}(t-t_{\mathbf{b}},x_{\mathbf{b}},v)\right|^p\mathrm{d}\gamma\mathrm{d}s\right]^{1/p},\end{aligned}\tag{52}$$

and

$$\begin{aligned}&\|\nabla_v f(t)\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}}\|_p\\ &\le t\|\nabla_x f_0\|_p + \|\nabla_v f_0\|_p + C\|f_0\|_p + Ct\left[\int_0^t\int_{\gamma_-}|g|^p\mathrm{d}\gamma\mathrm{d}s\right]\\ &+t\left[\int_0^t\int_{\gamma_-}\left|\Big\{\sum_{i=1}^2\tau_i\partial_{\tau_i}g - \frac{n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})}\big\{\partial_t g + \sum_{i=1}^2(v\cdot\tau_i)\partial_{\tau_i}g + \nu g - H(0,x,v)\big\}\Big\}(t-t_{\mathbf{b}},x_{\mathbf{b}},v)\right|^p\mathrm{d}\gamma\mathrm{d}s\right]^{1/p}\\ &+\left[\int_0^t\int_{\gamma_-}|\nabla_v g|^p\mathrm{d}\gamma\mathrm{d}s\right]^{1/p} + t\left[\int_0^t\int_{\gamma_-}|\langle v\rangle g|^p\mathrm{d}\gamma\mathrm{d}s\right]^{1/p} + t^{(p-1)/p}\left[\int_0^t\|\nabla_x H\|_p^p\mathrm{d}s\right]^{1/p}\\ &+t^{(p-1)/p}\left[\int_0^t\|\nabla_v H\|_p^p\mathrm{d}s\right]^{1/p} + Ct^{(p-1)/p}\left[\int_0^t\|H\|_p^p\mathrm{d}s\right]^{1/p}.\end{aligned}$$

From our hypothesis and assumption on f_0, g and H to have compact supports, these terms are bounded, therefore

$$\partial f\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}} \equiv \left[\partial_t f\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}}, \nabla_x f\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}}, \nabla_v f\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}}\right] \in L^\infty([0,T];L^p(\Omega\times\mathbb{R}^3)).$$

On the other hand, thanks to the compatibility condition, we need to show f has the same trace on the set

$$\mathcal{M} \equiv \{t = t_{\mathbf{b}}(x,v)\} \equiv \{(t_{\mathbf{b}}(x,v),x,v)\in[0,T]\times\Omega\times\mathbb{R}^3\}.\tag{53}$$

We claim : Let $\phi(t,x,v)\in C_c^\infty((0,T)\times\Omega\times\mathbb{R}^3)$ and we have

$$\int_0^T\iint_{\Omega\times\mathbb{R}^3} f\partial\phi = -\int_0^T\iint_{\Omega\times\mathbb{R}^3}\partial f\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}}\phi,\tag{54}$$

so that $f\in W^{1,p}$ with weak derivatives given by $\partial f\mathbf{1}_{\{t\neq t_{\mathbf{b}}\}}$.

Proof of claim. We first fix the test function $\phi(t,x,v)$. There exists $\delta = \delta_\phi > 0$ such that $\phi\equiv 0$ for $t\ge\frac{1}{\delta}$, or $\mathrm{dist}(x,\partial\Omega) < \delta$, or $|v|\ge\frac{1}{\delta}$. Let $\phi(t,x,v)\neq 0$ and $(t,x,v)\in\mathcal{M}$. If follows that $t = t_{\mathbf{b}}(x,v)$ so that $x_{\mathbf{b}} = x - t_{\mathbf{b}}v$. Hence $|x - x_{\mathbf{b}}| = t_{\mathbf{b}}|v|$ and

$$\mathrm{dist}(x,\Omega)\le|x-x_{\mathbf{b}}| = t_{\mathbf{b}}|v|.$$

Since $t_{\mathbf{b}}\le\frac{1}{\delta}$, this implies that

$$|v|\ge\frac{\delta}{t_{\mathbf{b}}}\ge\delta^2.$$

Otherwise $\mathrm{dist}(x,\partial\Omega) \le \delta$ so that $\phi(t,x,v)=0$. Furthermore, by the Velocity lemma and this lower bound of $|v|$, we conclude that there exists $\delta'(\delta,\Omega)>0$ such that

$$\begin{aligned}|v\cdot n(x_{\mathbf{b}})|^2 &\gtrsim_\Omega |v\cdot\nabla_x\xi(x_{\mathbf{b}})|^2 = \alpha(t-t_{\mathbf{b}};t,x,v) \ge e^{-C_\Omega\langle v\rangle t_{\mathbf{b}}}\alpha(t;t,x,v) \ge e^{-C_\Omega\langle v\rangle t_{\mathbf{b}}} C_\xi|v|^2|\xi(x)|\\ &\ge e^{-C_\Omega\delta^{-2}}C_\xi\delta^4 \min_{\mathrm{dist}(x,\partial\Omega)\ge\delta}|\xi(x)| = 2\delta'(\delta,\Omega)>0.\end{aligned}$$

In particular, this lower bound and a direct computation of (49) imply that $\{\phi\neq 0\}\cap\mathcal{M}$ is a smooth 6D hypersurface.

We next take C^1 approximation of f_0^l, H^l, and g^l (by partition of unity and localization) such that

$$||f_0^l - f_0||_{W^{1,p}} \to 0, \quad ||g^l-g||_{W^{1,p}([0,T]\times\gamma_-\backslash\gamma_-^{\delta'})}\to 0, \quad ||H^l-H||_{W^{1,p}([0,T]\times\Omega\times\mathbb{R}^3)}\to 0.$$

This implies, from the trace theorem, that

$$f_0^l(x,v)\to f_0(x,v) \quad \text{and} \quad g^l(0,x,v)\to g(0,x,v) \quad \text{in} \quad L^1(\gamma_-\backslash\gamma_-^{\delta'}).$$

We define accordingly, for $(t,x,v)\in[0,T]\times\Omega\times\mathbb{R}^3$,

$$f^l(t,x,v) = \mathbf{1}_{\{t<t_{\mathbf{b}}\}}e^{-\int_0^t\nu}f_0^l(x-tv,v)+\mathbf{1}_{\{t>t_{\mathbf{b}}\}}e^{-\int_0^{t_{\mathbf{b}}}\nu}g^l(t-t_{\mathbf{b}},x_{\mathbf{b}},v)+\int_0^{\min\{t,t_{\mathbf{b}}\}}e^{-\int_0^s\nu}H^l(t-s,x-sv,v)\mathrm{d}s, \tag{55}$$

and $f^l_\pm(t,x,v)\equiv \mathbf{1}_{\{t\gtrless t_{\mathbf{b}}\}}f^l$. Therefore for all $(x,v)\in\gamma_-$,

$$f^l_+(s,x+sv,v)-f^l_-(s,x+sv,v) = e^{-s\nu(v)}g^l(0,x,v)-e^{-s\nu(v)}f_0^l(x,v).$$

Since $\{\phi\neq0\}\cap\mathcal{M}$ is a smooth hypersurface, we apply the Gauss theorem to f^l to obtain

$$\iiint \partial_{\mathbf{e}}\phi f^l \mathrm{d}x\mathrm{d}v\mathrm{d}t = \iint [f^l_+-f^l_-]\phi\, \mathbf{e}\cdot\mathbf{n}_{\mathcal{M}}\mathrm{d}\mathcal{M} - \left\{\iiint_{t>t_{\mathbf{b}}}\phi\,\partial_{\mathbf{e}}f^l_+\mathrm{d}x\mathrm{d}v\mathrm{d}t + \iiint_{t<t_{\mathbf{b}}}\phi\,\partial_{\mathbf{e}}f^l_-\mathrm{d}x\mathrm{d}v\mathrm{d}t\right\}, \tag{56}$$

where $\partial_{\mathbf{e}} = [\partial_t,\nabla_x,\nabla_v] = [\partial_t,\partial_{x_1},\partial_{x_2},\partial_{x_3},\partial_{v_1},\partial_{v_2},\partial_{v_3}]$ and $\mathbf{n}_{\mathcal{M}} = \mathbf{e}_1\in\mathbb{R}^7$. We have used $(s,x+sv,v)$ and $(x,v)\in\gamma_-$ as our parametrization for the manifold $\mathcal{M}\cap\{\phi\neq0\}$, so that $n(x_{\mathbf{b}}(x,v))\cdot v\ge 2\delta'$ is equivalent to $n(x)\cdot v\ge2\delta'$. Therefore the above hypersurface integration over $\{t\neq t_{\mathbf{b}}\}$ is bounded by

$$\begin{aligned}&C_{\phi\delta}\int_0^{\frac{1}{\delta}}\int_{n(x)\cdot v\ge2\delta'}|f^l_+(s,x+sv,v)-f^l_-(s,x+sv,v)|\mathrm{d}S_x\mathrm{d}v\mathrm{d}s\\ &\lesssim_{\phi,\delta}\int_{n(x)\cdot v\ge 2\delta'}|g^l(0,x,v)-f_0^l(s,v)|\mathrm{d}S_x\mathrm{d}v\to0, \quad \text{as } l\to\infty,\end{aligned}$$

since the compatibility condition $f_0(x,v)=g(0,x,v)$ for $(x,v)\in\gamma_-$. Clearly, taking difference of (55) and (48), we deduce $f^l\to f$ strongly in $L^p(\{\phi\neq0\})$ due to the first estimate of (52). Furthermore, due to (52), we have a uniform-in-l bound of $f^l_\pm$ in $W^{1,p}(\{t\gtrless t_{\mathbf{b}},\ \phi\neq0\})$ such that, up to subsequence,

$$\partial_{\mathbf{e}}f^l_+ \rightharpoonup \partial_{\mathbf{e}}f\mathbf{1}_{\{t>t_{\mathbf{b}}\}}, \quad \partial_{\mathbf{e}}f^l_-\rightharpoonup\partial_{\mathbf{e}}f\mathbf{1}_{\{t<t_{\mathbf{b}}\}}, \quad \text{weakly in } L^p(\{\phi\neq0\}).$$

Finally we conclude the claim by letting $l\to\infty$ in (56).

Now notice that from its explicit form (48), and since all the data are compactly supported in velocity, f is itself compactly supported in velocity. Recall $\partial=[\partial_t,\nabla_x,\nabla_v]$. From this and the L^p bounds above, we conclude

$$\{\partial_t+v\cdot\nabla_x+\nu\}\partial f = \partial H - \partial v\cdot\nabla_x f - \partial\nu f\in L^p. \tag{57}$$

By the trace theorem (Lemma 7), traces of $\partial_t f,\nabla_x f,\nabla_v f$ exist. To evaluate these traces, we take derivatives along characteristics. Letting $t\to t_{\mathbf{b}}$ and $t\to0$, we deduce (44). From the Green's identity, Lemma 8, we have (45), (46) and (47), and therefore we conclude $\partial f\in C^0([0,T];L^p)$.

In order to remove the compact support assumption we employ the cut-off function χ used in (3). Define $f^m = \chi(|v|/m)f$ then f^m satisfies

$$\{\partial_t + v\cdot\nabla_x + \nu\}f^m = \chi(|v|/m)H, \tag{58}$$
$$f^m(0,x,v) = \chi(|v|/m)f_0, \quad f^m|_{\gamma_-} = \chi(|v|/m)g.$$

Note that $\nabla_v[\chi(|v|/m)g] = \chi(|v|/m)\nabla_v g + g\nabla_v\chi(|v|/m)$ and $\chi(|v|/m)f_0(x,v) = \chi(|v|/m)g(0,x,v)$ for $(x,v)\in\gamma_-$. Apply previous result to compute the traces of the derivatives of f^m. It is standard (using Green's identity) to show that $\partial_t f^m, \nabla_x f^m$ and $\nabla_v f^m$ are Cauchy and we can pass a limit. □

We now study weighted $W^{1,p}$ estimate. Recall (3). We first define an effective collision frequency:

$$\nu_{\varpi,\beta}(t,x,v) = \varpi\langle v\rangle - \beta\alpha^{\beta-1}(v\cdot\nabla_x\alpha). \tag{59}$$

Clearly $\nu_{\varpi,\beta}(t,x,v)\sim\beta\langle v\rangle$ from our choice (4), and from the definition, it is easy to verify that

$$[\partial_t + v\cdot\nabla_x + \nu_{\varpi,\beta}](e^{-\varpi\langle v\rangle t}\alpha^\beta f) = e^{\varpi\langle v\rangle t}\alpha^\beta[\partial_t + v\cdot\nabla_x]f. \tag{60}$$

Proposition 2. *Let f be a solution of (40). Assume (43) and $\langle v\rangle g\in L^\infty([0,T]\times\gamma_-)$, $\langle v\rangle H\in L^\infty([0,T]\times\Omega\times\mathbb{R}^3)$. For any fixed $p\in[2,\infty]$, assume*

$$\begin{aligned} e^{-\varpi\langle v\rangle t}\alpha^\beta\partial_t g,\ e^{-\varpi\langle v\rangle t}\alpha^\beta\nabla_\tau g &\in L^\infty([0,T];L^p(\gamma_-)),\\ e^{-\varpi\langle v\rangle t}\alpha^\beta\big\{|\nabla_\tau g| + \frac{1}{n(x)\cdot v}\big(|\partial_t g| + \langle v\rangle|\nabla_\tau g| + |H|\big)\big\} &\in L^\infty([0,T];L^p(\gamma_-)),\\ e^{-\varpi\langle v\rangle t}\alpha^\beta\big| - v\cdot\nabla_x f_0 - \nu(v)f_0 + H_0\big| &\in L^p(\Omega\times\mathbb{R}^3), \end{aligned}$$

and assume $1/p+1/q=1$ there exist $TC_T = O(T)$ and $\varepsilon\ll 1$ such that for all $t\in[0,T]$

$$\Big|\iint_{\Omega\times\mathbb{R}^3} e^{-\varpi\langle v\rangle t}\alpha^\beta\partial H(t)h(t)\Big| \le C_T\big\{||h(t)||_q + \varepsilon||\nu_{l,\beta}^{1/q}h(t)||_q\big\}.$$

Then $f(t,x,v)$ satisfies

$$||f(t)||_\infty \le ||f_0||_\infty + \sup_{0\le s\le t}||g(s)||_\infty + \Big|\Big|\int_0^t H(s)ds\Big|\Big|_\infty.$$

Recall $\partial = [\partial_t,\nabla_x,\nabla_v]$, then

$$\begin{aligned} \{\partial_t + v\cdot\nabla_x + \nu_{\varpi,\beta}\}[e^{-\varpi\langle v\rangle t}\alpha^\beta\partial f] &= e^{-\varpi\langle v\rangle t}\alpha^\beta\partial H - \partial v\cdot e^{-\varpi\langle v\rangle t}\alpha^\beta\nabla_x f - \partial\nu(v)e^{-\varpi\langle v\rangle t}\alpha^\beta f,\\ e^{-\varpi\langle v\rangle t}\alpha^\beta\partial f|_{t=0} &= e^{-\varpi\langle v\rangle t}\alpha^\beta\partial f_0, \quad e^{-\varpi\langle v\rangle t}\alpha^\beta\partial f|_{\gamma_-} = e^{-\varpi\langle v\rangle t}\alpha^\beta[\partial g|_{\gamma_-}], \end{aligned}$$

where $[\partial g|_{\gamma_-}]$ is given in (44). Moreover, recalling (41) and (42), we have for $2\le p<\infty$,

$$\begin{aligned} &\int_{\Omega\times\mathbb{R}^3}|e^{-\varpi\langle v\rangle t}\alpha^\beta\partial f(t)|^p + \int_0^t\int_{\Omega\times\mathbb{R}^3}\nu_{l,\beta}|e^{-\varpi\langle v\rangle t}\alpha^\beta\partial f|^p + \int_0^t\int_{\gamma_+}|e^{-\varpi\langle v\rangle t}\alpha^\beta\partial f|^p \qquad (61)\\ &\lesssim \int_{\Omega\times\mathbb{R}^3}|e^{-\varpi\langle v\rangle t}\alpha^\beta\partial f_0|^p + \int_0^t\int_{\gamma_-}|e^{-\varpi\langle v\rangle t}\alpha^\beta\partial g|^p\\ &+\int_0^t\int_{\Omega\times\mathbb{R}^3}|e^{-\varpi\langle v\rangle t}\alpha^\beta\partial H - \partial v\cdot e^{-\varpi\langle v\rangle t}\alpha^\beta\nabla_x f - \partial\nu e^{-\varpi\langle v\rangle t}\alpha^\beta f||e^{-\varpi\langle v\rangle t}\alpha^\beta\partial f|^{p-1},\\ &||e^{-\varpi\langle v\rangle t}\alpha^\beta\partial f(t)||_\infty \lesssim ||e^{-\varpi\langle v\rangle t}\alpha^\beta\partial f_0||_\infty + ||e^{-\varpi\langle v\rangle t}\alpha^\beta\partial g||_\infty\\ &+\int_0^t||e^{-\varpi\langle v\rangle t}\alpha^\beta\partial H - \partial v\cdot e^{-\varpi\langle v\rangle t}\alpha^\beta\nabla_x f - \partial\nu e^{-\varpi\langle v\rangle t}\alpha^\beta f||_\infty, \qquad \textit{for } p=\infty. \end{aligned}$$

Proof. First we assume f_0, g and H have compact supports in $\{v \in \mathbb{R}^3 : |v| < m\}$. We estimate ∂f in the bulk. From the velocity lemma (Lemma 1), we have

$$\sup_{t\leq t_{\mathbf{b}}} \frac{e^{-\varpi\langle v\rangle t}\alpha^\beta(x,v)}{\alpha^\beta(x-tv,v)} \leq e^{C_{m,\beta}t}, \quad \sup_{t\geq t_{\mathbf{b}}} \frac{e^{-\varpi\langle v\rangle t}\alpha^\beta}{e^{-\varpi\langle v\rangle(t-t_{\mathbf{b}})}\alpha^\beta(x_{\mathbf{b}},v)} \leq e^{C_{m,\beta}t_{\mathbf{b}}},$$
$$\sup_{\max\{t-t_{\mathbf{b}},0\}\leq s\leq t} \frac{e^{-\varpi\langle v\rangle t}\alpha^\beta}{e^{-\varpi\langle v\rangle(t-s)}\alpha(x-sv,v)^\beta} \leq e^{C_{m,\beta}s}.$$

Multiply $e^{-\varpi\langle v\rangle t}\alpha^\beta$ by the above direct computations and use the above inequalities to get

$$\begin{aligned}
&e^{-\varpi\langle v\rangle t}\alpha^\beta|\partial_t f(t,x,v)| \\
&\lesssim\ e^{C_{m,\beta}t}e^{-t\nu(v)}\alpha^\beta\big|[\nu f_0 + v\cdot\nabla_x f_0 - H_{|t=0}](x-tv,v)\big|\mathbf{1}_{\{t<t_{\mathbf{b}}\}} \\
&\quad + e^{C_{m,\beta}t_{\mathbf{b}}}e^{-t_{\mathbf{b}}\nu}e^{-\varpi\langle v\rangle(t-t_{\mathbf{b}})}\alpha^\beta\partial_t\big|g(t-t_{\mathbf{b}},x_{\mathbf{b}},v)\big|\mathbf{1}_{\{t>t_{\mathbf{b}}\}} \\
&\quad + \int_0^{\min(t,t_{\mathbf{b}})} e^{C_{m,\beta}s}e^{-s\nu}e^{-\varpi\langle v\rangle(t-s)}\alpha^\beta\big|\partial_t H(t-s,x-vs,v)\big|\mathrm{d}s, \\
&e^{-\varpi\langle v\rangle t}\alpha^\beta|\nabla_x f(t,x,v)| \\
&\lesssim\ e^{C_{m,\beta}t}e^{-t\nu}\alpha^\beta\big|\nabla_x f_0(x-tv,v)\big|\mathbf{1}_{\{t<t_{\mathbf{b}}\}} + e^{C_{m,\beta}t_{\mathbf{b}}}e^{-t_{\mathbf{b}}\nu}\sum_{i=1}^2\tau_i e^{-\varpi\langle v\rangle(t-t_{\mathbf{b}})}\alpha^\beta|\partial_{\tau_i}g(t-t_{\mathbf{b}},x_{\mathbf{b}},v)|\mathbf{1}_{\{t>t_{\mathbf{b}}\}} \\
&\quad + e^{C_{m,\beta}t_{\mathbf{b}}}e^{-t_{\mathbf{b}}\nu}n(x_{\mathbf{b}})\frac{e^{-\varpi\langle v\rangle(t-t_{\mathbf{b}})}\alpha^\beta(x_{\mathbf{b}},v)}{|v\cdot n(x_{\mathbf{b}})|}\Big|\Big\{\partial_t g + \sum_{i=1}^2(v\cdot\tau_i)\partial_{\tau_i}g + \nu g - H\Big\}(t-t_{\mathbf{b}},x_{\mathbf{b}},v)\Big|\mathbf{1}_{\{t>t_{\mathbf{b}}\}} \\
&\quad + \int_0^{\min(t,t_{\mathbf{b}})} e^{C_{m,\beta}s}e^{-s\nu}e^{-\varpi\langle v\rangle(t-s)}\alpha^\beta\big|\nabla_x H(t-s,x-vs,v)\big|\mathrm{d}s,
\end{aligned} \tag{62}$$

$$\begin{aligned}
&e^{-\varpi\langle v\rangle t}\alpha^\beta|\nabla_v f(t,x,v)| \\
&\lesssim\ e^{C_{m,\beta}t}e^{-t\nu}\alpha^\beta\big|[-t\nabla_x f_0 + \nabla_v f_0 - t\nabla_v\nu(v)f_0](x-tv,v)\big|\mathbf{1}_{\{t<t_{\mathbf{b}}\}} \\
&\quad + e^{C_{m,\beta}t_{\mathbf{b}}}e^{-t_{\mathbf{b}}\nu}\sum_{i=1}^2\tau_i e^{-\varpi\langle v\rangle(t-t_{\mathbf{b}})}\alpha^\beta|\partial_{\tau_i}g(t-t_{\mathbf{b}},x_{\mathbf{b}},v)|\mathbf{1}_{\{t>t_{\mathbf{b}}\}} \\
&\quad + e^{C_{m,\beta}t_{\mathbf{b}}}e^{-t_{\mathbf{b}}\nu}n(x_{\mathbf{b}})\frac{e^{-\varpi\langle v\rangle(t-t_{\mathbf{b}})}\alpha^\beta}{|v\cdot n(x_{\mathbf{b}})|}\Big|\Big\{\partial_t g + \sum_{i=1}^2(v\cdot\tau_i)\partial_{\tau_i}g + \nu g - H\Big\}(t-t_{\mathbf{b}},x_{\mathbf{b}},v)\Big|\mathbf{1}_{\{t>t_{\mathbf{b}}\}} \\
&\quad + e^{C_{m,\beta}t_{\mathbf{b}}}e^{-t_{\mathbf{b}}\nu}e^{-\varpi\langle v\rangle(t-t_{\mathbf{b}})}\alpha^\beta\big\{|\nabla_v g(t-t_{\mathbf{b}},x_{\mathbf{b}},v)| + |t_{\mathbf{b}}\nabla_v\nu(v)||g(t-t_{\mathbf{b}},x_{\mathbf{b}},v)|\big\}\mathbf{1}_{\{t>t_{\mathbf{b}}\}} \\
&\quad + \int_0^{\min(t,t_{\mathbf{b}})} e^{C_{m,\beta}s}e^{-s\nu}e^{-\varpi\langle v\rangle(t-s)}\alpha^\beta\big|\{\nabla_v H - s\nabla_x H - s\nabla\nu H\}(t-s,x-vs,v)\big|\mathrm{d}s.
\end{aligned}$$

Following (50) and (51) of Proposition 1 and using the condition of Proposition 2, we deduce

$$\begin{aligned}
||e^{-\varpi\langle v\rangle t}\alpha^\beta\partial_t f(t)||_p &\lesssim_{t,m,\beta} ||\alpha^\beta[v\cdot\nabla_x f_0 + \nu f_0 - H(0,\cdot,\cdot)]||_p + \left[\int_0^t ||e^{-\varpi\langle v\rangle s}\alpha^\beta\partial_t g(s)||^p_{\gamma,p}\mathrm{d}s\right]^{1/p} \\
&\quad + \left[\int_0^t ||e^{-\varpi\langle v\rangle s}\alpha^\beta\partial_t H(s)||^p_p\mathrm{d}s\right]^{1/p}, \\
||e^{-\varpi\langle v\rangle t}\alpha^\beta\nabla_x f(t)||_p &\lesssim_{t,m,\beta} ||\alpha^\beta\nabla_x f_0||_p + \sum_{i=1}^2\left[\int_0^t ||e^{-\varpi\langle v\rangle s}\alpha^\beta\partial_{\tau_i}g(s)||^p_{\gamma,p}\mathrm{d}s\right]^{1/p} \\
&\quad + \left[\int_0^t \Big|\Big|\frac{e^{-\varpi\langle v\rangle t}\alpha^\beta}{v\cdot n}\{\partial_t g + \sum(v\cdot\tau_i)\partial_{\tau_i}g + \nu g - H\}\Big|\Big|^p_{\gamma,p}\mathrm{d}s\right]^{1/p} \\
&\quad + \left[\int_0^t ||e^{-\varpi\langle v\rangle s}\alpha^\beta\nabla_x H(s)||^p_p\mathrm{d}s\right]^{1/p},
\end{aligned}$$

$$||e^{-\varpi\langle v\rangle t}\alpha^\beta\nabla_v f(t)||_p \lesssim_{t,m,\beta} ||\alpha^\beta \nabla_v f_0||_p + \sum_{i=1}^2\Big[\int_0^t ||e^{-\varpi\langle v\rangle s}\alpha^\beta \partial_{\tau_i} g(s)||^p_{\gamma,p}\mathrm{d}s\Big]^{1/p} + \sup_{0\le s\le t}||\langle v\rangle g(s)||_\infty$$

$$+\Big[\int_0^t \Big|\Big|\frac{e^{-\varpi\langle v\rangle t}\alpha^\beta}{v\cdot n}\{\partial_t g + \sum (v\cdot\tau_i)\partial_{\tau_i} g + \nu g - H\}\Big|\Big|^p_{\gamma,p}\mathrm{d}s\Big]^{1/p}$$

$$+\Big[\int_0^t ||e^{-\varpi\langle v\rangle s}\alpha^\beta\nabla_v g(s)||_p^p\mathrm{d}s\Big]^{1/p}$$

$$+\Big[\int_0^t ||e^{-\varpi\langle v\rangle s}\alpha^\beta\nabla_v H(s)||_p^p + ||e^{-\varpi\langle v\rangle s}\alpha^\beta\nabla_x H(s)||_p^p\mathrm{d}s\Big]^{1/p} + \sup_{0\le s\le t}||\langle v\rangle H(s)||_\infty.$$

By the hypothesis of Proposition 2 and assumption on f_0, g and H to have compact support, the right hand sides are bounded and hence $e^{-\varpi\langle v\rangle t}\alpha^\beta\partial_t f, e^{-\varpi\langle v\rangle t}\alpha^\beta\nabla_x f$, and $e^{-\varpi\langle v\rangle t}\alpha^\beta\nabla_v f$ are in $L^\infty([0,T];L^p(\Omega\times\mathbb{R}^3))$.

Since f_0, g and H are compactly supported on $\{v\in\mathbb{R}^3 : |v|\le m\}$, the derivatives $e^{-\varpi\langle v\rangle t}\alpha^\beta\partial_t f$, $e^{-\varpi\langle v\rangle t}\alpha^\beta\nabla_x f$ and $e^{-\varpi\langle v\rangle t}\alpha^\beta\nabla_v f$ are compactly supported on $\{v\in\mathbb{R}^3 : |v|\le m\}$ and hence from (60) and (57)

$$\{\partial_t + v\cdot\nabla_x + \nu_{l,\beta}\}[e^{-\varpi\langle v\rangle t}\alpha^\beta\partial f] = e^{-\varpi\langle v\rangle t}\alpha^\beta\partial H - \partial v\cdot e^{-\varpi\langle v\rangle t}\alpha^\beta\nabla_x f - \partial\nu(v)e^{-\varpi\langle v\rangle t}\alpha^\beta f.$$

Moreover, from the general definition of traces, by choosing a test function multiplied by $e^{-\varpi\langle v\rangle t}\alpha^\beta$, we deduce $e^{-\varpi\langle v\rangle t}\alpha^\beta\partial f$ has the same trace as $e^{-\varpi\langle v\rangle t}\alpha^\beta[\partial f|_\gamma]$.

Now we can apply Lemma 8 to have (61) which does not depend on the velocity cut-off. Therefore for the general case, we use (58) and pass a limit to conclude the proof. □

3. Dynamical Non-local to Local Estimate

The main purpose of this section is to prove Lemma 2 and its variants

Lemma 9. *Let $(t,x,v)\in[0,\infty)\times\bar\Omega\times\mathbb{R}^d$ for $d=2,3$. Let $\bar u = u$ if $d=3$ and $\bar u=(u_1,u_2,0)$ if $d=2$.*

(1) For $\frac12<\beta<1$ and $0<\kappa\le 1$, we have

$$\begin{aligned}&\int_0^{t_{\mathbf{b}}(x,v)}\int_{\mathbb{R}^3} e^{-l\langle v\rangle(t-s)}\frac{e^{-\theta|v-u|^2}}{|v-u|^{2-\kappa}[\alpha(x-(t_{\mathbf{b}}(x,v)-s)v,\bar u)]^\beta}\frac{|v|}{|\bar u|}\frac{\langle u\rangle^r}{\langle v\rangle^r}Z(s,x,v)\mathrm{d}u\mathrm{d}s\\ &\lesssim_{\theta,r}\quad \frac{O(\tilde\delta)}{|v|^2[\alpha(x,v)]^{\beta-1}}\sup_{s\in[0,t_{\mathbf{b}}(x,v)]}\{e^{-l\langle v\rangle(t-s)}Z(s,x,v)\}\\ &\qquad + \frac{C_{\tilde\delta}}{[\alpha(x,v)]^{\beta-1/2}}\int_0^{t_{\mathbf{b}}(x,v)} e^{-Cl\langle v\rangle(t-s)}Z(s,x,v)\mathrm{d}s.\end{aligned}\tag{63}$$

(2) Let $[X_{\mathbf{cl}}(s;t,x,v),V_{\mathbf{cl}}(s;t,x,v)]$ is the specular backward trajectory or the bounce-back trajectory. For $\frac12<\beta<1$ and $0<\kappa\le1$ and $\delta>0$ and $1\gg\tilde\delta>0$ and $r\in\mathbb{R}$, there exists $l\gg_\xi 1$ and $C_{l,\beta,\xi,r}>0, C_{\tilde\delta,\delta,\beta}>0$ such that

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^3} e^{-l\langle v\rangle(t-s)}\frac{e^{-\theta|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}}\frac{|v|}{|u|}\frac{\langle u\rangle^r}{\langle v\rangle^r}\frac{Z(s,x,v)}{\big[\alpha(X_{\mathbf{cl}}(s;t,x,v),u)\big]^\beta}\mathrm{d}u\mathrm{d}s\\ &\lesssim_{\xi,r}\frac{C_\delta O(\tilde\delta)+C_{\tilde\delta,\delta}O(l^{-1})}{\langle v\rangle\big[\alpha(x,v)\big]^{\beta-1/2}}\sup_{0\le s\le t}\big\{e^{-C_{l,\beta,\xi,r}\langle v\rangle(t-s)}Z(s,x,v)\big\}.\end{aligned}\tag{64}$$

For the specular cycles and the bounce-back cycles it is important to control the *number of bounces*,

$$\ell_*(s)=\ell_*(s;t,x,v)\in\mathbb{N}\quad\text{if}\quad t^{\ell_*+1}\le s<t^{\ell_*}.$$

Here we prove $t_{\mathbf{b}}(x,v) \simeq \alpha(x,v)^{1/2}/|v|^2$. Recall (40) of [8] or (2.4) of [2]: if Ω is bounded and $\partial\Omega$(i.e. ξ) is smooth then for $(x,v) \in \gamma_-$

$$t_{\mathbf{b}}(x,v) \gtrsim_\Omega \frac{\sqrt{\alpha(x,v)}}{|v|^2}. \tag{65}$$

It suffices to prove $t_{\mathbf{b}}(x,v) \gtrsim_\Omega \frac{|n(x)\cdot v|}{|v|^2}$. For $x \in \partial\Omega$ there exists $0<\delta \ll 1$ such that

$$\sup_{\substack{y\in\partial\Omega\\|x-y|<\delta}} \frac{|(x-y)\cdot n(x)|}{|x-y|^2} \lesssim \max_{\substack{y\in\partial\Omega\\|x-y|<\delta}} |\nabla^2\xi(x)|.$$

If $|x-y|\geq\delta$ then $\frac{|(x-y)\cdot n(x)|}{|x-y|^2} \leq \delta^{-2}|(x-y)\cdot n(x)| \lesssim_{\delta,\Omega} 1$. By the compactness of Ω and $\partial\Omega$ we have $|(x-y)\cdot n(x)| \lesssim |x-y|^2$ for all $x,y\in\partial\Omega$. Taking the inner product of $x - x_{\mathbf{b}}(x,v) = t_{\mathbf{b}}(x,v)v$ with $n(x)$ we have

$$t_{\mathbf{b}}(x,v)|v\cdot n(x)| = |(x-x_{\mathbf{b}}(x,v))\cdot n(x)| \lesssim |x-x_{\mathbf{b}}(x,v)|^2 = C_\Omega |v|^2 |t_{\mathbf{b}}(x,v)|^2,$$

and this proves (65).

If Ω is convex (2) then for $(x,v)\in\gamma_-$

$$t_{\mathbf{b}}(x,v) \lesssim_\xi \frac{\sqrt{\alpha(x,v)}}{|v|^2}. \tag{66}$$

It suffices to show $t_{\mathbf{b}}(x,v) \lesssim_\xi \frac{|n(x)\cdot v|}{|v|^2}$. Since $\xi(x) = 0 = \xi(x - t_{\mathbf{b}}(x,v)v)$ for $(x,v)\in\gamma_-$, we have

$$\begin{aligned} 0 = \xi(x - t_{\mathbf{b}}(x,v)v) &= \xi(x) + \int_0^{t_{\mathbf{b}}(x,v)} [-v\cdot\nabla_x\xi(x-sv)]\mathrm{d}s \\ &= [-v\cdot\nabla_x\xi(x)]t_{\mathbf{b}}(x,v) + \int_0^{t_{\mathbf{b}}(x,v)}\int_0^s \{v\cdot\nabla_x^2\xi(x-\tau v)\cdot v\}\mathrm{d}\tau\mathrm{d}s. \end{aligned}$$

By the convexity of ξ in (2) we have $[v\cdot\nabla_x\xi(x)]t_{\mathbf{b}}(x,v) \geq \frac{(t_{\mathbf{b}}(x,v))^2}{2}C_\xi|v|^2$, and therefore this proves (66).

An important consequence of Velocity lemma is that for the specular cycles

$$\alpha(X_{\mathbf{cl}}(s;t,x,v), V_{\mathbf{cl}}(s;t,x,v)) \gtrsim e^{-\mathfrak{C}|v||t-s|}\alpha(x,v),$$

and therefore for the specular cycles

$$\begin{aligned} \ell_*(s;t,x,v) &\leq \frac{|t-s|}{\min_{0\leq\ell\leq\ell_*(s;t,x,v)}|t^\ell - t^{\ell+1}|} \lesssim \frac{|t-s|}{\min_{0\leq\ell\leq\ell_*(s;t,x,v)}\frac{\sqrt{\alpha(x^\ell,v^\ell)}}{|v^\ell|^2}} \\ &\lesssim \frac{|t-s||v|^2}{\sqrt{\alpha(x,v)}}e^{\mathfrak{C}|v|(t-s)}. \end{aligned} \tag{67}$$

Remark that for the bounce-back cycles we do not have the growth term $e^{\mathfrak{C}|v|(t-s)}$. This is because of the fact $\alpha(X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))$ is either $\alpha(x^1,v^1)$ or $\alpha(x^2,v^2)$, and the fact $|t-t^2|\leq 2|t^1-t^2| \lesssim \frac{C_\Omega}{|v|}$ for the bounded domain.

We are ready to prove several versions of the non-local to local estimates. The first one is for the stochastic(diffuse) cycles:

Proof of (1) of Lemma 2. Since $\frac{\langle u\rangle^r}{\langle v\rangle^r} \lesssim \{1+|v-u|^2\}^{\frac{r}{2}}$ and $\langle V_{\mathbf{cl}}(s)-u\rangle^r e^{-\theta|V_{\mathbf{cl}}(s)-u|^2} \lesssim e^{-C_{\theta,r}|V_{\mathbf{cl}}(s)-u|^2}$, it suffices to consider $r=0$ case. We prove (19). *Step 1.* We show that

$$\int_{\mathbb{R}^3} \frac{e^{-\theta|v-u|^2}}{|v-u|^{2-\kappa}[\alpha(x-(t_{\mathbf{b}}(x,v)-s)v,u)]^\beta}\mathrm{d}u \lesssim \frac{1}{|v|^{2\beta-1}|\xi|^{\beta-\frac{1}{2}}}. \tag{68}$$

For fixed $s\in[0,t_{\mathbf{b}}(x,v))$ and therefore fixed $X_{\mathbf{cl}}(s) = x-(t_{\mathbf{b}}(x,v)-s)v \in \bar{\Omega}$.

Firstly, we consider the case of $|\xi(x)|\leq\delta_\Omega$. From the assumption, we have $\nabla\xi(x)\neq 0$ and therefore there is uniquely determined unit vector $n(X_{\mathbf{cl}}(s)) = \frac{\nabla\xi(X_{\mathbf{cl}}(s))}{|\nabla\xi(X_{\mathbf{cl}}(s))|}$. We choose two unit vector τ_1 and τ_2 so that $\{\tau_1,\tau_2,n(X_{\mathbf{cl}}(s))\}$ is an orthonormal basis of $\mathbb{R}^3$.

We decompose the velocity variables $u \in \mathbb{R}^3$ as

$$u = u_n n(X_{\mathbf{cl}}(s)) + u_\tau \cdot \tau = u_n n(X_{\mathbf{cl}}(s)) + \sum_{i=1}^{2} u_{\tau,i}\tau_i.$$

We note that $u_\tau \in \mathbb{R}^2$ and $u_n \in \mathbb{R}$ are completely free coordinates. Therefore using the Fubini's theorem we can rearrange the order of integration freely. Now we split, for $0 \le s \le t_{\mathbf{b}}(x,v)$,

$$\begin{aligned}
&\int_{\mathbb{R}^3} \frac{e^{-\theta|v-u|^2}}{|v-u|^{2-\kappa}} \frac{1}{[\alpha(x-(t_{\mathbf{b}}(x,v)-s)v,u)]^\beta} \mathrm{d}u \\
&\lesssim \int_{\mathbb{R}^2}\int_{\mathbb{R}} \frac{e^{-\theta|v-u|^2}}{|v-u|^{2-\kappa}\big[|u_n|^2 + |\xi(X_{\mathbf{cl}}(s))||u|^2\big]^\beta} \mathrm{d}u_n \mathrm{d}u_\tau \\
&= \int_{|u|\ge 5|v|} + \int_{|u|\le \frac{|v|}{2}} + \int_{\frac{|v|}{2}\le|u|\le 5|v|} = (\mathbf{I}) + (\mathbf{II}) + (\mathbf{III}).
\end{aligned}$$

For the first term $(\mathbf{I})$ we use, for $|u| \ge 5|v|$ (therefore $|v| \le \frac{|u|}{5}$),

$$|u-v|^2 = \frac{|u-v|^2}{2} + \frac{|u-v|^2}{2} \ge \frac{\frac{|u|^2}{2} - |v|^2}{2} + \frac{\frac{|u|^2}{2} - |v|^2}{2} \ge \frac{23}{4}|v|^2 + \frac{23}{100}|u|^2 \gtrsim |v|^2 + |u|^2,$$

and we use $\big[|u_n|^2 + |\xi||u|^2\big]^\beta \ge \big[|u_n|^2 + 25|\xi||v|^2\big]^\beta \gtrsim \big[|u_n|^2 + |\xi||v|^2\big]^\beta$ for $|u| \ge 5|v|$ to have

$$(\mathbf{I}) \lesssim e^{-C|v|^2} \int_{\mathbb{R}^2} \mathrm{d}u_\tau \frac{e^{-C|u_\tau|^2}}{|v_\tau - u_\tau|^{2-\kappa}} \int_{\mathbb{R}} \mathrm{d}u_n \frac{e^{-C|u_n|^2}}{\big[|u_n|^2 + |\xi||v|^2\big]^\beta}.$$

Since $\frac{1}{|v_\tau - u_\tau|^{2-\kappa}} \in L^1_{\mathrm{loc}}(\{u_\tau \in \mathbb{R}^2\})$ for $\kappa > 0$ we first integrate over u_τ is finite. Then

$$\begin{aligned}
(\mathbf{I}) &\lesssim e^{-C|v|^2} \int_{\mathbb{R}} \frac{e^{-C|u_n|^2}}{\big[|u_n|^2 + |\xi||v|^2\big]^\beta} \mathrm{d}u_n \\
&\lesssim e^{-C|v|^2}\Big\{ \int_{10}^{\infty} \frac{e^{-C|u_n|^2}}{|u_n|^{2\beta}\mathbf{1}_{\{|u_n|\ge 10\}}} \mathrm{d}|u_n| + \int_0^{10} \frac{\mathrm{d}|u_n|}{\big[|u_n|^2 + |\xi||v|^2\big]^\beta} \Big\} \\
&\lesssim \big(1 + \int_0^{10} \frac{\mathrm{d}|u_n|}{\big[|u_n|^2 + |\xi||v|^2\big]^\beta}\big) e^{-C|v|^2} \lesssim\ e^{-C|v|^2}\big(1 + \int_0^{10} \frac{\mathrm{d}[|\xi|^{\frac12}|v|\tan\theta]}{|\xi|^\beta|v|^{2\beta}(1+\tan^2\theta)^\beta}\big) \\
&\lesssim e^{-C|v|^2}\Big(1 + \frac{1}{|v|^{2\beta-1}}\frac{1}{|\xi|^{\beta-1/2}} \int_0^{\pi/2} (\cos\theta)^{2\beta-2}\mathrm{d}\theta\Big) \lesssim e^{-C|v|^2}\Big(1 + \frac{1}{|v|^{2\beta-1}}\frac{1}{|\xi|^{\beta-1/2}}\Big) \\
&\lesssim \frac{e^{-C_\theta|v|^2}}{|v|^{2\beta-1}}\frac{1}{|\xi|^{\beta-1/2}},
\end{aligned}$$

where we have used a change of variables: $|u_n| = |\xi|^{\frac12}|v|\tan\theta$ and $\mathrm{d}|u_n| = |\xi|^{\frac12}|v|\sec^2\theta \mathrm{d}\theta$ and $(\cos\theta)^{2\beta-2} \in L^1_{\mathrm{loc}}(\{\theta \in [0,\frac{\pi}{2}]\})$ for $\beta > \frac12$.

For the second term $(\mathbf{II})$, we use $|v-u| \ge |v| - |u| \ge |v| - \frac{|v|}{2} \ge \frac{|v|}{2}$ from $|u| \le \frac{|v|}{2}$, and apply the change of variables $u \mapsto |v|u$ to have

$$\begin{aligned}
(\mathbf{II}) \ &\lesssim\ \frac{1}{|v|^{2-\kappa}} \int_{|u_n|+|u_\tau|\le\frac{|v|}{2}} \frac{e^{-C|v|^2}\mathrm{d}u_n\mathrm{d}u_\tau}{\big[|u_n|^2 + |\xi||u_\tau|^2\big]^\beta} \\
&=\ \frac{1}{|v|^{2-\kappa}} \int_{|v|(|u_n|+|u_\tau|)\le\frac{|v|}{2}} \frac{e^{-C|v|^2}|v|\mathrm{d}u_n|v|^2\mathrm{d}u_\tau}{\big[|v|^2|u_n|^2 + |\xi||v|^2|u_\tau|^2\big]^\beta} \\
&\lesssim\ \frac{e^{-C|v|^2}}{|v|^{2\beta-\kappa-1}} \int_{|u_\tau|\le\frac12}\int_{|u_\tau|\le\frac12} \frac{1}{\big[|u_n|^2 + |\xi||u_\tau|^2\big]^\beta} \mathrm{d}u_n\mathrm{d}u_\tau.
\end{aligned}$$

Now we apply the change of variables $|u_n| = |\xi|^{\frac{1}{2}}|u_\tau|\tan\theta$ for $\theta \in [0, \frac{\pi}{2}]$ with $\mathrm{d}u_n = |\xi|^{\frac{1}{2}}|u_\tau|\sec^2\theta\mathrm{d}\theta$ to have

$$\begin{aligned}(\mathbf{II}) &\lesssim \frac{e^{-C|v|^2}}{|v|^{2\beta-\kappa-1}}\int_{|u_\tau|\leq\frac{1}{2}}\mathrm{d}u_\tau\int_0^{\frac{\pi}{2}}\frac{|\xi|^{\frac{1}{2}}|u_\tau|\sec^2\theta\mathrm{d}\theta}{\left[|\xi||u_\tau|^2\tan^2\theta+|\xi||u_\tau|^2\right]^\beta}\\ &\lesssim \frac{e^{-C|v|^2}}{|v|^{2\beta-\kappa-1}|\xi|^{\beta-1/2}}\int_{|u_\tau|\leq\frac{1}{2}}\frac{\mathrm{d}u_\tau}{|u_\tau|^{2\beta-1}}\int_0^{\pi/2}(\cos\theta)^{2\beta-2}\mathrm{d}\theta\\ &\lesssim \frac{e^{-C|v|^2}}{|v|^{2\beta-\kappa-1}|\xi|^{\beta-1/2}},\end{aligned}$$

where we have used $\frac{1}{|u_\tau|^{2\beta-1}} \in L^1_{\mathrm{loc}}(\{u_\tau \in \mathbb{R}^2\})$ for $\beta < \frac{3}{2}$ and $(\cos\theta)^{2\beta-2} \in L^1_{\mathrm{loc}}(\{\theta \in [0, \frac{\pi}{2}]\})$ for $\beta > \frac{1}{2}$.

For the last term $(\mathbf{III})$, we use the lower bound of $|u|$ ($|u| \geq \frac{|v|}{2}$) to have $\left[|u_n|^2 + |\xi||u|^2\right]^\beta \geq \left[|u_n|^2 + |\xi|\frac{|v|^2}{4}\right]^\beta \gtrsim \left[|u_n|^2 + |\xi||v|^2\right]^\beta$ and

$$\begin{aligned}\int_{\frac{|v|}{2}\leq|u|\leq5|v|} &\lesssim \int_{0\leq|u_\tau|\leq5|v|}\frac{e^{-\frac{C}{2}|v_\tau-u_\tau|^2}}{|v_\tau-u_\tau|^{2-\kappa}}\mathrm{d}u_\tau\int_0^{5|v|}\frac{1}{\left[|u_n|^2+|\xi||v|^2\right]^\beta}\mathrm{d}u_n\\ &\lesssim \int_0^{5|v|}\frac{1}{\left[|u_n|^2+|\xi||v|^2\right]^\beta}\mathrm{d}u_n,\end{aligned}$$

where we have used $\frac{1}{|u_\tau|^{2-\kappa}} \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ for $\kappa > 0$. We apply a change of variables: $|u_n| = |\xi|^{1/2}|v|\tan\theta$ for $\theta \in [0, \pi/2]$ with $\mathrm{d}|u_n| = |\xi|^{\frac{1}{2}}|v|\sec^2\theta\mathrm{d}\theta$. Hence

$$(\mathbf{III}) \lesssim \int_0^{5|v|}\frac{1}{\left[|u_n|^2+|\xi||v|^2\right]^\beta}\mathrm{d}u_n = \int_0^{\frac{\pi}{2}}\frac{(\cos\theta)^{2\beta-2}}{|\xi|^{\beta-\frac{1}{2}}|v|^{2\beta-1}}\mathrm{d}\theta \lesssim \frac{1}{|v|^{2\beta-1}}\frac{1}{|v|^{2\beta-1}},$$

where we used $(\cos\theta)^{2\beta-2} \in L^1_{\mathrm{loc}}(\{\theta \in [0, \frac{\pi}{2}]\})$ for $\beta > \frac{1}{2}$. Overall, we combine the estimates of $(\mathbf{I}), (\mathbf{II})$ and $(\mathbf{III})$ to conclude (68).

Secondly, we consider the case of $|\xi(x)| > \delta_\Omega$. Then we can choose any orthonormal basis, for example standard basis $\{\tau_1, \tau_2, n\} = (e_1, e_2, e_3)$, to decompose the velocity variables $u \in \mathbb{R}^3$ as $u = u_1e_1 + u_2e_2 + u_3e_3 := u_{\tau,1}e_1 + u_{\tau,2}e_2 + u_ne_3$. Then

$$\begin{aligned}\alpha(X_{\mathbf{cl}}(s), u) &= |u\cdot\nabla\xi(X_{\mathbf{cl}}(s))|^2 - 2\xi(X_{\mathbf{cl}}(s))\{u\cdot\nabla^2\xi(X_{\mathbf{cl}}(s))\cdot u\}\\ &\geq 2|\xi(X_{\mathbf{cl}}(s))|\{u\cdot\nabla^2\xi(X_{\mathbf{cl}}(s))\cdot u\}\\ &= |\xi(X_{\mathbf{cl}}(s))|\{u\cdot\nabla^2\xi(X_{\mathbf{cl}}(s))\cdot u\} + |\xi(X_{\mathbf{cl}}(s))|\{u\cdot\nabla^2\xi(X_{\mathbf{cl}}(s))\cdot u\}\\ &= \delta_\Omega C_\xi|u|^2 + |\xi(X_{\mathbf{cl}}(s))|\{u\cdot\nabla^2\xi(X_{\mathbf{cl}}(s))\cdot u\}\\ &\gtrsim |u_n|^2 + |\xi(X_{\mathbf{cl}}(s))||u|^2.\end{aligned}$$

Then we follow all the proof with the same decomposition for $v := v_{\tau,1}e_1 + v_{\tau,2}e_2 + v_ne_3$ as well to conclude (68) for $|\xi(x)| > \delta_\Omega$.

Step 2. We first assume $v\cdot\nabla\xi(x) \geq 0$ and

$$x \in \partial\Omega.$$

We claim that there exist $\sigma_1, \sigma_2 > 0$ such that

$$|v\cdot\nabla\xi(x-(t_{\mathbf{b}}(x,v)-s)v)| \gtrsim \sqrt{\alpha(x-(t_{\mathbf{b}}(x,v)-s)v, v)} \quad \text{for all } s \in [0,\sigma_1]\cup[t_{\mathbf{b}}(x,v)-\sigma_2, t_{\mathbf{b}}(x,v)], \tag{69}$$

and $|v|\sqrt{-\xi(x-(t_{\mathbf{b}}(x,v)-s)v)} \gtrsim \sqrt{\alpha(x-(t_{\mathbf{b}}(x,v)-s)v,v)}$ for all $s \in [\sigma_1, t_{\mathbf{b}}(x,v)-\sigma_2]$. The mapping $s \mapsto \xi(x-(t_{\mathbf{b}}(x,v)-s)v)$ is one-to-one and onto on $s \in [0,\sigma_1]$ or on $s \in [t_{\mathbf{b}}(x,v)-$

$\sigma_2, t_{\mathbf{b}}(x,v)]$. Moreover this mapping $s \mapsto \xi(x-(t_{\mathbf{b}}(x,v)-s)v)$ is diffeomorphism and we have a change of variables on $s\in[0,\sigma_1]$ or $s\in[t_{\mathbf{b}}(x,v)-\sigma_2, t_{\mathbf{b}}(x,v)]$.

$$\mathrm{d}s = \frac{\mathrm{d}|\xi|}{|\nabla\xi(x-(t_{\mathbf{b}}(x,v)-s)v)\cdot v|} \lesssim \frac{\mathrm{d}|\xi|}{\sqrt{\alpha(x-(t_{\mathbf{b}}(x,v)-s)v)}}. \tag{70}$$

Firstly we prove (69). It suffices to show

$$\begin{aligned}
&|v\cdot\nabla\xi(x-(t_{\mathbf{b}}(x,v)-s)v)| \geq |v|\sqrt{-\xi(x-(t_{\mathbf{b}}(x,v)-s)v)}, \quad s\in[0,\sigma_1]\cup[t_{\mathbf{b}}(x,v)-\sigma_2, t_{\mathbf{b}}(x,v)],\\
&|v\cdot\nabla\xi(x-(t_{\mathbf{b}}(x,v)-s)v)| \leq |v|\sqrt{-\xi(x-(t_{\mathbf{b}}(x,v)-s)v)}, \quad s\in[\sigma_1, t_{\mathbf{b}}(x,v)-\sigma_2].
\end{aligned}$$

If $v=0$ or $v\cdot\nabla\xi(x)=0$ then (69) holds clearly. Therefore we may assume $v\neq 0$ and $v\cdot\nabla\xi(x)>0$. First we choose $t^*\in(0,t_{\mathbf{b}}(x,v))$ solving $v\cdot\nabla\xi(x-(t_{\mathbf{b}}(x,v)-t^*)v)=0$ uniquely due to the mean value theorem and the facts $v\cdot\frac{\nabla\xi(x)}{|\nabla\xi(x)|}>0$ and $v\cdot\frac{\nabla\xi(x_{\mathbf{b}}(x,v))}{|\nabla\xi(x_{\mathbf{b}}(x,v))|}<0$(this is due to the Velocity lemma and $v\cdot\frac{\nabla\xi(x)}{|\nabla\xi(x)|}>0$), and due to

$$\frac{d}{ds}\Big(v\cdot\nabla\xi(x-(t_{\mathbf{b}}(x,v)-s)v)\Big) = v\cdot\nabla^2\xi(X_{\mathbf{cl}}(s))\cdot v \geq C_\xi|v|^2,$$

from the convexity of ξ. Clearly we have $v\cdot\nabla\xi(x-(t_{\mathbf{b}}(x,v)-s)v)\geq 0$ for $s\in[t^*, t_{\mathbf{b}}(x,v)]$ and $v\cdot\nabla\xi(x-(t_{\mathbf{b}}(x,v)-s)v)\leq 0$ for $s\in[0,t^*]$.

Define $\Phi(s)=\big\{|v\cdot\nabla\xi(x-(t_{\mathbf{b}}(x,v)-s)v)|^2+|v|^2\xi(x-(t_{\mathbf{b}}(x,v)-s)v)\big\}$. Since $2\big(v\cdot\nabla^2\xi(x-(t_{\mathbf{b}}(x,v)-s)v)\cdot v\big)+|v|^2>0$ we have

$$\frac{d}{ds}\Phi(s) = \big(v\cdot\nabla\xi(x-(t_{\mathbf{b}}(x,v)-s)v)\big)\Big\{2\big(v\cdot\nabla^2\xi(x-(t_{\mathbf{b}}(x,v)-s)v)\cdot v\big)+|v|^2\Big\},$$

is strictly negative for $s\in[0,t^*]$ and is strictly positive for $s\in[t^*, t_{\mathbf{b}}(x,v)]$. Note that $\Phi(0)>0$ and $\Phi(t_{\mathbf{b}}(x,v))>0$ from $v\cdot\frac{\nabla\xi(x)}{|\nabla\xi(x)|}>0$ and $v\cdot\frac{\nabla\xi(x_{\mathbf{b}}(x,v))}{|\nabla\xi(x_{\mathbf{b}}(x,v))|}<0$. Note that Φ is continuous function on the interval $[0,t_{\mathbf{b}}(x,v)]$ so that it has a minimum. If $\min_{[0,t_{\mathbf{b}}(x,v)]}\Phi(s)\leq 0$, there exist $\sigma_1,\sigma_2>0$ satisfying

$$\begin{aligned}
\Phi(t_{\mathbf{b}}(x,v)+\sigma_1) &= \Phi(t_{\mathbf{b}}(x,v)) + \int_0^{\sigma_1}\frac{d}{ds}\Phi(s)\mathrm{d}s = 0,\\
\Phi(t_{\mathbf{b}}(x,v)-\sigma_2) &= \Phi(t_{\mathbf{b}}(x,v)) - \int_{t_{\mathbf{b}}(x,v)-\sigma_2}^{t_{\mathbf{b}}(x,v)}\frac{d}{ds}\Phi(s)\mathrm{d}s = 0,
\end{aligned}$$

then $\sigma_1\leq t^*$ and $t_{\mathbf{b}}(x,v)-\sigma_2\geq t^*$ and there is no other $s\in[0,t_{\mathbf{b}}(x,v)]$ satisfying $\Phi(s)=0$. Moreover we have $\Phi(s)\leq 0$ for $s\in[\sigma_1, t_{\mathbf{b}}(x,v)-\sigma_2]$. If $\min_{[0,t_{\mathbf{b}}(x,v)]}\Phi(s)>0$, there does not exist such σ_1 and σ_2 then we let $\sigma_1=t^*$ and $\sigma_2=t_{\mathbf{b}}(x,v)-t^*$. This proves (69).

Secondly we prove (70). Together with the above proof and

$$\frac{d|\xi|}{ds} = -\frac{d}{ds}\xi(x-(t_{\mathbf{b}}(x,v)-s)v) = -v\cdot\nabla_x\xi(x-(t_{\mathbf{b}}(x,v)-s)v),$$

and the inverse function theorem we prove (70).

Step 3. For small $0<\tilde{\delta}\ll 1$, we define

$$\tilde{\sigma}_1 := \min\Big\{\sigma_1, \tilde{\delta}\frac{\sqrt{\alpha(x,v)}}{|v|^2}\Big\}, \quad \tilde{\sigma}_2 := \min\Big\{\sigma_2, \tilde{\delta}\frac{\sqrt{\alpha(x,v)}}{|v|^2}\Big\} \tag{71}$$

Then both of (69) and (70) hold on $s\in[0,\tilde{\sigma}_1]\cup[t_{\mathbf{b}}(x,v)-\tilde{\sigma}_2, t_{\mathbf{b}}(x,v)]$ without constant changing. Moreover, if $s\in[0,\tilde{\sigma}_1]\cup[t_{\mathbf{b}}(x,v)-\tilde{\sigma}_2, t_{\mathbf{b}}(x,v)]$ then

$$\max\{|\xi|\} := \max_{s\in[0,\tilde{\sigma}_1]\cup[t_{\mathbf{b}}(x,v)-\tilde{\sigma}_2, t_{\mathbf{b}}(x,v)]}|\xi(X_{\mathbf{cl}}(s))| \lesssim \tilde{\delta}\frac{\alpha(x,v)}{|v|^2}. \tag{72}$$

On $s\in[\tilde{\sigma}_1, t_{\mathbf{b}}(x,v)-\tilde{\sigma}_2]$ we have the following estimate with $\tilde{\delta}-$dependent constant:

$$|v|\sqrt{-\xi(x-(t_{\mathbf{b}}(x,v)-s)v)} \gtrsim_\xi (\tilde{\delta})^{-2}\sqrt{\alpha(x-(t_{\mathbf{b}}(x,v)-s)v,v)}. \tag{73}$$

The proof of (72) is due to, for $s \in [0, \tilde{\sigma}_1]$,

$$\begin{aligned} |\xi(x - (t_{\mathbf{b}}(x,v) - s)v)| &\leq \int_0^s |v \cdot \nabla \xi(x - (t_{\mathbf{b}}(x,v) - \tau)v)| \mathrm{d}\tau \\ &\lesssim \sqrt{\alpha(x,v)} |s| \lesssim \min \left\{ \sqrt{\alpha} t_Z, \frac{\tilde{\delta}\alpha}{|v|^2} \right\} \equiv B, \end{aligned} \tag{74}$$

where we have used $\alpha(X_{\mathbf{cl}}(\tau), V_{\mathbf{cl}}(\tau)) \lesssim_\xi \alpha(x,v)$ from the Velocity lemma(Lemma 1). The proof for $s \in [t_{\mathbf{b}}(x,v) - \tilde{\sigma}_2, t_{\mathbf{b}}(x,v)]$ is exactly same.

Now we prove (73). Recall that $t^* \in [0, t_{\mathbf{b}}(x,v)]$ in the previous step: $v \cdot \nabla \xi(x - (t_{\mathbf{b}}(x,v) - t^*)v) = 0$. Clearly $|\xi(X_{\mathbf{cl}}(s))|$ is an increasing function on $s \in [0, t^*]$ and a decreasing function on $s \in [t^*, t_{\mathbf{b}}(x,v)]$. This is due to the convexity of ξ:

$$\frac{d^2}{ds^2} [-\xi(s - (t_{\mathbf{b}}(x,v) - s)v)] = v \cdot \nabla \xi(x - (t_{\mathbf{b}}(x,v) - s)v) \cdot v \gtrsim_\xi |v|^2,$$

and $v \cdot \nabla \xi(x) > 0$ and $v \cdot \nabla \xi(x_{\mathbf{b}}(x,v)) < 0$.

Therefore

$$\begin{aligned} -\xi(x - (t_{\mathbf{b}}(x,v) - s)v) &= -\xi(x) - \int_{t_{\mathbf{b}}(x,v)}^s v \cdot \nabla \xi(x - (t_{\mathbf{b}}(x,v) - \tau)v) \mathrm{d}\tau \\ &= \int_s^{t_{\mathbf{b}}(x,v)} v \cdot \nabla \xi(x - (t_{\mathbf{b}}(x,v) - \tau)v) \mathrm{d}\tau \\ &\geq (t_{\mathbf{b}}(x,v) - s)(v \cdot \nabla \xi(x - (t_{\mathbf{b}}(x,v) - s)v)) \\ &\geq \tilde{\sigma}_2 |v \cdot \nabla \xi(x - \tilde{\sigma}_2 v)| \quad \text{for} \quad s \in [t^*, t_{\mathbf{b}}(x,v) - \tilde{\sigma}_2], \end{aligned}$$

$$\begin{aligned} -\xi(x - (t_{\mathbf{b}}(x,v) - s)v) &= -\xi(x_{\mathbf{b}}(x,v)) - \int_0^s v \cdot \nabla \xi(x - (t_{\mathbf{b}}(x,v) - \tau)v) \mathrm{d}\tau \\ &\geq s |v \cdot \nabla \xi(x - (t_{\mathbf{b}} - s)v)| \\ &\geq \tilde{\sigma}_1 |v \cdot \nabla \xi(x_{\mathbf{b}}(x,v) + \tilde{\sigma}_1 v)| \quad \text{for} \quad s \in [0, t^*]. \end{aligned}$$

Hence, for $s \in [\tilde{\sigma}_1, t_{\mathbf{b}}(x,v) - \tilde{\sigma}_2]$,

$$\begin{aligned} |\xi(x - (t_{\mathbf{b}} - s)v)| &\geq \min \Big\{ |\xi(x - \tilde{\sigma}_2 v)|, |\xi(x_{\mathbf{b}}(x,v) + \tilde{\sigma}_1 v)| \Big\} \\ &\geq \min \Big\{ \tilde{\sigma}_2 |v \cdot \nabla \xi(x - \tilde{\sigma}_2 v)|, \tilde{\sigma}_1 |v \cdot \nabla \xi(x_{\mathbf{b}}(x,v) + \tilde{\sigma}_1 v)| \Big\}. \end{aligned}$$

From the definition of $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ in (71) we have

$$|v|^2 |\xi(x - (t_{\mathbf{b}}(x,v) - s)v)| \geq \tilde{\delta} \sqrt{\alpha(x,v)} \min \Big\{ |v \cdot \nabla \xi(x - \tilde{\sigma}_2 v)|, |v \cdot \nabla \xi(x_{\mathbf{b}}(x,v) + \tilde{\sigma}_1 v)| \Big\}.$$

Without loss of generality we may assume $|v \cdot \nabla \xi(x - \tilde{\sigma}_2 v)| = \min \big\{ |v \cdot \nabla \xi(x - \tilde{\sigma}_2 v)|, |v \cdot \nabla \xi(x_{\mathbf{b}}(x,v) + \tilde{\sigma}_1 v)| \big\}$. Then by the Velocity lemma we have $\sqrt{\alpha(x,v)} \gtrsim_\xi |v| |\xi(x - \tilde{\sigma}_2 v)|^{1/2}$. Then we choose $s = t_{\mathbf{b}}(x,v) - \tilde{\sigma}_2$ to have $|v|^2 |\xi(x - \tilde{\sigma}_2 v)| \geq \tilde{\delta} |v| |\xi(x - \tilde{\sigma}_2 v)|^{1/2} \times |v \cdot \nabla \xi(x - \tilde{\sigma}_2 v)|$ and

$$|v| |\xi(x - \tilde{\sigma}_2 v)|^{1/2} \gtrsim \tilde{\delta} \times |v \cdot \nabla \xi(x - \tilde{\sigma}_2 v)|.$$

The left hand side is the lower bound of $|v|^2 |\xi(x - (t_{\mathbf{b}}(x,v) - s)v)|$ for $s \in [\tilde{\sigma}_1, t_{\mathbf{b}}(x,v) - \tilde{\sigma}_2]$ and the right hand side is bounded below by the Velocity lemma: $e^{-\mathfrak{C}|v| t_{\mathbf{b}}(x,v)} \alpha(x,v) \gtrsim_\xi \alpha(x,v)$. Therefore we conclude (73).

Step 4. We prove (19). From (68)

$$\begin{aligned} &\int_0^{t_{\mathbf{b}}(x,v)} \int_{\mathbb{R}^3} e^{-l\langle v \rangle (t-s)} \frac{e^{-\theta|v-u|^2}}{|v-u|^\kappa \alpha(x - (t_{\mathbf{b}}(x,v) - s)v, u)} Z(s,v) \mathrm{d}u \mathrm{d}s \\ &\lesssim \int_0^{t_{\mathbf{b}}(x,v)} e^{-l\langle v \rangle (t-s)} \frac{1}{|v|^{2\beta-1} |\xi|^{\beta - \frac{1}{2}}} Z(s,v) \mathrm{d}s. \end{aligned}$$

According to (71) we split the time integration as

$$\int_0^{t_{\mathbf{b}}(x,v)} e^{-l\langle v\rangle(t-s)} \frac{1}{|v|^{2\beta-1}|\xi|^{\beta-\frac{1}{2}}} Z(s,v)\mathrm{d}s = \underbrace{\int_0^{\tilde{\sigma}_1} + \int_{t_{\mathbf{b}}(x,v)-\tilde{\sigma}_2}^{t_{\mathbf{b}}(x,v)}}_{(\mathbf{IV})} + \underbrace{\int_{\tilde{\sigma}_1}^{t_{\mathbf{b}}(x,v)-\tilde{\sigma}_2}}_{(\mathbf{V})}.$$

For the first two terms (**IV**), we use the mapping of (70)

$$s \in [0,\tilde{\sigma}_1] \cup [t_{\mathbf{b}}(x,v)-\tilde{\sigma}_2, t_{\mathbf{b}}(x,v)] \mapsto |\xi(x-(t_{\mathbf{b}}(x,v)-s)v)| \in \big[0,B\big),$$

where the range of $|\xi|$ has been bounded in (72), and B is given by (74). By the change of variables of (70)

$$\begin{aligned}(\mathbf{IV}) &\lesssim \sup_{0\le s\le t_{\mathbf{b}}(x,v)} \{e^{-l\langle v\rangle(t-s)}Z(s,v)\} \frac{1}{|v|^{2\beta-1}} \int_0^{C\tilde{\delta}\frac{\alpha(x,v)}{|v|^2}} \frac{1}{|\xi|^{\beta-1/2}} \frac{\mathrm{d}|\xi|}{\sqrt{\alpha(x,v)}} \\ &\lesssim \sup_{0\le s\le t_{\mathbf{b}}(x,v)} \{e^{-l\langle v\rangle(t-s)}Z(s,v)\} \frac{1}{|v|^{2\beta-1}} \frac{1}{\sqrt{\alpha(x,v)}} \Big[|\xi|^{-\beta+\frac{3}{2}}\Big]_{|\xi|=0}^{|\xi|=B}\end{aligned}$$

where we have used $\beta<\frac{3}{2}$. The lemma follows with B given by (74).

For (**V**) we use $\sqrt{\alpha(X_{\mathbf{cl}}(s))} \lesssim_{\xi,\tilde{\delta}} |v|\sqrt{-\xi(X_{\mathbf{cl}}(s))}$ for $s\in[\tilde{\sigma}_1, t_{\mathbf{b}}(x,v)-\tilde{\sigma}_2]$, from (69), to have

$$\frac{1}{|v|^{2\beta-1}|\xi|^{\beta-\frac{1}{2}}} = \frac{1}{\big(|v|\sqrt{-\xi}\ \big)^{2(\beta-\frac{1}{2})}} \lesssim \frac{1}{[\alpha(x,v)]^{\beta-\frac{1}{2}}}.$$

Finally

$$(\mathbf{V}) \lesssim \frac{1}{[\alpha(x,v)]^{\beta-1/2}} \int_0^{t_{\mathbf{b}}(x,v)} e^{-l\langle v\rangle(t-s)}Z(s,v)\mathrm{d}s \lesssim \frac{O(l^{-1})}{\langle v\rangle[\alpha(x,v)]^{\beta-1/2}} \sup_{0\le s\le t}\{e^{-l\langle v\rangle(t-s)}Z(s,x,v)\}.$$

Now we assume $x\notin\partial\Omega$. We find $\bar{x}\in\partial\Omega$ and $\bar{t}_{\mathbf{b}}$ so that

$$x-(t_{\mathbf{b}}(x,v)-s)v = \bar{x}-(\bar{t}_{\mathbf{b}}-s)v.$$

Therefore, by the *Step 1* and the fact $\bar{x}\in\partial\Omega$, we have

$$\begin{aligned}&\int_0^{\bar{t}_{\mathbf{b}}}\int_{\mathbb{R}^3} e^{-l\langle v\rangle(t-s)} \frac{e^{-\theta|v-u|^2}}{|v-u|^{2-\kappa}\big[\alpha(\bar{x}-(\bar{t}_{\mathbf{b}}-s)v,u)\big]^{\beta}} Z(s,v)\mathrm{d}u\mathrm{d}s \\ \lesssim\ & \int_0^{\bar{t}_{\mathbf{b}}} e^{-l\langle v\rangle(t-s)} \frac{e^{-C|v-u|^2}}{|v|^{2\beta-1}|\xi|^{\beta-\frac{1}{2}}} Z(s,v)\mathrm{d}s \\ \lesssim_{z,\theta}\ & O(\tilde{\delta}) \sup_{0\le s\le\bar{t}_{\mathbf{b}}} \{e^{-l\langle v\rangle(t-s)}Z(s,v)\} \frac{e^{-C|v|^2}}{|v|^2}[\alpha(\bar{x},v)]^{-\beta+1} + \frac{1}{[\alpha(\bar{x},v)]^{\beta-1/2}} \int_0^{\bar{t}_{\mathbf{b}}} e^{-l\langle v\rangle(t-s)}Z(s,v)\mathrm{d}s.\end{aligned}$$

We then deduce our lemma since $\alpha(\bar{x},v)\backsim\alpha(x,v)$ via the Velocity Lemma with the fact $\bar{t}_{\mathbf{b}}|v|\lesssim_\Omega 1$. □

Now we prove a variant of (1) of Lemma 2 with extra $\frac{|v|}{|u|}$.

Proof of (1) of Lemma 9. We prove (63). Due to *Step 2* and *Step 3* in the proof of *(1)* of Lemma 2, it suffices to show

$$\int_{\mathbb{R}^3} \frac{|v|e^{-\theta|v-u|^2}\mathrm{d}u}{|v-u|^{2-\kappa}|u|\big[\alpha(x-(t_{\mathbf{b}}(x,v)-s)v,u)\big]^{\beta}} \lesssim \frac{1}{|v|^{2\beta-1}|\xi(x-(t_{\mathbf{b}}(x,v)-s)v)|^{\beta-\frac{1}{2}}}. \tag{75}$$

As *Step 1* in the proof of *(1)*, for fixed s and $x-(t_{\mathbf{b}}(x,v)-s)v$, we decompose $u=u_{\tau,1}\tau_1+u_{\tau,2}\tau_2+u_n n$ where $\{\tau_1,\tau_2,n\}$ is the orthonormal basis that we chose in the proof of *(1)*.

Now we split as

$$\int_{\mathbb{R}^3} \frac{|v| e^{-\theta|v-u|^2} \mathrm{d}u}{|v-u|^{2-\kappa}|u|\big[\alpha(x-(t_{\mathbf{b}}(x,v)-s)v,u)\big]^{\beta}} \lesssim_{\xi} \int_{\mathbb{R}^2}\int_{\mathbb{R}} \frac{|v|e^{-\theta|v-u|^2}\mathrm{d}u_n \mathrm{d}u_\tau}{|v-u|^{2-\kappa}|u|\big\{|u_n|^2+|\xi(X_{\mathbf{cl}}(s))||u|^2\big\}^{\beta}}$$
$$= \int_{|u|\geq \frac{|v|}{5}} + \int_{|u|\leq \frac{|v|}{5}}.$$

For the first term, we have $\frac{|v|}{|u|}\leq 5$ so we reduce it to the previous case (68)

$$\int_{|u|\geq \frac{|v|}{5}} \lesssim \int_{\mathbb{R}^3} \frac{e^{-\theta|v-u|^2}\mathrm{d}u_n\mathrm{d}u_\tau}{|v-u|^{2-\kappa}\big\{|u_n|^2+|\xi(X_{\mathbf{cl}}(s))||u|^2\big\}^{\beta}},$$

which is bounded by $\frac{1}{|v|^{2\beta-1}|\xi|^{\beta-1/2}}$.

Now we consider the case of $|u|\leq \frac{|v|}{5}$. For fixed $0<\kappa\leq 1$

$$\frac{|v|}{|v-u|^{2-\kappa}} \ \lesssim\ \frac{|v|}{|v|^{2-\kappa}} \ \lesssim\ |v|^{-1+\kappa},$$

and we have, from $|v-u|^2 = \frac{|v-u|^2}{2}+\frac{|v-u|^2}{2} \geq \frac{4^2}{2\cdot 5^2}|v|^2 + \frac{4^2}{2}|u|^2$,

$$e^{-\theta|v-u|^2} \leq e^{-C_\theta|v|^2}e^{-C_\theta|u|^2}.$$

We split $\int_{\mathbb{R}^3}\mathrm{d}u = \int_{|u_n|\geq|\xi|^{1/2}|u_\tau|} + \int_{|u_n|\leq|\xi|^{1/2}|u_\tau|}$ to have (Note $\frac{1}{2}<\beta<1$)

$$\begin{aligned}
\int_{|u_n|\geq|\xi|^{1/2}|u_\tau|} &\lesssim \frac{e^{-C|v|^2}}{|v|^{1-\kappa}}\int_{\mathbb{R}^2}\frac{e^{-C_\theta|u_\tau|^2}}{|u_\tau|}\int_{|\xi|^{1/2}|u_\tau|}^{|v|/5}|u_n|^{-2\beta}e^{-C_\theta|u_n|^2}\mathrm{d}|u_n|\mathrm{d}u_\tau\\
&\lesssim \frac{e^{-C|v|^2}}{|v|^{1-\kappa}}\int_{|u_\tau|\leq\frac{|v|}{5}}\frac{e^{-C_\theta|u_\tau|^2}}{|u_\tau|}\Big(1+\int_{|\xi|^{1/2}|u_\tau|}^{\frac{|v|}{5}}\frac{\mathrm{d}u_n}{|u_n|^{2\beta}}\Big)\mathrm{d}u_\tau\\
&\lesssim \frac{e^{-C|v|^2}}{|v|^{1-\kappa}}\Big\{|v|+|v|^{-2\beta+1}\int_{|u_\tau|\leq\frac{|v|}{5}}\frac{\mathrm{d}u_\tau}{|u_\tau|}+\frac{1}{|\xi|^{\beta-\frac{1}{2}}}\int_{|u_\tau|\leq\frac{|v|}{5}}|u_\tau|^{-2\beta}\mathrm{d}u_\tau\Big\}\\
&\lesssim e^{-C|v|^2}|v|^{\kappa}\big\{1+\frac{1}{|v|^{2\beta-1}}(1+\frac{1}{|\xi|^{\beta-\frac{1}{2}}})\big\}\\
&\lesssim \frac{e^{-C|v|^2}}{|v|^{2\beta-1}|\xi|^{\beta-\frac{1}{2}}},\\
\int_{|u_n|\leq|\xi|^{1/2}|u_\tau|} &\lesssim \frac{e^{-C|v|^2}}{|v|^{1-\kappa}}\int_{|u_\tau|\leq\frac{|v|}{5}}\frac{e^{-C_\theta|u_\tau|^2}}{|\xi|^{\beta}|u_\tau|^{2\beta}|u_\tau|}\int_{|u_n|\leq|\xi|^{1/2}|u_\tau|}\mathrm{d}u_n\mathrm{d}u_\tau\\
&\lesssim \frac{e^{-C|v|^2}}{|v|^{1-\kappa}}\int_{|u_\tau|\leq\frac{|v|}{5}}\frac{e^{-C_\theta|u_\tau|^2}}{|\xi|^{\beta-1/2}|u_\tau|^{2\beta}}\mathrm{d}u_\tau\\
&\lesssim \frac{|v|^{-2\beta+\kappa+1}e^{-C|v|^2}}{|\xi|^{\beta-1/2}} \lesssim \frac{e^{-C|v|^2}}{|v|^{2\beta-1}|\xi|^{\beta-\frac{1}{2}}}.
\end{aligned}$$

Therefore, combining the cases of $|u|\leq\frac{|v|}{5}$ and $|u|\geq\frac{|v|}{5}$, we conclude (75).

□

Now we proof he non-local to local estimate for the specular and bounce-back cycles.

Proof of (2) Lemma 2. It suffices to consider $r=0$ case. We consider the specular BC case first.

For fixed (x,v) we use the following notation $\alpha(s) := \alpha(s;t,x,v) := \alpha(X_{\mathbf{cl}}(s;t,x,v), V_{\mathbf{cl}}(s;t,x,v))$.

Firstly we consider the estimate (20) for $|v| < \delta$. Using (67),

$$
\begin{aligned}
&\mathbf{1}_{\{|v|\le\delta\}} \int_0^t \int_{\mathbb{R}^3} e^{-l\langle v\rangle(t-s)} \frac{e^{-\theta|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}} \frac{Z(s,x,v)}{\big[\alpha(X_{\mathbf{cl}}(s;t,x,v),u)\big]^{\beta}} \mathrm{d}u\mathrm{d}s \\
&\lesssim \sum_{\ell=0}^{\ell_*(0;t,x,v)} \int_{t^{\ell+1}}^{t^{\ell}} \int_{\mathbb{R}^3} e^{-l\langle v\rangle(t-s)} \frac{e^{-\theta|v^\ell-u|^2}}{|v^\ell-u|^{2-\kappa}} \frac{Z(s,x,v)}{\big[\alpha(x^\ell-(t^\ell-s)v^\ell,u)\big]^{\beta}} \mathrm{d}u\mathrm{d}s \\
&\lesssim \frac{t|v|^2 e^{\mathfrak{C}t\delta}}{\big[\alpha(x,v)\big]^{1/2}} \sup_{\ell} \Big\{ \frac{O(\tilde{\delta})e^{\mathfrak{C}t\delta}}{|v|^2[\alpha(x,v)]^{\beta-1}} \sup_{t^{\ell+1}\le s\le t^\ell} \{e^{-l\langle v\rangle(t-s)}Z(s,x,v)\} \\
&\qquad\qquad + \frac{C_{\tilde{\delta}}e^{\mathfrak{C}t\delta}}{[\alpha(x,v)]^{\beta-1/2}} \int_{t^{\ell+1}}^{t^\ell} e^{-l\langle v\rangle(t-s)}Z(s,x,v)\mathrm{d}s\Big\},
\end{aligned}
$$

where we have used (19). By (65) and (66) and the Velocity lemma(Lemma 1) we have $|t^\ell - t^{\ell+1}| \lesssim_\xi \frac{\sqrt{\alpha(x,v)}}{|v|^2} e^{\mathfrak{C}t|v|} \lesssim_{\xi,\delta} \frac{\sqrt{\alpha(x,v)}}{|v|^2} e^{\mathfrak{C}t\delta}$ and hence we bound (20) for $|v| < \delta$ by

$$
\mathbf{1}_{\{|v|<\delta\}} \int \cdots \lesssim_\xi \frac{O(\tilde{\delta}+l^{-1})te^{2\mathfrak{C}t\delta}}{\big[\alpha(x,v)\big]^{\beta-\frac12}} \sup_{0\le s\le t} \big\{e^{-l\langle v\rangle(t-s)}Z(s,x,v)\big\}.
$$

Now we consider $|v| \ge \delta$. We split the time interval as

$$
[0,t] \;=\; [t-\frac{1}{|v|},t] \;\cup\; \bigcup_{j=1}^{[t|v|]+1} [t-(j+1)\frac{1}{|v|}, t-j\frac{1}{|v|}]. \tag{76}
$$

Consider the first time section $[t-\frac{1}{|v|},t]$. Due to (67) we bound

$$
\sup_{s\in[t-\frac{1}{|v|},t]} \ell_*(s;t,x,v) \lesssim_\xi \frac{\frac{1}{|v|}|v|^2 e^{\mathfrak{C}\frac{1}{|v|}|v|}}{[\alpha(x,v)]^{1/2}} \lesssim \frac{|v|e^{\mathfrak{C}}}{[\alpha(x,v)]^{1/2}},
$$

and for $s\in[t-\frac{1}{|v|},t]$, $e^{-\mathfrak{C}}\alpha(x,v) \lesssim \alpha(X_{\mathbf{cl}}(s;t,x,v),V_{\mathbf{cl}}(s;t,x,v)) \lesssim e^{\mathfrak{C}}\alpha(x,v)$, and $|t^\ell-t^{\ell+1}| \lesssim \frac{[\alpha(x,v)]^{1/2}e^{\mathfrak{C}}}{|v|^2}$. Then we use (19) to have

$$
\begin{aligned}
&\int_{t-1/|v|}^t \int_{\mathbb{R}^3} e^{-l\langle v\rangle(t-s)} \frac{e^{-\theta|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}} \frac{Z(s,x,v)}{\big[\alpha(X_{\mathbf{cl}}(s;t,x,v),u)\big]^{\beta}} \mathrm{d}u\mathrm{d}s \\
&\lesssim \sum_{\ell=0}^{\ell_*(0;t,x,v)} \int_{t^{\ell+1}}^{t^\ell} \int_{\mathbb{R}^3} e^{-l\langle v\rangle(t-s)} \frac{e^{-\theta|v^\ell-u|^2}}{|v^\ell-u|^{2-\kappa}} \frac{Z(s,x,v)}{\big[\alpha(x^\ell-(t^\ell-s)v^\ell,u)\big]^{\beta}} \mathrm{d}u\mathrm{d}s \\
&\lesssim \frac{C_\xi|v|}{[\alpha(x,v)]^{1/2}} \sup_\ell \frac{O(\tilde{\delta})e^{C_\xi}}{|v|^2[\alpha(x,v)]^{\beta-1}} \sup_{t^{\ell+1}\le s\le t^\ell} \{e^{-l\langle v\rangle(t-s)}Z(s,x,v)\} \\
&\quad + \sum_{\ell=0}^{\ell_*(0,t,x,v)} \frac{C_{\tilde{\delta}}e^{C_\xi}}{[\alpha(x,v)]^{\beta-1/2}} \int_{t^{\ell+1}}^{t^\ell} e^{-l\langle v\rangle(t-s)}Z(s,x,v)\mathrm{d}s \\
&\lesssim \frac{O(\tilde{\delta})}{|v|[\alpha(x,v)]^{\beta-1/2}} \sup_{0\le s\le t} \{e^{-Cl\langle v\rangle(t-s)}Z(s,x,v)\} + \frac{C_{\tilde{\delta},\xi}}{[\alpha(x,v)]^{\beta-1/2}} \int_0^t e^{-Cl\langle v\rangle(t-s)}Z(s,x,v)\mathrm{d}s \\
&\lesssim \Big(\frac{O(\tilde{\delta})}{|v|[\alpha(x,v)]^{\beta-1/2}} + \frac{C_{\tilde{\delta},\xi}}{l\langle v\rangle[\alpha(x,v)]^{\beta-1/2}} \Big) \sup_{0\le s\le t} \{e^{-Cl\langle v\rangle(t-s)}Z(s,x,v)\}.
\end{aligned}
$$

Now we consider time sections $[t-(j+1)\frac{1}{|v|}, t-j\frac{1}{|v|}]$ for $j\ge1$. Assume that

$$
[t-(j+1)\frac{1}{|v|}, t-j\frac{1}{|v|}] \subset [t^{\ell_{j+1}-1}, t^{\ell_{j+1}}] \cup \cdots \cup [t^{\ell_j+1}, t^{\ell_j}],
$$

and $[t-(j+1)\frac{1}{|v|}, t-j\frac{1}{|v|}] \cap [t^{\ell_{j+1}-2}, t^{\ell_{j+1}-1}] = \emptyset$ and $[t-(j+1)\frac{1}{|v|}, t-j\frac{1}{|v|}] \cap [t^{\ell_j}, t^{\ell_j-1}] = \emptyset$.

Note that for all $s \in [t-(j+1)\frac{1}{|v|}, t-j\frac{1}{|v|}]$

$$e^{-C_\xi j}\alpha(t) \lesssim \alpha(s) \lesssim e^{C_\xi j}\alpha(t),$$

and

$$\ell_{j+1}-\ell_j \lesssim \frac{(j+1)\frac{1}{|v|}-j\frac{1}{|v|}}{\frac{\sqrt{\alpha(t-j\frac{1}{|v|})}}{|v|^2}} \lesssim \frac{|v|}{\sqrt{\alpha(t)}}e^{C_\xi j},$$

and for $\ell \in [\ell_{j+1}-1, \ell_j]$

$$|t^\ell - t^{\ell+1}| \lesssim \frac{\sqrt{\alpha(t-j\frac{1}{|v|})}}{|v|^2} \lesssim \frac{\sqrt{\alpha(t)}}{|v|^2}e^{C_\xi j}.$$

From (19), for all $\ell \in [\ell_{j+1}-1, \ell_j]$

$$\begin{aligned}&\int_{t^\ell}^{t^{\ell+1}}\int_{\mathbb{R}^3} e^{-l\langle v\rangle(t-s)}\frac{e^{-\theta|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}}\frac{Z(s,x,v)}{\big[\alpha(X_{\mathbf{cl}}(s;t,x,v),u)\big]^\beta}\mathrm{d}u\mathrm{d}s\\ &\lesssim \frac{\tilde{\delta}}{|v|^2\alpha(t-j/|v|)^{\beta-1}}\sup_{[t^\ell,t^{\ell+1}]}\{e^{-l\langle v\rangle(t-s)}Z\} + \frac{1}{\alpha(t-j/|v|)^{\beta-1/2}}\int_{t^{\ell+1}}^{t^\ell} e^{-l\langle v\rangle(t-s)}Z,\end{aligned}$$

is bounded by

$$\begin{aligned}&\frac{\tilde{\delta}e^{C_\xi j}}{|v|^2\alpha(t)^{\beta-1}}e^{-\frac{l}{2}j}\sup_{[t^\ell,t^{\ell+1}]}\{e^{-\frac{l}{2}\langle v\rangle(t-s)}Z\} + \frac{e^{C_\xi j}}{\alpha(t)^{\beta-1/2}}e^{-\frac{l}{2}j}\int_{t^{\ell+1}}^{t^\ell} e^{-\frac{l}{2}\langle v\rangle(t-s)}Z\\ &\lesssim \frac{e^{-Clj}}{|v|^2\alpha(t)^{\beta-1}}\sup_{0\le s\le t}\{e^{-\frac{l}{2}\langle v\rangle(t-s)}Z(s)\},\end{aligned}$$

where we have used the fact $t-s \ge j\frac{1}{|v|}$ for $s \in [t-(j+1)\frac{1}{|v|}, t-j\frac{1}{|v|}]$.

Therefore

$$\begin{aligned}&\int_{t-(j+1)\frac{1}{|v|}}^{t-j\frac{1}{|v|}}\int_{\mathbb{R}^3} e^{-l\langle v\rangle(t-s)}\frac{e^{-\theta|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}}\frac{Z(s,x,v)}{\big[\alpha(X_{\mathbf{cl}}(s;t,x,v),u)\big]^\beta}\mathrm{d}u\mathrm{d}s\\ &\lesssim |\ell_{j+1}-\ell_j|\sup_{\ell_{j+1}\le\ell\le\ell_j}\int_{t^\ell}^{t^{\ell+1}}\cdots\\ &\lesssim \frac{|v|}{\sqrt{\alpha(t)}}e^{C_\xi j}\times\frac{e^{-Clj}}{|v|^2\alpha(t)^{\beta-1}}\sup_{0\le s\le t}\{e^{-\frac{l}{2}\langle v\rangle(t-s)}Z(s,x,v)\}\\ &\lesssim \frac{e^{-C'lj}}{|v|\alpha(t)^{\beta-1/2}}\sup_{0\le s\le t}\{e^{-\frac{l}{2}\langle v\rangle(t-s)}Z(s,x,v)\}.\end{aligned}$$

Now we sum up all contributions of $[t-(j+1)\frac{1}{|v|}, t-j\frac{1}{|v|}]$ for $j\ge 1$:

$$\begin{aligned}\sum_{j=1}^{t|v|}\int_{t-(j+1)/|v|}^{t-j/|v|} &\le \sum_{j=1}^{t|v|}\frac{e^{-C'lj}}{|v|\alpha(t)^{\beta-1/2}}\sup_{0\le s\le t}\{e^{-\frac{l}{2}\langle v\rangle(t-s)}Z(s,x,v)\}\\ &\lesssim \frac{e^{-C_\xi l}}{|v|[\alpha(x,v)]^{\beta-1/2}}\sup_{0\le s\le t}\{e^{-\frac{l}{2}\langle v\rangle(t-s)}Z(s,x,v)\}.\end{aligned}$$

where we used $\sum_{j=1}^{t|v|}e^{-C'lj} = e^{-C'l}\sum_{j=2}^{t|v|}e^{-C'lj} \le e^{-C'l}$.

These prove (20). For the bounce-back case we set $\mathfrak{C}=0$ and we have same conclusion. □

The proof of (64), (2) of Lemma 9 is a direct consequence of (75) and the proof of (20)

4. Diffuse Reflection

4.1 $W^{1,p}(1 < p < 2)$ Estimate

Consider the iteration (34) with (36) and with $f^0 \equiv f_0$, and with the compatibility condition for the initial datum (10). Remark that the normalized Maxwellian is $\mu(v) = e^{-\frac{|v|^2}{2}}$. From Lemma 6, we have a uniform bound for $0 < T \ll 1$

$$\sup_m \sup_{0 \le t < T} ||\langle v\rangle^\zeta f^m(t)||_\infty \lesssim_\Omega ||\langle v\rangle^\zeta f_0||_\infty (1 + ||\langle v\rangle^\zeta f_0||_\infty),$$

for $\zeta + \kappa > 2$. We apply Proposition 1 for $m = 1, 2, ...$ with $\nu(F^m) \ge 0$

$$H = -Kf^m + \Gamma(f^m, f^m), \quad g = c_\mu \sqrt{\mu(v)} \int_{n\cdot u>0} f^m(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u.$$

Recall $\partial = [\partial_t, \nabla_x, \nabla_v]$. Then ∂f^m satisfies

$$\{\partial_t + v\cdot\nabla_x + \nu(F^m)\}\partial f^{m+1} = \mathcal{G}^m, \quad \partial f^{m+1}(0,x,v) = \partial f_0(x,v), \tag{77}$$

where

$$\begin{aligned}
\mathcal{G}^m &= -[\partial v]\cdot\nabla_x f^{m+1} - \partial[\nu(F^m)]f^{m+1} - \partial[Kf^m - \Gamma_{\text{gain}}(f^m,f^m)],\\
|\mathcal{G}^m| &\lesssim |\nabla_x f^{m+1}| + K|\nabla_{t,x}f^m| + |\Gamma(\partial f^m, f^m)| + |\Gamma(f^m,\partial f^m)| + |\nu(\nabla_{t,x}F^m)||f^{m+1}| \qquad (78)\\
&\quad + |\nu_v(F^m)||f^{m+1}| + |K_v f^m| + |\Gamma_{\text{gain},v}(f^m,f^m)|,
\end{aligned}$$

with $\nu_v(F^m)$ and $K_v f^m$ and $\Gamma_{\text{gain},v}(f^m,f^m)$ defined in *(3)* of Lemma 5. Furthermore using Lemma 5 yields

$$\begin{aligned}
&|\nu_v(F^m)||f^{m+1}| + |K_v f^m| + |\Gamma_{\text{gain},v}(f^m,f^m)|\\
&\lesssim \big\{|\nabla_v \nu(F^m)|\langle v\rangle^{-\beta} + |K_v\langle v\rangle^{-\beta}| + |\Gamma_v(\langle v\rangle^{-\beta},\langle v\rangle^{-\beta})|\big\}||\langle v\rangle^\beta f_0||_\infty \qquad (79)\\
&\lesssim \langle v\rangle^{-\beta}||\langle v\rangle^\beta f_0||_\infty(1 + ||\langle v\rangle^\beta f_0||_\infty).
\end{aligned}$$

For $(x,v)\in\kappa_-$, from (77) and (42), the boundary condition is bounded by

$$\begin{aligned}
|\partial f^{m+1}(t,x,v)| \;\lesssim\;& c_\mu\sqrt{\mu(v)}\left(1 + \frac{\langle v\rangle}{|n(x)\cdot v|}\right)\int_{n(x)\cdot u>0}|\partial f^m(t,x,u)|\langle u\rangle\sqrt{\mu}\{n(x)\cdot u\}\mathrm{d}u\\
&+\frac{1}{|n(x)\cdot v|}\Big\{\nu(F^m)|f^{m+1}| + |Kf^m| + |\Gamma_{\text{gain}}(f^m,f^m)|\Big\} \qquad (80)\\
\lesssim\;& \sqrt{\mu(v)}\left(1 + \frac{\langle v\rangle}{|n(x)\cdot v|}\right)\int_{n(x)\cdot u>0}|\partial f^m(t,x,u)|\mu^{1/4}\{n(x)\cdot u\}\mathrm{d}u\\
&+\frac{\langle v\rangle^{-\beta}}{|n(x)\cdot v|}||\langle v\rangle^\beta f_0||_\infty(1 + ||\langle v\rangle^\beta f_0||_\infty),
\end{aligned}$$

where we used the boundary condition for $f^{m+1}|_{\gamma_-}$ again, and (79) and Lemma 5 and $\langle u\rangle\sqrt{\mu(u)} \lesssim \mu(u)^{\frac14}$. Set $\partial f^0 = [\partial_t f^0, \nabla_x f^0, \nabla_v f^0] = [0,0,0]$.

Now we are ready to prove Theorem 1:

Proof of Thoerem 1. We claim that for $1 \le p < 2$, if $0 < T_* \ll 1$(therefore, for $\beta > 4$ from Lemma 6 $\sup_m \sup_{0\le t\le T} ||\langle v\rangle^\beta f(t)||_\infty \lesssim ||\langle v\rangle^\beta f_0||_\infty(1 + ||\langle v\rangle^\beta f_0||_\infty)$), and the compatibility condition (10) then uniformly-in-m,

$$\sup_{0\le t\le T_*} ||\partial f^m||_p^p + \int_0^{T_*} |\partial f^m|_{\gamma,p}^p \lesssim_{\Omega,T_*} ||\partial f_0||_p^p + ||\langle v\rangle^\beta f_0||_\infty^p (1 + ||\langle v\rangle^\beta f_0||_\infty)^p. \tag{81}$$

Recall that the time derivative of the initial datum is defined as $\partial_t f_0 \equiv -v\cdot\nabla_x f_0 - Lf_0 + \Gamma(f_0,f_0)$. We remark that the sequence (34) is the one used in Lemma 6 and shown to be Cauchy in L^∞. Therefore the limit function f is a solution of the Boltzmann equation with the diffuse boundary condition. On the other hand, due to the weak lower semi-continuity for L^p in the case of $p > 1$, once we have (81) then we pass a limit $\partial f^m \rightharpoonup \partial f$ weakly in $\sup_{t\in[0,T_*]}||\cdot||_p^p$ and $\partial f^m|_\gamma \rightharpoonup \partial f|_\gamma$

in $\int_0^{T_*} |\cdot|^p_{\gamma,p}$ to conclude that ∂f satisfies the same estimate of (81). Repeat the same procedure for $[T_*, 2T_*], [2T_*, 3T_*], \cdots$, to conclude Theorem 1.

We prove the claim (81) by induction. From Proposition 1, ∂f^1 exists. Because of our choice ∂f^0 the estimate (81) is valid for $m = 1$. Now assume that ∂f^i exists and (81) is valid for all $i = 1, 2, \cdots, m$. Applying Proposition 1 to show that ∂f^{m+1} exists and to get (45) $-$ (47), we have

$$\begin{aligned}
&\sup_{0\le s\le t} ||\partial f^{m+1}(s)||_p^p + \int_0^t |\partial f^{m+1}|^p_{\gamma_+,p} \\
\lesssim\ & ||\partial f_0||_p^p + \int_0^t |\partial f^{m+1}|^p_{\gamma_-,p} \\
&+ \int_0^t \iint_{\Omega\times\mathbb{R}^3} \Big\{|K\partial f^m| + |\Gamma_{\text{gain}}(f^m, \partial f^m)| + |\Gamma_{\text{gain}}(\partial f^m, f^m)|\Big\} |\partial f^{m+1}|^{p-1} \\
&+ t \sup_{0\le s\le t} ||\partial f^{m+1}(s)||_p^p + ||\langle v\rangle^\beta f_0||_\infty + ||\langle v\rangle^\beta f_0||_\infty \int_{\mathbb{R}^3} |\partial f^{m+1}|^p \mathrm{d}v,
\end{aligned} \tag{82}$$

where we have used (122) and (79) and

$$\begin{aligned}
\iint_{\Omega\times\mathbb{R}^3} \langle v\rangle^{-\beta} ||\langle v\rangle^\beta f_0||_\infty |\partial f^{m+1}|^{p-1} \mathrm{d}v \le & ||\langle v\rangle^\beta f_0||_\infty \{ \iint \langle v\rangle^{-\beta} \mathrm{d}v + \iint \langle v\rangle^{-\beta} |\partial f^{m+1}|^p \} \\
\lesssim & ||\langle v\rangle^\beta f_0||_\infty \{1 + \iint_{\Omega\times\mathbb{R}^3} |\partial f^{m+1}|^p \mathrm{d}v\}.
\end{aligned}$$

Firstly we consider K and Γ_{gain} contributions in (94). Use Lemma 4 and Lemma 5 to have

$$\begin{aligned}
&\int_0^t \iint_{\Omega\times\mathbb{R}^3} \Big\{|K\partial f^m| + |\Gamma_{\text{gain}}(f^m, \partial f^m)| + |\Gamma_{\text{gain}}(\partial f^m, f^m)|\Big\} |\partial f^{m+1}|^{p-1} \\
\lesssim & \int_0^t ||\partial f^m||_p^p + \int_0^t ||\partial f^{m+1}||_p^p + C_{\gamma,p,\beta} ||\langle v\rangle^\beta f_0||_\infty (1 + ||\langle v\rangle^\beta f_0||_\infty) \left\{ \int_0^t ||\partial f^m||_p^p + \int_0^t ||\partial f^{m+1}||_p^p \right\}.
\end{aligned} \tag{83}$$

Secondly we consider the boundary contributions. Recall (17). We use (80) to obtain

$$\begin{aligned}
&\int_0^t \int_{\gamma_-} |\partial f^{m+1}(s)|^p \\
&\lesssim_p \sup_{x\in\partial\Omega} \left(\int_{\gamma_-} \sqrt{\mu(v)}^p \left(|n\cdot v| + \frac{\langle v\rangle^p}{|n\cdot v|^{p-1}} \right) \mathrm{d}v \right) \\
&\qquad \times \int_0^t \int_{\partial\Omega} \left[\int_{u\cdot n(x)>0} |\partial f^m(s,x,u)| \mu^{1/4}(u) \{n\cdot u\} \mathrm{d}u \right]^p \mathrm{d}S_x \mathrm{d}s \\
&\quad + \sup_{x\in\partial\Omega} \left(\int_{\gamma_-} \langle v\rangle^{-p\beta} |n\cdot v|^{1-p} \mathrm{d}v \right) \times t ||\langle v\rangle^\beta f_0||_\infty^p \\
&\lesssim_p \int_0^t \int_{\partial\Omega} \left[\int_{u\cdot n(x)>0} |\partial f^m(s,x,u)| \mu^{1/4}(u) \{n\cdot u\} \mathrm{d}u \right]^p \mathrm{d}S_x \mathrm{d}s \\
&\quad + t ||\langle v\rangle^\beta f_0||_\infty^p (1 + ||\langle v\rangle^\beta f_0||_\infty)^p.
\end{aligned}$$

Now we focus on $\int_0^t \int_{\partial\Omega} \left[\int_{u\cdot n(x)>0} |\partial f^m(s,x,u)| \mu^{1/4}(u) \{n\cdot u\} \mathrm{d}u \right]^p \mathrm{d}S_x \mathrm{d}s$. We split the $\{u \in \mathbb{R}^3 : n(x)\cdot u > 0\}$ as

$$\int_0^t \int_{\partial\Omega} \left[\int_{n\cdot u>0} |\partial f^m| \mu^{1/4} \{n\cdot u\} \mathrm{d}u \right]^p \lesssim_p \int_0^t \int_{\partial\Omega} \left[\int_{(x,u)\in\gamma_+\setminus\gamma_+^\varepsilon} \mathrm{d}u \right]^p + \int_0^t \int_{\partial\Omega} \left[\int_{(x,u)\in\gamma_+^\varepsilon} \mathrm{d}u \right]^p. \tag{84}$$

We use Hölder's inequality to bound

$$\left[\int_{(x,u)\in\gamma_+^\varepsilon}\mathrm{d}u\right]^p \le \left[\int_{(x,u)\in\gamma_+^\varepsilon}\mu^{\frac{p}{4(p-1)}}\{n\cdot u\}\mathrm{d}u\right]^{p-1}\left[\int_{(x,u)\in\gamma_+^\varepsilon}|\partial f^m(s,x,u)|^p\{n(x)\cdot u\}\mathrm{d}u\right],$$

to bound the second term of (84)

$$\int_0^t\int_{\partial\Omega}\left[\int_{(x,u)\in\gamma_+^\varepsilon}\mathrm{d}u\right]^p \lesssim_p \varepsilon\int_0^t|\partial f^m(s)|^p_{\gamma_+,p}\mathrm{d}s. \tag{85}$$

For the first term (non-grazing part) of (84) we use Hölder's inequality and Lemma 7 and Lemma 4 and Lemma 5 for f^m to estimate

$$\begin{aligned}
&\int_0^t\int_{\partial\Omega}\left[\int_{(x,u)\in\gamma_+\setminus\gamma_+^\varepsilon}\mathrm{d}u\right]^p\\
\lesssim_\varepsilon\quad & ||\partial f_0||_p^p+\int_0^t||\partial f^m(s)||_p^p\mathrm{d}s+\int_0^t\iint_{\Omega\times\mathbb{R}^3}\big|[\partial_t+v\cdot\nabla_x+\nu(F^m)]\partial f^m\big||\partial f^m|^{p-1}\mathrm{d}x\mathrm{d}v\mathrm{d}s\\
\lesssim_\varepsilon\quad & ||\partial f_0||_p^p+\int_0^t||\partial f^m(s)||_p^p \qquad\qquad (86)\\
+\quad & \int_0^t\iint_{\Omega\times\mathbb{R}^3}\big\{\langle v\rangle^{-\beta+\gamma}||\langle v\rangle^\beta f_0||_\infty+|\partial f^m|+|K\partial f^{m-1}|\\
&\quad+|\Gamma_{\text{gain}}(f^{m-1},\partial f^{m-1})|+|\Gamma_{\text{gain}}(\partial f^{m-1},f^{m-1})|\big\}|\partial f^m|^{p-1}\\
\lesssim_\varepsilon\quad & ||\partial f_0||_p^p+(1+||\langle v\rangle^\beta f_0||_\infty(1+||\langle v\rangle^\beta f_0||_\infty))\sum_{i=m,m-1}\int_0^t||\partial f^i(s)||_p^p+t||\langle v\rangle^\beta f_0||^p_\infty(1+||\langle v\rangle^\beta f_0||_\infty)^p.
\end{aligned}$$

Putting together the estimates (83), (85), (86) and (83), and choosing sufficiently small $\varepsilon\ll 1, T_*\ll 1$,, we deduce that

$$\begin{aligned}
&\sup_{0\le t\le T_*}||\partial f^{m+1}(t)||_p^p+\int_0^{T_*}|\partial f^{m+1}|^p_{\gamma_+,p}\\
&\lesssim C_{T_*,\Omega}\{||\partial f_0||_p^p+\mathcal{P}(||\langle v\rangle^\beta f_0||_\infty)\}+\frac{1}{8}\max_{i=m,m-1}\Big\{\sup_{0\le t\le T_*}||\partial f^i(t)||_p^p+\int_0^{T_*}|\partial f^i|^p_{\gamma_+,p}\Big\}.
\end{aligned}$$

To conclude the proof we use the following fact from [2] : Suppose $a_i\ge 0, D\ge 0$ and $A_i=\max\{a_i,a_{i-1},\cdots,a_{i-(k-1)}\}$ for fixed $k\in\mathbb{N}$.

$$If\ \ a_{m+1}\le\frac{1}{8}A_m+D\ \text{ then }\ A_m\ \le\ \frac{1}{8}A_0+\left(\frac{8}{7}\right)^2 D,\ \text{ for }\frac{m}{k}\gg 1. \tag{87}$$

Proof of (87): In fact, we can iterate for $m, m-1, ...$ to get

$$\begin{aligned}
a_m\ &\le\ \frac{1}{8}\max\{\frac{1}{8}A_{m-2}+D,A_{m-2}\}+D\le\frac{1}{8}A_{m-2}+(1+\frac{1}{8})D\\
&\le\ \frac{1}{8}\max\{\frac{1}{8}A_{m-3}+D,A_{m-3}\}+(1+\frac{1}{8})D\le\frac{1}{8}A_{m-3}+(1+\frac{1}{8}+\frac{1}{8^2})D\\
&\le\ \frac{1}{8}A_{m-k}+\frac{8}{7}D.
\end{aligned}$$

Similarly $a_{m-i}\le\frac{1}{8}A_{m-k}+\frac{8}{7}D$ for all $i=0,1,\cdots,k-1$. Therefore if $1\ll m/k\in\mathbb{N}$,

$$\begin{aligned}
A_m\ &=\ \max\{a_m,a_{m-1},\cdots,a_{m-(k-1)}\}\le\frac{1}{8}A_{m-k}+\frac{8}{7}D\\
&\le\ \frac{1}{8^2}A_{m-2k}+\frac{8}{7}(1+\frac{1}{8})D\le\frac{1}{8^3}A_{m-3k}+\frac{8}{7}(1+\frac{1}{8}+\frac{1}{8^2})D\\
&\le\ \left(\frac{1}{8}\right)^{[\frac{m}{k}]}A_{m-[\frac{m}{k}]k}+\left(\frac{8}{7}\right)^2 D\le\left(\frac{1}{8}\right)^{\frac{m}{k}}A_0+\left(\frac{8}{7}\right)^2 D\ \le\ \frac{1}{8}A_0+\left(\frac{8}{7}\right)^2 D.
\end{aligned}$$

This completes the proof of (87).

In (87), setting $k=2$ and

$$a_i = \sup_{0\le t\le T_*} ||\partial f^i(t)||_p^p + \int_0^t |\partial f^i|^p_{\gamma_+,p}, \qquad D = C_{T_*,\Omega}\{||\partial f_0||_p^p + \mathcal{P}(||\langle v\rangle^\beta f_0||_\infty)\},$$

and applying (87), we complete the proof of the lemma. □

The following result indicates that Theorem 1 is optimal :

Lemma 10. *Let $\Omega = B(0;1)$ with $B(0;1) = \{x\in\mathbb{R}^3 : |x| < 1\}$. There exists an initial datum $f_0(x,v)\in C^\infty$ with $f_0 \subset\subset B(0;1)\times B(0;1)$ so that the solution f to*

$$\partial_t f + v\cdot\nabla_x f = 0, \quad f|_{t=0} = f_0, \quad f(t,x,v)|_{\gamma_-} = c_\mu\sqrt{\mu(v)}\int_{n(x)\cdot u>0} f(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u, \tag{88}$$

satisfies

$$\int_0^1\int_{\gamma_-} |\nabla_x f(s,x,v)|^2 \mathrm{d}\gamma\mathrm{d}s \;=\; +\infty,$$

so that the estimate (11) of Theorem 1 fails for $p=2$.

Proof. We prove by contradiction. Suppose $\int_0^1\int_{\gamma_-}|\partial f(s,x,v)|^2\mathrm{d}\gamma\mathrm{d}s < +\infty$. Then

$$\partial_n f(t,x,v) = \frac{1}{n\cdot v}\Big\{-\partial_t f - (\tau_1\cdot v)\partial_{\tau_1} f - (\tau_2\cdot v)\partial_{\tau_2} f\Big\}, \quad \mathit{for}\ (x,v)\in\gamma_-,$$

We use the boundary condition to define :

$$\begin{aligned}
\partial_t f(t,x,v)|_{\gamma_-} &= c_\mu\sqrt{\mu(v)}A(t,x) \equiv c_\mu\sqrt{\mu}\int_{n\cdot u>0}\partial_t f\sqrt{\mu}\{n\cdot u\}\mathrm{d}u,\\
\partial_{\tau_i} f(t,x,v)|_{\gamma_-} &= c_\mu\sqrt{\mu(v)}B_i(t,x)\\
&\equiv c_\mu\sqrt{\mu}\int_{n\cdot u>0}\partial_{\tau_i} f\sqrt{\mu}\{n\cdot u\}\mathrm{d}u + c_\mu\sqrt{\mu}\int_{n\cdot u>0}\nabla_v f\frac{\partial\mathcal{T}}{\partial\tau_i}\mathcal{T}^{-1}u\sqrt{\mu}\{n\cdot u\}\mathrm{d}u.
\end{aligned}$$

We make a change of variables $v_n = v\cdot n(x),\ v_{\tau_1} = v\cdot\tau_1(x),\ v_{\tau_2} = v\cdot\tau_2(x)$ to compute

$$\begin{aligned}
&\int_{\partial\Omega}\mathrm{d}S_x\int_0^\infty \mathrm{d}v_n\iint_{\mathbb{R}^2}\mathrm{d}v_{\tau_1}\mathrm{d}v_{\tau_2}\,\frac{\mu(v)}{v_\perp}\Big\{(A)^2 + (v_{\tau_1})^2(B_1)^2 + (v_{\tau_2})^2(B_2)^2 + 2v_{\tau_1}AB_1 + 2v_{\tau_2}AB_2 + 2v_{\tau_1}v_{\tau_2}B_1B_2\Big\}\\
&= \int_0^\infty \mathrm{d}v_n\frac{e^{-\frac{|v_n|^2}{2}}}{v_n}\int_{\partial\Omega}\mathrm{d}S_x\big\{(A)^2 + 2\pi(B_1)^2 + 2\pi(B_2)^2\big\}.
\end{aligned}$$

Note that the integration over $\partial\Omega$ is a function of t only (independent of v). Since $\int_0^\infty \frac{\mathrm{d}v_n}{v_n} = \infty$, we conclude that $A = B_1 = B_2 \equiv 0$ for $(t,x)\in[0,\infty)\times\partial\Omega$. In particular from $A(t,x) = 0$ we have for all $t\ge 0$

$$\int_{n(x)\cdot u>0} f(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u = \int_{n(x)\cdot u>0} f(0,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u. \tag{89}$$

We now choose the initial datum to vanish near $\partial\Omega$:

$$f_0(x,v) = \phi(|x|)\phi(|v|),$$

where $\phi\in C^\infty([0,\infty))$ and $\phi\ge 0$ and supp$\phi\subset\subset[0,1)$ and $\phi\equiv 1$ on $[0,\frac{1}{2}]$. Clearly

$$c_\mu\sqrt{\mu(v)}\int_{n(x)\cdot u>0} f_0(x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u = 0.$$

Hence $f(t,x,v)\ge 0$ from $f_0\ge 0$ and the zero inflow boundary condition from (89) and the above equality. Moreover following the backward trajectory to the initial plane for $t\in[\frac{1}{8},\frac{1}{4}]$ and $(x,v)\in\gamma_+$ and $|v-\frac{x}{|x|}|<\frac{1}{64}$, and $|v|\in[\frac{1}{8},\frac{1}{2}]$,

$$f(t,x,v) = f_0(x-tv,v) = 1,$$

which contradicts to $c_\mu\sqrt{\mu(v)}\int_{n\cdot u>0} f(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u = 0$ for $(t,x,v)\in[0,\infty)\times\gamma_-$ from (89). $\square$

4.2. Weighted $W^{1,p}$ $(2\leq p<\infty)$ Estimate

We now establish the weighted $W^{1,p}$ estimate for $2\leq p<\infty$ with the same iteration (34). From Lemma 6 for $\zeta+\kappa>2$ and $0<\theta<\frac{1}{4}$, we have a uniform bound (33). Recall the notation $\partial=[\partial_t,\nabla_x,\nabla_v]$. Then $e^{-\varpi\langle v\rangle t}[\alpha(x,v)]^\beta\partial f^m$ satisfies

$$\begin{aligned}[][\partial_t+v\cdot\nabla_x+\nu_{\varpi,\beta}+\nu(F^m)]e^{-\varpi\langle v\rangle t}[\alpha(x,v)]^\beta\partial f^{m+1}=e^{-\varpi\langle v\rangle t}[\alpha(x,v)]^\beta\mathcal{G}^m,\\ [\alpha(x,v)]^\beta\partial f^{m+1}(0,x,v)=[\alpha(x,v)]^\beta\partial f_0(x,v).\end{aligned}\tag{90}$$

Here

$$\nu_{\varpi,\beta}:=\varpi\langle v\rangle-\beta\alpha(x,v)^{-1}v\cdot\nabla_x\alpha\gtrsim\langle v\rangle\{\varpi-C_\xi\beta\},\tag{91}$$

and $\mathcal{G}^m$ is defined in (122). Using Lemma 4 and Lemma 5 and Lemma 6 we have

$$\begin{aligned}&e^{-\varpi\langle v\rangle t}\alpha(x,v)^\beta|\mathcal{G}^m|\\ &\lesssim e^{-\varpi\langle v\rangle t}\alpha(x,v)^\beta e^{-C_\theta|v|^2}P(||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty)\\ &\quad+e^{-\varpi\langle v\rangle t}\alpha(x,v)^\beta|\partial_x f^{m+1}|+P(||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty)e^{-\varpi\langle v\rangle t}\alpha(x,v)^\beta\int_{\mathbb{R}^3}\mathbf{k}_{\kappa,\theta/2}(v,u)|\partial f^m(u)|\mathrm{d}u.\end{aligned}$$

For $(x,v)\in\gamma$, from (42) and (80), the boundary condition is bounded for $\beta<\frac{p-1}{2p}$ by

$$\begin{aligned}&e^{-\varpi\langle v\rangle t}[\alpha(x,v)]^\beta|\partial f^{m+1}(t,x,v)|\\ &\lesssim\ e^{-\varpi\langle v\rangle t}[\alpha(x,v)]^\beta\sqrt{\mu}\Big(1+\frac{\langle v\rangle}{|n(x)\cdot v|}\Big)\int_{n\cdot u>0}|\partial f^m(t,x,u)|\langle u\rangle\sqrt{\mu}\{n\cdot u\}\mathrm{d}u\\ &\quad+\frac{e^{-\varpi\langle v\rangle t}[\alpha(x,v)]^\beta}{|n(x)\cdot v|}e^{-\frac{\theta}{4}|v|^2}||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty\mathcal{P}(||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty).\end{aligned}\tag{92}$$

Set $f^0=f_0$ and $\partial f^0=[\partial_t f^0,\nabla_x f^0,\nabla_v f^0]=[0,0,0]$. The main estimate is the following :

Proof of Lemma 2. Fix $p\geq 2$, $\frac{p-2}{2p}<\beta<\frac{p-1}{2p}$ and $\varpi\gg_\Omega 1$. We claim that there exists $0<T_*\ll 1$ such that we have the following uniformly-in-m,

$$\begin{aligned}&\sup_{0\leq t\leq T_*}||e^{-\varpi\langle v\rangle t}\alpha^\beta\partial f^m(t)||_p^p+\int_0^{T_*}|e^{-\varpi\langle v\rangle s}\alpha^\beta\partial f^m|_{\gamma,p}^p\\ &\lesssim_{\Omega,T_*}\ (1+P(||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty))||\alpha^\beta\partial f_0||_p^p,\end{aligned}\tag{93}$$

where $\mathcal{P}$ is some polynomial.

Once we have (93) then we pass to the limit, $e^{-\varpi\langle v\rangle t}\alpha^\beta\partial f^m\rightharpoonup e^{-\varpi\langle v\rangle t}\alpha^\beta\partial f$ weakly with norms $\sup_{t\in[0,T_*]}||\cdot||_p^p$ and $e^{-\varpi\langle v\rangle t}\alpha^\beta\partial f^m|_\gamma\rightharpoonup e^{-\varpi\langle v\rangle t}\alpha^\beta\partial f|_\gamma$ in $\int_0^{T_*}|\cdot|_{\gamma,p}^p$ and $e^{-\varpi\langle v\rangle t}\alpha^\beta\partial f$ satisfies (93). Repeat the same procedure for $[T_*,2T_*],[2T_*,3T_*],\cdots$, up to the local existence time interval $[0,T^*]$ in Lemma 6 to conclude Theorem 2.

We prove (93) by induction. From Proposition 2 ∂f^1 exits. More precisely we construct $\partial_t f^1,\nabla_x f^1$ first and then $\nabla_v f^1$. Because of our choice of ∂f^0 the estimate (93) is valid for $m=1$. Now assume that ∂f^i exists and (93) is valid for all $i=1,2,\cdots,m$. Applying the weighted inflow estimate(Proposition 2) we deduce that ∂f^{m+1} exists. From the Green's identity(Lemma 8) we have

$$\sup_{0\leq s\leq t}||e^{-\varpi\langle v\rangle s}\alpha^{\beta}\partial f^{m+1}(s)||_p^p+\int_0^t|e^{-\varpi\langle v\rangle s}\alpha^{\beta}\partial f^{m+1}|_{\gamma_+,p}^p+\int_0^{T_*}||\langle v\rangle^{1/p}e^{-\varpi\langle v\rangle s}\alpha^{\beta}\partial f^{m+1}||_p^p$$

$$\lesssim||\alpha^{\beta}\partial f_0||_p^p+\int_0^t|e^{-\varpi\langle v\rangle s}\alpha^{\beta}\partial f^{m+1}|_{\gamma_-,p}^p$$

$$+\{t+\varepsilon\}\sup_{0\leq s\leq t}||e^{-\varpi\langle v\rangle s}\alpha^{\beta}\partial f^{m+1}(s)||_p^p+t\mathcal{P}(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty^p)$$

$$+\int_0^t\iint_{\Omega\times\mathbb{R}^3}[e^{-\varpi\langle v\rangle t}\alpha^{\beta}]^p\Big\{|K\partial f^m|+|\Gamma_{\text{gain}}(f^m,\partial f^m)|+|\Gamma_{\text{gain}}(\partial f^m,f^m)|\Big\}|\partial f^{m+1}|^{p-1}. \tag{94}$$

Step 1. Estimate for $K\partial f^m$ and Γ_{gain}: The key estimate is the following : For $0<\beta<\frac{p-1}{2p}$, $0<\theta\leq\frac{1}{4}$, and some $C_{\varpi,\beta,p}>0$,

$$\sup_{x\in\Omega}\int_{\mathbb{R}^3}\mathbf{k}_{\kappa,\theta}(v,u)\frac{[e^{-\frac{\varpi}{\beta}\langle v\rangle t}\alpha(x,v)]^{\frac{\beta p}{p-1}}}{[e^{-\frac{\varpi}{\beta}\langle v\rangle t}\alpha(x,u)]^{\frac{\beta p}{p-1}}}\mathrm{d}u\ \lesssim_{\Omega,\theta}\ \langle v\rangle^{\frac{\beta p}{p-1}}e^{C_{\varpi,\beta,p}t^2}. \tag{95}$$

Recall the function $\mathbf{k}_{\kappa,\theta}(v,u)$ in (30) and remark that if $\zeta=\frac{1}{4}$ then $\mathbf{k}_{\kappa,\theta}(v,u)\geq\mathbf{k}(v,u)$. First we assume $|\xi(x)|<\delta_\Omega$ so that $n(x):=\frac{\nabla\xi(x)}{|\nabla\xi(x)|}$ is well-defined. We decompose $u_n=u\cdot n(x)=u\cdot\frac{\nabla\xi(x)}{|\nabla\xi(x)|}$ and $u_\tau=u-u_nn(x)$. For $0\leq\kappa\leq1$ is bounded by

$$\begin{aligned}
&|v|^{\frac{\beta p}{p-1}}\int_{\mathbb{R}^3}\frac{1}{|v-u|^{2-\kappa}}e^{-C_\theta|v-u|^2}\frac{e^{-\frac{\varpi p}{p-1}\langle v\rangle t}}{e^{-\frac{\varpi p}{p-1}\langle u\rangle t}}\frac{1}{|u\cdot\nabla\xi(x)|^{\frac{\beta p}{p-1}}}\mathrm{d}u\\
\lesssim_\Omega\ &|v|^{\frac{\beta p}{p-1}}\int_{\mathbb{R}^3}|v-u|^{-2+\kappa}e^{-\frac{C_\theta|v-u|^2}{2}}e^{\frac{\varpi p}{p-1}t|v-u|}|u_n|^{\frac{-\beta p}{p-1}}\mathrm{d}u\\
\lesssim_\Omega\ &|v|^{\frac{\beta p}{p-1}}e^{C_{\varpi,\beta,p}t^2}\int_{\mathbb{R}^2}\mathrm{d}u_\tau\int_{\mathbb{R}}\mathrm{d}u_n|v-u|^{-2+\kappa}e^{-\frac{C_\theta|v-u|^2}{4}}|u_n|^{-\frac{\beta p}{p-1}}\\
\lesssim_\Omega\ &C_\kappa|v|^{\frac{\beta p}{p-1}}e^{C_{\varpi,\beta,p}t^2}.
\end{aligned}$$

where we have used

$$e^{\frac{\varpi\beta p}{p-1}t|v-u|}\lesssim e^{C_{\varpi,\beta,p}t^2}\times e^{-\frac{C_\theta|v-u|^2}{4}}, \tag{96}$$

for some $C_{\varpi,\beta,p}>0$. Furthermore we split the last integration as $\int_{|u_n|/2\leq|v_n-u_n|}+\int_{|u_n|/2\geq|v_n-u_n|}$. Both of them are bounded by

$$C\left[\int\frac{e^{-\frac{C_\theta|v_n-u_n|^2}{8}}}{|u_n|^{\frac{\beta p}{p-1}}}\mathrm{d}u_n+\int\frac{e^{-\frac{C_\theta|v_n-u_n|^2}{8}}}{|v_n-u_n|^{\frac{\beta p}{p-1}}}\mathrm{d}u_n\right]\lesssim\langle v_n\rangle^{-\frac{\beta p}{p-1}}+1.$$

If $|\xi(x)|\geq\delta_\Omega$ then

$$\alpha(x,v)\geq2|\xi(x)|\{v\cdot\nabla^2\xi(x)\cdot v\}\gtrsim\delta_\Omega|v|^2\gtrsim\delta_\Omega|v_3|^2,$$

where $v=(v_1,v_2,v_3)$ is the standard coordinate. We set $v_3=v_n$ and $v_\tau=(v_1,v_2)$ and follow the exactly same proof. Therefore we conclude (95).

In order to estimate $K\partial f^m$ contribution in (94) recall, for $1/p+1/q=1$,

$$
\begin{aligned}
& e^{-\varpi\langle v\rangle t}[\alpha(x,v)]^{\beta}|K\partial f^m| \\
\leq\ & e^{-\varpi\langle v\rangle t}[\alpha(x,v)]^{\beta}\int |\mathbf{k}(v,u)||\partial f^m(u)|\mathrm{d}u \\
=\ & e^{-\varpi\langle v\rangle t}[\alpha(x,v)]^{\beta}\int |\mathbf{k}(v,u)|\frac{e^{-\varpi\langle v\rangle t}[\alpha(x,u)]^{\beta}|\partial f^m(u)|}{e^{-\varpi\langle u\rangle t}[\alpha(x,u)]^{\beta}}\mathrm{d}u \\
\leq\ & \left(\int |\mathbf{k}(v,u)|\frac{[e^{-\frac{\varpi}{\beta}\langle v\rangle t}\alpha(x,v)]^{\beta q}}{[e^{-\frac{\varpi}{\beta}\langle u\rangle t}\alpha(x,u)]^{\beta q}}\mathrm{d}u\right)^{\frac{1}{q}}\left(\int |\mathbf{k}(v,u)|\Big|[e^{-\frac{\varpi}{\beta}\langle u\rangle t}\alpha(x,u)]^{\beta}\partial f^m(u)\Big|^p\mathrm{d}u\right)^{\frac{1}{p}} \\
\leq\ & \langle v\rangle^{\beta}e^{C_{l,\beta,p}t^2}\left(\int |\mathbf{k}(v,u)|\Big|e^{-\varpi\langle u\rangle t}[\alpha(x,u)]^{\beta}\partial f^m(u)\Big|^p\mathrm{d}u\right)^{\frac{1}{p}}.
\end{aligned}
$$

The $K\partial f^m$ contribution in (94) is therefore bounded by

$$
\begin{aligned}
& \int_0^t\iint_{\Omega\times\mathbb{R}^3}[e^{-\frac{\varpi}{\alpha}\langle v\rangle s}\alpha]^{\beta p}|K\partial f^m||\partial f^{m+1}|^{p-1}\mathrm{d}v\mathrm{d}x\mathrm{d}s \\
\leq\ & \int_0^t\iint_{\Omega\times\mathbb{R}^3}\langle v\rangle^{\frac{\beta}{p}}e^{C_{\varpi,\beta,p}s^2}\Big(\int |\mathbf{k}(v,u)||[e^{-\frac{\varpi}{\beta}\langle u\rangle s}\alpha]^{\beta}\partial f^m(u)|^p\mathrm{d}u\Big)^{\frac{1}{p}} \\
& \qquad\qquad \times\langle v\rangle^{\frac{\beta}{q}}[e^{-\frac{\varpi}{\beta}\langle v\rangle s}\alpha]^{\beta(p-1)}|\partial f^{m+1}(v)|^{p-1} \\
\leq\ & C_{\varepsilon}\int_0^t\iint_{\Omega\times\mathbb{R}^3}e^{pC_{\varpi,\beta,p}s^2}\langle u\rangle^{\beta}|[e^{-\frac{\varpi}{\beta}\langle u\rangle s}\alpha]^{\beta}\partial f^m(u)|^p\Big(\int |\mathbf{k}(v,u)|\frac{\langle v\rangle^{\beta}}{\langle u\rangle^{\beta}}\mathrm{d}v\Big)\mathrm{d}x\mathrm{d}u\mathrm{d}s \\
& +\varepsilon\int_0^t\iint_{\Omega\times\mathbb{R}^3}\langle v\rangle^{\beta}[e^{-\frac{\varpi}{\beta}\langle v\rangle s}\alpha]^{\beta p}|\partial f^{m+1}|^p\mathrm{d}x\mathrm{d}v\mathrm{d}s \\
\leq\ & C\int_0^t\iint_{\Omega\times\mathbb{R}^3}e^{C_{\varpi,\beta,p}s^2}\langle v\rangle^{\beta}|e^{-\varpi\langle u\rangle s}\alpha^{\beta}\partial f^m|^p+\varepsilon\int_0^t\iint_{\Omega\times\mathbb{R}^3}\langle v\rangle^{\beta}|e^{-\varpi\langle v\rangle s}\alpha^{\beta}\partial f^{m+1}|^p \\
\leq\ & Cte^{C_{\varpi,\beta,p}t^2}\sup_{0\leq s\leq t}\iint_{\Omega\times\mathbb{R}^3}|e^{-\varpi\langle v\rangle s}\alpha^{\beta}\partial f^m|^p \\
& +(\delta+\varepsilon)\max_{i=m,m+1}\int_0^t\iint_{\Omega\times\mathbb{R}^3}\langle v\rangle|e^{-\varpi\langle v\rangle s}\alpha^{\beta}\partial f^i|^p, \qquad (97)
\end{aligned}
$$

where we used $\langle v\rangle^{\beta}\leq C_{\delta}+\delta\langle v\rangle$ and (31) in Lemma 4 and $e^{C_{\varpi,\beta,p}s^2}$ factor comes from (95).

Now we consider Γ_{gain} contributions. Recall $\mathbf{k}_{\kappa,\theta}(v,u)$ in (30), (2) of Lemma 5 and (95). In order to estimate the nonlinear terms in (94) we apply (95) to have

$$
\begin{aligned}
& e^{-\varpi\langle v\rangle s}\alpha^{\beta}\big\{|\Gamma_{\text{gain}}(f^m,\partial f^m)|+|\Gamma_{\text{gain}}(\partial f^m,f^m)|\big\}(s,x,v) \\
\lesssim_{\theta}\ & ||\langle v\rangle^{\zeta}e^{\theta|v|^2}f^m||_{\infty}e^{-\varpi\langle v\rangle s}\alpha^{\beta}\Big|\int_{\mathbb{R}^3}\mathbf{k}_{\kappa,\theta/4}(v,u)\partial f^m(u)\mathrm{d}u\Big| \\
\lesssim_{\theta}\ & ||\langle v\rangle^{\zeta}e^{\theta|v|^2}f^m||_{\infty}\Big(\int_{\mathbb{R}^3}\mathbf{k}_{\kappa,\theta/4}(v,u)\frac{[e^{-\frac{\varpi}{\beta}\langle u\rangle s}\alpha]^{\beta q}}{[e^{-\frac{\varpi}{\beta}\langle u\rangle s}\alpha]^{\beta q}}\mathrm{d}u\Big)^{\frac{1}{q}}\Big(\int_{\mathbb{R}^3}\mathbf{k}_{\kappa,\theta/4}(v,u)|[e^{-\varpi\langle u\rangle s}\alpha]^{\beta}\partial f^m(u)|^p\mathrm{d}u\Big)^{\frac{1}{p}} \\
\lesssim_{\theta}\ & ||\langle v\rangle^{\zeta}e^{\theta|v|^2}f^m||_{\infty}\langle v\rangle^{\beta}e^{Cs^2}\Big(\int_{\mathbb{R}^3}\mathbf{k}_{\kappa,\theta}(v,u)|e^{-\varpi\langle u\rangle s}\alpha^{\beta}\partial f^m(u)|^p\mathrm{d}u\Big)^{\frac{1}{p}},
\end{aligned}
$$

where at the last line we used $\frac{p-2}{2p}<\beta<\frac{p-1}{2p}$ so that $\langle v\rangle^{\beta}\leq\langle v\rangle$.

Therefore the nonlinear contributions in (94) are bounded by

$$\begin{aligned}&\int_0^t \iint [e^{-\varpi\langle v\rangle s}\alpha^\beta]^p |\Gamma_{\text{gain}}(f^m,\partial f^m)+\Gamma_{\text{gain}}(\partial f^m,f^m)||\partial f^{m+1}|^{p-1}\mathrm{d}v\mathrm{d}x\mathrm{d}s\\&\leq Cte^{C_{\varpi,\beta,p}t^2}P(||\langle v\rangle^\zeta e^{\theta|v|^2}f^m||_\infty)\sup_{0\leq s\leq t}\iint_{\Omega\times\mathbb{R}^3}|e^{-\varpi\langle v\rangle s}\alpha^\beta\partial f^m|^p\\&\quad+(\delta+\varepsilon)P(||\langle v\rangle^\zeta e^{\theta|v|^2}f^m||_\infty)\max_{i=m,m+1}\int_0^t\iint_{\Omega\times\mathbb{R}^3}\langle v\rangle|e^{-\varpi\langle v\rangle s}\alpha^\beta\partial f^i|^p,\end{aligned} \tag{98}$$

where we have used Lemma 1.

Step 2. Boundary Estimate: Recall (17). We use (92) to estimate the contribution of γ_-

$$\begin{aligned}&\int_0^t\int_{\gamma_-}|e^{-\varpi\langle v\rangle s}\alpha(x,v)^\beta\partial f^{m+1}(s,x,v)|^p\\&\lesssim_p \int_0^t\int_{\gamma_-}[e^{-\varpi\langle v\rangle s}\alpha(x,v)^\beta]^p\sqrt{\mu}^p\Big(1+\frac{\langle v\rangle}{|n(x)\cdot v|}\Big)^p\left[\int_{n(x)\cdot u>0}|\partial f^m(s,x,u)|\mu^{1/4}\{n\cdot u\}\mathrm{d}u\right]^p\\&\quad+\mathcal{P}(||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty)\int_0^t\int_{\gamma_-}\frac{[e^{-\varpi\langle v\rangle s}\alpha(x,v)^\beta]^p}{|n(x)\cdot v|^p}e^{-\frac{\theta p}{4}|v|^2}\mathrm{d}\gamma\mathrm{d}s.\end{aligned} \tag{99}$$

Using $e^{-\varpi\langle v\rangle s}\alpha(x,v)\leq e^{-\frac{\varpi\langle v\rangle}{2}s}|\nabla_x\xi(x)\cdot v|$ for $x\in\partial\Omega$, the last term is bounded by

$$C_\Omega\mathcal{P}(||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty)\int_0^t\int_{\partial\Omega}\int_{\mathbb{R}^3}|n(x)\cdot v|^{\beta p-p+1}e^{-\frac{\theta p}{4}|v|^2}\mathrm{d}v\mathrm{d}S_x\mathrm{d}s\lesssim_{\Omega,p,\zeta}t\mathcal{P}(||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty),$$

for $\beta>\frac{p-2}{2p}$ so that $2\beta p-p+1>-1$.

For the first term in (99) we split as

$$\left[\int_{n(x)\cdot u>0}\cdots\mathrm{d}u\right]^p\lesssim_p\left[\int_{(x,u)\in\gamma_+^\varepsilon}\cdots\mathrm{d}u\right]^p+\left[\int_{(x,u)\in\gamma_+\setminus\gamma_+^\varepsilon}\cdots\mathrm{d}u\right]^p.$$

The γ_+^ε contribution (grazing part) of (99) is bounded by

$$\begin{aligned}&C_p\int_0^t\int_{\gamma_-}[e^{-\varpi\langle v\rangle s}\alpha(x,v)^\beta]^p\sqrt{\mu}^p\Big(|n\cdot v|+\frac{\langle v\rangle^p}{|n\cdot v|^{p-1}}\Big)\\&\quad\times\left|\int_{(x,u)\in\gamma_+^\varepsilon}e^{-\varpi\langle u\rangle s}\alpha(x,u)^\beta\partial f^m\{n\cdot u\}^{1/p}\frac{\{n\cdot u\}^{1/q}\mu^{1/4}}{e^{-\varpi\langle u\rangle s}\alpha(x,u)^\beta}\mathrm{d}u\right|^p\mathrm{d}v\mathrm{d}S_x\mathrm{d}s\\&\lesssim_{\Omega,p}\int_0^t\int_{\gamma_-}[e^{-\varpi\langle v\rangle s}\alpha(v)^\beta]^p\Big(|n\cdot v|+\frac{\langle v\rangle^p}{|n\cdot v|^{p-1}}\Big)\sqrt{\mu}^p\\&\quad\times\left[\int_{(x,u)\in\gamma_+}[e^{-\varpi\langle v\rangle s}\alpha(u)^\beta]^p|\partial f^m|^p\{n\cdot u\}\mathrm{d}u\right]\left[\int_{(x,u)\in\gamma_+^\varepsilon}[e^{-\varpi\langle u\rangle s}\alpha(u)^\beta]^{-q}\mu^{q/4}\{n\cdot u\}\mathrm{d}u\right]^{p/q}\mathrm{d}v\mathrm{d}S_x\mathrm{d}s,\\&\lesssim_{\Omega,p,\varpi,\beta}\varepsilon^a e^{C_{\varpi,\beta,p}t^2}\int_0^t|e^{-\varpi\langle v\rangle s}\alpha^\beta\partial f^m(s)|^p_{\gamma_+,p}\mathrm{d}s,\end{aligned}$$

where we used $[e^{-\varpi\langle v\rangle s}\alpha(x,v)]\leq|\nabla\xi(x)\cdot v|\lesssim_\Omega|n(x)\cdot v|$ and, for $\beta>\frac{p-2}{2p}$,

$$\begin{aligned}&[e^{-\varpi\langle v\rangle s}\alpha(x,v)^\beta]^p\Big(|n\cdot v|+\frac{\langle v\rangle^p}{|n\cdot v|^{p-1}}\Big)\sqrt{\mu}^p\\&\lesssim_\Omega\Big(|n(x)\cdot v|^{1+\beta p}+\langle v\rangle^p|n(x)\cdot v|^{(\beta-1)p+1}\Big)\sqrt{\mu(v)}^p\in L^1(\{v\in\mathbb{R}^3\}),\end{aligned}$$

and, here, $a > 0$ is determined via, with $\frac{p-1}{p} = \frac{1}{q}$,

$$
\begin{aligned}
&\int_{\gamma_+^\varepsilon} [e^{-\frac{\varpi}{\beta}\langle u\rangle s}\alpha(x,u)]^{-\frac{\beta p}{p-1}} \mu^{\frac{p}{4(p-1)}} \{n\cdot u\}\mathrm{d}u \lesssim_\Omega \int_{\gamma_+^\varepsilon} \Big[e^{-\frac{\frac{\varpi}{\beta}\langle u\rangle s}{2}}|u\cdot\nabla\xi(x)|\Big]^{-\frac{\beta p}{p-1}} e^{-\frac{p}{4(p-1)}|u|^2}|n\cdot u|\mathrm{d}u \\
&\lesssim_\Omega \int_{\gamma_+^\varepsilon} |u\cdot n|^{1-\frac{\beta p}{p-1}} e^{\frac{\varpi}{2(p-1)}\langle u\rangle s} e^{-\frac{p}{4(p-1)}|u|^2}\mathrm{d}u \lesssim_\Omega e^{C_{\varpi,\beta,p}s^2}\int_{\gamma_+^\varepsilon} |u\cdot n|^{1-\frac{\beta p}{p-1}} e^{-\frac{p}{8(p-1)}|u|^2}\mathrm{d}u \\
&\lesssim_{\Omega,p} \varepsilon^a e^{C_{\varpi,\beta,p}t^2},
\end{aligned}
$$

for some $a > 0$ since $1 - \frac{2\beta p}{p-1} > -1$.

On the other hand, for the non-grazing contribution $\gamma_+\backslash\gamma_+^\varepsilon$, we use a similar estimate to get

$$
\begin{aligned}
&\int_0^t \int_{\gamma_-} [e^{-\varpi\langle v\rangle s}\alpha(x,v)^\beta]^p \sqrt{\mu}^p\Big(1+\frac{\langle v\rangle}{|n(x)\cdot v|}\Big)^p \left[\int_{\gamma_+\backslash\gamma_+^\varepsilon} |\partial f^m(s,x,u)|\mu(u)^{1/4}\{n(x)\cdot u\}\mathrm{d}u\right]^p \mathrm{d}\gamma\mathrm{d}s \\
&\lesssim_\Omega \int_0^t\int_{\partial\Omega}\int_{\mathbb{R}^3} [e^{-\varpi\langle v\rangle s}\alpha(x,v)^\beta]^p\Big(|n\cdot v| + \frac{\langle v\rangle^p}{|n\cdot v|^{p-1}}\Big)\sqrt{\mu}^p \\
&\quad\times\left[\int_{\gamma_+\backslash\gamma_+^\varepsilon} e^{-\varpi\langle v\rangle s}\alpha(x,v)^\beta|\partial f^m(s,x,u)|\{n\cdot u\}^{1/p}\frac{\{n\cdot u\}^{1/q}\mu(u)^{1/4}}{[e^{-\varpi\langle u\rangle s}\alpha(x,u)]^\beta}\mathrm{d}u\right]^p \mathrm{d}v\mathrm{d}S_x\mathrm{d}s \\
&\lesssim_\Omega \int_0^t\int_{\gamma_-} [e^{-\varpi\langle v\rangle s}\alpha(x,v)^\beta]^p\Big(|n\cdot v| + \frac{\langle v\rangle^p}{|n\cdot v|^{p-1}}\Big)\sqrt{\mu}^p \\
&\quad\times\left[\int_{\gamma_+\backslash\gamma_+^\varepsilon} [e^{-\varpi\langle u\rangle s}\alpha(x,u)^\beta]^p|\partial f^m|^p\{n\cdot u\}\mathrm{d}u\right]\left[\int_{\gamma_+} [e^{-\varpi\langle u\rangle s}\alpha(x,u)^\beta]^{-q}\mu^{q/4}\{n\cdot u\}\mathrm{d}u\right]^{p/q} \mathrm{d}v\mathrm{d}S_x\mathrm{d}s \\
&\lesssim_\Omega e^{C_{\varpi,\beta,p}t^2}\int_0^t\int_{\gamma_+\backslash\gamma_+^\varepsilon} [e^{-\varpi\langle u\rangle s}\alpha(x,u)^\beta]^p|\partial f^m(s)|^p\mathrm{d}\gamma\mathrm{d}s,
\end{aligned}
$$

where we used $\frac{p-2}{2p} < \beta < \frac{p-1}{2p}$ and

$$
\begin{aligned}
&\int_{\gamma_+} [e^{-\varpi\langle u\rangle s}\alpha(x,u)^\beta]^{-q}\mu(u)^{q/4}\{n(x)\cdot u\}\mathrm{d}u \\
&= \int_{\gamma_+} [e^{-\varpi\langle u\rangle s}\alpha(x,u)^\beta]^{-\frac{p}{p-1}}\mu(u)^{\frac{p}{4(p-1)}}\{n\cdot u\}\mathrm{d}u \lesssim_{\Omega,p} e^{C_{\varpi,\beta,p}t^2}.
\end{aligned}
$$

By Lemma 7 and (90), the non-grazing part is further bounded by

$$
\begin{aligned}
&\int_0^t\int_{\gamma_+\backslash\gamma_+^\varepsilon} \\
&\lesssim_\varepsilon \int_0^t ||\alpha^\beta\partial f_0||_p^p + \int_0^t ||e^{-\varpi\langle v\rangle s}\alpha^\beta\partial f^m||_p^p + \int_0^t\iint_{\Omega\times\mathbb{R}^3} |\{\partial_t + v\cdot\nabla_x + \langle v\rangle\}(e^{-\varpi\langle v\rangle s}\alpha^\beta\partial f^m)^p| \\
&\lesssim \int_0^t ||\alpha^\beta\partial f_0||_p^p + \int_0^t ||e^{-\varpi\langle v\rangle s}\alpha^\beta\partial f^m||_p^p + \int_0^t\iint [e^{-\varpi\langle v\rangle s}\alpha^\beta]^p|K\partial f^{m-1}||\partial f^m|^{p-1} \\
&\quad+\int_0^t\iint [e^{-\varpi\langle v\rangle s}\alpha^\beta]^p\{|\Gamma(f^{m-1},\partial f^{m-1})| + |\Gamma(\partial f^{m-1},f^{m-1})|\}|\partial f^m|^{p-1} \\
&\quad+ t\sup_{0\le s\le t}||e^{-\varpi\langle v\rangle s}\alpha^\beta\partial f^m(s)||_p^p + (1+t)\mathcal{P}(||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty).
\end{aligned}
$$

In summary, for small $T_* \ll 1$, the boundary contribution of (94) is controlled by, for all $0 \leq t \leq T_*$,

$$
\begin{aligned}
&\int_0^t |e^{-\varpi\langle v\rangle s}\alpha^\beta \partial f^m(s)|_{\gamma_-,p}^p \mathrm{d}s \\
\lesssim &\int_0^{T_*} ||\alpha(v)^\beta \partial f_0||_p^p + \varepsilon^a \int_0^{T_*} |e^{-\varpi\langle v\rangle s}\alpha^\beta \partial f^m|_{\gamma_+,p}^p \\
&+ T_* \max_{i=m-1,m} \sup_{0\leq t\leq T_*} ||e^{-\varpi\langle v\rangle t}\alpha^\beta \partial f^i(t)||_p^p + \mathcal{P}(||\langle v\rangle^\zeta e^{\theta|v|^2} f_0||_\infty) \\
&+ C_{\Omega,T_*}\Big\{ \int_0^{T_*} ||e^{-\varpi\langle v\rangle s}\alpha^\beta \partial f^m||_p^p + \int_0^{T_*} \iint [e^{-\varpi\langle v\rangle s}\alpha^\beta]^p |K\partial f^{m-1}||\partial f^m|^{p-1} \\
&\qquad + \int_0^{T_*} \iint [e^{-\varpi\langle v\rangle s}\alpha^\beta]^p \{|\Gamma_{\mathrm{gain}}(f^{m-1},\partial f^{m-1})| + |\Gamma_{\mathrm{gain}}(\partial f^{m-1}, f^{m-1})|\}|\partial f^m|^{p-1}\Big\}.
\end{aligned}
$$

Applying (97) and (98) to the boundary estimates for $m-1$, then putting together the estimates (100), (97) and (98) we deduce from (94)

$$
\begin{aligned}
&\sup_{0\leq t\leq T_*} ||e^{-\varpi\langle v\rangle t}\alpha^\beta \partial f^{m+1}(t)||_p^p + \int_0^{T_*} ||\langle v\rangle^{1/p} e^{-\varpi\langle v\rangle s}\alpha^\beta \partial f^{m+1}||_p^p \\
&\quad + \int_0^{T_*} |e^{-\varpi\langle v\rangle s}\alpha^\beta \partial f^{m+1}|_{\gamma_+,p}^p \mathrm{d}s \\
&\leq C_{T_*,\Omega}\big\{||\alpha^\beta \partial f_0||_p^p + \mathcal{P}(||\langle v\rangle^\zeta e^{\theta|v|^2} f_0||_\infty)\big\} + \big\{\varepsilon + \delta + T_* e^{C_{\varpi,\beta,p}(T^*)^2}\big\}\mathcal{P}(||\langle v\rangle^\zeta e^{\theta|v|^2} f_0||_\infty^p) \\
&\quad \times \max_{i=m,m-1}\Big\{ \sup_{0\leq t\leq T_*} ||\alpha^\beta \partial f^i(t)||_p^p + \int_0^{T_*} |e^{-\varpi\langle v\rangle s}\alpha^\beta \partial f^i|_{\gamma_+,p}^p + \int_0^{T_*} ||\langle v\rangle^{1/p} e^{-\varpi\langle v\rangle t}\alpha^\beta \partial f^i||_p^p\Big\}.
\end{aligned}
$$

Recall $C_{\varpi,\beta,p}$ from (95). Choose $T_* \ll 1$, and $\varepsilon \ll 1$, $\delta \ll 1$ and hence

$$
\begin{aligned}
&\sup_{0\leq t\leq T_*} ||e^{-\varpi\langle v\rangle t}\alpha^\beta \partial f^{m+1}(t)||_p^p + \int_0^{T_*} |e^{-\varpi\langle v\rangle t}\alpha^\beta \partial f^{m+1}|_{\gamma_+,p}^p \\
\leq\ & C_{T_*,\Omega}\big\{||\alpha^\beta \partial f_0||_p^p + \mathcal{P}(||\langle v\rangle^\zeta e^{\theta|v|^2} f_0||_\infty)\big\} \\
&+\frac{1}{8}\max_{i=m,m-1}\Big\{ \sup_{0\leq t\leq T_*} ||e^{-\varpi\langle v\rangle t}\alpha^\beta \partial f^i(t)||_p^p + \int_0^{T_*} |e^{-\varpi\langle v\rangle t}\alpha^\beta \partial f^i|_{\gamma_+,p}^p\Big\}.
\end{aligned}
$$

Set

$$
\begin{aligned}
a_i &= \sup_{0\leq t\leq T_*} ||e^{-\varpi\langle v\rangle t}\alpha^\beta \partial f^{m+1}(t)||_p^p + \int_0^{T_*} |e^{-\varpi\langle v\rangle t}\alpha^\beta \partial f^{m+1}|_{\gamma_+,p}^p, \\
D &= C_{T_*,\Omega}\big\{||\alpha^\beta \partial f_0||_p^p + \mathcal{P}(||\langle v\rangle^\zeta e^{\theta|v|^2} f_0||_\infty)\big\}.
\end{aligned}
$$

Apply (87) with $k=2$ to complete the proof. □

4.3. Weighted $\mathbf{C^1}$ Estimate

We start with the same iterative sequences (90) with $\beta = \frac{1}{2}$. For $(x,v) \in \gamma$, note that $\sqrt{\alpha(x,v)} = |n(x)\cdot v|$. From (92) with $\beta = \frac{1}{2}$, we have, for $(x,v)\in\gamma_-$,

$$
\begin{aligned}
&e^{-\varpi\langle v\rangle t}|\sqrt{\alpha(x,v)}\partial f^{m+1}(t,x,v)| \\
\lesssim\ &\langle v\rangle c_\mu \sqrt{\mu(v)} \int_{n(x)\cdot u>0} e^{-\varpi\langle u\rangle t}\sqrt{\alpha(x,u)}|\partial f^m(t,x,u)|e^{\varpi\langle u\rangle t}\langle u\rangle\sqrt{\mu(u)}\mathrm{d}u \\
&+ e^{-\frac{\theta}{4}|v|^2} P(||\langle v\rangle^\zeta e^{\theta|v|^2} f_0||_\infty).
\end{aligned} \tag{100}
$$

Recall the stochastic cycles in Definition 1 : For (t,x,v) with $(x,v)\notin\gamma_0$ and let $(t^0,x^0,v^0) = (t,x,v)$. For $v^\ell\cdot n(x^{\ell+1}) > 0$ we define the $(\ell+1)-$component of the back-time cycle as

$$(t^{\ell+1}, x^{\ell+1}, v^{\ell+1}) = (t^k - t_{\mathbf{b}}(x^\ell,v^\ell), x_{\mathbf{b}}(x^\ell,v^\ell), v^{\ell+1}).$$

Lemma 11. *If $t^1 < 0$ then*

$$|e^{-\varpi\langle v\rangle t}\alpha(x,v)^{1/2}\partial f^{m+1}(t,x,v)| \lesssim \ ||\alpha(x,v)^{1/2}\partial f_0||_\infty + \int_0^t |\mathcal{N}^m(s,x-(t-s)v,v)|ds. \quad (101)$$

If $t^1 > 0$ then

$$\begin{aligned}
&|e^{-\varpi\langle v\rangle t}\alpha(x,v)^{1/2}\partial f^{m+1}(t,x,v)| \\
\lesssim\ & \int_{t^1}^t |\mathcal{N}^m(s,x-(t-s)v,v)|\mathrm{d}s + e^{-\frac{\theta}{4}|v|^2}\mathcal{P}(||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty) \\
&+ \frac{1}{w(v)}\int_{\prod_{j=1}^{\ell-1}\mathcal{V}_j}\sum_{i=1}^{\ell-1} 1_{\{t^{\ell+1}<0<t^\ell\}}\ |\alpha^{1/2}\partial f^{m+1-i}(0,x^i-t^iv^i,v^i)|\ \mathrm{d}\Sigma_i^{\ell-1} \\
&+ \frac{1}{w(v)}\int_{\prod_{j=1}^{\ell-1}\mathcal{V}_j}\sum_{i=1}^{\ell-1} 1_{\{t^{i+1}<0<t^i\}}\int_0^{t^i} |\mathcal{N}^{m-i}(s,x^i-(t^i-s)v^i,v^i)|\ \mathrm{d}s\ \mathrm{d}\Sigma_i^{\ell-1} \\
&+ \frac{1}{w(v)}\int_{\prod_{j=1}^{\ell-1}\mathcal{V}_j}\sum_{i=1}^{\ell-1} 1_{\{t^{i+1}<0\}}\int_{t^{i+1}}^{t^i} |\mathcal{N}^{m-i}(s,x^i-(t^i-s)v^i,v^i)|\ \mathrm{d}s\ \mathrm{d}\Sigma_i^{\ell-1} \qquad (102) \\
&+ \frac{1}{w(v)}\int_{\prod_{j=1}^{\ell-1}\mathcal{V}_j}\sum_{i=2}^{\ell-1} 1_{\{t^{i-1}<0\}}e^{-\frac{\theta}{4}|v^{i-1}|^2}\mathcal{P}(||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty)\ \mathrm{d}\Sigma_{i-1}^{\ell-1} \\
&+ \frac{1}{w(v)}\int_{\prod_{j=1}^{\ell-1}\mathcal{V}_j}\mathbf{1}_{\{t^\ell>0\}}|e^{-\varpi\langle v^{\ell-1}\rangle t^\ell}\alpha(x^\ell,v^{\ell-1})^{1/2}\partial f^{m+1-\ell}(t^\ell,x^\ell,v^{\ell-1})|\ \mathrm{d}\Sigma_{\ell-1}^{\ell-1},
\end{aligned}$$

where $\mathcal{V}_j = \{v^j\in\mathbb{R}^3 : n(x^j)\cdot v^j > 0\}$ and

$$w(v) = \frac{c_\mu}{\langle v\rangle\sqrt{\mu(v)}},$$

and

$$\mathrm{d}\Sigma_i^{\ell-1} = \Big\{\Pi_{j=i+1}^{\ell-1}\mu(v^j)c_\mu|n(x^j)\cdot v^j|\mathrm{d}v^j\Big\}\Big\{w(v^i)e^{\varpi\langle v^i\rangle t^i}\langle v^i\rangle^2 c_\mu\mu(v^i)\mathrm{d}v^i\Big\}\Big\{\Pi_{j=1}^{i-1}e^{\varpi\langle v^j\rangle t^j}\langle v^j\rangle^2 c_\mu\mu(v^j)\mathrm{d}v^j\Big\}.$$

Remark that $\mathrm{d}\Sigma_i^{\ell-1}$ is not a probability measure!

Proof. For $t^1<0$ we use (90) with $\beta=1$ to obtain

$$e^{-\varpi\langle v\rangle t}\alpha(x,v)^{1/2}\partial f^{m+1}(t,x,v) \lesssim \alpha(x-tv,v)^{1/2}\partial f_0(x-tv,v) + \int_0^t e^{-\nu_{\varpi,1}(v)(t-s)}\mathcal{N}^m(s,x-(t-s)v,v)\mathrm{d}s.$$

Consider the case of $t^1>0$. We prove by the induction on ℓ, the number of iterations. First for $\ell=1$, along the characteristics, for $t^1>0$, we have

$$\begin{aligned}
&e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f^{m+1}(t,x,v) \\
\lesssim\ & e^{-\nu_{\varpi,1}(t-t_1)}e^{-\varpi\langle v\rangle t^1}\alpha^{1/2}\partial f^{m+1}(t^1,x^1,v) + \int_{t^1}^t e^{-\nu_{\varpi,1}(t-s)}\mathcal{N}^m(s,x-(t-s)v,v)\mathrm{d}s.
\end{aligned}$$

Now we apply (100) to the first term above to further estimate

$$\begin{aligned}
&e^{-\varpi\langle v\rangle t}\alpha^{1/2}|\partial f^{m+1}(t,x,v)| \\
&\lesssim \; e^{-\nu_{\varpi,1}(v)(t-t^1)}e^{-\frac{\theta}{4}|v|^2}||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty(1+||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty) \\
&\quad + e^{-\nu_{\varpi,1}(v)(t-t^1)}\langle v\rangle c_\mu\sqrt{\mu(v)}\int_{\mathcal{V}_1} e^{-\varpi\langle v^1\rangle t^1}\alpha^{1/2}|\partial f^m(t^1,x^1,v^1)|e^{\varpi\langle v^1\rangle t^1}\langle v^1\rangle\sqrt{\mu(v^1)}\mathrm{d}v^1 \\
&\quad + \int_{t^1}^t e^{-\nu_{\varpi,1}(v)(t-s)}|\mathcal{N}^m(s,x-(t-s)v,v)|\mathrm{d}s \\
&\lesssim \; e^{-\frac{\theta}{4}|v|^2}||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty(1+||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty) \\
&\quad + \frac{c_\mu}{w(v)}\int_{\mathcal{V}_1} e^{-\varpi\langle v^1\rangle t^1}\alpha^{1/2}|\partial f^m(t^1,x^1,v^1)|e^{\varpi\langle v^1\rangle t^1}w(v^1)\langle v^1\rangle^2\mu(v^1)\mathrm{d}v^1 \\
&\quad + \int_{t^1}^t |\mathcal{N}^m(s,x-(t-s)v,v)|,
\end{aligned} \tag{103}$$

where $w(v)$ is defined in (111). Now we continue to express $\partial f^m(t^1,x^1,v^1)$ via backward trajectory to get

$$\begin{aligned}
&e^{-\varpi\langle v^1\rangle t^1}\alpha(x^1,v^1)^{1/2}|\partial f^m(t^1,x^1,v^1)| \\
&\leq \; \mathbf{1}_{\{t^2<0<t^1\}}\Big\{\alpha^{1/2}|\partial f^m(0,x^1-t^1v^1,v^1)| + \int_0^{t^1} |\mathcal{N}^{m-1}(s,x^1-(t^1-s)v^1,v^1)|\mathrm{d}s\Big\} \\
&\quad + \mathbf{1}_{\{t^2>0\}}\Big\{e^{-\varpi\langle v^1\rangle t^2}\alpha^{1/2}|\partial f^m(t^2,x^2,v^1)| + \int_{t^2}^{t^1} |\mathcal{N}^{m-1}(s,x^1-(t^1-s)v^1,v^1)|\mathrm{d}s\Big\}.
\end{aligned}$$

Therefore we conclude from (103) that

$$\begin{aligned}
&e^{-\varpi\langle v\rangle t}\alpha(x,v)^{1/2}|\partial f^{m+1}(t,x,v)| \\
&\lesssim \int_{t^1}^t |\mathcal{N}^m(s,x-(t-s)v,v)|\mathrm{d}s + e^{-\frac{\theta}{4}|v|^2}||\langle v\rangle^\zeta e^{\theta|v|^2}f^m||_\infty(1+||\langle v\rangle^\zeta e^{\theta|v|^2}f^m||_\infty) \\
&\quad + \frac{1}{w(v)}\int_{\mathcal{V}_1}\mathbf{1}_{\{t^2<0<t^1\}}\alpha(x^1-t^1v^1,v^1)^{1/2}|\partial f_0(x^1-t^1v^1,v^1)|e^{\varpi\langle v^1\rangle t^1}w(v^1)\langle v^1\rangle^2c_\mu\mu(v^1)\mathrm{d}v^1 \\
&\quad + \frac{1}{w(v)}\int_{\mathcal{V}_1}\mathbf{1}_{\{t^2<0<t^1\}}\int_0^{t^1} |\mathcal{N}^{m-1}(s,x^1-(t^1-s)v^1,v^1)|\mathrm{d}s e^{\varpi\langle v^1\rangle t^1}w(v^1)\langle v^1\rangle^2c_\mu\mu(v^1)\mathrm{d}v^1 \\
&\quad + \frac{1}{w(v)}\int_{\mathcal{V}_1}\mathbf{1}_{\{t^2>0\}}\int_{t^2}^{t^1} |\mathcal{N}^{m-1}(s,x^1-(t^1-s)v^1,v^1)|\mathrm{d}s e^{\varpi\langle v^1\rangle t^1}w(v^1)\langle v^1\rangle^2c_\mu\mu(v^1)\mathrm{d}v^1 \\
&\quad + \frac{1}{w(v)}\int_{\mathcal{V}_1}\mathbf{1}_{\{t^2>0\}}e^{-\varpi\langle v^1\rangle t^2}\alpha(x^2,v^1)^{1/2}|\partial f^m(t^2,x^2,v^1)|e^{\varpi\langle v^1\rangle t^1}w(v^1)\langle v^1\rangle^2c_\mu\mu(v^1)\mathrm{d}v^1,
\end{aligned}$$

and it equals (110) for $\ell=2$.

Assume (110) is valid for $\ell\in\mathbb{N}$. We use (100) and express the last term of (110) as

$$\begin{aligned}
&\mathbf{1}_{\{t^\ell>0\}}e^{-\varpi\langle v^{\ell-1}\rangle t^\ell}\alpha(x^\ell,v^{\ell-1})|\partial f^{m+1-k}(t^\ell,x^\ell,v^{\ell-1})| \\
&\lesssim \langle v^{\ell-1}\rangle c_\mu\sqrt{\mu(v^{\ell-1})}\int_{\mathcal{V}_\ell}\mathbf{1}_{\{t^\ell>0\}}e^{-\varpi\langle v^\ell\rangle t^\ell}\alpha^{1/2}|\partial f^{m+1-(k+1)}(t^\ell,x^\ell,v^\ell)|e^{\varpi\langle v^\ell\rangle t^\ell}\langle v^\ell\rangle\sqrt{\mu(v^\ell)}\mathrm{d}v^\ell \\
&\quad + e^{-\frac{\theta}{4}|v_{k-1}|^2}||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty(1+||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty).
\end{aligned} \tag{104}$$

Then we decompose $\mathbf{1}_{\{t^\ell>0\}}e^{-\varpi\langle v^\ell\rangle t^\ell}\alpha^{1/2}|\partial f^{m+1-(\ell+1)}(t^\ell,x^\ell,v^\ell)| = \mathbf{1}_{\{t^{\ell+1}<0<t^\ell\}}+\mathbf{1}_{\{t^{\ell+1}>0\}}$, where the first part hits the initial plane as

$$\begin{aligned}&\mathbf{1}_{\{t^{\ell+1}<0<t^\ell\}}e^{-\varpi\langle v^\ell\rangle t^\ell}\alpha^{1/2}|\partial f^{m+1-(\ell+1)}(t^\ell,x^\ell,v^\ell)|\\ &\lesssim \alpha^{1/2}|\partial f_0(x^\ell-t^\ell v^\ell,v^\ell)|+\int_0^{t^\ell}|\mathcal{N}^{m+1-(\ell+2)}(s,x^\ell-(t^\ell-s)v^\ell,v^\ell)|\mathrm{d}s,\end{aligned}\tag{105}$$

and the second part hits at the boundary as

$$\begin{aligned}&\mathbf{1}_{\{t^{\ell+1}>0\}}e^{-\varpi\langle v\rangle t}\alpha^{1/2}|\partial f^{m+1-(\ell+1)}(t^\ell,x^\ell,v^\ell)|\\ &\lesssim e^{-\varpi\langle v^\ell\rangle t^{\ell+1}}\alpha^{1/2}|\partial f^{m+1-(\ell+1)}(t^{\ell+1},x^{\ell+1},v^\ell)|+\int_{t^{\ell+1}}^{t^\ell}|\mathcal{N}^{m+1-(\ell+2)}(s,x^\ell-(t^\ell-s)v^\ell,v^\ell)|\mathrm{d}s.\end{aligned}\tag{106}$$

To summarize, from (104) upon integrating over $\prod_{j=1}^{\ell-1}\mathcal{V}_j$, we obtain a bound for the last term of (110) as

$$\begin{aligned}&\frac{1}{w(v)}\int_{\prod_{j=1}^{\ell-1}\mathcal{V}_j}\mathbf{1}_{\{t^\ell>0\}}|e^{-\varpi\langle v^{\ell-1}\rangle t^\ell}\alpha^{1/2}\partial f^{m+1-\ell}(t^\ell,x^\ell,v^{\ell-1})|\mathrm{d}\Sigma_{\ell-1}^{\ell-1}\\ &\lesssim \frac{1}{w(v)}\int_{\prod_{j=1}^{\ell-1}\mathcal{V}_j}\mathbf{1}_{\{t^\ell>0\}}e^{-\frac{\theta}{4}|v^{\ell-1}|^2}||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty(1+||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty)\mathrm{d}\Sigma_{\ell-1}^{\ell-1}\\ &\quad+\frac{1}{w(v)}\int_{\prod_{j=1}^{\ell}\mathcal{V}_j}\mathbf{1}_{\{t^\ell>0\}}\mathbf{d}|\partial f^{m+1-(\ell+1)}(t^\ell,x^\ell,v^\ell)|\mathrm{d}\Sigma_\ell^\ell,\end{aligned}$$

where by (105) and (106), the last term is bounded by

$$\begin{aligned}&\frac{1}{w(v)}\int_{\prod_{j=1}^{\ell}\mathcal{V}_j}\langle v^{\ell-1}\rangle c_\mu\sqrt{\mu(v^{\ell-1})}\sqrt{\mu(v^\ell)}\langle v^\ell\rangle e^{\varpi\langle v^\ell\rangle t^\ell}\mathrm{d}v^\ell\\ &\times\prod_{j=1}^{\ell-2}\Big\{e^{\varpi\langle v^j\rangle t^j}\langle v^j\rangle^2c_\mu\mu(v^j)\mathrm{d}v^j\Big\}\Big\{w(v^{\ell-1})e^{\varpi\langle v^{\ell-1}\rangle t^{\ell-1}}\langle v^{\ell-1}\rangle^2\mu(v^{\ell-1})\mathrm{d}v^{\ell-1}\Big\}\\ &\times\Big\{\mathbf{1}_{\{t^{\ell+1}<0<t^\ell\}}\Big[\alpha^{1/2}|\partial f(0,x^\ell-t^\ell v^\ell,v^\ell)|+\int_0^{t^\ell}|\mathcal{N}^{m-\ell-2}(s,x^\ell-(t^\ell-s)v^\ell,v^\ell)|\mathrm{d}s\Big]\\ &\quad+\mathbf{1}_{\{t^{\ell+1}>0\}}\Big[e^{-\varpi\langle v^\ell\rangle t^{\ell+1}}\alpha^{1/2}|\partial f^{m-\ell-1}(t^{\ell+1},x^{\ell+1},v^\ell)|+\int_{t^{\ell+1}}^{t^\ell}|\mathcal{N}^{m-\ell-2}(s,x^\ell-(t^\ell-s)v^\ell,v^\ell)|\mathrm{d}s\Big]\Big\}.\end{aligned}$$

Now we use (111) to conclude Lemma 11. □

Lemma 12. *There exists $\ell_0(\varepsilon)>0$ such that for $\ell\geq\ell_0$ and for all $(t,x,v)\in[0,1]\times\bar{\Omega}\times\mathbb{R}^3$, we have*

$$\int_{\prod_{j=1}^{\ell-1}\mathcal{V}_j}\mathbf{1}_{\{t^\ell(t,x,v,v^1,\cdots,v^{\ell-1})>0\}}\mathrm{d}\Sigma_{\ell-1}^{\ell-1}\quad\lesssim_\Omega\quad\left(\frac{1}{2}\right)^{-\ell/5}.$$

Proof. The proof is based on Lemma 23 of [8]. We note that, for some fixed constant $C_0>0$,

$$\begin{aligned}d\Sigma_{\ell-1}^{\ell-1}\quad&\leq\quad w(v^{\ell-1})e^{\varpi\langle v^{\ell-1}\rangle t^{\ell-1}}\langle v^{\ell-1}\rangle^2c_\mu\mu(v^{\ell-1})\Pi_{j=1}^{\ell-2}e^{\varpi\langle v^j\rangle t^j}\langle v^j\rangle^2c_\mu\mu(v^j)\mathrm{d}v^1\ldots\mathrm{d}v^{\ell-1}\\ &\leq\quad\Pi_{j=1}^{\ell-1}\{C'e^{C't^2}\mu(v^j)^{\frac14}\}\ \mathrm{d}v^1\ldots\mathrm{d}v^{\ell-1}\leq\{C_0\}^\ell\Pi_{j=1}^{\ell-1}\mu(v^j)^{\frac14}\mathrm{d}v^j.\end{aligned}$$

Choose $\delta=\delta(C_0)>0$ small and define

$$\mathcal{V}_j^\delta\equiv\{v^j\in\mathcal{V}_j:v^j\cdot n(x^j)\geq\delta,\ |v^j|\leq\delta^{-1}\},$$

where we have $\int_{\mathcal{V}_j\setminus\mathcal{V}_j^\delta}C_0\mu(v^j)^{\frac14}\lesssim\delta$ for some $C_0>0$. Choose sufficiently small $\delta>0$.

On the other hand if $v^j\in\mathcal{V}_j^\delta$ then by Lemma 6 of [8], $(t^j-t^{j+1})\ \geq\ \delta^3/C_\Omega$. Therefore if $t^\ell\geq0$ then there can be at most $\left\{\left[\frac{C_\Omega}{\delta^3}\right]+1\right\}$ numbers of $v^m\in\mathcal{V}_m^\delta$ for $1\leq m\leq\ell-1$. Equivalently there are at least $\ell - 2 - \left[\frac{C_\Omega}{\delta^3}\right]$ numbers of $v^{m_i} \in \mathcal{V}_{m_i}\backslash\mathcal{V}^\delta_{m_i}$. Hence from $\{C_0\}^\ell = \{C_0\}^m \times \{C_0\}^{\ell-1-m}$, we have

$$\begin{aligned}
&\int_{\prod_{j=1}^{\ell-1}\mathcal{V}_j} 1_{\{t^\ell(t,x,v,v^1,\cdots,v^{\ell-1})>0\}} d\Sigma^{\ell-1}_{\ell-1} \\
\leq\ & \sum_{m=1}^{\left[\frac{C_\Omega}{\delta^3}\right]+1} \int_{\left\{\begin{array}{l}\text{there are exactly } m \text{ of } v_{m_i}\in\mathcal{V}^\delta_{m_i} \\ \text{and } \ \ell-1-m \ of \ v_{m_i}\in\mathcal{V}_{m_i}\backslash\mathcal{V}^\delta_{m_i}\end{array}\right\}} \prod_{j=1}^{\ell-1} C_0\mu(v^j)^{1/4}\mathrm{d}v^j \\
\leq\ & \sum_{m=1}^{\left[\frac{C_\Omega}{\delta^3}\right]+1} \left(\begin{array}{c}\ell-1\\ m\end{array}\right) \left\{\int_{\mathcal{V}} C_0\mu(v)^{1/4}\mathrm{d}v\right\}^m \left\{\int_{\mathcal{V}\backslash\mathcal{V}^\delta} C_0\mu(v)^{1/4}\mathrm{d}v\right\}^{\ell-1-m} \\
\leq\ & \left(\left[\frac{C_\Omega}{\delta^3}\right]+1\right)\{\ell-1\}^{\left[\frac{C_\Omega}{\delta^3}\right]+1}\{\delta\}^{\ell-2-\left[\frac{C_\Omega}{\delta^3}\right]}\left\{\int_{\mathcal{V}} C_0\mu(v)^{1/4}dv\right\}^{\left[\frac{C_\Omega}{\delta^3}\right]+1} \lesssim \frac{\ell}{N}\{Ck\}^{\frac{\ell}{N}}\left(\frac{\ell}{N}\right)^{-\frac{N\ell}{10}} \\
\leq\ & \{CN\}^{\frac{\ell}{N}}\left(\frac{\ell}{N}\right)^{\frac{\ell}{N}}\left(\frac{\ell}{N}\right)^{-\frac{\ell}{N}\frac{N^2}{20}} \leq \left(\frac{\ell}{N}\right)^{\frac{\ell}{N}\left(-\frac{N^2}{20}+3\right)} \leq \left(\frac{1}{\ell/N}\right)^{\frac{-\frac{N^2}{20}+3}{N}\ell} \leq \left(\frac{1}{2}\right)^{-\ell},
\end{aligned}$$

where we have chosen $\ell = N \times \left(\left[\frac{C_\Omega}{\delta^3}\right]+1\right)$ and $N = \left(\left[\frac{C_\Omega}{\delta^3}\right]+1\right) \gg C > 1$. $\square$

Now we are ready to prove the weighted C^1 part of the main theorem :

Proof of weighted C^1 part in Theorem 2. First we show $W^{1,\infty}$ estimate. Recall that we use the same sequences (90) with $\beta = 1$ used for the weighted $W^{1,p}$ estimate ($2 \leq p < \infty$). We estimate along the stochastic cycles with (101) and (110). For $t^1 < 0$, the backward trajectory first hits $t = 0$. From Lemma 11 and Lemma 5 for (92), we deduce

$$\begin{aligned}
&\sup_{0\leq t\leq T_*} ||\mathbf{1}_{\{t_1<0\}}e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f^{m+1}(t)||_\infty \\
&\lesssim ||\alpha^{1/2}\partial f_0||_\infty + P(||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty) + T_*\sup_{0\leq t\leq T_*}||e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f^{m+1}(t)||_\infty \\
&+ \underbrace{\int_{t^1}^t\int_{\mathbb{R}^3} e^{-\varpi\langle v\rangle(t-s)}\mathbf{k}_{\kappa,\theta/4}(v,u)\frac{\alpha(x,v)^{1/2}}{\alpha(x,u)^{1/2}}\mathrm{d}u\mathrm{d}s} \times P(||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty)\max_m\sup_{0\leq t\leq T_*}||e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f^{m+1}(t)||_\infty,
\end{aligned}$$

where we have used (166). Note that, for any $\beta > \frac{1}{2}$,

$$\frac{1}{\alpha(x,u)^{1/2}} \lesssim \frac{1}{\alpha(x,u)^\beta} + 1 \tag{107}$$

We apply (19) to bound the underbrace term as, for $1 \geq \beta > \frac{1}{2}$,

$$\begin{aligned}
&\left\{\mathbf{1}_{|v|\lesssim 1}\frac{\alpha(x,v)^{\frac{1}{2}+\frac{3}{4}-\frac{\beta}{2}}t_Z^{\frac{3}{2}-\beta}}{|v|^{2\beta-1}} + \mathbf{1}_{|v|\gtrsim 1}\frac{\varepsilon^{\frac{3}{2}-\beta}\alpha(x,v)^{\frac{1}{2}}}{|v|^2\alpha(x,v)^{\beta-1}}\right\} + \frac{\alpha(x,v)^{\frac{1}{2}}}{\varpi\langle v\rangle\varepsilon^2\alpha(x,v)^{\beta-\frac{1}{2}}} \\
&\lesssim\ t^{\frac{3}{2}-\beta} + \varepsilon^{\frac{3}{2}-\beta} + \frac{1}{\varepsilon^2\varpi},
\end{aligned} \tag{108}$$

where we used $\alpha(x,v) \lesssim |v|^2$.

$$\begin{aligned}
&\int_0^{t_{\mathbf{b}}(x,v)}\int_{\mathbb{R}^3} e^{-l\langle v\rangle(t-s)}\frac{e^{-\theta|v-u|^2}}{|v-u|^{2-\kappa}[\alpha(x-(t_{\mathbf{b}}(x,v)-s)v,\bar{u})]^\beta}\frac{\langle u\rangle^r}{\langle v\rangle^r}Z(s,x,v)\mathrm{d}u\mathrm{d}s \\
&\lesssim_{\theta,r}\quad \min\left\{\frac{\varepsilon^{\frac{3}{2}-\beta}}{|v|^2\{\alpha(x,v)\}^{\beta-1}}, \frac{\{\alpha(x,v)\}^{\frac{3}{4}-\frac{\beta}{2}}t_Z^{\frac{3}{2}-\beta}}{|v|^{2\beta-1}}\right\}\sup_{s\in[0,t_{\mathbf{b}}(x,v)]}\{e^{-l\langle v\rangle(t-s)}Z(s,x,v)\} \\
&\qquad + \frac{1}{\varepsilon^2\{\alpha(x,v)\}^{\beta-1/2}}\int_0^{t_{\mathbf{b}}(x,v)} e^{-Cl\langle v\rangle(t-s)}Z(s,x,v)\mathrm{d}s.
\end{aligned} \tag{109}$$

where $t_Z = \sup\{s : Z(s,x,v) \neq 0\}$.

$$
\begin{aligned}
&|e^{-\varpi\langle v\rangle t}\alpha(x,v)^{1/2}\partial f^{m+1}(t,x,v)| \\
\lesssim\ & \int_{t^1}^t |\mathcal{N}^m(s, x-(t-s)v, v)|\mathrm{d}s + e^{-\frac{\theta}{4}|v|^2}\mathcal{P}(||\langle v\rangle^\zeta e^{\theta|v|^2} f_0||_\infty) \\
&+ \frac{1}{w(v)}\int_{\prod_{j=1}^{\ell-1}\mathcal{V}_j} \sum_{i=1}^{\ell-1} 1_{\{t^{\ell+1}<0<t^\ell\}}\ |\alpha^{1/2}\partial f^{m+1-i}(0, x^i - t^i v^i, v^i)|\ \mathrm{d}\Sigma_i^{\ell-1} \\
&+ \frac{1}{w(v)}\int_{\prod_{j=1}^{\ell-1}\mathcal{V}_j} \sum_{i=1}^{\ell-1} 1_{\{t^{i+1}<0<t^i\}} \int_0^{t^i} |\mathcal{N}^{m-i}(s, x^i-(t^i-s)v^i, v^i)|\ \mathrm{d}s\ \mathrm{d}\Sigma_i^{\ell-1} \\
&+ \frac{1}{w(v)}\int_{\prod_{j=1}^{\ell-1}\mathcal{V}_j} \sum_{i=1}^{\ell-1} 1_{\{t^{i+1}<0\}} \int_{t^{i+1}}^{t^i} |\mathcal{N}^{m-i}(s, x^i-(t^i-s)v^i, v^i)|\ \mathrm{d}s\ \mathrm{d}\Sigma_i^{\ell-1} \\
&+ \frac{1}{w(v)}\int_{\prod_{j=1}^{\ell-1}\mathcal{V}_j} \sum_{i=2}^{\ell-1} 1_{\{t^{i-1}<0\}} e^{-\frac{\theta}{4}|v^{i-1}|^2}\mathcal{P}(||\langle v\rangle^\zeta e^{\theta|v|^2} f_0||_\infty)\ \mathrm{d}\Sigma_{i-1}^{\ell-1} \\
&+ \frac{1}{w(v)}\int_{\prod_{j=1}^{\ell-1}\mathcal{V}_j} \mathbf{1}_{\{t^\ell>0\}}|e^{-\varpi\langle v^{\ell-1}\rangle t^\ell}\alpha(x^\ell, v^{\ell-1})^{1/2}\partial f^{m+1-\ell}(t^\ell, x^\ell, v^{\ell-1})|\ \mathrm{d}\Sigma_{\ell-1}^{\ell-1},
\end{aligned}
\tag{110}
$$

where $\mathcal{V}_j = \{v^j \in \mathbb{R}^3 : n(x^j)\cdot v^j > 0\}$ and

$$
w(v) = \frac{c_\mu}{\langle v\rangle\sqrt{\mu(v)}},
$$

and

$$
\mathrm{d}\Sigma_i^{\ell-1} = \Big\{\Pi_{j=i+1}^{\ell-1}\mu(v^j)c_\mu|n(x^j)\cdot v^j|\mathrm{d}v^j\Big\}\Big\{w(v^i)e^{\varpi\langle v^i\rangle t^i}\langle v^i\rangle^2 c_\mu\mu(v^i)\mathrm{d}v^i\Big\}\Big\{\Pi_{j=1}^{i-1}e^{\varpi\langle v^j\rangle t^j}\langle v^j\rangle^2 c_\mu\mu(v^j)\mathrm{d}v^j\Big\}.
$$

If $t^1(t,x,v)\geq 0$, the backward trajectory first hits the boundary, then from (110) we have the following line-by-line estimate

$$
\begin{aligned}
&|\mathbf{1}_{\{t_1>0\}}e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f^{m+1}(t,x,v)| \\
\leq\ & \underbrace{\int_{t^1}^t\int_{\mathbb{R}^3} e^{-\varpi\langle v\rangle(t-s)}\mathbf{k}_{\kappa,\frac{\theta}{4}}(V_{\mathbf{cl}}(s),u)\frac{\alpha(X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))^{\frac12}}{\alpha(X_{\mathbf{cl}}(s),u)^{\frac12}}\mathrm{d}u\mathrm{d}s}\,||e^{-\varpi\langle v\rangle s}\alpha^{1/2}\partial f^m(s)||_\infty + P(||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty) \\
&+ \ell(Ce^{Ct^2})^\ell \max_{1\leq i\leq \ell-1}||\alpha^{\frac12}\partial f_0^{m+1-i}||_\infty \\
&+ \ell(Ce^{Ct^2})^\ell\langle v\rangle\sqrt{\mu(v)}\times\underbrace{\max_i\int_0^{t^i}\int_{\mathbb{R}^3} e^{-\varpi\langle v^i\rangle(t-s)}\mathbf{k}_{\kappa,\frac{\theta}{4}}(V_{\mathbf{cl}}(s),u)\frac{\alpha(X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))^{\frac12}}{\alpha(X_{\mathbf{cl}}(s),u)^{\frac12}}\mathrm{d}u\mathrm{d}s} \\
&\times \max_{1\leq i\leq k-1}\sup_{0\leq s\leq t}||e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f^{m+1-i}(t)||_\infty \\
&+ \ell(Ce^{Ct^2})^\ell P(||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty) \\
&+ \left(\frac12\right)^{-\frac{\ell}{5}}\sup_{0\leq s\leq t}||e^{-\varpi\langle v\rangle s}\alpha^{1/2}\partial f^{m+1-\ell}(s)||_\infty,
\end{aligned}
$$

where we have used (90), Lemma 12, and Lemma 5 for (92) and (166). For the underbraced terms we apply (107) and (108). Therefore

$$
\begin{aligned}
&|\mathbf{1}_{\{t_1>0\}}e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f^{m+1}(t,x,v)| \\
\lesssim\ & C_\ell C^{C\ell t^2}\big\{t^{\frac32-\beta}+\varepsilon^{\frac32-\beta}+\frac{1}{\varepsilon^2\varpi}\big\}\times\max_{0\leq i\leq m}\sup_{0\leq s\leq t}||e^{-\varpi\langle v\rangle s}\alpha^{1/2}\partial f^i(s)||_\infty \\
&+ C_\ell C^{C\ell t^2}\max_{0\leq i\leq m}||\alpha^{1/2}\partial f_0^i||_\infty + \left(\frac12\right)^{-\frac{\ell}{5}}\max_{0\leq i\leq m}\sup_{0\leq s\leq t}||e^{-\varpi\langle v\rangle s}\alpha^{1/2}\partial f^i(s)||_\infty.
\end{aligned}
$$

We choose a large ℓ and then small t and then small ε and then finally large ϖ to conclude

$$\sup_{0\le t\le T_*}||e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f^{m+1}(t)||_\infty \le \frac{1}{8}\max_{m-\ell\le i\le m}\sup_{0\le t\le T_*}||e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f^{i}(t)||_\infty + ||\alpha^{1/2}\partial f_0||_\infty + P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}\partial f_0||_\infty).$$

Set $D=||\alpha^{1/2}\partial f_0||_\infty + P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}\partial f_0||_\infty)$,

$$a_i=\sup_{0\le t\le T_*}||e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f^{i}(t)||_\infty,\quad A_i=\max\{a_i,a_{i-1},\cdots,a_{i-(\ell-1)}\},$$

then we have $a_{m+1}\le \frac{1}{8}A_m+D$. Use (87) to conclude

$$\sup_{0\le t\le T_*}||e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f(t)||_\infty \lesssim ||\alpha^{1/2}\partial f_0||_\infty + P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty).$$

The existence and uniqueness and the estimate in Theorem 2 are clear for short time $T_*>0$. We follow the same procedure for $t\in[T_*,2T_*]$ to conclude

$$\sup_{T_*\le t\le 2T_*}||e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f(t)||_\infty \lesssim_{\Omega,T_*} ||e^{-\varpi\langle v\rangle T_*}\partial f(T_*)||_\infty + P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty.$$

Then we conclude the weighted $W^{1,\infty}$ part of Theorem 2 following the same procedure for $[T_*,2T_*],[2T_*,3T_*],\cdots$.

Now we consider the continuity of $e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f$. Remark that for each step $e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f^m$ satisfies the condition of Proposition 2. Therefore we conclude $e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f^m\in C^1([0,T_*]\times\bar\Omega\times\mathbb{R}^3)$. Now we follow $W^{1,\infty}$ estimate part for $e^{-\varpi\langle v\rangle t}\alpha^{1/2}[\partial f^{m+1}-\partial f^m]$ to show that $e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f^m$ is Cauchy in L^∞. Then $e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f^m\to e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f$ strongly in L^∞ so that $e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f\in C^0([0,T_*]\times\bar\Omega\times\mathbb{R}^3)$. □

5. Specular Reflection

We denote the standard spherical coordinate for $\omega\in\mathbb{S}^2$

$$\omega=(\cos\theta(\omega)\sin\phi(\omega),\sin\theta(\omega)\sin\phi(\omega),\cos\phi(\omega)),$$

where $\phi(\omega)\in[0,\pi)$ is the inclination and $\theta(\omega)\in[0,2\pi)$ is the azimuth. We define an orthonormal basis of $\mathbb{R}^3$

$$\begin{aligned}\hat{\mathbf{r}}(\omega)&:=(\cos\theta(\omega)\sin\phi(\omega),\sin\theta(\omega)\sin\phi(\omega),\cos\phi(\omega)),\\ \hat\phi(\omega)&:=(\cos\theta(\omega)\cos\phi(\omega),\sin\theta(\omega)\cos\phi(\omega),-\sin\phi(\omega)),\\ \hat\theta(\omega)&:=(-\sin\theta(\omega),\cos\theta(\omega),0).\end{aligned}$$

Moreover

$$\hat{\mathbf{r}}\times\hat\phi=\hat\theta,\quad \hat\phi\times\hat\theta=\hat{\mathbf{r}},\quad \hat\theta\times\hat{\mathbf{r}}=\hat\phi,\quad\text{and}\quad \partial_\theta\hat{\mathbf{r}}=\sin\phi\,\hat\theta,\quad \partial_\phi\hat{\mathbf{r}}=\hat\phi.$$

Lemma 13. *Assume $\mathbf{0}\in\Omega$ and Ω is convex (2). Fix*

$$\mathbf{p}=(z,w)\in\partial\Omega\times\mathbb{S}^2\quad\textit{with}\quad n(z)\cdot w=0.$$

We define the north pole $\mathcal{N}_{\mathbf{p}}\in\partial\Omega$ and the south pole $\mathcal{S}_{\mathbf{p}}\in\partial\Omega$ as

$$\mathcal{N}_{\mathbf{p}}=|\mathcal{N}_{\mathbf{p}}|(\frac{z}{|z|}\times w)\in\partial\Omega,\quad \mathcal{S}_{\mathbf{p}}=-|\mathcal{S}_{\mathbf{p}}|(\frac{z}{|z|}\times w)\in\partial\Omega,$$

and the straight-line $\mathcal{L}_{\mathbf{p}}$ passing both poles

$$\mathcal{L}_{\mathbf{p}}=\{\tau\mathcal{N}_{\mathbf{p}}+(1-\tau)\mathcal{S}_{\mathbf{p}}:\tau\in\mathbb{R}\}.$$

(1) There exists a smooth map

$$\begin{aligned}\eta_{\mathbf{p}}:[0,2\pi)\times(0,\pi)&\to\partial\Omega\backslash\{\mathcal{N}_{\mathbf{p}},\mathcal{S}_{\mathbf{p}}\},\\ \mathbf{x}_{\parallel_{\mathbf{p}}}:=(\mathbf{x}_{\parallel_{\mathbf{p}},1},\mathbf{x}_{\parallel_{\mathbf{p}},2})&\mapsto\eta_{\mathbf{p}}(\mathbf{x}_{\parallel_{\mathbf{p}}}),\end{aligned}\tag{111}$$

which is one-to-one and onto. Here on $[0,2\pi)\times(0,\pi)$ *we have* $\partial_i\eta_{\mathbf{p}} := \frac{\partial \eta_{\mathbf{p}}}{\partial \mathbf{x}_{\|_{\mathbf{p}},i}} \neq 0$ *and*

$$\frac{\partial \eta_p}{\partial \mathbf{x}_{\|_p,1}}(\mathbf{x}_{\|_p}) \times \frac{\partial \eta_p}{\partial \mathbf{x}_{\|_p,2}}(\mathbf{x}_{\|_p}) \neq 0. \tag{112}$$

We define

$$\mathbf{n}_{\mathbf{p}} := n\circ \eta_{\mathbf{p}} : [0,2\pi)\times(0,\pi)\to \mathbb{S}^2.$$

(2) We define the $\mathbf{p}-$*spherical coordinate:*
For $\delta>0,\ \delta_1>0,\ C>0$, *we have a smooth one-to-one and onto map*

$$\Phi_{\mathbf{p}} : [0,C\delta)\times[0,2\pi)\times(\delta_1,\pi-\delta_1)\times\mathbb{R}\times\mathbb{R}^2 \to \{x\in\bar{\Omega} : |\xi(x)|<\delta\}\backslash B_{C\delta_1}(\mathcal{L}_{\mathbf{p}})\times\mathbb{R}^3,$$

where the $\delta-$*neighborhood* $B_{C\delta_1}(\mathcal{L}_{\mathbf{p}}) := \{x\in\mathbb{R}^3 : |x-y|<C\delta_1 \text{ for some } y\in\mathcal{L}_{\mathbf{p}}\}$. *More precisely*

$$\Phi_{\mathbf{p}}(\mathbf{x}_{\perp_{\mathbf{p}}},\mathbf{x}_{\|_{\mathbf{p}}},\mathbf{v}_{\perp_{\mathbf{p}}},\mathbf{v}_{\|_{\mathbf{p}}}) := \begin{bmatrix} \mathbf{x}_{\perp_{\mathbf{p}}}[-\mathbf{n}_{\mathbf{p}}(\mathbf{x}_{\|_p})] + \eta_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}}) \\ \mathbf{v}_{\perp_{\mathbf{p}}}[-\mathbf{n}_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}})] + \mathbf{v}_{\|_{\mathbf{p}}}\cdot\nabla\eta_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}}) + \mathbf{x}_{\perp_{\mathbf{p}}}\mathbf{v}_{\|_{\mathbf{p}}}\cdot\nabla[-\mathbf{n}_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}})] \end{bmatrix}, \tag{113}$$

where $\nabla\eta_{\mathbf{p}} = (\partial_1\eta_{\mathbf{p}},\partial_2\eta_{\mathbf{p}}) = (\frac{\partial\eta_{\mathbf{p}}}{\partial\mathbf{x}_{\|_{\mathbf{p}},1}},\frac{\partial\eta_{\mathbf{p}}}{\partial\mathbf{x}_{\|_{\mathbf{p}},2}})$ *and* $\nabla\mathbf{n}_{\mathbf{p}} = (\partial_1\mathbf{n}_{\mathbf{p}},\partial_2\mathbf{n}_{\mathbf{p}}) = (\frac{\partial\mathbf{n}_{\mathbf{p}}}{\partial\mathbf{x}_{\|_{\mathbf{p}},1}},\frac{\partial\mathbf{n}_{\mathbf{p}}}{\partial\mathbf{x}_{\|_{\mathbf{p}},2}})$

We fix an inverse map

$$\Phi_{\mathbf{p}}^{-1} : \{x\in\bar{\Omega} : |\xi(x)|<\delta\}\backslash B_{C\delta'}(\mathcal{L}_{\mathbf{p}})\times\mathbb{R}^3 \to [0,C\delta)\times[0,2\pi)\times(\delta_1,\pi-\delta_1)\times\mathbb{R}\times\mathbb{R}^2.$$

In general this choice is not unique but once we fix the range as above then an inverse map is uniquely determined.

Denote

$$(\mathbf{x}_{\perp_{\mathbf{p}}},\mathbf{x}_{\|_{\mathbf{p}},1},\mathbf{x}_{\|_{\mathbf{p}},2},\mathbf{v}_{\perp_{\mathbf{p}}},\mathbf{v}_{\|_{\mathbf{p}},1},\mathbf{v}_{\|_{\mathbf{p}},2}) = \Phi_{\mathbf{p}}^{-1}(x,v).$$

(3) For $|\xi(X_{\mathbf{cl}}(s;t,x,v))|<\delta$ *and* $|X_{\mathbf{cl}}(s;t,x,v)-\mathcal{L}_{\mathbf{p}}|>C\delta_1$ *we define*

$$\begin{aligned}(\mathbf{X}_{\mathbf{p}}(s;t,x,v),\mathbf{V}_{\mathbf{p}}(s;t,x,v)) &:= \Phi_{\mathbf{p}}^{-1}(X_{\mathbf{cl}}(s;t,x,v),V_{\mathbf{cl}}(s;t,x,v))\\ &:= (\mathbf{x}_{\perp_{\mathbf{p}}}(s;t,x,v),\mathbf{x}_{\|_{\mathbf{p}}}(s;t,x,v),\mathbf{v}_{\perp_{\mathbf{p}}}(s;t,x,v),\mathbf{v}_{\|_{\mathbf{p}}}(s;t,x,v)).\end{aligned}$$

Then $|v|\simeq|\mathbf{V}_{\mathbf{p}}|$ *and*

$$\begin{aligned}&\dot{\mathbf{x}}_{\perp_{\mathbf{p}}}(s;t,x,v) = \mathbf{v}_{\perp_{\mathbf{p}}}(s;t,x,v),\\ &\dot{\mathbf{x}}_{\|_{\mathbf{p}}}(s;t,x,v) = \mathbf{v}_{\|_{\mathbf{p}}}(s;t,x,v),\\ &\dot{\mathbf{v}}_{\perp_{\mathbf{p}}}(s;t,x,v) = F_{\perp_{\mathbf{p}}}(\mathbf{x}_{\mathbf{p}}(s;t,x,v),\mathbf{v}_{\mathbf{p}}(s;t,x,v)),\\ &\dot{\mathbf{v}}_{\|_{\mathbf{p}}}(s;t,x,v) = F_{\|_{\mathbf{p}}}(\mathbf{x}_{\mathbf{p}}(s;t,x,v),\mathbf{v}_{\mathbf{p}}(s;t,x,v)).\end{aligned} \tag{114}$$

Here

$$\begin{aligned}F_{\perp_{\mathbf{p}}} &= F_{\perp_{\mathbf{p}}}(\mathbf{x}_{\perp_{\mathbf{p}}},\mathbf{x}_{\|_{\mathbf{p}}},\mathbf{v}_{\|_{\mathbf{p}}})\\ &= \sum_{j,k=1}^{2}\mathbf{v}_{\|_{\mathbf{p}},k}\mathbf{v}_{\|_{\mathbf{p}},j}\ \partial_j\partial_k\eta_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}})\cdot\mathbf{n}_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}}) - \mathbf{x}_{\perp_{\mathbf{p}}}\sum_{k=1}^{2}\mathbf{v}_{\|_{\mathbf{p}},k}(\mathbf{v}_{\|_{\mathbf{p}}}\cdot\nabla)\partial_k\mathbf{n}_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}})\cdot\mathbf{n}_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}}),\end{aligned} \tag{115}$$

where

$$\sum_{j,k=1}^{2}\mathbf{v}_{\|_{\mathbf{p}},k}\mathbf{v}_{\|_{\mathbf{p}},j}\ \partial_j\partial_k\eta_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}})\cdot\mathbf{n}_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}}) \lesssim_\xi -|\mathbf{v}_\||^2,$$

and

$$\begin{aligned}F_{\|_{\mathbf{p}}} &= F_{\|_{\mathbf{p}}}(\mathbf{x}_{\perp_{\mathbf{p}}},\mathbf{x}_{\|_{\mathbf{p}}},\mathbf{v}_{\perp_{\mathbf{p}}},\mathbf{v}_{\|_{\mathbf{p}}})\\ &= \sum_i G_{\mathbf{p},ij}(\mathbf{x}_{\perp_{\mathbf{p}}},\mathbf{x}_{\|_{\mathbf{p}}})\frac{(-1)^i}{\mathbf{n}_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}})\cdot(\partial_1\eta_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}})\times\partial_2\eta_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}}))}\\ &\quad\times\big\{2\mathbf{v}_{\perp_{\mathbf{p}}}\mathbf{v}_{\|_{\mathbf{p}}}\cdot\nabla\mathbf{n}_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}}) - \mathbf{v}_{\|_{\mathbf{p}}}\cdot\nabla^2\eta_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}})\cdot\mathbf{v}_{\|_{\mathbf{p}}} + \mathbf{x}_{\perp_{\mathbf{p}}}\mathbf{v}_{\|_{\mathbf{p}}}\cdot\nabla^2\mathbf{n}_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}})\cdot\mathbf{v}_{\|_{\mathbf{p}}}\big\}\cdot\big\{\mathbf{n}_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}})\times\partial_{i+1}\eta_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}}})\big\},\end{aligned} \tag{116}$$

where a smooth bounded function $G_{\mathbf{p},ij}(\mathbf{x}_{\perp_{\mathbf{p}}},\mathbf{x}_{\|_{\mathbf{p}}})$ *is specified in (122).*

(4) Let $\mathbf{q} = (y,u) \in \partial\Omega \times \mathbb{S}^2$ *with* $n(y)\cdot u = 0$ *and* $\mathbf{p} \sim \mathbf{q}$. *Let* $x \sim y \sim z$ *and*

$$\Phi_{\mathbf{p}}(\mathbf{x}_{\perp_{\mathbf{p}}}, \mathbf{x}_{\|_{\mathbf{p}}}, \mathbf{v}_{\perp_{\mathbf{p}}}, \mathbf{v}_{\|_{\mathbf{p}}}) = (x,v) = \Phi_{\mathbf{q}}(\mathbf{x}_{\perp_{\mathbf{q}}}, \mathbf{x}_{\|_{\mathbf{q}}}, \mathbf{v}_{\perp_{\mathbf{q}}}, \mathbf{v}_{\|_{\mathbf{q}}}).$$

Then

$$\frac{\partial(\mathbf{x}_{\perp_{\mathbf{p}}}, \mathbf{x}_{\|_{\mathbf{p}}}, \mathbf{v}_{\perp_{\mathbf{p}}}, \mathbf{v}_{\|_{\mathbf{p}}})}{\partial(\mathbf{x}_{\perp_{\mathbf{q}}}, \mathbf{x}_{\|_{\mathbf{q}}}, \mathbf{v}_{\perp_{\mathbf{q}}}, \mathbf{v}_{\|_{\mathbf{q}}})} = \nabla\Phi_{\mathbf{q}}^{-1}\nabla\Phi_{\mathbf{p}} = \mathbf{Id}_{6,6} + O_\xi(|\mathbf{p}-\mathbf{q}|)\left[\begin{array}{ccc|ccc} 0 & 0 & 0 & & & \\ 0 & 1 & 1 & & \mathbf{0}_{3,3} & \\ 0 & 1 & 1 & & & \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & |v| & |v| & 0 & 1 & 1 \\ 0 & |v| & |v| & 0 & 1 & 1 \end{array}\right]. \quad (117)$$

Proof. Proof of (1) Denote

$$\begin{aligned} \tfrac{z}{|z|} &= \hat{\mathbf{r}}(\tfrac{z}{|z|}) := (\cos\theta(\tfrac{z}{|z|})\sin\phi(\tfrac{z}{|z|}), \sin\theta(\tfrac{z}{|z|})\sin\phi(\tfrac{z}{|z|}), \cos\phi(\tfrac{z}{|z|})), \\ w &= w_\phi\hat{\phi}(\tfrac{z}{|z|}) + w_\theta\hat{\theta}(\tfrac{z}{|z|}) := (w\cdot\hat{\phi}(\tfrac{z}{|z|}))\hat{\phi}(\tfrac{z}{|z|}) + (w\cdot\hat{\theta}(\tfrac{z}{|z|}))\hat{\theta}(\tfrac{z}{|z|}), \\ \tfrac{z}{|z|}\times w &= w_\phi\hat{\theta}(\tfrac{z}{|z|}) - w_\theta\hat{\phi}(\tfrac{z}{|z|}). \end{aligned}$$

We define the rotational matrix which maps $\{\frac{z}{|z|}, w, \frac{z}{|z|}\times w\} \mapsto \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$:

$$\mathcal{O}_p = \begin{bmatrix} \frac{z}{|z|} \\ w \\ \frac{z}{|z|}\times w \end{bmatrix}_{3\times 3} = \begin{bmatrix} \hat{\mathbf{r}}(\frac{z}{|z|}) \\ w_\phi\hat{\phi}(\frac{z}{|z|}) + w_\theta\hat{\theta}(\frac{z}{|z|}) \\ -w_\theta\hat{\phi}(\frac{z}{|z|}) + w_\phi\hat{\theta}(\frac{z}{|z|}) \end{bmatrix}_{3\times 3}.$$

For $x \in \partial\Omega$ with $x \neq \mathcal{N}_{\mathbf{p}}$ and $x \neq \mathcal{S}_{\mathbf{p}}$ we define

$$(\mathbf{x}_{\|_{\mathbf{p}},1}, \mathbf{x}_{\|_{\mathbf{p}},2}) \in [0,2\pi)\times(0,\pi), \quad \text{such that} \quad \hat{\mathbf{r}}(\mathbf{x}_{\|_{\mathbf{p}},1}, \mathbf{x}_{\|_{\mathbf{p}},2}) = \mathcal{O}_{\mathbf{p}}\Big(\frac{x}{|x|}\Big).$$

Now we define $R_{\mathbf{p}} : [0,2\pi)\times[0,\pi) \to (0,\infty)$ such that

$$\xi(R_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}},1}, \mathbf{x}_{\|_{\mathbf{p}},2})\mathcal{O}_{\mathbf{p}}^{-1}\hat{\mathbf{r}}(\mathbf{x}_{\|_{\mathbf{p}},1}, \mathbf{x}_{\|_{\mathbf{p}},2})) = 0.$$

We also define $\eta_{\mathbf{p}} : [0,2\pi)\times[0,\pi) \to \partial\Omega$ such that

$$\eta_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}},1}, \mathbf{x}_{\|_{\mathbf{p}},2}) = R_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}},1}, \mathbf{x}_{\|_{\mathbf{p}},2})\mathcal{O}_{\mathbf{p}}^{-1}\hat{\mathbf{r}}(\mathbf{x}_{\|_{\mathbf{p}},1}, \mathbf{x}_{\|_{\mathbf{p}},2}).$$

Directly, with fixed $\mathbf{p} = (z,w)$,

$$\begin{aligned} \frac{\partial R_{\mathbf{p}}}{\partial \mathbf{x}_{\|_{\mathbf{p}},1}}(\mathbf{x}_{\|_{\mathbf{p}},1}, \mathbf{x}_{\|_{\mathbf{p}},2}) &= \frac{-\sin(\mathbf{x}_{\|_{\mathbf{p}},2})R_{\mathbf{p}}\nabla\xi(\eta_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}},1}, \mathbf{x}_{\|_{\mathbf{p}},2}))\cdot\mathcal{O}_{\mathbf{p}}^{-1}\hat{\theta}(\mathbf{x}_{\|_{\mathbf{p}},1}, \mathbf{x}_{\|_{\mathbf{p}},2})}{\nabla\xi(\eta_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}},1}, \mathbf{x}_{\|_{\mathbf{p}},2}))\cdot\mathcal{O}_{\mathbf{p}}^{-1}\hat{\mathbf{r}}(\mathbf{x}_{\|_{\mathbf{p}},1}, \mathbf{x}_{\|_{\mathbf{p}},2})} \\ &= \frac{-\sin(\mathbf{x}_{\|_{\mathbf{p}},2})[R_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}},1}, \mathbf{x}_{\|_{\mathbf{p}},2})]^2\nabla\xi(\eta_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}},1}, \mathbf{x}_{\|_{\mathbf{p}},2}))\cdot\mathcal{O}_{\mathbf{p}}^{-1}\hat{\theta}(\mathbf{x}_{\|_{\mathbf{p}},1}, \mathbf{x}_{\|_{\mathbf{p}},2})}{\nabla\xi(\eta_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}},1}, \mathbf{x}_{\|_{\mathbf{p}},2}))\cdot\eta_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}},1}, \mathbf{x}_{\|_{\mathbf{p}},2})}, \\ \frac{\partial R_{\mathbf{p}}}{\partial \mathbf{x}_{\|_{\mathbf{p}},2}}(\mathbf{x}_{\|_{\mathbf{p}},1}, \mathbf{x}_{\|_{\mathbf{p}},2}) &= \frac{-R_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}},1}, \mathbf{x}_{\|_{\mathbf{p}},2})\nabla\xi(\eta_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}},1}, \mathbf{x}_{\|_{\mathbf{p}},2}))\cdot\mathcal{O}_{\mathbf{p}}^{-1}\hat{\phi}(\mathbf{x}_{\|_{\mathbf{p}},1}, \mathbf{x}_{\|_{\mathbf{p}},2})}{\nabla\xi(\eta_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}},1}, \mathbf{x}_{\|_{\mathbf{p}},2}))\cdot\mathcal{O}_{\mathbf{p}}^{-1}\hat{\mathbf{r}}(\mathbf{x}_{\|_{\mathbf{p}},1}, \mathbf{x}_{\|_{\mathbf{p}},2})} \\ &= \frac{-[R_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}},1}, \mathbf{x}_{\|_{\mathbf{p}},2})]^2\nabla\xi(\eta_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}},1}, \mathbf{x}_{\|_{\mathbf{p}},2}))\cdot\mathcal{O}_{\mathbf{p}}^{-1}\hat{\phi}(\mathbf{x}_{\|_{\mathbf{p}},1}, \mathbf{x}_{\|_{\mathbf{p}},2})}{\nabla\xi(\eta_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}},1}, \mathbf{x}_{\|_{\mathbf{p}},2}))\cdot\eta_{\mathbf{p}}(\mathbf{x}_{\|_{\mathbf{p}},1}, \mathbf{x}_{\|_{\mathbf{p}},2})}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial\eta_{\mathbf{p}}}{\partial \mathbf{x}_{\|_{\mathbf{p}},1}}(\mathbf{x}_{\|_{\mathbf{p}},1}, \mathbf{x}_{\|_{\mathbf{p}},2}) &= \frac{\partial R_{\mathbf{p}}}{\partial \mathbf{x}_{\|_{\mathbf{p}},1}}\mathcal{O}_{\mathbf{p}}^{-1}\hat{\mathbf{r}} + \sin(\mathbf{x}_{\|_{\mathbf{p}},2})R_p\mathcal{O}_{\mathbf{p}}^{-1}\hat{\theta}, \\ \frac{\partial\eta_{\mathbf{p}}}{\partial \mathbf{x}_{\|_{\mathbf{p}},2}}(\mathbf{x}_{\|_{\mathbf{p}},1}, \mathbf{x}_{\|_{\mathbf{p}},2}) &= \frac{\partial R_{\mathbf{p}}}{\partial \mathbf{x}_{\|_{\mathbf{p}},2}}\mathcal{O}_{\mathbf{p}}^{-1}\hat{\mathbf{r}} + R_{\mathbf{p}}\mathcal{O}_{\mathbf{p}}^{-1}\hat{\phi}. \end{aligned}$$

Directly we check a non-degenerate condition (112)

$$\frac{\partial\eta_p}{\partial \mathbf{x}_{\|_p,1}}(\mathbf{x}_{\|_p}) \times \frac{\partial\eta_p}{\partial \mathbf{x}_{\|_p,2}}(\mathbf{x}_{\|_p}) = R_p\frac{\partial R_p}{\partial \mathbf{x}_{\|_p,1}}\mathcal{O}_p^{-1}\hat{\theta} + \sin(\mathbf{x}_{\|_p,2})R_p\frac{\partial R_p}{\partial \mathbf{x}_{\|_p,2}}\mathcal{O}_p^{-1}\hat{\phi} - \sin(\mathbf{x}_{\|_p,2})R_p^2\mathcal{O}_p^{-1}\hat{\mathbf{r}} \neq 0.$$

Proof of (2) of Lemma 13. We fix $\mathbf{p} = (z, w)$ and drop $\mathbf{p}-$index in this step. Define

$$\Phi_1 : [0,\infty) \times [0, 2\pi) \times (0, \pi) \to \bar{\Omega}\backslash \mathcal{L}_{\mathbf{p}}, \quad \Phi_1(\mathbf{x}_\perp, \mathbf{x}_\parallel) = \mathbf{x}_\perp[-\mathbf{n}(\mathbf{x}_\parallel)] + \eta(\mathbf{x}_\parallel).$$

Note that this mapping is surjective: for any $x \in \bar{\Omega}$ there exists $\mathbf{x}_\parallel^* = \min_{\mathbf{y}_\parallel \in \partial\Omega} |x - \eta(\mathbf{y}_\parallel)|^2 (\partial\Omega$ is compact) and therefore $(x - \eta(\mathbf{x}_\parallel^*)) \cdot \frac{\partial \eta}{\partial \mathbf{x}_{\parallel,i}}(\mathbf{x}_\parallel^*) = 0$ for $i = 1, 2$. Since $\nabla \eta(\mathbf{x}_\parallel) \neq 0$ from (112) and $\xi(\eta(\mathbf{x}_\parallel)) = 0$, we have $0 \equiv \nabla\xi(\eta(\mathbf{x}_\parallel)) \cdot \frac{\partial \eta}{\partial \mathbf{x}_{\parallel,i}}(\mathbf{x}_\parallel)$. Due to (112), we conclude $\frac{x - \eta(\mathbf{x}_\parallel^*)}{|x - \eta(\mathbf{x}_\parallel^*)|} = [-\mathbf{n}(\mathbf{x}_\parallel^*)]$ and $x = \eta(\mathbf{x}_\parallel^*) + (x - \eta(\mathbf{x}_\parallel^*)) = \eta(\mathbf{x}_\parallel^*) + |x - \eta(\mathbf{x}_\parallel^*)| [-\mathbf{n}(\mathbf{x}_\parallel^*)]$.

Since η and ξ(therefore n and $\mathbf{n}$) are smooth, the Φ_1 is smooth. The Jacobian matrix is

$$\frac{\partial \Phi_1(\mathbf{x}_\perp, \mathbf{x}_\parallel)}{\partial(\mathbf{x}_\perp, \mathbf{x}_\parallel)} = \begin{bmatrix} -\mathbf{n}(\mathbf{x}_\parallel) & \begin{matrix} \frac{\partial \eta}{\partial \mathbf{x}_{\parallel,1}}(\mathbf{x}_\parallel) \\ +\mathbf{x}_\perp \frac{-\partial \mathbf{n}}{\partial \mathbf{x}_{\parallel,1}}(\mathbf{x}_\parallel) \end{matrix} & \begin{matrix} \frac{\partial \eta}{\partial \mathbf{x}_{\parallel,2}}(\mathbf{x}_\parallel) \\ +\mathbf{x}_\perp \frac{-\partial \mathbf{n}}{\partial \mathbf{x}_{\parallel,2}}(\mathbf{x}_\parallel) \end{matrix} \end{bmatrix}_{3\times 3}, \tag{118}$$

where $\Big[-\mathbf{n}(\mathbf{x}_\parallel) \Big], \begin{bmatrix} \frac{\partial \eta}{\partial \mathbf{x}_{\parallel,1}}(\mathbf{x}_\parallel) \\ +\mathbf{x}_\perp \frac{-\partial \mathbf{n}}{\partial \mathbf{x}_{\parallel,1}}(\mathbf{x}_\parallel) \end{bmatrix}, \begin{bmatrix} \frac{\partial \eta}{\partial \mathbf{x}_{\parallel,2}}(\mathbf{x}_\parallel) \\ +\mathbf{x}_\perp \frac{-\partial \mathbf{n}}{\partial \mathbf{x}_{\parallel,2}}(\mathbf{x}_\parallel) \end{bmatrix}$ are column vectors in $\mathbb{R}^3$. By the basic linear algebra, the Jacobian(a determinant of the Jacobian matrix) equals

$$-\mathbf{n} \cdot (\partial_1 \eta \times \partial_2 \eta) + \mathbf{x}_\perp \mathbf{n} \cdot (\partial_1 \mathbf{n} \times \partial_2 \eta) + \mathbf{x}_\perp \mathbf{n} \cdot (\partial_2 \eta \times \partial_2 \mathbf{n}) - |\mathbf{x}_\perp|^2 \mathbf{n} \cdot (\partial_1 \mathbf{n} \times \partial_2 \mathbf{n}).$$

We use the facts $\nabla \eta(\mathbf{x}_\parallel) \neq 0$ and $\xi(\eta(\mathbf{x}_\parallel)) = 0$ and

$$0 \equiv \nabla \xi(\eta(\mathbf{x}_\parallel)) \cdot \frac{\partial \eta}{\partial \mathbf{x}_{\parallel,i}}(\mathbf{x}_\parallel) = |\nabla \xi(\eta(\mathbf{x}_\parallel))| \left(\mathbf{n}(\mathbf{x}_\parallel) \cdot \frac{\partial \eta}{\partial \mathbf{x}_{\parallel,i}}(\mathbf{x}_\parallel) \right),$$

and therefore

$$-\mathbf{n}(\mathbf{x}_\parallel) \cdot \left(\frac{\partial \eta}{\partial \mathbf{x}_{\parallel,1}}(\mathbf{x}_\parallel) \times \frac{\partial \eta}{\partial \mathbf{x}_{\parallel,2}}(\mathbf{x}_\parallel) \right) \neq 0, \quad \text{for all } \mathbf{x}_\parallel \in [0, 2\pi) \times (0, \pi),$$

to conclude that there exists small $\delta > 0$ such that if $|\mathbf{x}_\perp| \leq \delta$ then

$$\det \left(\frac{\partial \Phi_1(\mathbf{x}_\perp, \mathbf{x}_\parallel)}{\partial(\mathbf{x}_\perp, \mathbf{x}_\parallel)} \right) = -\mathbf{n}(\mathbf{x}_\parallel) \cdot \left(\frac{\partial \eta}{\partial \mathbf{x}_{\parallel,1}}(\mathbf{x}_\parallel) \times \frac{\partial \eta}{\partial \mathbf{x}_{\parallel,2}}(\mathbf{x}_\parallel) \right) + O_\xi(|\mathbf{x}_\perp|) \neq 0.$$

We use the inverse function theorem and we choose an inverse map

$$\Phi_1^{-1} : \Phi_1([0, \delta) \times [0, 2\pi) \times (0, \pi)) \to [0, \delta) \times [0, 2\pi) \times (0, \pi).$$

Note that in general there are infinitely many inverse maps.

If $x \in \Phi_1([0, \delta) \times [0, 2\pi) \times (0, \pi))$ then

$$\Phi^{-1}(x) := (\mathbf{x}_\perp, \mathbf{x}_\parallel) \quad \text{and} \quad x = \eta(\mathbf{x}_\parallel) + \mathbf{x}_\perp[-\mathbf{n}(\mathbf{x}_\parallel)].$$

Since Φ_1 is surjective onto $\bar{\Omega}\backslash\mathcal{L}_{\mathbf{p}}$, for $x \in \bar{\Omega}\backslash\mathcal{L}_{\mathbf{p}}$ and $\mathbf{x}_\perp \geq 0$,

$$\begin{aligned} \xi(x) &= \xi(\eta(\mathbf{x}_\parallel) + \mathbf{x}_\perp[-\mathbf{n}(\mathbf{x}_\parallel)]) \\ &= \xi(\eta(\mathbf{x}_\parallel)) + \int_0^{\mathbf{x}_\perp} \frac{d}{ds} \xi(\eta(\mathbf{x}_\parallel) + s[-\mathbf{n}(\mathbf{x}_\parallel)]) \mathrm{d}s \\ &= \int_0^{\mathbf{x}_\perp} [-\mathbf{n}(\mathbf{x}_\parallel)] \cdot \nabla \xi(\eta(\mathbf{x}_\parallel) + s[-\mathbf{n}(\mathbf{x}_\parallel)]) \mathrm{d}s \\ &= \int_0^{\mathbf{x}_\perp} \Big\{ [-\mathbf{n}(\mathbf{x}_\parallel)] \cdot \nabla \xi(\eta(\mathbf{x}_\parallel)) + \int_0^s \mathbf{n}(\mathbf{x}_\parallel) \cdot \nabla^2 \xi(\eta(\mathbf{x}_\parallel) + \tau[-\mathbf{n}(\mathbf{x}_\parallel)]) \cdot \mathbf{n}(\mathbf{x}_\parallel) \mathrm{d}\tau \Big\} \mathrm{d}s, \end{aligned}$$

and by the convexity of ξ we have the following equivalent relation:

For all $x \in \bar{\Omega}$ there exists(not uniquely) $(\mathbf{x}_\perp, \mathbf{x}_\parallel) \in [0, \infty) \times [0, 2\pi) \times (0, \pi)$ satisfying $x = \mathbf{x}_\perp[-\mathbf{n}(\mathbf{x}_\parallel)] + \eta(\mathbf{x}_\parallel)$. Then for all $(\mathbf{x}_\perp, \mathbf{x}_\parallel)$ with $x = \mathbf{x}_\perp[-\mathbf{n}(\mathbf{x}_\parallel)] + \eta(\mathbf{x}_\parallel)$ we have

$$\begin{aligned} |\nabla \xi(\eta(\mathbf{x}_\parallel))||\mathbf{x}_\perp| + C_\xi \frac{|\mathbf{x}_\perp|^2}{2} &\leq |\xi(x)| = |\xi(\eta(\mathbf{x}_\parallel) + \mathbf{x}_\perp[-\mathbf{n}(\mathbf{x}_\parallel)])| \\ &\leq |\nabla \xi(\eta(\mathbf{x}_\parallel))||\mathbf{x}_\perp| + \sup_{x \in \bar{\Omega}} |\nabla^2 \xi(x)| \frac{|\mathbf{x}_\perp|^2}{2}. \end{aligned} \tag{119}$$

Therefore there exists $0 < C_1 \ll 1$ such that if $|\xi(x)| \leq C_1\delta$ then $|\mathbf{x}_\perp| < \delta$ and hence there exists unique $(\mathbf{x}_\perp, \mathbf{x}_\|)$ and all the above computations hold.

Next we define

$$\Phi(\mathbf{x}_\perp, \mathbf{x}_\|, \mathbf{v}_\perp, \mathbf{v}_\|) = \begin{pmatrix} \mathbf{x}_\perp[-\mathbf{n}(\mathbf{x}_\|)] + \eta(\mathbf{x}_\|) \\ \mathbf{v}_\perp[-\mathbf{n}(\mathbf{x}_\|)] + \mathbf{v}_\| \cdot \nabla_{\mathbf{x}_\|}\eta(\mathbf{x}_\|) - \mathbf{x}_\perp \mathbf{v}_\| \cdot \nabla_{\mathbf{x}_\|}\mathbf{n}(\mathbf{x}_\|) \end{pmatrix}.$$

The Jacobian matrix is

$$\frac{\partial \Phi(\mathbf{x}_\perp, \mathbf{x}_\|, \mathbf{v}_\perp, \mathbf{v}_\|)}{\partial(\mathbf{x}_\perp, \mathbf{x}_\|, \mathbf{v}_\perp, \mathbf{v}_\|)} = \left[\begin{array}{ccc|c} & \frac{\partial \Phi_1(\mathbf{x}_\perp,\mathbf{x}_\|)}{\partial(\mathbf{x}_\perp,\mathbf{x}_\|)} & & \mathbf{0}_{3,3} \\ \hline -\mathbf{v}_\| \cdot \nabla_{\mathbf{x}_\|}\mathbf{n}(\mathbf{x}_\|) & \begin{matrix} -\mathbf{v}_\perp \frac{\partial \mathbf{n}}{\partial \mathbf{x}_{\|,1}}(\mathbf{x}_\|) \\ +\mathbf{v}_\| \cdot \nabla_{\mathbf{x}_\|} \frac{\partial \eta}{\partial \mathbf{x}_{\|,1}}(\mathbf{x}_\|) \\ -\mathbf{x}_\perp \mathbf{v}_\| \cdot \nabla_{\mathbf{x}_\|} \frac{\partial \mathbf{n}}{\partial \mathbf{x}_{\|,1}}(\mathbf{x}_\|) \end{matrix} & \begin{matrix} -\mathbf{v}_\perp \frac{\partial \mathbf{n}}{\partial \mathbf{x}_{\|,2}}(\mathbf{x}_\|) \\ +\mathbf{v}_\| \cdot \nabla_{\mathbf{x}_\|} \frac{\partial \eta}{\partial \mathbf{x}_{\|,2}}(\mathbf{x}_\|) \\ -\mathbf{x}_\perp \mathbf{v}_\| \cdot \nabla_{\mathbf{x}_\|} \frac{\partial \mathbf{n}}{\partial \mathbf{x}_{\|,2}}(\mathbf{x}_\|) \end{matrix} & \frac{\partial \Phi_1(\mathbf{x}_\perp,\mathbf{x}_\|)}{\partial(\mathbf{x}_\perp,\mathbf{x}_\|)} \end{array}\right]. \tag{120}$$

The Jacobian(a determinant of the Jacobian matrix) equals

$$\det\left(\frac{\partial \Phi(\mathbf{x}_\perp, \mathbf{x}_\|, \mathbf{v}_\perp, \mathbf{v}_\|)}{\partial(\mathbf{x}_\perp, \mathbf{x}_\|, \mathbf{v}_\perp, \mathbf{v}_\|)}\right) = \left(\det\left(\frac{\partial \Phi_1(\mathbf{x}_\perp, \mathbf{x}_\|)}{\partial(\mathbf{x}_\perp, \mathbf{x}_\|)}\right)\right)^2 \neq 0,$$

for $|\xi(x)| \leq \delta$(and therefore $|\mathbf{x}_\perp| \leq C\delta$). By the inverse function theorem we have the inverse mapping Φ^{-1}.

Proof of (3) of Lemma 13. From $\dot{v} = 0$ and the second equation of (113) equals

$$\begin{aligned} 0 =& \dot{\mathbf{v}}_\perp(s)[-\mathbf{n}(\mathbf{x}_\|(s))] - 2\mathbf{v}_\perp(s)\mathbf{v}_\| \cdot \nabla\mathbf{n}(\mathbf{x}_\|(s)) + \dot{\mathbf{v}}_\|(s) \cdot \nabla\eta(\mathbf{x}_\|(s)) \\ &+ \mathbf{v}_\| \cdot \nabla^2\eta(\mathbf{x}_\|) \cdot \mathbf{v}_\| - \mathbf{x}_\perp \dot{\mathbf{v}}_\| \cdot \nabla\mathbf{n}(\mathbf{x}_\|) + \mathbf{x}_\perp \mathbf{v}_\| \cdot \nabla^2\mathbf{n}(\mathbf{x}_\|) \cdot \mathbf{v}_\|. \end{aligned} \tag{121}$$

We take the inner product with $\mathbf{n}(\mathbf{x}_\|(s))$ to the above equation to have

$$\dot{\mathbf{v}}_\perp(s) = [\mathbf{v}_\| \cdot \nabla^2\eta(\mathbf{x}_\|) \cdot \mathbf{v}_\|] \cdot \mathbf{n}(\mathbf{x}_\|) + \mathbf{x}_\perp[\mathbf{v}_\| \cdot \nabla^2\mathbf{n}(\mathbf{x}_\|) \cdot \mathbf{v}_\|] \cdot \mathbf{n}(\mathbf{x}_\|) := F_\perp(\mathbf{v}_\perp, \mathbf{v}_\|, \mathbf{x}_\|),$$

where we have used the fact $\nabla\mathbf{n} \perp \mathbf{n}$ and $\nabla\eta \perp \mathbf{n}$.

Since $0 = \xi(\eta(\mathbf{x}_\|))$ we take $\mathbf{x}_{\|,i}$ and $\mathbf{x}_{\|,j}$ to have

$$0 = \partial_{\mathbf{x}_{\|,j}}\big[\sum_k \partial_k\xi \partial_{\mathbf{x}_{\|,i}}\eta_k\big] = \sum_{k,m} \partial_k\partial_m\xi \partial_{\mathbf{x}_{\|,j}}\eta_m \partial_{\mathbf{x}_{\|,i}}\eta_k + \sum_k \partial_k\xi \partial_{\mathbf{x}_{\|,i}}\partial_{\mathbf{x}_{\|,j}}\eta_k,$$

we have from (2)

$$\big[\mathbf{v}_\| \cdot \nabla^2\eta \cdot \mathbf{v}_\|\big] \cdot \mathbf{n} = \sum_{i,j,k} \frac{\mathbf{v}_{\|,i}\partial_k\xi\partial_i\partial_j\eta_k\mathbf{v}_{\|,j}}{|\nabla\xi|} = -\sum_{i,j,k,m} \frac{\{\mathbf{v}_{\|,i}\partial_i\eta_m\}\partial_k\partial_m\xi\{\partial_j\eta_m\mathbf{v}_{\|,j}\}}{|\nabla\xi|} \lesssim_\xi -|\mathbf{v}_\||^2.$$

Define $a_{ij}(\mathbf{x}_\|)$ via

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \partial_1\mathbf{n} \cdot \partial_1\mathbf{n} & \partial_1\mathbf{n} \cdot \partial_2\mathbf{n} \\ \partial_2\mathbf{n} \cdot \partial_1\mathbf{n} & \partial_2\mathbf{n} \cdot \partial_2\mathbf{n} \end{bmatrix} \begin{bmatrix} \partial_1\eta \cdot \partial_1\eta & \partial_1\eta \cdot \partial_2\eta \\ \partial_2\eta \cdot \partial_1\eta & \partial_2\eta \cdot \partial_2\eta \end{bmatrix}^{-1},$$

where $\det(\partial_i\eta \cdot \partial_j\eta) = |\partial_1\eta \times \partial_2\eta|^2 \neq 0$ due to (112). Then $\nabla\mathbf{n}$ is generated by $\nabla\eta$:

$$-\partial_i\mathbf{n}(\mathbf{x}_\|) = \sum_k a_{ik}(\mathbf{x}_\|)\partial_k\eta(\mathbf{x}_\|).$$

We take the inner product (121) with $(-1)^{i+1}(\mathbf{n}(\mathbf{x}_\|) \times \partial_i\mathbf{n}(\mathbf{x}_\|))$ to have

$$\begin{aligned} &\sum_k (\delta_{ki} + \mathbf{x}_\perp a_{ki})\dot{\mathbf{v}}_{\|,k} \\ &= \frac{(-1)^{i+1}}{-\mathbf{n}(\mathbf{x}_\|) \cdot (\partial_1\eta(\mathbf{x}_\|) \times \partial_2\eta(\mathbf{x}_\|))} \\ &\quad \times \Big\{ - 2\mathbf{v}_\perp\mathbf{v}_\| \cdot \nabla\mathbf{n}(\mathbf{x}_\|) + \mathbf{v}_\| \cdot \nabla^2\eta(\mathbf{x}_\|) \cdot \mathbf{v}_\| - \mathbf{x}_\perp\mathbf{v}_\| \cdot \nabla^2\mathbf{n}(\mathbf{x}_\|) \cdot \mathbf{v}_\| \Big\} \cdot (-\mathbf{n}(\mathbf{x}_\|) \times \partial_{i+1}\eta(\mathbf{x}_\|)), \end{aligned}$$

where we used the notational convention for $\partial_{i+1}\eta$, the index $i+1$ mod 2 . For $|\xi(x)| \ll 1$(and therefore $|\mathbf{x}_\perp| \ll 1$) the matrix $\delta_{ki} + \mathbf{x}_\perp a_{ki}$ is invertible: there exists the inverse matrix G_{ij} such that $\sum_i (\delta_{ki} + \mathbf{x}_\perp a_{ki}(\mathbf{x}_\parallel))G_{ij}(\mathbf{x}_\perp, \mathbf{x}_\parallel) = \delta_{kj}$. Therefore

$$\begin{aligned}\dot{\mathbf{v}}_{\parallel,j} &= \sum_i G_{ij}(\mathbf{x}_\perp, \mathbf{x}_\parallel) \frac{(-1)^{i+1}}{-\mathbf{n}(\mathbf{x}_\parallel)\cdot(\partial_1\eta(\mathbf{x}_\parallel)\times\partial_2\eta(\mathbf{x}_\parallel))} \\ &\quad\times\Big\{ -2\mathbf{v}_\perp\mathbf{v}_\parallel\cdot\nabla\mathbf{n}(\mathbf{x}_\parallel) + \mathbf{v}_\parallel\cdot\nabla^2\eta(\mathbf{x}_\parallel)\cdot\mathbf{v}_\parallel - \mathbf{x}_\perp\mathbf{v}_\parallel\cdot\nabla^2\mathbf{n}(\mathbf{x}_\parallel)\cdot\mathbf{v}_\parallel\Big\}\cdot(-\mathbf{n}(\mathbf{x}_\parallel)\times\partial_{i+1}\eta(\mathbf{x}_\parallel)) \\ &:= F_{\parallel,j}(\mathbf{x}_\perp, \mathbf{x}_\parallel, \mathbf{v}_\perp, \mathbf{v}_\parallel).\end{aligned}$$

Here

$$\begin{aligned}\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22}\end{bmatrix} &= \frac{1}{1+\mathbf{x}_\perp(a_{11}+a_{22}) + (\mathbf{x}_\perp)^2(a_{11}a_{22}-a_{12}a_{21})}\begin{bmatrix} 1+\mathbf{x}_\perp a_{22} & -\mathbf{x}_\perp a_{12} \\ -\mathbf{x}_\perp a_{21} & 1+\mathbf{x}_\perp a_{11}\end{bmatrix}, \\ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix} &= \frac{1}{|\partial_1\eta|^2|\partial_2\eta|^2 - (\partial_1\eta\cdot\partial_2\eta)^2} \\ &\quad\times\begin{bmatrix} |\partial_1\mathbf{n}|^2|\partial_2\eta|^2 - (\partial_1\mathbf{n}\cdot\partial_2\mathbf{n})(\partial_1\eta\cdot\partial_2\eta) & -|\partial_1\mathbf{n}|^2(\partial_1\eta\cdot\partial_2\eta) + (\partial_1\mathbf{n}\cdot\partial_2\mathbf{n})|\partial_1\eta|^2 \\ (\partial_1\mathbf{n}\cdot\partial_2\mathbf{n})|\partial_2\eta|^2 - |\partial_2\mathbf{n}|^2(\partial_1\eta\cdot\partial_2\eta) & -(\partial_1\mathbf{n}\cdot\partial_2\mathbf{n})(\partial_1\eta\cdot\partial_2\eta) + |\partial_2\mathbf{n}|^2|\partial_1\eta|^2\end{bmatrix}.\end{aligned} \tag{122}$$

Proof of (4) of Lemma 13. Let $\mathbf{q} = (y,u) \in \partial\Omega\times\mathbb{S}^2$ with $n(y)\cdot u = 0$ and $\mathbf{p}\sim\mathbf{q}$. By the definition (113)

$$\begin{aligned}\mathbf{x}_{\perp_\mathbf{p}} &= \mathbf{x}_{\perp_\mathbf{q}}, \\ \eta_\mathbf{p}(\mathbf{x}_{\parallel_\mathbf{p}}) &= \eta_\mathbf{q}(\mathbf{x}_{\parallel_\mathbf{q}}), \\ \mathbf{v}_{\perp_\mathbf{p}} &= \mathbf{v}_{\perp_\mathbf{q}}, \\ \mathbf{v}_{\parallel_\mathbf{p}}\cdot\nabla\eta_\mathbf{p}(\mathbf{x}_{\parallel_\mathbf{p}}) - \mathbf{x}_{\perp_\mathbf{p}}\mathbf{v}_{\parallel_\mathbf{p}}\cdot\nabla\mathbf{n}_\mathbf{p}(\mathbf{x}_{\parallel_\mathbf{p}}) &= \mathbf{v}_{\parallel_\mathbf{q}}\cdot\nabla\eta_\mathbf{q}(\mathbf{x}_{\parallel_\mathbf{q}}) - \mathbf{x}_{\perp_\mathbf{q}}\mathbf{v}_{\parallel_\mathbf{q}}\cdot\nabla\mathbf{n}_\mathbf{q}(\mathbf{x}_{\parallel_\mathbf{q}}).\end{aligned} \tag{123}$$

From the above second identity and (111)

$$\hat{\mathbf{r}}(\mathbf{x}_{\parallel_\mathbf{q},1}, \mathbf{x}_{\parallel_\mathbf{q},2}) = \mathcal{O}_\mathbf{q}\mathcal{O}_\mathbf{p}^{-1}\hat{\mathbf{r}}(\mathbf{x}_{\parallel_\mathbf{p},1}, \mathbf{x}_{\parallel_\mathbf{p},2}).$$

Therefore for $i=1,2$,

$$\sum_{j=1,2}\frac{\partial\hat{\mathbf{r}}}{\partial\mathbf{x}_{\parallel_\mathbf{q},j}}(\mathbf{x}_{\parallel_\mathbf{q}})\frac{\partial\mathbf{x}_{\parallel_\mathbf{q},j}}{\partial\mathbf{x}_{\parallel_\mathbf{p},i}} = \mathcal{O}_\mathbf{q}\mathcal{O}_\mathbf{p}^{-1}\frac{\partial\hat{\mathbf{r}}}{\partial\mathbf{x}_{\parallel_\mathbf{p},i}}(\mathbf{x}_{\parallel_\mathbf{p}}),$$

and

$$\begin{aligned}&\left[\begin{array}{c|c|c} -\sin(\mathbf{x}_{\parallel_\mathbf{q},2})\hat{\mathbf{r}}(\mathbf{x}_{\parallel_\mathbf{q}}) & \sin(\mathbf{x}_{\parallel_\mathbf{q},2})\hat{\theta}(\mathbf{x}_{\parallel_\mathbf{q}}) & \hat{\phi}(\mathbf{x}_{\parallel_\mathbf{q}})\end{array}\right]\left[\begin{array}{c|c} 0 & \mathbf{0}_{1,2} \\ \hline \mathbf{0}_{2,1} & \frac{\partial\mathbf{x}_{\parallel_\mathbf{q}}}{\partial\mathbf{x}_{\parallel_\mathbf{p}}}\end{array}\right]_{3\times3} \\ &= \mathcal{O}_\mathbf{q}\mathcal{O}_\mathbf{p}^{-1}\left[\begin{array}{c|c|c}\mathbf{0}_{3,1} & \sin(\mathbf{x}_{\parallel_\mathbf{p},2})\hat{\theta}(\mathbf{x}_{\parallel_\mathbf{p}}) & \hat{\phi}(\mathbf{x}_{\parallel_\mathbf{p}})\end{array}\right],\end{aligned}$$

where we used $\hat{\theta}\times\hat{\phi} = -\hat{\mathbf{r}}$.

Directly

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\partial\mathbf{x}_{\parallel_\mathbf{q},1}}{\partial\mathbf{x}_{\parallel_\mathbf{p},1}} & \frac{\partial\mathbf{x}_{\parallel_\mathbf{q},1}}{\partial\mathbf{x}_{\parallel_\mathbf{p},2}} \\ 0 & \frac{\partial\mathbf{x}_{\parallel_\mathbf{q},2}}{\partial\mathbf{x}_{\parallel_\mathbf{p},1}} & \frac{\partial\mathbf{x}_{\parallel_\mathbf{q},2}}{\partial\mathbf{x}_{\parallel_\mathbf{p},2}}\end{bmatrix} = \begin{bmatrix}\frac{-1}{\sin(\mathbf{x}_{\parallel_\mathbf{q},2})}\hat{\mathbf{r}}(\mathbf{x}_{\parallel_\mathbf{q}})^T \\ \frac{1}{\sin(\mathbf{x}_{\parallel_\mathbf{q},2})}\hat{\theta}(\mathbf{x}_{\parallel_\mathbf{q}})^T \\ \hat{\phi}(\mathbf{x}_{\parallel_\mathbf{q}})^T\end{bmatrix}\mathcal{O}_\mathbf{q}\mathcal{O}_\mathbf{p}^{-1}\left[\begin{array}{c|c|c}\mathbf{0}_{3,1} & \sin(\mathbf{x}_{\parallel_\mathbf{p},2})\hat{\theta}(\mathbf{x}_{\parallel_\mathbf{p}}) & \hat{\phi}(\mathbf{x}_{\parallel_\mathbf{p}})\end{array}\right].$$

Here $\mathcal{O}_\mathbf{q} = \mathcal{O}_\mathbf{p} + O_\xi(|\mathbf{p}-\mathbf{q}|)$, and $\sin(\mathbf{x}_{\parallel_\mathbf{p},2})\hat{\theta}(\mathbf{x}_{\parallel_\mathbf{p}}) = \sin(\mathbf{x}_{\parallel_\mathbf{q},2})\hat{\theta}(\mathbf{x}_{\parallel_\mathbf{q}}) + O_\xi(|\mathbf{p}-\mathbf{q}|)$ and $\hat{\phi}(\mathbf{x}_{\parallel_\mathbf{p}}) = \hat{\phi}(\mathbf{x}_{\parallel_\mathbf{q}}) + O_\xi(|\mathbf{p}-\mathbf{q}|)$. Therefore

$$\left[\frac{\partial\mathbf{x}_{\parallel_\mathbf{q}}}{\partial\mathbf{x}_{\parallel_\mathbf{p}}}\right]_{2\times2} \lesssim \mathbf{Id}_{2,2} + O_\xi(|\mathbf{p}-\mathbf{q}|).$$

From the third equality of (123)

$$\left[\begin{array}{c|c|c} -\mathbf{n}_{\mathbf{q}}(\mathbf{x}_{\|\mathbf{q}}) & \begin{smallmatrix} \partial_1 \eta_{\mathbf{q}}(\mathbf{x}_{\|\mathbf{q}}) \\ -\mathbf{x}_{\perp \mathbf{q}} \partial_1 \mathbf{n}_{\mathbf{q}}(\mathbf{x}_{\|\mathbf{q}}) \end{smallmatrix} & \begin{smallmatrix} \partial_2 \eta_{\mathbf{q}}(\mathbf{x}_{\|\mathbf{q}}) \\ -\mathbf{x}_{\perp \mathbf{q}} \partial_2 \mathbf{n}_{\mathbf{q}}(\mathbf{x}_{\|\mathbf{q}}) \end{smallmatrix} \end{array} \right] \left[\begin{array}{c|c} 0 & \mathbf{0}_{1,2} \\ \hline \mathbf{0}_{2,1} & \frac{\partial \mathbf{v}_{\|\mathbf{q}}}{\partial \mathbf{x}_{\|\mathbf{p}}} \end{array} \right] = \left[\begin{array}{c|c|c} \mathbf{0}_{3,1} & Z_1 & Z_2 \end{array} \right],$$

where $Z_i = \sum_{j=1}^2 \mathbf{v}_{\|\mathbf{p},j} \sum_{m=1}^2 \Big(\partial_m \partial_j \eta_{\mathbf{p}} - \mathbf{x}_{\perp \mathbf{p}} \partial_m \partial_j \mathbf{n}_{\mathbf{p}} \Big) \Big(\delta_{mi} - \frac{\partial \mathbf{x}_{\|\mathbf{q},m}}{\partial \mathbf{x}_{\|\mathbf{p},i}} \Big)$. Therefore

$$\begin{aligned} \left[\begin{array}{c|c} 0 & \mathbf{0}_{1,2} \\ \hline \mathbf{0}_{2,1} & \frac{\partial \mathbf{v}_{\|\mathbf{q}}}{\partial \mathbf{x}_{\|\mathbf{p}}} \end{array} \right] &= \frac{1}{[-\mathbf{n}_{\mathbf{q}}] \cdot \big([\partial_1 \eta_{\mathbf{q}} - \mathbf{x}_{\perp \mathbf{q}} \partial_1 \mathbf{n}_{\mathbf{q}}] \times [\partial_2 \eta_{\mathbf{q}} - \mathbf{x}_{\perp \mathbf{q}} \partial_2 \mathbf{n}_{\mathbf{q}}] \big)} \\ &\quad \times \left[\begin{array}{c} (\partial_1 \eta_{\mathbf{q}} - \mathbf{x}_{\perp \mathbf{q}} \partial_1 \mathbf{n}_{\mathbf{q}}) \times (\partial_2 \eta_{\mathbf{q}} - \mathbf{x}_{\perp \mathbf{q}} \partial_2 \mathbf{n}_{\mathbf{q}}) \\ (\partial_2 \eta_{\mathbf{q}} - \mathbf{x}_{\perp \mathbf{q}} \partial_2 \mathbf{n}_{\mathbf{q}}) \times (-\mathbf{n}_{\mathbf{q}}) \\ (-\mathbf{n}_{\mathbf{q}}) \times (\partial_1 \eta_{\mathbf{q}} - \mathbf{x}_{\perp \mathbf{q}} \partial_1 \mathbf{n}_{\mathbf{q}}) \end{array} \right] \left[\begin{array}{c|c|c} \mathbf{0}_{3,1} & Z_1 & Z_2 \end{array} \right], \end{aligned}$$

and hence

$$\left[\frac{\partial \mathbf{v}_{\|\mathbf{q}}}{\partial \mathbf{x}_{\|\mathbf{p}}} \right]_{2\times 2} \lesssim |\mathbf{v}_{\|}| |\mathbf{p} - \mathbf{q}|.$$

Again from the third equality of (123)

$$\begin{aligned} &\left[\begin{array}{c|c|c} -\mathbf{n}_{\mathbf{q}}(\mathbf{x}_{\|\mathbf{q}}) & \begin{smallmatrix} \partial_1 \eta_{\mathbf{q}}(\mathbf{x}_{\|\mathbf{q}}) \\ -\mathbf{x}_{\perp \mathbf{q}} \partial_1 \mathbf{n}_{\mathbf{q}}(\mathbf{x}_{\|\mathbf{q}}) \end{smallmatrix} & \begin{smallmatrix} \partial_2 \eta_{\mathbf{q}}(\mathbf{x}_{\|\mathbf{q}}) \\ -\mathbf{x}_{\perp \mathbf{q}} \partial_2 \mathbf{n}_{\mathbf{q}}(\mathbf{x}_{\|\mathbf{q}}) \end{smallmatrix} \end{array} \right] \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \frac{\partial \mathbf{v}_{\|\mathbf{q},1}}{\partial \mathbf{v}_{\|\mathbf{p},1}} & \frac{\partial \mathbf{v}_{\|\mathbf{q},1}}{\partial \mathbf{v}_{\|\mathbf{p},2}} \\ 0 & \frac{\partial \mathbf{v}_{\|\mathbf{q},2}}{\partial \mathbf{v}_{\|\mathbf{p},1}} & \frac{\partial \mathbf{v}_{\|\mathbf{q},2}}{\partial \mathbf{v}_{\|\mathbf{p},2}} \end{array} \right] \\ &= \left[\begin{array}{c|c|c} -\mathbf{n}_{\mathbf{p}}(\mathbf{x}_{\|\mathbf{p}}) & \begin{smallmatrix} \partial_1 \eta_{\mathbf{p}}(\mathbf{x}_{\|\mathbf{p}}) \\ -\mathbf{x}_{\perp \mathbf{p}} \partial_1 \mathbf{n}_{\mathbf{p}}(\mathbf{x}_{\|\mathbf{p}}) \end{smallmatrix} & \begin{smallmatrix} \partial_2 \eta_{\mathbf{p}}(\mathbf{x}_{\|\mathbf{p}}) \\ -\mathbf{x}_{\perp \mathbf{p}} \partial_2 \mathbf{n}_{\mathbf{p}}(\mathbf{x}_{\|\mathbf{p}}) \end{smallmatrix} \end{array} \right]. \end{aligned}$$

Since

$$\left[\begin{array}{c|c|c} -\mathbf{n}_{\mathbf{p}}(\mathbf{x}_{\|\mathbf{p}}) & \begin{smallmatrix} \partial_1 \eta_{\mathbf{p}}(\mathbf{x}_{\|\mathbf{p}}) \\ -\mathbf{x}_{\perp \mathbf{p}} \partial_1 \mathbf{n}_{\mathbf{p}}(\mathbf{x}_{\|\mathbf{p}}) \end{smallmatrix} & \begin{smallmatrix} \partial_2 \eta_{\mathbf{p}}(\mathbf{x}_{\|\mathbf{p}}) \\ -\mathbf{x}_{\perp \mathbf{p}} \partial_2 \mathbf{n}_{\mathbf{p}}(\mathbf{x}_{\|\mathbf{p}}) \end{smallmatrix} \end{array} \right] = \left[\begin{array}{c|c|c} -\mathbf{n}_{\mathbf{q}}(\mathbf{x}_{\|\mathbf{q}}) & \begin{smallmatrix} \partial_1 \eta_{\mathbf{q}}(\mathbf{x}_{\|\mathbf{q}}) \\ -\mathbf{x}_{\perp \mathbf{q}} \partial_1 \mathbf{n}_{\mathbf{q}}(\mathbf{x}_{\|\mathbf{q}}) \end{smallmatrix} & \begin{smallmatrix} \partial_2 \eta_{\mathbf{q}}(\mathbf{x}_{\|\mathbf{q}}) \\ -\mathbf{x}_{\perp \mathbf{q}} \partial_2 \mathbf{n}_{\mathbf{q}}(\mathbf{x}_{\|\mathbf{q}}) \end{smallmatrix} \end{array} \right] + O_\xi(|\mathbf{p} - \mathbf{q}|),$$

so that

$$\left[\begin{array}{cc} \frac{\partial \mathbf{v}_{\|\mathbf{q},1}}{\partial \mathbf{v}_{\|\mathbf{p},1}} & \frac{\partial \mathbf{v}_{\|\mathbf{q},1}}{\partial \mathbf{v}_{\|\mathbf{p},2}} \\ \frac{\partial \mathbf{v}_{\|\mathbf{q},2}}{\partial \mathbf{v}_{\|\mathbf{p},1}} & \frac{\partial \mathbf{v}_{\|\mathbf{q},2}}{\partial \mathbf{v}_{\|\mathbf{p},2}} \end{array} \right] = \mathbf{Id}_{2,2} + O_\xi(|\mathbf{p} - \mathbf{q}|).$$

□

We are ready to prove Theorem 4:

Proof of Theorem 4. First consider the case of $t < t_{\mathbf{b}}(x,v)$. In this case

$$(X_{\mathbf{cl}}(s;t,x,v), V_{\mathbf{cl}}(s;t,x,v)) = (x - (t-s)v, v).$$

Directly

$$\frac{\partial (X_{\mathbf{cl}}(s;t,x,v), V_{\mathbf{cl}}(s,t,x,v))}{\partial (t,x,v)} = \left[\begin{array}{ccc} -v & \mathbf{Id}_{3,3} & -(t-s)\mathbf{Id}_{3,3} \\ \mathbf{0}_{3,1} & \mathbf{0}_{3,3} & \mathbf{Id}_{3,3} \end{array} \right]_{6\times 7},$$

where $\mathbf{Id}_{m,m}$ is the m by m identity matrix and $\mathbf{0}_{m,n}$ is the m by n zero matrix.

Now we consider the case of $t \geq t_{\mathbf{b}}(x,v)$. We need several steps:

Step 1. Moving frames and grouping with respect to the scaling $t|v| = L_\xi$, with fixed $0 < L_\xi \ll 1$.

Fix $(t,x,v) \in [0,\infty) \times \bar{\Omega} \times \mathbb{R}^3$. Also we fix small constant $\delta_\xi > 0$ which depends on the domain. We define, at the boundary,

$$\mathbf{r}^\ell := \frac{|\mathbf{v}_\perp^\ell|}{|v^\ell|} = \frac{|v \cdot n(x^\ell)|}{|v|} = \frac{|V_{\mathbf{cl}}(t^\ell;t,x,v) \cdot n(X_{\mathbf{cl}}(t^\ell;t,x,v))|}{|v|}. \tag{124}$$

Bounces ℓ(and (t^ℓ, x^ℓ, v^ℓ)) are categorized as *Type I* or *Type II*:

$$\begin{aligned} &\text{a bounce } \ell \text{ is } \textit{Type I (almost grazing)} \text{ if and only if } \mathbb{R}^\ell \leq \sqrt{\delta}, \\ &\text{a bounce } \ell \text{ is } \textit{Type II (non-grazing)} \text{ if and only if } \mathbb{R}^\ell > \sqrt{\delta}. \end{aligned} \tag{125}$$

Let $s_* \in [t^{\ell+1}, t^\ell]$ such that $|\xi(X_{\mathbf{cl}}(s_*; t^\ell, x^\ell, v^\ell))| = \max_{t^{\ell+1}\leq \tau \leq t^\ell} |\xi(X_{\mathbf{cl}}(\tau; t^\ell, x^\ell, v^\ell))|$. Since $\frac{d^2}{ds^2}\xi(X_{\mathbf{cl}}(s; t^\ell, x^\ell, v^\ell)) = \frac{d^2}{ds^2}\xi(x^\ell - (t^\ell - s)v^\ell) = v^\ell \cdot \nabla_x^2 \xi(x^\ell - (t^\ell - s)v^\ell) \cdot v^\ell > 0$ there exists a unique s solving $\frac{d}{ds}\xi(X_{\mathbf{cl}}(s; t^\ell, x^\ell, v^\ell)) = v^\ell \cdot \nabla_x \xi(X_{\mathbf{cl}}(s; t^\ell, x^\ell, v^\ell)) = 0$ which is s_*. Note that $v^\ell \cdot \nabla\xi(x^\ell - (t^\ell - s)v^\ell)$ is monotone in either $(t^{\ell+1}, s_*)$ or (s_*, t^ℓ). Therefore

$$\begin{aligned}|\xi(X_{\mathbf{cl}}(s_*; t^\ell, x^\ell, v^\ell))| &= \Big| \int_{s_*}^{t^\ell} v^\ell \cdot \nabla \xi(x^\ell - (t^\ell - s)v^\ell, v^\ell) \Big| = \Big| \int_{s_*}^{t^\ell}\int_{s}^{t^\ell} v^\ell \cdot \nabla^2 \xi(x^\ell - (t^\ell - \tau)v^\ell, v^\ell) \cdot v^\ell \Big| \\ &\simeq_\xi \frac{|v^\ell|^2 |t^\ell - s_*|^2}{2} \simeq_\xi \Big(\sup_{s\in[t^{\ell+1}, t^\ell]} \frac{|v^\ell \cdot n(X_{\mathbf{cl}}(s))|}{|v^\ell|} \Big)^2,\end{aligned}$$

where we used (65) and (66) and the Velocity lemma(Lemma 1).

Therefore if a bounce ℓ is *Type I* then $\max_{t^{\ell+1}\leq\tau\leq t^\ell} |\xi(X_{\mathbf{cl}}(\tau; t, x, v))| \leq C\delta$. If a bounce ℓ is *Type II* then $|\xi(X_{\mathbf{cl}}(\tau; t, x, v))| > C\delta$ for some $\tau \in [t^{\ell+1}, t^\ell]$.

Now we assign a coordinate chart for each bounce ℓ (moving frames).

For *Type I* bounce ℓ in (125), we assign $\mathbf{p}^\ell \in \partial\Omega \times \mathbb{S}^2$ and $\mathbf{p}^\ell-$spherical coordinates in Lemma 13 and (113): we choose $\mathbf{p}^\ell := (z^\ell, w^\ell)$ on $\partial\Omega \times \mathbb{S}^2$ with $n(z^\ell)\cdot w^\ell = 0$ such that z^ℓ and w^ℓ do not depends on (t, x, v) and

$$|z^\ell - x^\ell| < \mathbf{r}^\ell, \quad \Big| w^\ell - \frac{v^\ell - (v^\ell \cdot n(z^\ell))n(z^\ell)}{|v^\ell - (v^\ell \cdot n(z^\ell))n(z^\ell)|} \Big| < \mathbf{r}^\ell. \tag{126}$$

Note that, by the definition of *Type I bounce*, $|v^\ell - (v^\ell \cdot n(z^\ell) n(z^\ell))|^2 = |v|^2 - |\mathbf{v}_\perp^\ell|^2 \gtrsim |v|^2(1-\delta) \gtrsim_\delta |v|^2$ and hence w^ℓ is well-defined.

Moreover, for $|v||t^\ell - s| \leq \frac{1}{100}\min_{x\in\partial\Omega}|x|$,

$$|X_{\mathbf{cl}}(s; t, x, v) - \mathcal{L}_{\mathbf{p}^\ell}| \gtrsim C_\delta > 0, \tag{127}$$

This is due to the fact that the projection of $V_{\mathbf{cl}}(s)$ on the plane passing z^ℓ and perpendicular to $n(z^\ell) \times w^\ell$ is at most $|v|$ but the distance from z^ℓ to the origin(the projection of poles $\mathcal{N}_{\mathbf{p}^\ell}$ and $\mathcal{S}_{\mathbf{p}^\ell}$) has lower bound $\frac{1}{10}\min_{x\in\partial\Omega}|x|$.

For *Type II* bounce $\ell(t^\ell, x^\ell, v^\ell)$, we choose $\mathbf{p}^\ell = (z^\ell, w^\ell)$ with $|z^\ell - x^\ell| \leq \sqrt{\delta}$ but we choose arbitrary $w^\ell \in \mathbb{S}^2$ satisfying $n(z^\ell)\cdot w^\ell = 0$. We choose $\mathbf{p}^\ell-$spherical coordinate in Lemma 13 and (113) with this $\mathbf{p}^\ell$. Note that unlike *Type I*, this $\mathbf{p}^\ell-$spherical coordinate might not be defined for $s \in [t^{\ell+1}, t^\ell]$ but only defined near the boundary.

Whenever the moving frame is defined(for all $\tau \in [t^{\ell+1}, t^\ell]$ when ℓ is *Type I*, and $\tau \sim t^\ell$ when ℓ is *Type II*) we denote

$$(\mathbf{X}_\ell(\tau), \mathbf{V}_\ell(\tau)) = (\mathbf{x}_{\perp_\ell}(\tau), \mathbf{x}_{\parallel_\ell}(\tau), \mathbf{v}_{\perp_\ell}(\tau), \mathbf{v}_{\parallel_\ell}(\tau)) := \Phi_{\mathbf{p}^\ell}^{-1}(X_{\mathbf{cl}}(\tau), V_{\mathbf{cl}}(\tau)).$$

Especially at the boundary we denote

$$(\mathbf{x}_{\perp_\ell}^\ell, \mathbf{x}_{\parallel_\ell}^\ell, \mathbf{v}_{\perp_\ell}^\ell, \mathbf{v}_{\parallel_\ell}^\ell) := (\mathbf{x}_{\perp_{\mathbf{p}_\ell}}^\ell, \mathbf{x}_{\parallel_{\mathbf{p}_\ell}}^\ell, \mathbf{v}_{\perp_{\mathbf{p}_\ell}}^\ell, \mathbf{v}_{\parallel_{\mathbf{p}_\ell}}^\ell), \quad \text{with } \mathbf{x}_{\perp_\ell}^\ell = 0.$$

Now we regroup the indices of the specular cycles, without order changing, as

$$\{0, 1, 2, \cdots, \ell_* - 1, \ell_*\} = \{0\} \cup \mathcal{G}_1 \cup \mathcal{G}_2 \cup \cdots \cup \mathcal{G}_{[\frac{|t-s||v|}{L_\xi}]} \cup \mathcal{G}_{[\frac{|t-s||v|}{L_\xi}]+1},$$

where $\big[a\big] \in \mathbb{N}$ is the greatest integer less than or equal to a. Each group is

$$\begin{aligned}\mathcal{G}_1 &= \{1, \cdots, \ell_1 - 1, \ell_1\}, \\ \mathcal{G}_2 &= \{\ell_1, \ell_1 + 1, \cdots, \ell_2 - 1, \ell_2\}, \\ &\vdots \\ \mathcal{G}_{[\frac{|t-s||v|}{L_\xi}]} &= \{\ell_{[\frac{|t-s||v|}{L_\xi}]-1}, \ell_{[\frac{|t-s||v|}{L_\xi}]-1} + 1, \cdots, \ell_{[\frac{|t-s||v|}{L_\xi}]} - 1, \ell_{[\frac{|t-s||v|}{L_\xi}]}\}, \\ \mathcal{G}_{[\frac{|t-s||v|}{L_\xi}]+1} &= \{\ell_{[\frac{|t-s||v|}{L_\xi}]}, \ell_{[\frac{|t-s||v|}{L_\xi}]} + 1, \cdots, \ell_*\},\end{aligned} \tag{128}$$

where $\ell_1 = \inf\{\ell \in \mathbb{N} : |v| \times |t^0 - t^{\ell_1}| \geq L_\xi\}$ and inductively

$$\ell_i = \inf\{\ell \in \mathbb{N} : |v| \times |t^{\ell_i} - t^{\ell_{i+1}}| \geq L_\xi\}, \tag{129}$$

and we have denoted $\ell_* = \ell_{[\frac{|t-s||v|}{L_\xi}]+1}$.

By the chain rule, with the assigned $\mathbf{p}^\ell-$spherical coordinate(moving frame), for fixed $0 \leq s \leq t$,

$$\begin{aligned}
&\frac{\partial(s, X_{\mathbf{cl}}(s;t,x,v), V_{\mathbf{cl}}(s;t,x,v))}{\partial(t,x,v)} \\
= \quad & \underbrace{\frac{\partial(s, X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))}{\partial(t^{\ell_*}, 0, \mathbf{x}^{\ell_*}_{\|_{\ell_*}}, \mathbf{v}^{\ell_*}_{\perp_{\ell_*}}, \mathbf{v}^{\ell_*}_{\|_{\ell_*}})}}_{\text{from the last bounce to the } s-\text{plane}} \\
\times \quad & \underbrace{\prod_{i=1}^{[\frac{|t-s||v|}{L_*}]} \underbrace{\frac{\partial(t^{\ell_{i+1}}, 0, \mathbf{x}^{\ell_{i+1}}_{\|_{\ell_{i+1}}}, \mathbf{v}^{\ell_{i+1}}_{\perp_{\ell_{i+1}}}, \mathbf{v}^{\ell_{i+1}}_{\|_{\ell_{i+1}}})}{\partial(t^{\ell_{i+1}-1}, 0, \mathbf{x}^{\ell_{i+1}-1}_{\|_{\ell_{i+1}-1}}, \mathbf{v}^{\ell_{i+1}-1}_{\perp_{\ell_{i+1}-1}}, \mathbf{v}^{\ell_{i+1}-1}_{\|_{\ell_{i+1}-1}})} \times \cdots \times \frac{\partial(t^{\ell_i+1}, 0, \mathbf{x}^{\ell_i+1}_{\|_{\ell_i+1}}, \mathbf{v}^{\ell_i+1}_{\perp_{\ell_i+1}}, \mathbf{v}^{\ell_i+1}_{\|_{\ell_i+1}})}{\partial(t^{\ell_i}, 0, \mathbf{x}^{\ell_i}_{\|_{\ell_i}}, \mathbf{v}^{\ell_i}_{\perp_{\ell_i}}, \mathbf{v}^{\ell_i}_{\|_{\ell_i}})}}_{i-\text{th intermediate group}}}_{\text{whole intermediate groups}} \\
\times \quad & \underbrace{\frac{\partial(t^1, 0, \mathbf{x}^1_{\|_1}, \mathbf{v}^1_{\perp_1}, \mathbf{v}^1_{\|_1})}{\partial(t,x,v)}}_{\text{from the } t-\text{plane to the first bounce}} \quad .
\end{aligned} \tag{130}$$

Step 2. From the last bounce ℓ_* to the $s-$plane

We choose $s^{\ell_*} \in (\frac{t^{\ell_*}+s}{2}, t^{\ell_*}) \subset (s, t^{\ell_*})$ such that $|v||t^{\ell_*} - s^{\ell_*}| \ll 1$ and the ℓ_*-spherical coordinate $(\mathbf{X}_{\ell_*}(s^{\ell_*}), \mathbf{V}_{\ell_*}(s^{\ell_*}))$ is well-defined regardless of the *Type* of ℓ_* in (125). Notice that s^{ℓ_*} is independent of t^{ℓ_*} so that $\frac{\partial s^{\ell_*}}{\partial t^{\ell_*}} = 0$.

By the chain rule,

$$\begin{aligned}
&\frac{\partial(s, X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))}{\partial(t^{\ell_*}, 0, \mathbf{x}^{\ell_*}_{\|_{\ell_*}}, \mathbf{v}^{\ell_*}_{\perp_{\ell_*}}, \mathbf{v}^{\ell_*}_{\|_{\ell_*}})} \\
&= \frac{\partial(s, X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))}{\partial(s^{\ell_*}, \mathbf{X}_{\ell_*}(s^{\ell_*}), \mathbf{V}_{\ell_*}(s^{\ell_*}))} \frac{\partial(s^{\ell_*}, \mathbf{x}_{\perp_{\ell_*}}(s^{\ell_*}), \mathbf{x}_{\|_{\ell_*}}(s^{\ell_*}), \mathbf{v}_{\perp_{\ell_*}}(s^{\ell_*}), \mathbf{v}_{\|_{\ell_*}}(s^{\ell_*}))}{\partial(t^{\ell_*}, 0, \mathbf{x}^{\ell_*}_{\|_{\ell_*}}, \mathbf{v}^{\ell_*}_{\perp_{\ell_*}}, \mathbf{v}^{\ell_*}_{\|_{\ell_*}})} \\
&= \frac{\partial(s, X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))}{\partial(s^{\ell_*}, X_{\mathbf{cl}}(s^{\ell_*}), V_{\mathbf{cl}}(s^{\ell_*}))} \frac{\partial(s^{\ell_*}, X_{\mathbf{cl}}(s^{\ell_*}), V_{\mathbf{cl}}(s^{\ell_*}))}{\partial(s^{\ell_*}, \mathbf{X}_{\ell_*}(s^{\ell_*}), \mathbf{V}_{\ell_*}(s^{\ell_*}))} \frac{\partial(s^{\ell_*}, \mathbf{x}_{\perp_{\ell_*}}(s^{\ell_*}), \mathbf{x}_{\|_{\ell_*}}(s^{\ell_*}), \mathbf{v}_{\perp_{\ell_*}}(s^{\ell_*}), \mathbf{v}_{\|}(s^{\ell_*}))}{\partial(t^{\ell_*}, 0, \mathbf{x}^{\ell_*}_{\|_{\ell_*}}, \mathbf{v}^{\ell_*}_{\perp_{\ell_*}}, \mathbf{v}^{\ell_*}_{\|_{\ell_*}})}.
\end{aligned}$$

Firstly, we claim

$$\frac{\partial(s, X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))}{\partial(s^{\ell_*}, \mathbf{X}_{\ell_*}(s^{\ell_*}), \mathbf{V}_{\ell_*}(s^{\ell_*}))} = \begin{bmatrix} 0 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ -V_{\mathbf{cl}}(s^{\ell_*}) & O_\xi(1)(1+|v||s^{\ell_*}-s|) & O_\xi(1)|s^{\ell_*}-s| \\ \mathbf{0}_{3,1} & O_\xi(1)|v| & O_\xi(1) \end{bmatrix}. \tag{131}$$

Since

$$X_{\mathbf{cl}}(s) = X_{\mathbf{cl}}(s^{\ell_*}) - (s^{\ell_*} - s)V_{\mathbf{cl}}(s^{\ell_*}), \quad V_{\mathbf{cl}}(s) = V_{\mathbf{cl}}(s^{\ell_*}),$$

we have

$$\frac{\partial(s, X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))}{\partial(s^{\ell_*}, X_{\mathbf{cl}}(s^{\ell_*}), V_{\mathbf{cl}}(s^{\ell_*}))} = \begin{bmatrix} 0 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ -V_{\mathbf{cl}}(s^{\ell_*}) & \mathbf{Id}_{3,3} & -(s^{\ell_*}-s)\mathbf{Id}_{3,3} \\ \mathbf{0}_{3,1} & \mathbf{0}_{3,3} & \mathbf{Id}_{3,3} \end{bmatrix}.$$

Due to Lemma 13

$$
\begin{aligned}
&\frac{\partial(s^{\ell_*}, X_{\mathbf{cl}}(s^{\ell_*}), V_{\mathbf{cl}}(s^{\ell_*}))}{\partial(s^{\ell_*}, \mathbf{X}_{\ell_*}(s^{\ell_*}), \mathbf{V}_{\ell_*}(s^{\ell_*}))} \\
&= \left[\begin{array}{c|ccc|ccc}
1 & & \mathbf{0}_{1,3} & & & \mathbf{0}_{1,3} & \\ \hline
\mathbf{0}_{3,1} & \mathbf{n}_{\ell_*} & \begin{matrix}\partial_1\eta_{\ell_*} \\ +\mathbf{x}_{\perp_{\ell_*}}\partial_1\mathbf{n}_{\ell_*}\end{matrix} & \begin{matrix}\partial_2\eta_{\ell_*} \\ +\mathbf{x}_{\perp_{\ell_*}}\partial_2\mathbf{n}_{\ell_*}\end{matrix} & & \mathbf{0}_{3,3} & \\ \hline
\mathbf{0}_{3,1} & \mathbf{v}_{\|_{\ell_*}} \cdot \nabla_{\mathbf{x}_{\|_{\ell_*}}} \mathbf{n}_{\ell_*} & \begin{matrix}\mathbf{v}_{\|_{\ell_*}} \cdot \nabla\partial_1\eta_{\ell_*} \\ +\mathbf{v}_{\perp_{\ell_*}}\partial_1\mathbf{n}_{\ell_*} \\ +\mathbf{x}_{\perp_{\ell_*}}\mathbf{v}_{\|_{\ell_*}} \cdot \nabla\partial_1\mathbf{n}_{\ell_*}\end{matrix} & \begin{matrix}\mathbf{v}_{\|_{\ell_*}} \cdot \nabla\partial_2\eta_{\ell_*} \\ +\mathbf{v}_{\perp_{\ell_*}}\partial_2\mathbf{n}\ell_* \\ +\mathbf{x}_{\perp_{\ell_*}}\mathbf{v}_{\|_{\ell_*}} \cdot \nabla\partial_2\mathbf{n}_{\ell_*}\end{matrix} & \mathbf{n}_{\ell_*} & \begin{matrix}\partial_1\eta_{\ell_*} \\ +\mathbf{x}_{\perp_{\ell_*}}\partial_1\mathbf{n}_{\ell_*}\end{matrix} & \begin{matrix}\partial_2\eta_{\ell_*} \\ +\mathbf{x}_{\perp_{\ell_*}}\partial_2\mathbf{n}_{\ell_*}\end{matrix}
\end{array}\right],
\end{aligned}
$$

where all entries are evaluated at $(\mathbf{X}_{\ell_*}(s^{\ell_*}), \mathbf{V}_{\ell_*}(s^{\ell_*}))$. The multiplication of above two matrices gives (131).

Secondly, we claim that whenever $\mathbf{p}^\ell-$spherical coordinate is defined for $\tau \in [s^\ell, t^\ell]$

$$
\begin{aligned}
&\frac{\partial(s^\ell, \mathbf{x}_{\perp_\ell}(s^\ell), \mathbf{x}_{\|_\ell}(s^\ell), \mathbf{v}_{\perp_\ell}(s^\ell), \mathbf{v}_{\|_\ell}(s^\ell))}{\partial(t^\ell, 0, \mathbf{x}^\ell_{\|_\ell}, \mathbf{v}^\ell_{\perp_\ell}, \mathbf{v}^\ell_{\|_\ell})} \\
&= \left[\begin{array}{c|cc|cc}
0 & 0 & \mathbf{0}_{1,2} & 0 & \mathbf{0}_{1,2} \\ \hline
-\mathbf{v}_\perp(s^\ell) & 0 & O_\xi(1)|v|^2|t^\ell - s^\ell|^2 & O_\xi(1)|t^\ell - s^\ell| & O_\xi(1)|v||t^\ell - s^\ell|^2 \\
-\mathbf{v}_\|(s^\ell) & \mathbf{0}_{2,1} & \mathbf{Id}_{2,2} + O_\xi(1)|v|^2|t^\ell - s^\ell|^2 & O_\xi(1)|v||t^\ell - s^\ell|^2 & O_\xi(1)|t^\ell - s^\ell|(\mathbf{Id}_{2,2} + |v||t^\ell - s^\ell|) \\ \hline
O_\xi(1)|v|^2 & 0 & O_\xi(1)|v|^2|t^\ell - s^\ell| & 1 + O_\xi(1)|v||t^\ell - s^\ell| & O_\xi(1)|v||t^\ell - s^\ell| \\
O_\xi(1)|v|^2 & \mathbf{0}_{2,1} & O_\xi(1)|v|^2|t^\ell - s^\ell| & O_\xi(1)|v||t^\ell - s^\ell| & \mathbf{Id}_{2,2} + O_\xi(1)|v||t^\ell - s^\ell|
\end{array}\right].
\end{aligned}
\tag{132}
$$

In this step we just need (132) for $\ell = \ell_*$. In Step 8 we need (132) for general ℓ.

For the first column(temporal derivatives), we use the fact that the characteristics ODE (114) is autonomous:

$$
\begin{aligned}
&\frac{\partial}{\partial t^\ell}(\mathbf{X}_\ell(s^\ell; t^\ell, x^\ell, v^\ell), \mathbf{V}_\ell(s^\ell; t^\ell, x^\ell, v^\ell)) \\
&= \frac{\partial}{\partial t^\ell}(\mathbf{X}_\ell(s^\ell - t^\ell; 0, x^\ell, v^\ell), \mathbf{V}_\ell(s^\ell - t^\ell; 0, x^\ell, v^\ell)) \\
&= -\frac{\partial}{\partial s^\ell}(\mathbf{X}_\ell(s^\ell; t^\ell, x^\ell, v^\ell), \mathbf{V}_\ell(s^\ell; t^\ell, x^\ell, v^\ell)) \\
&= -(\mathbf{V}_\ell(s^\ell; t^\ell, x^\ell, v^\ell), F(\mathbf{X}_\ell(s^\ell; t^\ell, x^\ell, v^\ell), \mathbf{V}_\ell(s^\ell; t^\ell, x^\ell, v^\ell)).
\end{aligned}
$$

For the remainder, firstly we show for $t^{\ell+1} < \tau < t^\ell$ where τ is independent of t^ℓ and the $\mathbf{p}^\ell-$spherical coordinate $(\mathbf{X}_\ell(\tau), \mathbf{V}_\ell(\tau))$ is well-defined and

$$
\begin{aligned}
|\partial_{\mathbf{x}^\ell_{\|_\ell}} \mathbf{X}_\ell(\tau)| &\lesssim e^{C_\xi|v||\tau - t^\ell|} \lesssim 1, \\
|\partial_{\mathbf{v}^\ell_\ell} \mathbf{X}_\ell(\tau)| &\lesssim |\tau - t^\ell| e^{C_\xi|v||\tau - t^\ell|} \lesssim |\tau - t^\ell|, \\
|\partial_{\mathbf{x}^\ell_{\|_\ell}} \mathbf{V}_\ell(\tau)| &\lesssim |v| \times |v||\tau - t^\ell| e^{C_\xi|v||\tau - t^\ell|} \lesssim |v|^2|\tau - t^\ell|, \\
|\partial_{\mathbf{v}^\ell_\ell} \mathbf{V}_\ell(\tau)| &\lesssim e^{C_\xi|v||\tau - t^\ell|} \lesssim 1,
\end{aligned}
\tag{133}
$$

where $\partial_{\mathbf{v}^\ell_\ell} = [\partial_{\mathbf{v}^\ell_{\perp_\ell}}, \partial_{\mathbf{v}^\ell_{\|_\ell}}]$.

From (115) and (116), $\dot{\mathbf{x}}_{\|_\ell} = \mathbf{v}_{\|_\ell}$, $\dot{\mathbf{x}}_{\perp_\ell} = \mathbf{v}_{\perp_\ell}$ and $\dot{\mathbf{v}}_{\perp_\ell} = F_{\perp_\ell}$ and $\dot{\mathbf{v}}_{\|_\ell} = F_{\|_\ell}$ in (116). Hence, for $\partial = [\frac{\partial}{\partial \mathbf{x}^\ell_{\|_\ell}}, \frac{\partial}{\partial \mathbf{v}^\ell_{\perp_\ell}}, \frac{\partial}{\partial \mathbf{v}^\ell_{\|_\ell}}]$,

$$
\begin{aligned}
\frac{d}{d\tau}|\partial \mathbf{v}_{\perp_\ell}(\tau)| &\lesssim |v||\mathbf{v}_{\|_\ell}(\tau)||\partial \mathbf{x}_{\|_\ell}(\tau)| + |\mathbf{v}_{\|_\ell}(\tau)|^2|\partial \mathbf{x}_{\perp_\ell}(\tau)| + |v||\partial \mathbf{v}_{\|_\ell}(\tau)| + |\mathbf{v}_{\|_\ell}(\tau)||\partial \mathbf{v}_{\perp_\ell}(\tau)| \\
&\lesssim |v|^2\big(|\partial \mathbf{x}_{\perp_\ell}(\tau)| + |\partial \mathbf{x}_{\|_\ell}(\tau)|\big) + |v|\big(|\partial \mathbf{v}_{\perp_\ell}(\tau)| + |\partial \mathbf{v}_{\|_\ell}(\tau)|\big), \\
\frac{d}{d\tau}|\partial \mathbf{v}_{\|_\ell}(\tau)| &\lesssim \big(|\mathbf{v}_{\|_\ell}(\tau)|^2 + |\mathbf{v}_{\perp_\ell}(\tau)||\mathbf{v}_{\|_\ell}(\tau)|\big)\big(|\partial \mathbf{x}_{\|_\ell}(\tau)| + |\partial \mathbf{x}_{\perp_\ell}(\tau)|\big) + |\mathbf{v}_{\|_\ell}(\tau)||\partial \mathbf{v}_{\perp_\ell}(\tau)| + |v||\partial \mathbf{v}_{\|_\ell}(\tau)| \\
&\lesssim |v|^2\big(|\partial \mathbf{x}_{\perp_\ell}(\tau)| + |\partial \mathbf{x}_{\|_\ell}(\tau)|\big) + |v|\big(|\partial \mathbf{v}_{\perp_\ell}(\tau)| + |\partial \mathbf{v}_{\|_\ell}(\tau)|\big).
\end{aligned}
$$

Hence

$$
\frac{d}{d\tau}\begin{bmatrix} |\partial \mathbf{x}_{\perp_\ell}(\tau)| + |\partial \mathbf{x}_{\|_\ell}(\tau)| \\ |\partial \mathbf{v}_{\perp_\ell}(\tau)| + |\partial \mathbf{v}_{\|_\ell}(\tau)| \end{bmatrix} \lesssim_\Omega \begin{bmatrix} 0 & 1 \\ |v|^2 & |v| \end{bmatrix} \begin{bmatrix} |\partial \mathbf{x}_{\perp_\ell}(\tau)| + |\partial \mathbf{x}_{\|_\ell}(\tau)| \\ |\partial \mathbf{v}_{\perp_\ell}(\tau)| + |\partial \mathbf{v}_{\|_\ell}(\tau)| \end{bmatrix}.
$$

We diagonalize the matrix as

$$\begin{bmatrix} 0 & 1 \\ |v|^2 & |v| \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2}|v| & \frac{1-\sqrt{5}}{2}|v| \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2}|v| & 0 \\ 0 & \frac{1-\sqrt{5}}{2}|v| \end{bmatrix} \begin{bmatrix} -\frac{1-\sqrt{5}}{2\sqrt{5}} & \frac{1}{|v|\sqrt{5}} \\ \frac{1+\sqrt{5}}{2\sqrt{5}} & \frac{-1}{|v|\sqrt{5}} \end{bmatrix} := PDP^{-1}.$$

Now

$$\begin{bmatrix} |\partial \mathbf{x}_{\parallel}(\tau)| + |\partial \mathbf{x}_{\perp}(\tau)| \\ |\partial \mathbf{v}_{\parallel}(\tau)| + |\partial \mathbf{v}_{\perp}(\tau)| \end{bmatrix} \leq P e^{C_\xi |\tau - t^\ell| D} P^{-1} \begin{bmatrix} |\partial \mathbf{x}_{\parallel}(t^\ell)| + |\partial \mathbf{x}_{\perp}(t^\ell)| \\ |\partial \mathbf{v}_{\parallel}(t^\ell)| + |\partial \mathbf{v}_{\perp}(t^\ell)| \end{bmatrix},$$

and by matrix multiplication

$$\begin{aligned}
&\leq \begin{bmatrix} -\frac{1-\sqrt{5}}{2\sqrt{5}} e^{C_\xi \frac{1+\sqrt{5}}{2}|v||\tau - t^\ell|} + \frac{1+\sqrt{5}}{2\sqrt{5}} e^{C_\xi \frac{1-\sqrt{5}}{2}|v||\tau - t^\ell|} & \frac{1}{\sqrt{5}|v|} e^{\frac{C_\xi}{2}|v||\tau-t^\ell|} \Big\{ e^{\frac{C_\xi\sqrt{5}}{2}|v||\tau-t^\ell|} - e^{C_\xi \frac{-\sqrt{5}}{2}|v||\tau - t^\ell|} \Big\} \\ \frac{|v|}{\sqrt{5}} e^{C_\xi \frac{|v|}{2}|\tau-t^\ell|} \Big\{ e^{C_\xi \frac{\sqrt{5}}{2}|v||\tau-t^\ell|} - e^{-C_\xi \frac{\sqrt{5}}{2}|v||\tau-t^\ell|} \Big\} & \frac{1+\sqrt{5}}{2\sqrt{5}} e^{C_\xi \frac{1+\sqrt{5}}{2}|v||\tau-t^\ell|} - \frac{1-\sqrt{5}}{2\sqrt{5}} e^{C_\xi \frac{1-\sqrt{5}}{2}|v||\tau-t^\ell|} \end{bmatrix} \\
&\quad \times \begin{bmatrix} |\partial \mathbf{x}_{\parallel}(t^\ell)| + |\partial \mathbf{x}_{\perp}(t^\ell)| \\ |\partial \mathbf{v}_{\parallel}(t^\ell)| + |\partial \mathbf{v}_{\perp}(t^\ell)| \end{bmatrix} \\
&\leq \begin{bmatrix} e^{C_\xi |v||\tau - t^\ell|} \big\{ |\partial \mathbf{x}_{\parallel}(t^\ell)| + |\partial \mathbf{x}_{\perp}(t^\ell)| \big\} + |\tau - t^\ell| e^{C_\xi |v||\tau - t^\ell|} \big\{ |\partial \mathbf{v}_{\parallel}(t^\ell)| + |\partial \mathbf{v}_{\perp}(t^\ell)| \big\} \\ |v|^2 |\tau - t^\ell| e^{C_\xi |v||\tau - t^\ell|} \big\{ |\partial \mathbf{x}_{\parallel}(t^\ell)| + |\partial \mathbf{x}_{\perp}(t^\ell)| \big\} + e^{C_\xi |v||\tau - t^\ell|} \big\{ |\partial \mathbf{v}_{\parallel}(t^\ell)| + |\partial \mathbf{v}_{\perp}(t^\ell)| \big\} \end{bmatrix},
\end{aligned}$$

and this proves (133).

From the characteristics ODE (114) in the $\mathbf{p}^\ell-$spherical coordinate,

$$\begin{aligned}
\mathbf{x}_{\perp_\ell}(s^\ell) &= \mathbf{v}^\ell_{\perp_\ell}(s^\ell - t^\ell) + \int_{t^\ell}^{s^\ell} \int_{t^\ell}^{\tau} F_{\perp_\ell}(s'; t^\ell, \mathbf{x}^\ell_{\parallel_\ell}, \mathbf{v}^\ell_{\parallel_\ell}, \mathbf{v}^\ell_{\perp_\ell}) \mathrm{d}s' \mathrm{d}\tau, \\
\mathbf{x}_{\parallel_\ell}(s^\ell) &= \mathbf{x}^\ell_{\parallel_\ell} + \mathbf{v}^\ell_{\parallel_\ell}(s^\ell - t^\ell) + \int_{t^\ell}^{s^\ell} \int_{t^\ell}^{\tau} F_{\parallel}(s'; t^\ell, \mathbf{x}^\ell_{\parallel_\ell}, \mathbf{v}^\ell_{\parallel_\ell}, \mathbf{v}^\ell_{\perp_\ell}) \mathrm{d}s' \mathrm{d}\tau, \\
\mathbf{v}_{\perp_\ell}(s^\ell) &= \mathbf{v}^\ell_{\perp_\ell} + \int_{t^\ell}^{s^\ell} F_{\perp_\ell}(\tau; t^n, \mathbf{x}^\ell_{\parallel_\ell}, \mathbf{v}^\ell_{\parallel_\ell}, \mathbf{v}^\ell_{\perp \ell}) \mathrm{d}\tau, \\
\mathbf{v}_{\parallel_\ell}(s^\ell) &= \mathbf{v}^\ell_{\parallel_\ell} + \int_{t^\ell}^{s^\ell} F_{\parallel_\ell}(\tau; t^\ell, \mathbf{x}^\ell_{\parallel_\ell}, \mathbf{v}^\ell_{\parallel_\ell}, \mathbf{v}^\ell_{\perp_\ell}) \mathrm{d}\tau.
\end{aligned}$$

From (133),

$$\begin{aligned}
\frac{\partial \mathbf{x}_{\perp_\ell}(s^\ell)}{\partial \mathbf{x}^\ell_{\parallel_\ell}} &\leq C_\Omega |v|^2 |s^\ell - t^\ell|^2 (1 + |v||s^\ell - t^\ell|) e^{|v||s^\ell - t^\ell|} \lesssim_\Omega |v|^2 |s^\ell - t^\ell|^2, \\
\frac{\partial \mathbf{x}_{\perp_\ell}(s^\ell)}{\partial \mathbf{v}^\ell_{\perp_\ell}} &\leq |s^\ell - t^\ell| + C_\Omega |v||s^\ell - t^\ell|^2 (1 + |v||s^\ell - t^\ell|) e^{|v||s^\ell - t^\ell|} \lesssim_\Omega |s^\ell - t^\ell|, \\
\frac{\partial \mathbf{x}_{\perp_\ell}(s^\ell)}{\partial \mathbf{v}^\ell_{\parallel_\ell}} &\leq C_\Omega |v||s^\ell - t^\ell|^2 (1 + |v||s^\ell - t^\ell|) e^{|v||s^\ell - t^\ell|} \lesssim_\Omega |v||s^\ell - t^\ell|, \\
\frac{\partial \mathbf{x}_{\parallel_\ell}(s^\ell)}{\partial \mathbf{x}^\ell_{\parallel_\ell}} &\leq \mathbf{Id}_{2,2} + C_\Omega |v|^2 |s^\ell - t^\ell|^2 (1 + |v||s^\ell - t^\ell|) e^{|v||s^\ell - t^\ell|} \leq \mathbf{Id}_{2,2} + O_\Omega(1) |v|^2 |s^\ell - t^\ell|^2, \\
\frac{\partial \mathbf{x}_{\parallel_\ell}(s^\ell)}{\partial \mathbf{v}^\ell_{\perp_\ell}} &\leq C_\Omega |s^\ell - t^\ell|^2 |v| (1 + |v||s^\ell - t^\ell|) e^{|v||s^\ell - t^\ell|} \lesssim_\Omega |v||s^\ell - t^\ell|^2, \\
\frac{\partial \mathbf{x}_{\parallel_\ell}(s^\ell)}{\partial \mathbf{v}^\ell_{\parallel_\ell}} &\leq |s^\ell - t^\ell| \Big\{ \mathbf{Id}_{2,2} + C_\Omega |v||s^\ell - t^\ell| (1 + |v||s^\ell - t^\ell|) e^{|v||s^\ell - t^\ell|} \Big\} \\
&\leq |s^\ell - t^\ell| \mathbf{Id}_{2,2} + O_\Omega(1) |v||s^\ell - t^\ell|^2,
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial \mathbf{v}_{\perp_\ell}(s^\ell)}{\partial \mathbf{x}^\ell_{\|_\ell}} &\leq C_\Omega |s^\ell - t^\ell||v|^2(1+|v||s^\ell - t^\ell|)e^{|v||s^\ell - t^\ell|} \lesssim_\Omega |v|^2|s^\ell - t^\ell|,\\
\frac{\partial \mathbf{v}_{\perp_\ell}(s^\ell)}{\partial \mathbf{v}^\ell_{\perp_\ell}} &\leq 1 + C_\Omega |s^\ell - t^\ell||v|(1+|v||s^\ell -^\ell|)e^{|v||s^\ell - t^\ell|} \leq 1 + O_\Omega(1)|v||s^\ell - t^\ell|,\\
\frac{\partial \mathbf{v}_{\perp_\ell}(s^\ell)}{\partial \mathbf{v}^\ell_{\|_\ell}} &\leq C_\Omega |s^\ell - t^\ell||v|(1+|v||s^\ell - t^\ell|)e^{|v||s^\ell - t^\ell|} \lesssim_\Omega |v||s^\ell - t^\ell|,\\
\frac{\partial \mathbf{v}_{\|_\ell}(s^\ell)}{\partial \mathbf{x}^\ell_{\|_\ell}} &\leq C_\Omega |s^\ell - t^\ell||v|^2(1+|v||s^\ell - t^\ell|)e^{|v||s^\ell - t^\ell|} \lesssim_\Omega |v|^2|s^\ell - t^\ell|,\\
\frac{\partial \mathbf{v}_{\|_\ell}(s^\ell)}{\partial \mathbf{v}^\ell_{\perp_\ell}} &\leq C_\Omega |s^\ell - t^\ell||v|(1+|v||s^\ell - t^\ell|)e^{|v||s^\ell - t^\ell|} \lesssim_\Omega |v||s^\ell - t^\ell|,\\
\frac{\partial \mathbf{v}_{\|_\ell}(s^\ell)}{\partial \mathbf{v}^\ell_{\|_\ell}} &\leq \mathbf{Id}_{2,2} + C_\Omega |s^\ell - t^\ell||v|(1+|v||s^\ell - t^\ell|)e^{|v||s^\ell - t^\ell|} \leq \mathbf{Id}_{2,2} + O_\Omega(1)|v||s^\ell - t^\ell|,
\end{aligned}
$$

and this proves the claim (132).

Step 3. From $t-$plane to the first bounce

We choose $s^1 \in (t^1, \frac{t^1+t}{2}) \subset (t^1, t)$ such that $|v||t^1 - s^1| \ll 1$ and the polar coordinate $(\mathbf{X}_1(s^1), \mathbf{V}_1(s^1))$ is well-defined. More precisely we choose $0 < \Delta$ such that $|v||t - \Delta - t^1| \ll 1$ and define

$$
s^1 := t - \Delta.
$$

Then, by the chain rule,

$$
\begin{aligned}
&\frac{\partial(t^1, 0, \mathbf{x}^1_{\|_1}, \mathbf{v}^1_{\perp_1}, \mathbf{v}^1_{\|_1})}{\partial(t,x,v)}\\
&= \frac{\partial(t^1, 0, \mathbf{x}^1_{\|_1}, \mathbf{v}^1_{\perp_1}, \mathbf{v}^1_{\|_1})}{\partial(s^1, X_{\mathbf{cl}}(s^1), V_{\mathbf{cl}}(s^1))}\frac{\partial(s^1, X_{\mathbf{cl}}(s^1), V_{\mathbf{cl}}(s^1))}{\partial(t,x,v)}\\
&= \frac{\partial(t^1, 0, \mathbf{x}^1_{\|_1}, \mathbf{v}^1_{\perp_1}, \mathbf{v}^1_{\|_1})}{\partial(s^1, \mathbf{x}_{\perp_1}(s^1), \mathbf{x}_{\|_1}(s^1), \mathbf{v}_{\perp_1}(s^1), \mathbf{v}_{\|_1}(s^1))}\frac{\partial(s^1, \mathbf{X}_1(s^1), \mathbf{V}_1(s^1))}{\partial(s^1, X_{\mathbf{cl}}(s^1), V_{\mathbf{cl}}(s^1))}\frac{\partial(s^1, X_{\mathbf{cl}}(s^1), V_{\mathbf{cl}}(s^1))}{\partial(t,x,v)}.
\end{aligned}
$$

We fix $\mathbf{p}^1-$spherical coordinate and drop the index of the chart.

Firstly, we claim

$$
\begin{aligned}
&\frac{\partial(t^1, 0, \mathbf{x}^1_{\|}, \mathbf{v}^1_{\perp}, \mathbf{v}^1_{\|})}{\partial(s^1, \mathbf{x}_{\perp}(s^1), \mathbf{x}_{\|}(s^1), \mathbf{v}_{\perp}(s^1), \mathbf{v}_{\|}(s^1))}\\
&\lesssim_\Omega \left[\begin{array}{c|cc|cc}
1 & \frac{1}{|\mathbf{v}^1_\perp|} & \frac{|v|^2|s^1-t^1|^2}{|\mathbf{v}^1_\perp|} & \frac{|s^1-t^1|}{|\mathbf{v}^1_\perp|} & \frac{|v||s^1-t^1|^2}{|\mathbf{v}^1_\perp|} \\ \hline
0 & 0 & \mathbf{0}_{1,2} & 0 & \mathbf{0}_{1,2} \\
\mathbf{0}_{2,1} & \frac{|v|}{|\mathbf{v}^1_\perp|} + |v|^2|s^1-t^1|^2 & \mathbf{Id}_{2,2} + |v||s^1-t^1| & \frac{|s^1-t^1||v|}{|\mathbf{v}^1_\perp|} + |s^1-t^1|^2|v| & |s^1-t^1| \\ \hline
0 & \frac{|v|^2}{|\mathbf{v}^1_\perp|} + |v|^2|s^1-t^1| & \frac{|v|^2}{|\mathbf{v}^1_\perp|} + |v|^2|s^1-t^1| & 1+|v||s^1-t^1| & |v||s^1-t^1| \\
\mathbf{0}_{2,1} & \frac{|v|^2}{|\mathbf{v}^1_\perp|} + |v|^2|s^1-t^1| & |v|^2|s^1-t^1| & 1+|v||s^1-t^1| & \mathbf{Id}_2 + |v||s^1-t^1|
\end{array}\right].
\end{aligned}
\tag{134}
$$

The t^1 is determined via $\mathbf{x}_\perp(t^1) = 0$, i.e.

$$
0 = \mathbf{x}_\perp(s^1) - \mathbf{v}_\perp(s^1)(s^1 - t^1) + \int_{t^1}^{s^1}\int_s^{s^1} F_\perp(\mathbf{X}_{\mathbf{cl}}(\tau), \mathbf{V}_{\mathbf{cl}}(\tau))\mathrm{d}\tau \mathrm{d}s, \tag{135}
$$

where $\mathbf{X}_{\mathbf{cl}}(\tau) = \mathbf{X}_{\mathbf{cl}}(\tau; s^1, \mathbf{X}_{\mathbf{cl}}(s^1;t,x,v), \mathbf{V}_{\mathbf{cl}}(s^1;t,x,v)), \mathbf{V}_{\mathbf{cl}}(\tau) = \mathbf{V}_{\mathbf{cl}}(\tau; s^1, \mathbf{X}_{\mathbf{cl}}(s^1;t,x,v), \mathbf{V}_{\mathbf{cl}}(s^1;t,x,v))$.

Recall that

$$\mathbf{v}_\perp^1 = -\lim_{s\downarrow t^1}\mathbf{v}_\perp(s) = -\mathbf{v}_\perp(s^1) + \int_{t^1}^{s^1} F_\perp(\mathbf{X_{cl}}(\tau),\mathbf{V_{cl}}(\tau))\mathrm{d}\tau,$$

$$\mathbf{x}_\parallel^1 = \mathbf{x}_\parallel(s^1) - (s^1-t^1)\mathbf{v}_\parallel(s^1) + \int_{t^1}^{s^1}\int_{\tau}^{s^1} F_\parallel(\mathbf{X_{cl}}(\tau),\mathbf{V_{cl}}(\tau))\mathrm{d}\tau\mathrm{d}s^1,$$

$$\mathbf{v}_\parallel^1 = \mathbf{v}_\parallel(s^1) - \int_{t^1}^{s^1} F_\parallel(\mathbf{X_{cl}}(\tau),\mathbf{V_{cl}}(\tau))\mathrm{d}\tau.$$

We use the fact that the ODE is autonomous to have $\frac{\partial t^1}{\partial s}=1$, $\frac{\partial(\mathbf{x}^1,\mathbf{v}^1)}{\partial s^1}=0$ and $|s^1-t^1|\lesssim_\xi \min\{\frac{|\mathbf{v}_\perp^1|}{|v|^2},t\}$ from (65) and (66), and (133) to have

$$\begin{aligned}
\frac{\partial t^1}{\partial \mathbf{x}_\perp(s^1)} &= \frac{1}{\mathbf{v}_\perp^1}\left\{1+\int_{t^1}^{s^1}\int_s^{s^1}\frac{\partial}{\partial \mathbf{x}_\perp(s^1)}F_\perp(\mathbf{X_{cl}}(\tau),\mathbf{V_{cl}}(\tau))\mathrm{d}\tau\mathrm{d}s\right\}\\
&\lesssim_\xi \frac{1}{|\mathbf{v}_\perp^1|}\left\{1+\int_{t^1}^{s^1}\int_s^{s^1}\big[1+|v|(s^1-\tau)\big]|v|^2e^{C_\xi|v|(s^1-\tau)}\mathrm{d}\tau\mathrm{d}s\right\}\\
&\lesssim_\xi \frac{1}{|\mathbf{v}_\perp^1|}\Big\{1+\big[1+|v||s^1-t^1|\big]|v|^2|s^1-t^1|^2e^{C_\xi|v||s^1-t^1|}\Big\}\lesssim_{\xi,t}\frac{1}{|\mathbf{v}_\perp^1|},\\
\frac{\partial t^1}{\partial \mathbf{x}_\parallel(s^1)} &= \frac{1}{\mathbf{v}_\perp^1}\int_{t^1}^{s^1}\int_s^{s^1}\frac{\partial}{\partial \mathbf{x}_\parallel(s^1)}F_\perp(\mathbf{X_{cl}}(\tau),\mathbf{V_{cl}}(\tau))\mathrm{d}\tau\mathrm{d}s\\
&\lesssim_\xi \frac{1}{|\mathbf{v}_\perp^1|}\left\{\int_{t^1}^{s^1}\int_s^{s^1}\big[1+|v|(s^1-\tau)\big]|v|^2e^{C_\xi|v|(s^1-\tau)}\mathrm{d}\tau\mathrm{d}s\right\}\\
&\lesssim_\xi \frac{1}{|\mathbf{v}_\perp^1|}\big[1+|v||s^1-t^1|\big]|v|^2|s^1-t^1|^2e^{C_\xi|v||s^1-t^1|}\lesssim_{\xi,t}\frac{|v|^2|s^1-t^1|^2}{|\mathbf{v}_\perp^1|},\\
\frac{\partial t^1}{\partial \mathbf{v}_\perp(s^1)} &= \frac{1}{\mathbf{v}_\perp^1}\left\{(t^1-s^1)+\int_{t^1}^{s^1}\int_s^{s^1}\frac{\partial}{\partial \mathbf{v}_\perp(s^1)}F_\perp(\mathbf{X_{cl}}(\tau),\mathbf{V_{cl}}(\tau))\mathrm{d}\tau\mathrm{d}s\right\}\\
&\lesssim_\xi \frac{|s^1-t^1|}{|\mathbf{v}_\perp^1|}+\frac{1}{|\mathbf{v}_\perp^1|}\int_{t^1}^{s^1}\int_s^{s^1}|v|\big[1+|v||s^1-\tau|\big]e^{C_\xi|v|(s^1-\tau)}\mathrm{d}\tau\mathrm{d}s\\
&\lesssim_\xi \frac{|s^1-t^1|}{|\mathbf{v}_\perp^1|}\Big\{1+|v||s^1-t^1|e^{C_\xi|v||s^1-t^1|}\Big\}\lesssim_{\xi,t}\frac{|s^1-t^1|}{|\mathbf{v}_\perp^1|},\\
\frac{\partial t^1}{\partial \mathbf{v}_\parallel(s_1)} &= \frac{1}{\mathbf{v}_\perp^1}\int_{t^1}^{s_1}\int_s^{s_1}\frac{\partial}{\partial \mathbf{v}_\parallel(s_1)}F_\perp(\mathbf{X_{cl}}(\tau),\mathbf{V_{cl}}(\tau))\mathrm{d}\tau\mathrm{d}s\\
&\lesssim_\xi \frac{1}{|\mathbf{v}_\perp^1|}\int_{t^1}^{s^1}\int_s^{s^1}|v|\big[1+|v||s^1-\tau|\big]e^{C_\xi|v|(s^1-\tau)}\mathrm{d}\tau\mathrm{d}s\lesssim_\xi\frac{|s^1-t^1|}{|\mathbf{v}_\perp^1|}|v||s^1-t^1|e^{C_\xi|v||s^1-t^1|}\\
&\lesssim_{\xi,t}\frac{|v||s^1-t^1|^2}{|\mathbf{v}_\perp^1|},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \mathbf{x}_\parallel^1}{\partial \mathbf{x}_\perp(s^1)} &= \frac{\partial t^1}{\partial \mathbf{x}_\perp(s^1)}\mathbf{v}_\parallel^1+\int_{t^1}^{s^1}\int_s^{s^1}\frac{\partial}{\partial \mathbf{x}_\perp(s^1)}F_\parallel(\mathbf{X_{cl}}(\tau),\mathbf{V_{cl}}(\tau))\mathrm{d}\tau\mathrm{d}s\\
&\lesssim_\xi \frac{|v|}{|\mathbf{v}_\perp^1|}+\Big(1+\frac{|v|}{|\mathbf{v}_\perp^1|}\Big)\big[1+|v||s^1-t^1|\big]|v|^2|s^1-t^1|^2e^{C_\xi|v||s^1-t^1|}\lesssim_{\xi,t}\frac{|v|}{|\mathbf{v}_\perp^1|}+|v|^2|s^1-t^1|^2,\\
\frac{\partial \mathbf{x}_\parallel^1}{\partial \mathbf{x}_\parallel(s^1)} &= \mathbf{Id}_{2,2}+\mathbf{v}_\parallel^1\frac{\partial t^1}{\partial \mathbf{x}_\parallel(s^1)}+\int_{t^1}^{s^1}\int_s^{s^1}\frac{\partial}{\partial \mathbf{x}_\parallel(s^1)}F_\parallel(\mathbf{X_{cl}}(\tau),\mathbf{V_{cl}}(\tau))\mathrm{d}\tau\mathrm{d}s\\
&\lesssim_{\xi,t}\mathbf{Id}_{2,2}+\Big(1+\frac{|v|}{|\mathbf{v}_\perp^1|}\Big)|v|^2|s^1-t^1|^2\lesssim_{\xi,t}\mathbf{Id}_{2,2}+|v||s^1-t^1|,
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial \mathbf{x}^1_{\parallel}}{\partial \mathbf{v}_{\perp}(s^1)} &= \frac{\partial t^1}{\partial \mathbf{v}_{\perp}(s^1)}\mathbf{v}^1_{\parallel} + \int_{t^1}^{s^1}\int_{s}^{s^1} \frac{\partial}{\partial \mathbf{v}_{\perp}(s^1)} F_{\parallel}(\mathbf{X_{cl}}(\tau), \mathbf{V_{cl}}(\tau)) \mathrm{d}\tau \mathrm{d}s \\
&\lesssim_{\xi,t} \frac{|s^1-t^1||v|}{|\mathbf{v}^1_{\perp}|} + |s^1-t^1|^2|v|, \\
\frac{\partial \mathbf{x}^1_{\parallel}}{\partial \mathbf{v}_{\parallel}(s^1)} &= -(s^1-t^1)\mathbf{Id}_{2,2} + \mathbf{v}^1_{\parallel}\frac{\partial t^1}{\partial \mathbf{v}_{\parallel}(s^1)} + \int_{t^1}^{s^1}\int_{s}^{s^1} \frac{\partial}{\partial \mathbf{v}_{\parallel}(s^1)} F_{\parallel}(\mathbf{X_{cl}}(\tau), \mathbf{V_{cl}}(\tau)) \mathrm{d}\tau \mathrm{d}s \\
&\lesssim_{\xi} -(s^1-t^1)\mathbf{Id}_{2,2} + |v|\frac{|s^1-t^1|}{|\mathbf{v}^1_{\perp}|}|v||s^1-t^1| + |v||s^1-t^1|^2\big[1+|v||s^1-t^1|\big] \\
&\lesssim_{\xi,t} |s^1-t^1|\Big(1+\frac{|v|^2|s^1-t^1|}{|\mathbf{v}^1_{\perp}|}\Big) \lesssim_{\xi,t} |s^1-t^1|,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial \mathbf{v}^1_{\perp}}{\partial \mathbf{x}_{\perp}(s^1)} &= \frac{-F_{\perp}(x^1,v)}{\mathbf{v}^1_{\perp}} - \frac{F_{\perp}(x^1,v)}{\mathbf{v}^1_{\perp}}\int_{t^1}^{s^1}\int_{s}^{s^1} \frac{\partial F_{\perp}(\mathbf{X_{cl}}(\tau), \mathbf{V_{cl}}(\tau))}{\partial \mathbf{x}_{\perp}(s^1)}\mathrm{d}\tau\mathrm{d}s + \int_{t^1}^{s^1} \frac{\partial F_{\perp}(\mathbf{X_{cl}}(\tau), \mathbf{V_{cl}}(\tau))}{\partial \mathbf{x}_{\perp}(s^1)}\mathrm{d}\tau \\
&\lesssim \frac{F_{\perp}(x^1,v)}{|\mathbf{v}^1_{\perp}|} + \Big(|v| + \frac{F_{\perp}(x^1,v)}{|\mathbf{v}^1_{\perp}|}|v||s^1-t^1|\Big)\big[1+|v||s^1-t^1|\big]|v||s^1-t^1|e^{C_{\xi}|v||s^1-t^1|} \\
&\lesssim \frac{|v|^2}{|\mathbf{v}^1_{\perp}|} + |v|^2|s^1-t^1|, \\
\frac{\partial \mathbf{v}^1_{\perp}}{\partial \mathbf{x}_{\parallel}(s^1)} &= \frac{-F_{\perp}(x^1,v)}{\mathbf{v}^1_{\perp}} + \int_{t^1}^{s^1} \frac{\partial}{\partial \mathbf{x}_{\parallel}(s^1)} F_{\perp}(\mathbf{X_{cl}}(\tau), \mathbf{V_{cl}}(\tau))\mathrm{d}\tau \\
&\lesssim \frac{|F_{\perp}(x^1,v)|}{|\mathbf{v}^1_{\perp}|} + |v|^2|s^1-t^1|\big(1+|v||s^1-t^1|\big)e^{C_{\xi}|v||s^1-t^1|} \lesssim \frac{|v|^2}{|\mathbf{v}^1_{\perp}|} + |v|^2|s^1-t^1|, \\
\frac{\partial \mathbf{v}^1_{\perp}}{\partial \mathbf{v}_{\perp}(s^1)} &= -1 + \frac{(s^1-t^1)F_{\perp}(x^1,v)}{\mathbf{v}^1_{\perp}} - \frac{F_{\perp}(x^1,v)}{\mathbf{v}^1_{\perp}}\int_{t^1}^{s^1}\int_{s}^{s^1} \frac{\partial F_{\perp}(\mathbf{X_{cl}}(\tau), \mathbf{V_{cl}}(\tau))}{\partial \mathbf{v}_{\perp}(s^1)} - \int_{s_1}^{t^1} \frac{\partial F_{\perp}(\mathbf{X_{cl}}(s), \mathbf{V_{cl}}(s))}{\partial \mathbf{v}_{\perp}(s_1)}\mathrm{d}s \\
&\lesssim -1 + \frac{|s^1-t^1||F_{\perp}(x^1,v)|}{|\mathbf{v}^1_{\perp}|}\Big\{1+|v||s^1-t^1|e^{C_{\xi}|v||s^1-t^1|}\Big\} + |v||s^1-t^1|e^{C_{\xi}|v||s^1-t^1|} \\
&\lesssim_{\xi} 1 + |v||s^1-t^1|, \\
\frac{\partial \mathbf{v}^1_{\perp}}{\partial \mathbf{v}_{\parallel}(s^1)} &= \frac{-F_{\perp}(x^1,v)}{\mathbf{v}^1_{\perp}}\int_{s^1}^{t^1}\int_{s^1}^{s} \frac{\partial F_{\perp}(\mathbf{X_{cl}}(\tau), \mathbf{V_{cl}}(\tau))}{\partial \mathbf{v}_{\parallel}(s^1)}\mathrm{d}\tau\mathrm{d}s - \int_{s^1}^{t^1} \frac{\partial F_{\perp}(\mathbf{X_{cl}}(\tau), \mathbf{V_{cl}}(\tau))}{\partial \mathbf{v}_{\parallel}(s^1)}\mathrm{d}\tau \\
&\lesssim \frac{|v|^2}{|\mathbf{v}^1_{\perp}|}|s^1-t^1|^2|v| + |s^1-t^1||v| \lesssim |v||s^1-t^1|\Big(1+\frac{|s^1-t^1||v|^2}{|\mathbf{v}^1_{\perp}|}\Big) \lesssim |v||s^1-t^1|,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial \mathbf{v}^1_{\parallel}}{\partial \mathbf{x}_{\perp}(s^1)} &= \frac{\partial t^1}{\partial \mathbf{x}_{\perp}(s^1)}F_{\parallel}(x^1,v) - \int_{t^1}^{s^1} \frac{\partial}{\partial \mathbf{x}_{\perp}(s^1)} F_{\parallel}(\mathbf{X_{cl}}(\tau), \mathbf{V_{cl}}(\tau))\mathrm{d}\tau \\
&\lesssim_{\xi} \frac{|F_{\parallel}(x^1,v)|}{|\mathbf{v}^1_{\perp}|}\big\{1+[1+|v||s^1-t^1|]|v|^2|s^1-t^1|^2e^{C_{\xi}|v||s^1-t^1|}\big\} + |v|^2|s^1-t^1|[1+|v||s^1-t^1|]e^{C_{\xi}|v||s^1-t^1|} \\
&\lesssim_{\xi,t} \frac{|v|^2}{|\mathbf{v}^1_{\perp}|} + |v|^2|s^1-t^1|, \\
\frac{\partial \mathbf{v}^1_{\parallel}}{\partial \mathbf{x}_{\parallel}(s^1)} &= \frac{\partial t^1}{\partial \mathbf{x}_{\parallel}(s^1)}F_{\parallel}(x^1,v) - \int_{t^1}^{s^1} \frac{\partial}{\partial \mathbf{x}_{\parallel}(s^1)} F_{\parallel}(\mathbf{X_{cl}}(\tau), \mathbf{V_{cl}}(\tau))\mathrm{d}\tau \\
&\lesssim \frac{|F_{\parallel}(x^1,v)|}{|\mathbf{v}^1_{\perp}|}\big[1+|v||s^1-t^1|\big]|v|^2|s^1-t^1|^2e^{C_{\xi}|v||s^1-t^1|} + |v|^2|s^1-t^1|\big[1+|v||s^1-t^1|\big]e^{C_{\xi}|v||s^1-t^1|} \\
&\lesssim_{\xi,t} |v|^2|s^1-t^1|,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathbf{v}_{\|}^1}{\partial \mathbf{v}_{\perp}(s^1)} &= \frac{\partial t^1}{\partial \mathbf{v}_{\perp}(s^1)} F_{\|}(x^1, v) - \int_{t^1}^{s^1} \frac{\partial}{\partial \mathbf{v}_{\perp}(s^1)} F_{\|}(\mathbf{X}_{\mathbf{cl}}(\tau), \mathbf{V}_{\mathbf{cl}}(\tau)) \mathrm{d}\tau \\
&\lesssim \frac{|s^1 - t^1||v|^2}{|\mathbf{v}_{\perp}^1|} + |v||s^1 - t^1| \lesssim 1 + |v||s^1 - t^1|, \\
\frac{\partial \mathbf{v}_{\|}^1}{\partial \mathbf{v}_{\|}(s^1)} &= \mathbf{Id}_{2,2} + \frac{\partial t^1}{\partial \mathbf{v}_{\|}(s^1)} F_{\|}(x^1, v) - \int_{t^1}^{s^1} \frac{\partial}{\partial \mathbf{v}_{\|}(s^1)} F_{\|}(\mathbf{X}_{\mathbf{cl}}(\tau), \mathbf{V}_{\mathbf{cl}}(\tau) \mathrm{d}\tau \\
&\lesssim \mathbf{Id}_{2,2} + \frac{|v|^3 |s^1 - t^1|^2}{|\mathbf{v}_{\perp}^1|} + |s^1 - t^1||v| \big[1 + |v||s^1 - t^1|\big] \lesssim_{\xi, t} \mathbf{Id}_{2,2} + |v||s^1 - t^1|.
\end{aligned}
$$

Secondly, we claim

$$
\begin{aligned}
&\frac{\partial(s^1, \mathbf{X}_1(s^1), \mathbf{V}_1(s^1))}{\partial(t, x, v)} = \frac{\partial(s^1, \mathbf{X}_1(s^1), \mathbf{V}_1(s^1))}{\partial(s^1, X_{\mathbf{cl}}(s^1), V_{\mathbf{cl}}(s^1))} \frac{\partial(s^1, X_{\mathbf{cl}}(s^1), V_{\mathbf{cl}}(s^1))}{\partial(t, x, v)} \\
&= \left[\begin{array}{c|c|c}
1 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ \hline
 & \frac{(\partial_1 \eta \times \partial_2 \eta)^T}{\mathbf{n} \cdot (\partial_1 \eta \times \partial_2 \eta)} + O_\xi(|v||t^1 - s^1|) & O_\xi(|t - s^1|) \\
\mathbf{0}_{3,1} & \frac{(\partial_2 \eta \times \mathbf{n})^T}{\mathbf{n} \cdot (\partial_1 \eta \times \partial_2 \eta)} + O_\xi(|v||t^1 - s^1|) & O_\xi(|t - s^1|) \\
 & \frac{(\mathbf{n} \times \partial_2 \eta)^T}{\mathbf{n} \cdot (\partial_1 \eta \times \partial_2 \eta)} + O_\xi(|v||t^1 - s^1|) & O_\xi(|t - s^1|) \\ \hline
 & O_\xi(|v|) & \frac{(\partial_1 \eta \times \partial_2 \eta)^T}{\mathbf{n} \cdot (\partial_1 \eta \times \partial_2 \eta)} + O_\xi(|v||t - s^1|) \\
\mathbf{0}_{3,1} & O_\xi(|v|) & \frac{(\partial_2 \eta \times \mathbf{n})^T}{\mathbf{n} \cdot (\partial_1 \eta \times \partial_2 \eta)} + O_\xi(|v||t - s^1|) \\
 & O_\xi(|v|) & \frac{(\mathbf{n} \times \partial_1 \eta)^T}{\mathbf{n} \cdot (\partial_1 \eta \times \partial_2 \eta)} + O_\xi(|v||t - s^1|)
\end{array}\right],
\end{aligned} \tag{136}
$$

where the entries are evaluated at $(\mathbf{X}_1(s^1), \mathbf{V}_1(s^1))$. Note that $|v||t^1 - s^1| \lesssim_\xi 1$.

Clearly

$$
\left[\begin{array}{c}
\partial s^1 / \partial(s^1, X_{\mathbf{cl}}(s^1), V_{\mathbf{cl}}(s^1)) \\
\partial \mathbf{X}_{\mathbf{cl}}(s^1) / \partial(s^1, X_{\mathbf{cl}}(s^1), V_{\mathbf{cl}}(s^1)) \\
\partial \mathbf{V}_{\mathbf{cl}}(s^1) / \partial(s^1, X_{\mathbf{cl}}(s^1), V_{\mathbf{cl}}(s^1))
\end{array}\right] = \left[\begin{array}{c|c}
1 & \mathbf{0}_{1,6} \\ \hline
\mathbf{0}_{6,1} & \frac{\partial(\mathbf{X}_{\mathbf{cl}}(s^1), \mathbf{V}_{\mathbf{cl}}(s^1))}{\partial(X_{\mathbf{cl}}(s^1), V_{\mathbf{cl}}(s^1))}
\end{array}\right].
$$

Now we consider the right lower 6 by 6 submatrix. Recall, from (120)

$$
\frac{\partial(X_{\mathbf{cl}}(s^1), V_{\mathbf{cl}}(s^1))}{\partial(\mathbf{X}_{\mathbf{cl}}(s^1), \mathbf{V}_{\mathbf{cl}}(s^1))} = \frac{\partial \Phi(\mathbf{X}_{\mathbf{cl}}(s^1), \mathbf{V}_{\mathbf{cl}}(s))}{\partial(\mathbf{X}_{\mathbf{cl}}(s^1), \mathbf{V}_{\mathbf{cl}}(s))} := \left[\begin{array}{c|c} A & \mathbf{0}_{3,3} \\ \hline B & A \end{array}\right] + \mathbf{x}_\perp \left[\begin{array}{c|c} C & \mathbf{0}_{3,3} \\ \hline D & C \end{array}\right].
$$

Note that, from (118) and (112),

$$
\begin{aligned}
\det(A) &= \det\Big[\ [-\mathbf{n}(\mathbf{x}_\|)] \quad \partial_{\mathbf{x}_{\|,1}} \eta(\mathbf{x}_\|) \quad \partial_{\mathbf{x}_{\|,2}} \eta(\mathbf{x}_\|)\ \Big] = [-\mathbf{n}(\mathbf{x}_\|)] \cdot \big(\partial_{\mathbf{x}_{\|,1}} \eta(\mathbf{x}_\|) \times \partial_{\mathbf{x}_{\|,2}} \eta(x_\|)\big) \neq 0, \\
A^{-1} &= \frac{1}{[-\mathbf{n}] \cdot (\partial_{\mathbf{x}_{\|,1}} \eta \times \partial_{\mathbf{x}_{\|,2}} \eta)} \Big[(\partial_{\mathbf{x}_{\|,1}} \eta \times \partial_{\mathbf{x}_{\|,2}} \eta)^T, (\partial_{\mathbf{x}_{\|,2}} \eta \times [-\mathbf{n}])^T, ([-\mathbf{n}] \times \partial_{\mathbf{x}_{\|,1}} \eta)^T\Big].
\end{aligned}
$$

Since

$$
\begin{aligned}
\det(A + \mathbf{x}_\perp C) &= [-\mathbf{n}(\mathbf{x}_\|)] \cdot \big((\partial_1 \eta + \mathbf{x}_\perp[-\partial_1 \mathbf{n}]) \times (\partial_2 \eta + \mathbf{x}_\perp[-\partial_2 \mathbf{n}])\big) \\
&= [-\mathbf{n}] \cdot (\partial_1 \eta \times \partial_2 \eta) + O_\xi(\mathbf{x}_\perp) \neq 0, \quad \text{for} \quad |\mathbf{x}_\perp| \ll_\Omega 1,
\end{aligned}
$$

the matrix $A+\mathbf{x}_\perp C$ is invertible for $|\mathbf{x}_\perp|\ll_\Omega 1$ and directly

$$\begin{aligned}
(A+\mathbf{x}_\perp C)^{-1} &= \frac{1}{\det(A+\mathbf{x}_\perp C)}\begin{bmatrix} \big((\partial_1\eta+\mathbf{x}_\perp[-\partial_1\mathbf{n}])\times(\partial_2\eta+\mathbf{x}_\perp[-\partial_2\mathbf{n}])\big)^T \\ \big((\partial_2\eta+\mathbf{x}_\perp[-\partial_2\mathbf{n}])\times[-\mathbf{n}]\big)^T \\ \big([-\mathbf{n}]\times(\partial_1\eta+\mathbf{x}_\perp[-\partial_1\mathbf{n}])\big)^T \end{bmatrix}\\
&= \Big\{\frac{1}{[-\mathbf{n}]\cdot(\partial_1\eta\times\partial_2\eta)}+\frac{O_\xi(\mathbf{x}_\perp)}{[-\mathbf{n}\cdot(\partial_1\eta\times\partial_2\eta)]^2}\Big\}\begin{bmatrix} (\partial_1\eta\times\partial_2\eta)^T+O_\xi(\mathbf{x}_\perp) \\ (\partial_2\eta\times[-\mathbf{n}])^T \\ ([-\mathbf{n}]\times\partial_1\eta)^T \end{bmatrix}\\
&= \frac{1}{[-\mathbf{n}]\cdot(\partial_1\eta\times\partial_2\eta)}\begin{bmatrix} (\partial_1\eta\times\partial_2\eta)^T \\ (\partial_2\eta\times[-\mathbf{n}])^T \\ (-\mathbf{n}\times\partial_1\eta)^T \end{bmatrix}+O_\xi(\mathbf{x}_\perp)\\
&= A^{-1}+O_\xi(\mathbf{x}_\perp), \quad \text{for } \ |\mathbf{x}_\perp|\ll_\Omega 1.
\end{aligned}$$

From basic linear algebra

$$\begin{aligned}
&\det\left(\frac{\partial(X_{\mathbf{cl}}(s^1),V_{\mathbf{cl}}(s^1))}{\partial(\mathbf{X}_{\mathbf{cl}}(s^1),\mathbf{V}_{\mathbf{cl}}(s^1))}\right)=\det\left[\begin{array}{c|c} A+\mathbf{x}_\perp C & \mathbf{0}_{3,3} \\ \hline B+\mathbf{x}_\perp D & A+\mathbf{x}_\perp C\end{array}\right]=\{\det(A+\mathbf{x}_\perp C)\}^2\\
&=\Big\{[-\mathbf{n}]\cdot(\partial_1\eta\times\partial_2\eta)+\mathbf{x}_\perp[-\mathbf{n}]\cdot(\partial_1[-\mathbf{n}]\times\partial_2\eta)\\
&\qquad +\mathbf{x}_\perp[-\mathbf{n}]\cdot(\partial_1\eta\times\partial_2[-\mathbf{n}])+(\mathbf{x}_\perp)^2[-\mathbf{n}]\cdot(\partial_1\mathbf{n}\times\partial_2\mathbf{n})\Big\}^2\\
&=\big\{[-\mathbf{n}]\cdot(\partial_1\eta\times\partial_2\eta)\big\}^2+O_\xi(\mathbf{x}_\perp), \quad \text{for } \ |\mathbf{x}_\perp|\ll_\Omega 1,
\end{aligned}$$

and $\left(\frac{\partial(X_{\mathbf{cl}}(s^1),V_{\mathbf{cl}}(s^1))}{\partial(\mathbf{X}_{\mathbf{cl}}(s^1),\mathbf{V}_{\mathbf{cl}}(s^1))}\right)$ is invertible. By the basic linear algebra

$$\begin{aligned}
&\frac{\partial(\mathbf{X}_{\mathbf{cl}}(s^1),\mathbf{V}_{\mathbf{cl}}(s^1))}{\partial(X_{\mathbf{cl}}(s^1),V_{\mathbf{cl}}(s^1))}=\left[\frac{\partial(X_{\mathbf{cl}}(s^1),V_{\mathbf{cl}}(s^1))}{\partial(\mathbf{X}_{\mathbf{cl}}(s^1),\mathbf{V}_{\mathbf{cl}}(s^1))}\right]^{-1}=\left[\begin{array}{c|c} A+\mathbf{x}_\perp C & \mathbf{0}_{3,3} \\ \hline B+\mathbf{x}_\perp D & A+\mathbf{x}_\perp C\end{array}\right]^{-1}\\
&=\left[\begin{array}{c|c} (A+\mathbf{x}_\perp C)^{-1} & \mathbf{0}_{3,3} \\ \hline -(A+\mathbf{x}_\perp C)^{-1}(B+\mathbf{x}_\perp D)(A+\mathbf{x}_\perp C)^{-1} & (A+\mathbf{x}_\perp C)^{-1}\end{array}\right],\\
&=\begin{bmatrix} A^{-1}+O_\xi(\mathbf{x}_\perp) & \mathbf{0}_{3,3} \\ -A^{-1}BA^{-1}+O_\xi(\mathbf{x}_\perp) & A^{-1}+O_\xi(x_\perp)\end{bmatrix}\\
&=\begin{bmatrix} A^{-1}(\mathbf{x}_\parallel)+O_\xi(\mathbf{x}_\perp) & \mathbf{0}_{3,3} \\ |v|+O_\xi(\mathbf{x}_\perp) & A^{-1}(\mathbf{x}_\parallel)+O_\xi(\mathbf{x}_\perp)\end{bmatrix},
\end{aligned} \tag{137}$$

and we obtain

$$\frac{\partial(s^1,\mathbf{X}_{\mathbf{cl}}(s^1),\mathbf{V}_{\mathbf{cl}}(s^1))}{\partial(s^1,X_{\mathbf{cl}}(s^1),V_{\mathbf{cl}}(s^1))}=\left[\begin{array}{c|c|c} 1 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ \hline 0 & \frac{(\partial_1\eta\times\partial_2\eta)^T}{[-\mathbf{n}]\cdot(\partial_1\eta\times\partial_2\eta)}+O_\xi(\mathbf{x}_\perp) & \\ 0 & \frac{(\partial_2\eta\times[-\mathbf{n}])^T}{[-\mathbf{n}]\cdot(\partial_1\eta\times\partial_2\eta)}+O_\xi(\mathbf{x}_\perp) & \mathbf{0}_{3,3} \\ 0 & \frac{([-\mathbf{n}]\times\partial_1\eta)^T}{[-\mathbf{n}]\cdot(\partial_1\eta\times\partial_2\eta)}+O_\xi(\mathbf{x}_\perp) & \\ \hline 0 & O_\xi(1)(|v|) & \frac{(\partial_1\eta\times\partial_2\eta)^T}{[-\mathbf{n}]\cdot(\partial_1\eta\times\partial_2\eta)}+O_\xi(\mathbf{x}_\perp) \\ 0 & O_\xi(1)(|v|) & \frac{(\partial_2\eta\times[-\mathbf{n}])^T}{[-\mathbf{n}]\cdot(\partial_1\eta\times\partial_2\eta)}+O_\xi(\mathbf{x}_\perp) \\ 0 & O_\xi(1)(|v|) & \frac{([-\mathbf{n}]\times\partial_1\eta)^T}{[-\mathbf{n}]\cdot(\partial_1\eta\times\partial_2\eta)}+O_\xi(\mathbf{x}_\perp) \end{array}\right].$$

From $X_{\mathbf{cl}}(s_1;t,x,v)=x-(t-s_1)v=x-\Delta\times v$ and $V_{\mathbf{cl}}(s_1;t,x,v)=v$,

$$\frac{\partial(s_1,X_{\mathbf{cl}}(s_1),V_{\mathbf{cl}}(s_1))}{\partial(t,x,v)}=\begin{bmatrix} 1 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ \mathbf{0}_{3,1} & \mathbf{Id}_{3,3} & -(t-s^1)\mathbf{Id}_{3,3} \\ \mathbf{0}_{3,1} & \mathbf{0}_{3,3} & \mathbf{Id}_{3,3}\end{bmatrix}.$$

Finally we multiply above two matrices and use $|\mathbf{x}_\perp(s^1)|\lesssim|v||t^1-s^1|$ to conclude the second claim.

Step 4. Estimate of $\partial(t^{\ell+1},0,\mathbf{x}^{\ell+1}_{\parallel_{\ell+1}},\mathbf{v}^{\ell+1}_{\perp_{\ell+1}},\mathbf{v}^{\ell+1}_{\parallel_{\ell+1}})/\partial(t^\ell,0,\mathbf{x}^\ell_{\parallel_\ell},\mathbf{v}^\ell_{\perp_\ell},\mathbf{v}^\ell_{\parallel_\ell})$

Recall $\mathbf{r}^\ell$ from (124). We show that there exists $M = M_{\xi,t} \gg 1$, which is only depending on Ω, such that for all $\ell \in \mathbb{N}$ and $0 \le t^{\ell+1} \le t^\ell \le t$ and $v \in \mathbb{R}^3$,

$$
\begin{aligned}
J_\ell^{\ell+1} &:= \frac{\partial(t^{\ell+1}, 0, \mathbf{x}^{\ell+1}_{\|_{\ell+1}}, \mathbf{v}^{\ell+1}_{\perp_{\ell+1}}, \mathbf{v}^{\ell+1}_{\|_{\ell+1}})}{\partial(t^\ell, 0, \mathbf{x}^{\ell}_{\|_{\ell}}, \mathbf{v}^{\ell}_{\perp_{\ell}}, \mathbf{v}^{\ell}_{\|_{\ell}})} \\
&\le \left[\begin{array}{c|cccc|ccc}
1 & 0 & \frac{M}{|v|}\mathbf{r}^{\ell+1} & \frac{M}{|v|}\mathbf{r}^{\ell+1} & & \frac{M}{|v|^2} & \frac{M}{|v|^2}\mathbf{r}^{\ell+1} & \frac{M}{|v|^2}\mathbf{r}^{\ell+1} \\
\hline
0 & 0 & 0 & 0 & & 0 & 0 & 0 \\
0 & 0 & 1+M\mathbf{r}^{\ell+1} & M\mathbf{r}^{\ell+1} & & \frac{M}{|v|} & \frac{M}{|v|}\mathbf{r}^{\ell+1} & \frac{M}{|v|}\mathbf{r}^{\ell+1} \\
0 & 0 & M\mathbf{r}^{\ell+1} & 1+M\mathbf{r}^{\ell+1} & & \frac{M}{|v|} & \frac{M}{|v|}\mathbf{r}^{\ell+1} & \frac{M}{|v|}\mathbf{r}^{\ell+1} \\
\hline
0 & 0 & M|v|(\mathbf{r}^{\ell+1})^2 & M|v|(\mathbf{r}^{\ell+1})^2 & & 1+M\mathbf{r}^{\ell+1} & M(\mathbf{r}^{\ell+1})^2 & M(\mathbf{r}^{\ell+1})^2 \\
0 & 0 & M|v|\mathbf{r}^{\ell+1} & M|v|\mathbf{r}^{\ell+1} & & M & 1+M\mathbf{r}^{\ell+1} & M\mathbf{r}^{\ell+1} \\
0 & 0 & M|v|\mathbf{r}^{\ell+1} & M|v|\mathbf{r}^{\ell+1} & & M & M\mathbf{r}^{\ell+1} & 1+M\mathbf{r}^{\ell+1}
\end{array}\right] \\
&:= \underbrace{J(\mathbf{r}^{\ell+1})}_{\text{Definition of } J(\mathbf{r}^{\ell+1})} .
\end{aligned} \tag{138}
$$

Note this bound holds for both *Type I* and *Type II* in (125). We split the proof for each *Type*:

Proof of (138) when $\mathbb{R}^\ell < \sqrt{\delta}$ *and* $\mathbb{R}^{\ell+1} < \sqrt{\delta}$*:* Note that $\mathbf{p}^\ell-$spherical coordinate is well-defined of all $\tau \in [t^{\ell+1}, t^\ell]$. Due to the chart changing

$$
\frac{\partial(t^{\ell+1}, 0, \mathbf{x}^{\ell+1}_{\|_{\ell+1}}, \mathbf{v}^{\ell+1}_{\perp_{\ell+1}}, \mathbf{v}^{\ell+1}_{\|_{\ell+1}})}{\partial(t^\ell, 0, \mathbf{x}^{\ell}_{\|_{\ell}}, \mathbf{v}^{\ell}_{\perp_{\ell}}, \mathbf{v}^{\ell}_{\|_{\ell}})} = \left[\begin{array}{c|c} 1 & \mathbf{0}_{1,6} \\ \hline \mathbf{0}_{6,1} & \nabla\Phi^{-1}_{\mathbf{p}^\ell}\nabla\Phi_{\mathbf{p}^{\ell+1}} \end{array}\right] \underbrace{\frac{\partial(t^{\ell+1}, 0, \mathbf{x}^{\ell+1}_{\|_{\ell}}, \mathbf{v}^{\ell+1}_{\perp_{\ell}}, \mathbf{v}^{\ell+1}_{\|_{\ell}})}{\partial(t^\ell, 0, \mathbf{x}^{\ell}_{\|_{\ell}}, \mathbf{v}^{\ell}_{\perp_{\ell}}, \mathbf{v}^{\ell}_{\|_{\ell}})}}_{\tilde{J}_\ell^{\ell+1}} .
$$

Note that

$$
\begin{aligned}
|\mathbf{p}^\ell - \mathbf{p}^{\ell+1}| &\le |z^\ell - z^{\ell+1}| + |u^\ell - u^{\ell+1}| \\
&\lesssim |z^\ell - x^\ell| + |x^\ell - x^{\ell+1}| + |z^{\ell+1} - x^{\ell+1}| \\
&\quad + \left|u^\ell - \frac{v^\ell - (v^\ell\cdot n(z^\ell))n(z^\ell)}{|v^\ell - (v^\ell\cdot n(z^\ell))n(z^\ell)|}\right| + \left|u^{\ell+1} - \frac{v^{\ell+1} - (v^{\ell+1}\cdot n(z^{\ell+1}))n(z^{\ell+1})}{|v^{\ell+1} - (v^{\ell+1}\cdot n(z^{\ell+1}))n(z^{\ell+1})|}\right| \\
&\quad + \frac{|\mathbf{v}^\ell_\perp| + |\mathbf{v}^{\ell+1}_\perp| + |v||x^\ell - z^\ell| + |v||x^{\ell+1} - z^{\ell+1}|}{|v^\ell - (v^\ell\cdot n(z^\ell))n(z^\ell)|} \\
&\lesssim_\xi \mathbf{r}^\ell.
\end{aligned}
$$

where we have used $\mathbf{r}^\ell \le C\sqrt{\delta}$(therefore$|v^\ell - (v^\ell\cdot n(z^\ell))n(z^\ell)| \gtrsim (1-\sqrt{\delta})|v|$) and (126) and (65).

From (117)

$$
\begin{aligned}
\left[\begin{array}{c|c} 1 & \mathbf{0}_{1,6} \\ \hline \mathbf{0}_{6,1} & \nabla\Phi^{-1}_{\mathbf{p}^\ell}\nabla\Phi_{\mathbf{p}^{\ell+1}} \end{array}\right] \tilde{J}_\ell^{\ell+1} &\le \left[\begin{array}{c|ccc|ccc}
1 & & \mathbf{0}_{1,3} & & & \mathbf{0}_{1,3} & \\
\hline
 & 1 & 0 & 0 & & & \\
\mathbf{0}_{3,1} & 0 & 1+C\mathbf{r}^{\ell+1} & C\mathbf{r}^{\ell+1} & & \mathbf{0}_{3,3} & \\
 & 0 & C\mathbf{r}^{\ell+1} & 1+C\mathbf{r}^{\ell+1} & & & \\
\hline
 & 0 & 0 & 0 & 1 & 0 & 0 \\
\mathbf{0}_{3,1} & 0 & C\mathbf{r}^{\ell+1}|v| & C\mathbf{r}^{\ell+1}|v| & 0 & 1+C\mathbf{r}^{\ell+1} & C\mathbf{r}^{\ell+1} \\
 & 0 & C\mathbf{r}^{\ell+1}|v| & C\mathbf{r}^{\ell+1}|v| & 0 & C\mathbf{r}^{\ell+1} & 1+C\mathbf{r}^{\ell+1}
\end{array}\right] \tilde{J}_\ell^{\ell+1} \\
&\le J(C\mathbf{r}^{\ell+1}).
\end{aligned}
$$

Therefore in order to show (138) it suffices to show that $\tilde{J}_\ell^{\ell+1}$ is bounded as (138):

$$
\tilde{J}_\ell^{\ell+1} \ \le \ J(\mathbf{r}^{\ell+1}). \tag{139}
$$

In the proof of the claim (139), we fix the $\mathbf{p}^\ell-$spherical coordinate and drop the index ℓ for the chart.

The $t^{\ell+1}$ is determined through

$$0=\mathbf{v}_\perp^\ell(t^{\ell+1}-t^\ell)+\int_{t^{\ell+1}}^{t^\ell}\int_s^{t^\ell}F_\perp(\mathbf{X_{cl}}(\tau),\mathbf{V_{cl}}(\tau))\mathrm{d}\tau\mathrm{d}s. \tag{140}$$

Now we use

$$\mathbf{v}_\perp^{\ell+1}=-\lim_{s\downarrow t^{\ell+1}}\big\{\mathbf{v}_\perp^\ell+\int_{t^\ell}^s F_\perp(\mathbf{X_{cl}}(\tau),\mathbf{V_{cl}}(\tau))\mathrm{d}\tau\big\}=-\mathbf{v}_\perp^\ell+\int_{\ell+1}^{t^\ell}F_\perp(\mathbf{X_{cl}}(\tau),\mathbf{V_{cl}}(\tau))\mathrm{d}\tau,$$

and $|t^\ell-t^{\ell+1}|\lesssim_{\xi,t}\min\{\frac{|\mathbf{v}_\perp^{\ell+1}|}{|v|^2},1\}$, and (133) to have $\frac{\partial t^{\ell+1}}{\partial t^\ell}=1$ and

$$\begin{aligned}
\frac{\partial t^{\ell+1}}{\partial \mathbf{x}_\parallel^\ell}&=\frac{1}{\mathbf{v}_\perp^{\ell+1}}\int_{t^\ell}^{t^{\ell+1}}\int_{t^\ell}^s\frac{\partial}{\partial \mathbf{x}_\parallel^\ell}F_\perp(\mathbf{X_{cl}}(\tau),\mathbf{V_{cl}}(\tau))\mathrm{d}\tau\mathrm{d}s\\
&\lesssim\frac{|t^\ell-t^{\ell+1}|^2}{|\mathbf{v}_\perp^{\ell+1}|}|v^\ell|^2[1+|v^\ell||t^\ell-t^{\ell+1}|]\lesssim\frac{|v^\ell|^2|t^\ell-t^{\ell+1}|^2}{|\mathbf{v}_\perp^{\ell+1}|}\lesssim_{\xi,t}|t^\ell-t^{\ell+1}|\lesssim_{\xi,t}\frac{1}{|v|}\frac{|\mathbf{v}_\perp^{\ell+1}|}{|v|},\\
\frac{\partial t^{\ell+1}}{\partial \mathbf{v}_\perp^\ell}&=\frac{1}{\mathbf{v}_\perp^{\ell+1}}\left\{(t^{\ell+1}-t^\ell)+\int_{t^\ell}^{t^{\ell+1}}\int_{t^\ell}^s\frac{\partial}{\partial \mathbf{v}_\perp^\ell}F_\perp(\mathbf{X_{cl}}(\tau),\mathbf{V_{cl}}(\tau))\mathrm{d}\tau\mathrm{d}s\right\}\\
&\lesssim\{1+|v^\ell||t^\ell-t^{\ell+1}|\}\frac{|t^\ell-t^{\ell+1}|}{|\mathbf{v}_\perp^{\ell+1}|}\lesssim_{\xi,t}\frac{|t^\ell-t^{\ell+1}|}{|\mathbf{v}_\perp^{\ell+1}|}\lesssim_{\xi,t}\frac{1}{|v|^2},\\
\frac{\partial t^{\ell+1}}{\partial \mathbf{v}_\parallel^\ell}&=\frac{1}{\mathbf{v}_\perp^{\ell+1}}\int_{t^{\ell+1}}^{t^\ell}\int_s^{t^\ell}\frac{\partial}{\partial \mathbf{v}_\parallel^\ell}F_\perp(\mathbf{X_{cl}}(\tau),\mathbf{V_{cl}}(\tau))\mathrm{d}\tau\mathrm{d}s\lesssim\frac{|v^\ell||t^\ell-t^{\ell+1}|^2}{|\mathbf{v}_\perp^{\ell+1}|}\lesssim_{\xi,t}\frac{1}{|v|^2}\frac{|\mathbf{v}_\perp^{\ell+1}|}{|v|},
\end{aligned}$$

and $\frac{\partial \mathbf{x}_\parallel^{\ell+1}}{\partial t^\ell}=0$,

$$\begin{aligned}
\frac{\partial \mathbf{x}_\parallel^{\ell+1}}{\partial \mathbf{x}_\parallel^\ell}&=\mathbf{Id}_{2,2}+\frac{\mathbf{v}_\parallel^{\ell+1}}{\mathbf{v}_\perp^{\ell+1}}\int_{t^\ell}^{t^{\ell+1}}\int_{t^\ell}^s\frac{\partial}{\partial \mathbf{x}_\parallel^\ell}F_\perp(\mathbf{X_{cl}}(\tau),\mathbf{V_{cl}}(\tau))\mathrm{d}\tau\mathrm{d}s+\int_{t^\ell}^{t^{\ell+1}}\int_{t^\ell}^s\frac{\partial}{\partial \mathbf{x}_\parallel^\ell}F_\parallel(\mathbf{X_{cl}}(\tau),\mathbf{V_{cl}}(\tau))\mathrm{d}\tau\mathrm{d}s\\
&\lesssim\mathbf{Id}_{2,2}+\Big(1+\frac{|v^\ell|}{|\mathbf{v}_\perp^{\ell+1}|}\Big)|t^{\ell+1}-t^\ell|^2|v^\ell|^2\lesssim\mathbf{Id}_{2,2}+\frac{|\mathbf{v}_\perp^\ell|}{|v|},\\
\frac{\partial \mathbf{x}_\parallel^{\ell+1}}{\partial \mathbf{v}_\perp^\ell}&=\Big\{\frac{t^{\ell+1}-t^\ell}{\mathbf{v}_\perp^{\ell+1}}+\frac{1}{\mathbf{v}_\perp^{\ell+1}}\int_{t^\ell}^{t^{\ell+1}}\int_{t^\ell}^s\frac{\partial}{\partial \mathbf{v}_\perp^\ell}F_\perp(\mathbf{X_{cl}}(\tau),\mathbf{V_{cl}}(\tau))\mathrm{d}\tau\mathrm{d}s\Big\}\mathbf{v}_\parallel^{\ell+1}\\
&\quad+\int_{t^\ell}^{t^{\ell+1}}\int_{t^\ell}^s\frac{\partial}{\partial \mathbf{v}_\perp^\ell}F_\parallel(\mathbf{X_{cl}}(\tau),\mathbf{V_{cl}}(\tau))\mathrm{d}\tau\mathrm{d}s\\
&\lesssim|t^\ell-t^{\ell+1}|\frac{|v^\ell|}{|\mathbf{v}_\perp^{\ell+1}|}+|t^\ell-t^{\ell+1}|^2\frac{|v^\ell|^2}{|\mathbf{v}_\perp^{\ell+1}|}+|t^\ell-t^{\ell+1}|^2|v^\ell|\lesssim\frac{1}{|v|},\\
\frac{\partial \mathbf{x}_\parallel^{\ell+1}}{\partial \mathbf{v}_\parallel^\ell}&=(t^{\ell+1}-t^\ell)+\frac{\mathbf{v}_\parallel^{\ell+1}}{\mathbf{v}_\perp^{\ell+1}}\int_{t^\ell}^{t^{\ell+1}}\int_{t^\ell}^s\frac{\partial}{\partial \mathbf{v}_\parallel^\ell}F_\perp(\mathbf{X_{cl}}(\tau),\mathbf{V_{cl}}(\tau))\mathrm{d}\tau\mathrm{d}s\\
&\quad+\int_{t^\ell}^{t^{\ell+1}}\int_{t^\ell}^s\frac{\partial}{\partial \mathbf{v}_\parallel^\ell}F_\perp(\mathbf{X_{cl}}(\tau),\mathbf{V_{cl}}(\tau))\mathrm{d}\tau\mathrm{d}s\\
&\lesssim|t^\ell-t^{\ell+1}|+\Big(1+\frac{|v^\ell|}{|\mathbf{v}_\perp^{\ell+1}|}\Big)|t^\ell-t^{\ell+1}|^2|v^\ell|\big[1+|v^\ell||t^\ell-t^{\ell+1}|\big]\\
&\lesssim|t^\ell-t^{\ell+1}|+\Big(1+\frac{|v^\ell|}{|\mathbf{v}_\perp^{\ell+1}|}\Big)|t^\ell-t^{\ell+1}|^2|v^\ell|\lesssim\frac{1}{|v|}\frac{|\mathbf{v}_\perp^{\ell+1}|}{|v|}.
\end{aligned}$$

Since $\mathbf{v}_\perp^{\ell+1}=-\lim_{s\downarrow t^{\ell+1}}\mathbf{v}_\perp(s)$ and

$\dot{\mathbf{v}}_\perp(s)=F_\perp(\mathbf{X_{cl}}(s;t^\ell,\mathbf{x}^\ell,\mathbf{v}^\ell),\mathbf{V_{cl}}(s;t^\ell,\mathbf{x}^\ell,\mathbf{v}^\ell))=F_\perp(\mathbf{X_{cl}}(s;t^{\ell+1},\mathbf{x}^{\ell+1},\mathbf{v}^{\ell+1}),\mathbf{V_{cl}}(s;t^{\ell+1},\mathbf{x}^{\ell+1},\mathbf{v}^{\ell+1})),$

we have

$$0 = -\mathbf{v}_\perp^{\ell+1}(t^\ell - t^{\ell+1}) + \int_{t^{\ell+1}}^{t^\ell}\int_{t^{\ell+1}}^{s} F_\perp(\mathbf{X_{cl}}(\tau;t^{\ell+1},\mathbf{x}^{\ell+1},\mathbf{v}^{\ell+1}),\mathbf{V_{cl}}(\tau;t^{\ell+1},\mathbf{x}^{\ell+1},\mathbf{v}^{\ell+1}))\mathrm{d}\tau\mathrm{d}s.$$

From

$$\begin{aligned}
&F_\perp(\mathbf{X_{cl}}(\tau;t^\ell,\mathbf{x}^\ell,\mathbf{v}^\ell),\mathbf{V_{cl}}(\tau;t^\ell,\mathbf{x}^\ell,\mathbf{v}^\ell)) = F_\perp(\mathbf{x}^\ell,\mathbf{v}^\ell) + \int_{t^\ell}^{\tau}\frac{\partial}{\partial\tau}F_\perp(\mathbf{X_{cl}}(\tau;t^\ell,\mathbf{x}^\ell,\mathbf{v}^\ell),\mathbf{V_{cl}}(\tau;t^\ell,\mathbf{x}^\ell,\mathbf{v}^\ell))\mathrm{d}\tau,\\
&F_\perp(\mathbf{X_{cl}}(\tau;t^{\ell+1},\mathbf{x}^{\ell+1},\mathbf{v}^{\ell+1}),\mathbf{V_{cl}}(\tau;t^{\ell+1},\mathbf{x}^{\ell+1},\mathbf{v}^{\ell+1}))\\
&= F_\perp(\mathbf{x}^{\ell+1},\mathbf{v}^\ell) + \int_{t^{\ell+1}}^{\tau}\frac{\partial}{\partial\tau}F_\perp(\mathbf{X_{cl}}(\tau;t^\ell,\mathbf{x}^\ell,\mathbf{v}^\ell),\mathbf{V_{cl}}(\tau;t^\ell,\mathbf{x}^\ell,\mathbf{v}^\ell))\mathrm{d}\tau,
\end{aligned}$$

we have

$$F_\perp(\mathbf{X_{cl}}(\tau;t^\ell,\mathbf{x}^\ell,\mathbf{v}^\ell),\mathbf{V_{cl}}(\tau;t^\ell,\mathbf{x}^\ell,\mathbf{v}^\ell)) = F_\perp(\mathbf{x}^{\ell+1},\mathbf{v}^\ell) + O_\xi(1)|t^{\ell+1}-t^\ell||v^\ell|^3.$$

Therefore

$$0 = -\mathbf{v}_\perp^{\ell+1}(t^\ell - t^{\ell+1}) + \frac{1}{2}(t^\ell - t^{\ell+1})^2 F_\perp(\mathbf{x}^{\ell+1},\mathbf{v}^\ell) + O_\xi(1)|t^\ell - t^{\ell+1}|^3|v^\ell|^3,$$

so that

$$(t^\ell - t^{\ell+1})F_\perp(\mathbf{x}^{\ell+1},\mathbf{v}^\ell) = 2\mathbf{v}_\perp^{\ell+1} + O_\xi(1)|t^\ell - t^{\ell+1}|^2|v^\ell|^3. \tag{141}$$

Using (141), we can find an extra cancellation in terms of order of $t^\ell - t^{\ell+1}$ to get

$$\begin{aligned}
&\frac{\partial\mathbf{v}_\perp^{\ell+1}}{\partial t^\ell} = 0,\\
&\frac{\partial\mathbf{v}_\perp^{\ell+1}}{\partial\mathbf{x}_\parallel^\ell} = \frac{-F_\perp(\mathbf{x}^{\ell+1},\mathbf{v}^\ell)}{\mathbf{v}_\perp^{\ell+1}}\int_{t^{\ell+1}}^{t^\ell}\int_s^{t^\ell}\frac{\partial}{\partial\mathbf{x}_\parallel^\ell}F_\perp(\mathbf{X_{cl}}(\tau),\mathbf{V_{cl}}(\tau))\mathrm{d}\tau\mathrm{d}s + \int_{t^{\ell+1}}^{t^\ell}\frac{\partial}{\partial\mathbf{x}_\parallel^\ell}F_\perp(\mathbf{X_{cl}}(\tau),\mathbf{V_{cl}}(\tau))\mathrm{d}\tau\\
&\qquad = \Big\{\frac{(t^\ell - t^{\ell+1})F_\perp(\mathbf{x}^{k+1},\mathbf{v}^\ell)}{-2\mathbf{v}_\perp^{\ell+1}} + 1\Big\}(t^\ell - t^{\ell+1})\frac{\partial}{\partial\mathbf{x}_\parallel^\ell}F_\perp(\mathbf{x}^\ell,\mathbf{v}^\ell)\\
&\qquad\quad + O_\xi(1)\Big\{|t^\ell - t^{\ell+1}|^2|v^\ell|^3 + \frac{|t^\ell - t^{\ell+1}|^3|v^\ell|^3}{|\mathbf{v}_\perp^{\ell+1}|}|\frac{\partial}{\partial\mathbf{x}_\parallel^\ell}F_\perp(\mathbf{x}^\ell,\mathbf{v}^\ell)|\Big\}\\
&\qquad \lesssim_\xi \Big\{-1 + O_\xi(1)\frac{|t^\ell - t^{\ell+1}|^2|v^\ell|^3}{|\mathbf{v}_\perp^{\ell+1}|} + 1\Big\}|t^\ell - t^{\ell+1}||v^\ell|^2 + |t^\ell - t^{\ell+1}|^2|v^\ell|^3\Big\{1 + \frac{|t^\ell - t^{\ell+1}||v^\ell|^2}{|\mathbf{v}_\perp^{\ell+1}|}\Big\}\\
&\qquad \lesssim_\xi |t^\ell - t^{\ell+1}|^2|v^\ell|^3\Big(1 + \frac{|t^\ell - t^{\ell+1}||v^\ell|^2}{|\mathbf{v}_\perp^{\ell+1}|}\Big) \lesssim_\xi |t^\ell - t^{\ell+1}|^2|v^\ell|^3,\\
&\qquad \lesssim_{\xi,t} \frac{|\mathbf{v}_\perp^{\ell+1}|^2}{|v^\ell|},
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial \mathbf{v}_\perp^{\ell+1}}{\partial \mathbf{v}_\perp^{\ell}} &= -1 - \frac{\partial t^{\ell+1}}{\partial \mathbf{v}_\perp^{\ell}} F_\perp(\mathbf{x}^{\ell+1}, \mathbf{v}^\ell) + \int_{t^{\ell+1}}^{t^\ell} \frac{\partial}{\partial \mathbf{v}_\perp^\ell} F_\perp(\mathbf{X_{cl}}(\tau), \mathbf{V_{cl}}(\tau)) \mathrm{d}\tau \\
&= -1 + \frac{F_\perp(\mathbf{x}^{\ell+1}, \mathbf{v}^\ell)}{\mathbf{v}_\perp^{\ell+1}} (t^\ell - t^{\ell+1}) - \frac{F_\perp(\mathbf{x}^{\ell+1}, \mathbf{v}^\ell)}{\mathbf{v}_\perp^{\ell+1}} \int_{t^\ell}^{t^{\ell+1}} \int_{t^\ell}^{s} \frac{\partial}{\partial \mathbf{v}_\perp^\ell} F_\perp(\mathbf{X_{cl}}(\tau), \mathbf{V_{cl}}(\tau)) \mathrm{d}\tau \mathrm{d}s \\
&\quad + \int_{t^{\ell+1}}^{t^\ell} \frac{\partial}{\partial \mathbf{v}_\perp^\ell} F_\perp(\mathbf{X_{cl}}(\tau), \mathbf{V_{cl}}(\tau)) \mathrm{d}\tau \\
&= -1 + 2 + O_\xi(1) \frac{|t^\ell - t^{\ell+1}|^2 |v^\ell|^3}{\mathbf{v}_\perp^{\ell+1}} \\
&\quad - \frac{F_\perp(\mathbf{x}^{\ell+1}, \mathbf{v}^\ell)}{\mathbf{v}_\perp^{\ell+1}} \frac{(t^\ell - t^{\ell+1})^2}{2} \Big\{ \lim_{s \uparrow t^\ell} \frac{\partial}{\partial \mathbf{v}_\perp^\ell} F_\perp(\mathbf{X_{cl}}(\tau), \mathbf{V_{cl}}(\tau)) + O_\xi(1) |t^\ell - t^{\ell+1}| |v^\ell|^2 \Big\} \\
&\quad + (t^\ell - t^{\ell+1}) \Big\{ \lim_{s \uparrow t^\ell} \frac{\partial}{\partial \mathbf{v}_\perp^\ell} F_\perp(\mathbf{X_{cl}}(\tau), \mathbf{V_{cl}}(\tau)) + O_\xi(1) |t^\ell - t^{\ell+1}| |v^\ell|^2 \Big\} \\
&= 1 + O_\xi(1) \Big\{ \frac{|t^\ell - t^{\ell+1}|^2 |v^\ell|^3}{|\mathbf{v}_\perp^{\ell+1}|} + \frac{|t^\ell - t^{\ell+1}|^3}{|\mathbf{v}_\perp^{\ell+1}|} |v^\ell|^3 \Big| \lim_{s \uparrow t^\ell} \frac{\partial}{\partial \mathbf{v}_\perp^\ell} F_\perp(\mathbf{X_{cl}}(\tau), \mathbf{V_{cl}}(\tau)) \Big| + |t^\ell - t^{\ell+1}|^2 |v^\ell|^2 \Big\} \\
&\lesssim 1 + |t^\ell - t^{\ell+1}|^2 |v^\ell|^2 \Big\{ 1 + \frac{|v^\ell|}{|\mathbf{v}_\perp^{\ell+1}|} + \frac{|t^\ell - t^{\ell+1}| |v^\ell|^2}{|\mathbf{v}_\perp^{\ell+1}|} \Big\} \\
&\lesssim_{\xi,t} 1 + \frac{|\mathbf{v}_\perp^{\ell+1}|}{|v^\ell|}, \\
\frac{\partial \mathbf{v}_\perp^{\ell+1}}{\partial \mathbf{v}_\parallel^{\ell}} &= \frac{-F_\perp(\mathbf{x}^{\ell+1}, v^\ell)}{\mathbf{v}_\perp^{\ell+1}} \int_{t^{\ell+1}}^{t^\ell} \int_{s}^{t^\ell} \frac{\partial}{\partial \mathbf{v}_\parallel^\ell} F_\perp(\mathbf{X_{cl}}(\tau), \mathbf{V_{cl}}(\tau)) \mathrm{d}\tau \mathrm{d}s - \int_{t^{\ell+1}}^{t^\ell} \frac{\partial}{\partial \mathbf{v}_\parallel^\ell} F_\perp(\mathbf{X_{cl}}(\tau), \mathbf{V_{cl}}(\tau)) \mathrm{d}\tau \\
&= \Big\{ \frac{(t^\ell - t^{\ell+1}) F_\perp(\mathbf{x}^{\ell+1}, \mathbf{v}^\ell)}{-2 \mathbf{v}_\perp^{\ell+1}} + 1 \Big\} (t^\ell - t^{\ell+1}) \frac{\partial}{\partial \mathbf{v}_\parallel^\ell} F_\perp(\mathbf{x}^\ell, \mathbf{v}^\ell) \\
&\quad + O_\xi(1) |t^\ell - t^{\ell+1}|^2 |v^\ell|^2 \Big\{ \frac{|F_\perp(\mathbf{x}^{\ell+1}, \mathbf{v}^\ell)| |t^\ell - t^{\ell+1}|}{|\mathbf{v}_\perp^{\ell+1}|} + 1 \Big\} \\
&\lesssim_\xi |t^\ell - t^{\ell+1}|^2 |v^\ell|^2 \Big\{ 1 + \frac{|t^\ell - t^{\ell+1}| |v^\ell|^2}{|\mathbf{v}_\perp^{\ell+1}|} \Big\} \lesssim_{\xi,t} \frac{|\mathbf{v}_\perp^{\ell+1}|^2}{|v^\ell|^2}, \\
\frac{\partial \mathbf{v}_\parallel^{\ell+1}}{\partial t^\ell} &= 0, \\
\frac{\partial \mathbf{v}_\parallel^{\ell+1}}{\partial \mathbf{x}_\parallel^{\ell}} &= \frac{F_\parallel(\mathbf{x}^{\ell+1}, \mathbf{v}^\ell)}{\mathbf{v}_\perp^{\ell+1}} \int_{t^\ell}^{t^{\ell+1}} \int_{t^\ell}^{s} \frac{\partial}{\partial \mathbf{x}_\parallel^\ell} F_\perp(\mathbf{X_{cl}}(\tau), \mathbf{V_{cl}}(\tau)) \mathrm{d}\tau \mathrm{d}s \\
&\quad + \int_{t^\ell}^{t^{\ell+1}} \frac{\partial}{\partial \mathbf{x}_\parallel^\ell} F_\parallel(\mathbf{X_{cl}}(\tau), \mathbf{V_{cl}}(\tau)) \mathrm{d}\tau \lesssim |t^\ell - t^{\ell+1}| |v^\ell|^2 \Big\{ 1 + \frac{|t^\ell - t^{\ell+1}| |v^\ell|^2}{|\mathbf{v}_\perp^{\ell+1}|} \Big\} \\
&\lesssim_\xi |t^\ell - t^{\ell+1}| |v^\ell|^2 \lesssim_{\xi,t} |\mathbf{v}_\perp^{\ell+1}|, \\
\frac{\partial \mathbf{v}_\parallel^{\ell+1}}{\partial \mathbf{v}_\perp^{\ell}} &= \frac{-(t^\ell - t^{\ell+1}) F_\parallel(\mathbf{x}^{\ell+1}, \mathbf{v}^\ell)}{\mathbf{v}_\perp^{\ell+1}} + \frac{F_\parallel(\mathbf{x}^{\ell+1}, \mathbf{v}^\ell)}{\mathbf{v}_\perp^{\ell+1}} \int_{t^\ell}^{t^{\ell+1}} \int_{t^\ell}^{s} \frac{\partial}{\partial \mathbf{v}_\perp^\ell} F_\perp(\mathbf{X_{cl}}(\tau), \mathbf{V_{cl}}(\tau)) \mathrm{d}\tau \mathrm{d}s \\
&\quad + \int_{t^\ell}^{t^{\ell+1}} \frac{\partial}{\partial \mathbf{v}_\perp^\ell} F_\parallel(\mathbf{X_{cl}}(\tau), \mathbf{V_{cl}}(\tau)) \mathrm{d}\tau \\
&\lesssim_{\xi,t} \Big(1 + \frac{|v^\ell|}{|\mathbf{v}_\perp^{\ell+1}|} \Big) \min\{ |v^\ell|, \frac{|\mathbf{v}_\perp^{\ell+1}|}{|v^\ell|} \} \lesssim_{\xi,t} 1 + \frac{|\mathbf{v}_\perp^{\ell+1}|}{|v^\ell|},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathbf{v}_{\parallel}^{\ell+1}}{\partial \mathbf{v}_{\parallel}^{\ell}} &= \mathbf{Id}_{2,2} + \frac{F_{\parallel}(\mathbf{x}^{\ell+1}, \mathbf{v}^{\ell})}{\mathbf{v}_{\perp}^{\ell+1}} \int_{t^{\ell}}^{t^{\ell+1}} \int_{t^{\ell}}^{s} \frac{\partial}{\partial \mathbf{v}_{\parallel}^{\ell}} F_{\perp}(\mathbf{X}_{\mathbf{cl}}(\tau), \mathbf{V}_{\mathbf{cl}}(\tau)) \mathrm{d}\tau \mathrm{d}s \\
&\quad + \int_{t^{\ell}}^{t^{\ell+1}} \frac{\partial}{\partial \mathbf{v}_{\parallel}^{\ell}} F_{\parallel}(\mathbf{X}_{\mathbf{cl}}(\tau), \mathbf{V}_{\mathbf{cl}}(\tau)) \mathrm{d}\tau \\
&\lesssim_{\xi} \mathbf{Id}_{2,2} + |t^{\ell} - t^{\ell+1}||v^{\ell}| \big\{ 1 + \frac{|v^{\ell}|^2 |t^{\ell} - t^{\ell+1}|}{|\mathbf{v}_{\perp}^{\ell+1}|} \big\} \\
&\lesssim_{\xi,t} \mathbf{Id}_{2,2} + \frac{|\mathbf{v}_{\perp}^{\ell+1}|}{|v^{\ell}|}.
\end{aligned}
$$

These estimates prove the claim (139).

Proof of (138) for either $\mathbf{r}^{\ell} \geq \sqrt{\delta}$ *or* $\mathbf{r}^{\ell+1} \geq \sqrt{\delta}$*:* Without loss of generality we assume $\mathbf{r}^{\ell} > C\sqrt{\delta}$ in (125). Recall that we chose a $\mathbf{p}^{\ell}$−spherical coordinate as $\mathbf{p}^{\ell} = (z^{\ell}, w^{\ell})$ with $|z^{\ell} - x^{\ell}| \leq \sqrt{\delta}$ and any $w^{\ell} \in \mathbb{S}^2$ with $n(z^{\ell}) \cdot w^{\ell} = 0$.

Fix ℓ. Let $\Delta_1, \Delta_2 > 0$ such that $|v|\Delta_1 \ll 1$ and $|v||t^{\ell+1} - (t^{\ell} - \Delta_1 - \Delta_2)| \ll 1$ so that

$$
s^{\ell} \equiv t^{\ell} - \Delta_1, \quad s^{\ell+1} \equiv s^{\ell} - \Delta_2 = t^{\ell} - \Delta_1 - \Delta_2,
$$

satisfying $|v||t^{\ell+1} - s^{\ell+1}| = |v||t^{\ell+1} - (t^{\ell} - \Delta_1 - \Delta_2)| \ll 1$ and $|v||t^{\ell} - s^{\ell}| = |v||\Delta_1| \ll 1$ so that the spherical coordinates are well-defined for $s \in [t^{\ell+1}, s^{\ell+1}]$ and $s \in [s^{\ell}, t^{\ell}]$. Notice that

$$
\frac{\partial t^{\ell+1}}{\partial s^{\ell+1}} = \frac{\partial(s^{\ell+1} + \Delta_1 + \Delta_2 - t_{\mathbf{b}}(x^{\ell}, v^{\ell}))}{\partial s^{\ell+1}} = 1, \quad \frac{\partial s^{\ell+1}}{\partial s^{\ell}} = \frac{\partial(s^{\ell} - \Delta_1)}{\partial s^{\ell}} = 1, \quad \frac{\partial s^{\ell}}{\partial t^{\ell}} = \frac{\partial(t^{\ell} - \Delta_1)}{\partial t^{\ell}} = 1.
$$

By the chain rule,

$$
\begin{aligned}
&\frac{\partial(t^{\ell+1}, 0, \mathbf{x}_{\parallel_{\ell+1}}^{\ell+1}, \mathbf{v}_{\perp_{\ell+1}}^{\ell+1}, \mathbf{v}_{\parallel_{\ell+1}}^{\ell+1})}{\partial(t^{\ell}, 0, \mathbf{x}_{\parallel_{\ell}}^{\ell}, \mathbf{v}_{\perp_{\ell}}^{\ell}, \mathbf{v}_{\parallel_{\ell}}^{\ell})} \\
=& \frac{\partial(t^{\ell+1}, 0, \mathbf{x}_{\parallel_{\ell+1}}^{\ell+1}, \mathbf{v}_{\perp_{\ell+1}}^{\ell+1}, \mathbf{v}_{\parallel_{\ell+1}}^{\ell+1})}{\partial(s^{\ell+1}, \mathbf{x}_{\perp_{\ell+1}}(s^{\ell+1}), \mathbf{x}_{\parallel_{\ell+1}}(s^{\ell+1}), \mathbf{v}_{\perp_{\ell+1}}(s^{\ell+1}), \mathbf{v}_{\parallel_{\ell+1}}(s^{\ell+1}))} \frac{\partial(s^{\ell+1}, \mathbf{X}_{\mathbf{p}^{\ell+1}}(s^{\ell+1}), \mathbf{V}_{\mathbf{p}^{\ell+1}}(s^{\ell+1}))}{\partial(s^{\ell+1}, X_{\mathbf{cl}}(s^{\ell+1}), V_{\mathbf{cl}}(s^{\ell+1}))} \\
&\times \frac{\partial(s^{\ell+1}, X_{\mathbf{cl}}(s^{\ell+1}), V_{\mathbf{cl}}(s^{\ell+1}))}{\partial(s^{\ell}, X_{\mathbf{cl}}(s^{\ell}), V_{\mathbf{cl}}(s^{\ell}))} \frac{\partial(s^{\ell}, X_{\mathbf{cl}}(s^{\ell}), V_{\mathbf{cl}}(s^{\ell}))}{\partial(s^{\ell}, \mathbf{X}_{\mathbf{p}^{\ell}}(s^{\ell}), \mathbf{V}_{\mathbf{p}^{\ell}}(s^{\ell}))} \frac{\partial(s^{\ell}, \mathbf{x}_{\perp_{\ell}}(s^{\ell}), \mathbf{x}_{\parallel_{\ell}}(s^{\ell}), \mathbf{v}_{\perp_{\ell}}(s^{\ell}), \mathbf{v}_{\parallel_{\ell}}(s^{\ell}))}{\partial(t^{\ell}, 0, \mathbf{x}_{\parallel_{\ell}}^{\ell}, \mathbf{v}_{\perp_{\ell}}^{\ell}, \mathbf{v}_{\parallel_{\ell}}^{\ell})}.
\end{aligned}
$$

Recall (134) to have

$$
\frac{\partial(t^{\ell+1}, 0, \mathbf{x}_{\parallel}^{\ell+1}, \mathbf{v}_{\perp}^{\ell+1}, \mathbf{v}_{\parallel}^{\ell+1})}{\partial(s_2, \mathbf{x}_{\perp}(s_2), \mathbf{x}_{\parallel}(s_2), \mathbf{v}_{\perp}(s_2), \mathbf{v}_{\parallel}(s_2))} \lesssim_{\xi} \left[\begin{array}{c|c|c} 1 & O_{\delta,\xi}(1)\frac{1}{|v|} & O_{\delta,\xi}(1)\frac{1}{|v|^2} \\ \hline 0 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ \mathbf{0}_{2,1} & O_{\delta,\xi}(1) & O_{\delta,\xi}(1)\frac{1}{|v|} \\ \hline \mathbf{0}_{3,1} & O_{\delta,\xi}(1)|v| & O_{\delta,\xi}(1) \end{array}\right].
$$

From (137)

$$
\frac{\partial(s_2, \mathbf{X}_{\mathbf{cl}}(s_2), \mathbf{V}_{\mathbf{cl}}(s_2))}{\partial(s_2, X_{\mathbf{cl}}(s_2), V_{\mathbf{cl}}(s_2))} \lesssim_{\xi} \left[\begin{array}{c|c|c} 1 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ \hline \mathbf{0}_{3,1} & O_{\xi}(1) & \mathbf{0}_{3,3} \\ \hline \mathbf{0}_{3,1} & O_{\xi}(1)|v| & O_{\xi}(1) \end{array}\right],
$$

and from $s_2 = s_1 - \Delta_2,\ X_{\mathbf{cl}}(s_2) = X_{\mathbf{cl}}(s_1) - (s_2 - s_1)V_{\mathbf{cl}}(s_1),\ V_{\mathbf{cl}}(s_2) = V_{\mathbf{cl}}(s_2)$,

$$
\frac{\partial(s_2, X_{\mathbf{cl}}(s_2), V_{\mathbf{cl}}(s_2))}{\partial(s_1, X_{\mathbf{cl}}(s_1), V_{\mathbf{cl}}(s_1))} \lesssim_{\xi} \begin{bmatrix} 1 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ \mathbf{0}_{3,1} & \mathbf{Id}_{3,3} & |s_1 - s_2|\mathbf{Id}_{3,3} \\ \mathbf{0}_{3,1} & \mathbf{0}_{3,3} & \mathbf{Id}_{3,3} \end{bmatrix},
$$

and from (120)

$$
\frac{\partial(s_1, X_{\mathbf{cl}}(s_1), V_{\mathbf{cl}}(s_1))}{\partial(s_1, \mathbf{X}_{\mathbf{cl}}(s_1), \mathbf{V}_{\mathbf{cl}}(s_1))} \lesssim_{\xi} \begin{bmatrix} 1 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ \mathbf{0}_{3,1} & O_{\xi}(1) & \mathbf{0}_{3,3} \\ \mathbf{0}_{3,1} & |v| & O_{\xi}(1) \end{bmatrix},
$$

Recall (132) to have

$$\frac{\partial(s_1, \mathbf{x}_\perp(s_1), \mathbf{x}_\parallel(s_1), \mathbf{v}_\perp(s_1), \mathbf{v}_\parallel(s_1))}{\partial(t^\ell, 0, \mathbf{x}_\parallel^\ell, \mathbf{v}_\perp^\ell, \mathbf{v}_\parallel^\ell)} \lesssim_\xi \left[\begin{array}{c|c|c|c} 1 & 0 & \mathbf{0}_{1,2} & \mathbf{0}_{1,3} \\ \hline \mathbf{0}_{3,1} & 0 & O_\xi(1) & O_\xi(1)|t^\ell - s_1| \\ \hline \mathbf{0}_{3,1} & 0 & O_\xi(1)|v| & O_\xi(1) \end{array}\right].$$

By direct matrix multiplication

$$\frac{\partial(t^{\ell+1}, 0, \mathbf{x}_{\parallel_{\ell+1}}^{\ell+1}, \mathbf{v}_{\perp_{\ell+1}}^{\ell+1}, \mathbf{v}_{\parallel_{\ell+1}}^{\ell+1})}{\partial(t^\ell, 0, \mathbf{x}_{\parallel_\ell}^\ell, \mathbf{v}_{\perp_\ell}^\ell, \mathbf{v}_{\parallel_\ell}^\ell)} \lesssim_{t,\xi} \left[\begin{array}{c|c|c|c} 1 & 0 & \frac{1}{|v|} & \frac{1}{|v|^2} \\ \hline 0 & 0 & \mathbf{0}_{1,2} & \mathbf{0}_{1,3} \\ \mathbf{0}_{2,1} & \mathbf{0}_{2,1} & 1 & \frac{1}{|v|} \\ \hline \mathbf{0}_{3,1} & \mathbf{0}_{3,1} & |v| & 1 \end{array}\right].$$

Note that for *Type II* we have $\mathbf{r}^{\ell+1} \gtrsim \sqrt{\delta}$ so that from (138)

$$J(\mathbf{r}^{\ell+1}) \gtrsim \left[\begin{array}{c|c|c|c} 1 & 0 & \frac{M}{|v|}\sqrt{\delta} & \frac{M}{|v|^2}\min\{1, \sqrt{\delta}\} \\ \hline 0 & 0 & \mathbf{0}_{1,2} & \mathbf{0}_{1,3} \\ \mathbf{0}_{2,1} & \mathbf{0}_{2,1} & M\sqrt{\delta} & \frac{M}{|v|}\min\{1, \sqrt{\delta}\} \\ \hline \mathbf{0}_{3,1} & \mathbf{0}_{3,1} & M|v|\min\{\delta, \sqrt{\delta}\} & M\min\{\delta, \sqrt{\delta}\} \end{array}\right] \gtrsim_{\delta,t,\xi} \frac{\partial(t^{\ell+1}, 0, \mathbf{x}_{\parallel_{\ell+1}}^{\ell+1}, \mathbf{v}_{\perp_{\ell+1}}^{\ell+1}, \mathbf{v}_{\parallel_{\ell+1}}^{\ell+1})}{\partial(t^\ell, 0, \mathbf{x}_{\parallel_\ell}^\ell, \mathbf{v}_{\perp_\ell}^\ell, \mathbf{v}_{\parallel_\ell}^\ell)}.$$

This proves our claim (138) for *Type II.*

Step 5. Eigenvalues and diagonalization of of (138)

By a basic linear algebra(row and column operations), the characteristic polynomial of (138) equals, with $\mathbf{r} = \mathbf{r}^{\ell+1}$,

$$\det \begin{bmatrix} 1-\lambda & 0 & \frac{M}{|v|}\mathbf{r} & \frac{M}{|v|}\mathbf{r} & \frac{M}{|v|^2} & \frac{M}{|v|^2}\mathbf{r} & \frac{M}{|v|^2}\mathbf{r} \\ 0 & -\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+M\mathbf{r}-\lambda & M\mathbf{r} & \frac{M}{|v|} & \frac{M}{|v|}\mathbf{r} & \frac{M}{|v|}\mathbf{r} \\ 0 & 0 & M\mathbf{r} & 1+M\mathbf{r}-\lambda & \frac{M}{|v|} & \frac{M}{|v|}\mathbf{r} & \frac{M}{|v|}\mathbf{r} \\ 0 & 0 & M|v|\mathbf{r}^2 & M|v|\mathbf{r}^2 & 1+M\mathbf{r}-\lambda & M\mathbf{r}^2 & M\mathbf{r}^2 \\ 0 & 0 & M|v|\mathbf{r} & M|v|\mathbf{r} & M & 1+M\mathbf{r}-\lambda & M\mathbf{r} \\ 0 & 0 & M|v|\mathbf{r} & M|v|\mathbf{r} & M & M\mathbf{r} & 1+M\mathbf{r}-\lambda \end{bmatrix}$$
$$= -\lambda(\lambda-1)^5[\lambda - (1+5M\mathbf{r})].$$

Therefore eigenvalues are

$$\begin{aligned} &\lambda_0 = 0, \quad \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 1 \\ &\lambda_7 = 1 + 5M\mathbf{r}^{\ell+1} = 1 + 5M\frac{|\mathbf{v}_\perp^{\ell+1}|}{|v^{\ell+1}|}. \end{aligned} \tag{141}$$

Corresponding eigenvectors are

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -|v|\mathbf{r} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -|v| \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -|v| \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ |v| \\ |v| \\ |v|^2\mathbf{r} \\ |v|^2 \\ |v|^2 \end{pmatrix}.$$

Write $P = P(\mathbf{r}^\ell)$ as a block matrix of above column eigenvectors. Then

$$\mathcal{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & |v| \\ 0 & 0 & -1 & 0 & 0 & 0 & |v| \\ 0 & 0 & 0 & -|v|\mathbf{r} & 0 & 0 & |v|^2\mathbf{r} \\ 0 & 0 & 0 & 0 & -|v| & 0 & |v|^2 \\ 0 & 0 & 0 & 0 & 0 & -|v| & |v|^2 \end{bmatrix}, \ \mathcal{P}^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & \frac{-1}{5|v|} & \frac{-1}{5|v|} & \frac{-1}{5|v|^2\mathbf{r}} & \frac{-1}{5|v|^2} & \frac{-1}{5|v|^2} \\ 0 & 0 & \frac{1}{5} & \frac{-4}{5} & \frac{1}{5|v|\mathbf{r}} & \frac{1}{5|v|} & \frac{1}{5|v|} \\ 0 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{-4}{5|v|\mathbf{r}} & \frac{1}{5|v|} & \frac{1}{5|v|} \\ 0 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5|v|\mathbf{r}} & \frac{-4}{5|v|} & \frac{1}{5|v|} \\ 0 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5|v|\mathbf{r}} & \frac{1}{5|v|} & \frac{-4}{5|v|} \\ 0 & 0 & \frac{1}{5|v|} & \frac{1}{5|v|} & \frac{1}{5|v|^2\mathbf{r}} & \frac{1}{5|v|^2} & \frac{1}{5|v|^2} \end{bmatrix}. \tag{142}$$

Therefore

$$J(\mathbf{r}) = \mathcal{P}(\mathbf{r})\Lambda(\mathbf{r})\mathcal{P}^{-1}(\mathbf{r}),,$$

and

$$\Lambda(\mathbf{r}) := \text{diag}\Big[0, 1, 1, 1, 1, 1, 1 + 5M\mathbf{r}\Big],$$

where the notation $\text{diag}[a_1, \cdots, a_m]$ is a $m \times m-$matrix with $a_{ii} = a_i$ and $a_{ij} = 0$ for all $i \neq j$.

Step 6. The $i-th$ intermediate group

We claim that, for $i = 1, 2, \cdots, [\frac{|t-s||v|}{L_\xi}]$,

$$\begin{aligned} J^{\ell_{i+1}}_{\ell_{i+1}-1} \times \cdots \times J^{\ell_i+1}_{\ell_i} &= \frac{\partial(t^{\ell_{i+1}}, 0, \mathbf{x}^{\ell_{i+1}}_{\|_{\ell_{i+1}}}, \mathbf{v}^{\ell_{i+1}}_{\perp_{\ell_{i+1}}}, \mathbf{v}^{\ell_{i+1}}_{\|_{\ell_{i+1}}})}{\partial(t^{\ell_{i+1}-1}, 0, \mathbf{x}^{\ell_{i+1}-1}_{\|_{\ell_{i+1}-1}}, \mathbf{v}^{\ell_{i+1}-1}_{\perp_{\ell_{i+1}-1}}, \mathbf{v}^{\ell_{i+1}-1}_{\|_{\ell_{i+1}-1}})} \times \cdots \times \frac{\partial(t^{\ell_i+1}, 0, \mathbf{x}^{\ell_i+1}_{\|_{\ell_{i+1}}}, \mathbf{v}^{\ell_i+1}_{\perp_{\ell_{i+1}}}, \mathbf{v}^{\ell_i+1}_{\|_{\ell_{i+1}}})}{\partial(t^{\ell_i}, 0, \mathbf{x}^{\ell_i}_{\|_{\ell_i}}, \mathbf{v}^{\ell_i}_{\perp_{\ell_i}}, \mathbf{v}^{\ell_i}_{\|_{\ell_i}})} \\ &\leq \mathcal{P}(\mathbf{r}_i)(\Lambda(\mathbf{r}_i))^{\frac{C_\xi}{\mathbf{r}_i}}\mathcal{P}^{-1}(\mathbf{r}_i). \end{aligned} \tag{143}$$

By the definition of the group, $L_\xi \leq |v||t^{\ell_i} - t^{\ell_{i+1}}| \leq C_1 < +\infty$ for all i. By the Velocity lemma(Lemma 1),

$$\frac{1}{\mathfrak{C}_1}e^{-\frac{\mathfrak{C}}{2}C_1}\mathbf{r}^{\ell_i} \leq \mathbf{r}^{\ell_{i+1}} \equiv \frac{|\mathbf{v}^{\ell_{i+1}}_\perp|}{|v|}, \mathbf{r}^{\ell_{i+1}-1} \equiv \frac{|\mathbf{v}^{\ell_{i+1}-1}_\perp|}{|v|}, \cdots, \mathbf{r}^{\ell_i+1} \equiv \frac{|\mathbf{v}^{\ell_i+1}_\perp|}{|v|}, \mathbf{r}^{\ell_i} \equiv \frac{|\mathbf{v}^{\ell_i}_\perp|}{|v|} \leq \mathfrak{C}_1 e^{\frac{\mathfrak{C}}{2}C_1}\mathbf{r}^{\ell_i},$$

and define

$$\mathbf{r}_i \equiv \mathfrak{C}_1 e^{\frac{\mathfrak{C}}{2}C_1}\mathbf{r}^{\ell_i}.$$

Then we have

$$\frac{1}{(\mathfrak{C}_1)^2}e^{-\mathfrak{C}C_1}\mathbf{r}_i \ \leq \ \mathbf{r}^j \ \leq \ \mathbf{r}_i \quad \text{for all} \quad \ell_{i+1} \leq j \leq \ell_i. \tag{144}$$

From (138), we have a uniform bound for all $\ell_{i+1} \leq j \leq \ell_i$

$$J^{j+1}_j \lesssim J(\mathbf{r}_i) = \mathcal{P}(\mathbf{r}_i)\Lambda(\mathbf{r}_i)\mathcal{P}^{-1}(\mathbf{r}_i).$$

Therefore

$$J^{\ell_{i+1}}_{\ell_{i+1}-1} \times \cdots \times J^{\ell_i+1}_{\ell_i} \leq \mathcal{P}(\mathbf{r}_i)[\Lambda(\mathbf{r}_i)]^{|\ell_{i+1}-\ell_i|}\mathcal{P}^{-1}(\mathbf{r}_i).$$

Now we only left to prove $|\ell_{i+1} - \ell_i| \lesssim_\Omega \frac{1}{\mathbf{r}_i}$: For any $\ell_{i+1} \leq j \leq \ell_i$, we have $\xi(x^j) = 0 = \xi(x^{j+1}) = \xi(x^j - (t^j - t^{j+1})v^j)$. We expand $\xi(x^j - (t^j - t^{j+1})v^j)$ in time to have

$$\begin{aligned} \xi(x^{j+1}) &= \xi(x^j) + \int_{t^j}^{t^{j+1}} \frac{d}{ds}\xi(X_{\mathbf{cl}}(s))\mathrm{d}s \\ &= \xi(x^j) + (v^j \cdot \nabla\xi(x^j))(t^{j+1} - t^j) + \int_{t^j}^{t^{j+1}}\int_{t^j}^{s} \frac{d^2}{d\tau^2}\xi(X_{\mathbf{cl}}(\tau))\mathrm{d}\tau\mathrm{d}s, \\ 0 &= (v^j \cdot \nabla\xi(x^j))(t^{j+1} - t^j) + \frac{(t^j - t^{j+1})^2}{2}(v^j \cdot \nabla^2\xi(X_{\mathbf{cl}}(\tau_*)) \cdot v^j), \quad \text{for some } \tau_* \in [t^{j+1}, t^j], \end{aligned}$$

and therefore

$$\frac{v^j \cdot \nabla\xi(x^j)}{|v|} = (t^j - t^{j+1})|v|\frac{v^j \cdot \nabla^2\xi(X_{\mathbf{cl}}(\tau_*)) \cdot v^j}{2|v|^2}.$$

From the convexity, there exists $C_2 \gg 1$

$$\frac{1}{C_2}|t^j - t^{j+1}||v| \leq |\mathbf{r}^j| = \frac{|\mathbf{v}^j_\perp|}{|v|} = \frac{|v^j \cdot \nabla \xi(x^j)|}{|v|} \leq C_2 |t^j - t^{j+1}||v|. \tag{145}$$

Therefore we have a lower bound of $|v||t^j - t^{j+1}|$: $|v||t^j - t^{j+1}| \geq \frac{1}{C_2}|\mathbf{r}^j| \geq \frac{1}{(\mathfrak{C}_1)^2 C_2} e^{-\mathfrak{C} C_1} \mathbf{r}_i$, where we have used (144). Finally, using the definition of one group($1 \leq |v||t^{\ell_i} - t^{\ell_{i+1}}| \leq C_1$), we have the following upper bound of the number of bounces in this one group($i-$th intermediate group)

$$|\ell_i - \ell_{i+1}| \leq \frac{|v||t^{\ell_i} - t^{\ell_{i+1}}|}{\min_{\ell_i \leq j \leq \ell_{i+1}} |v||t^j - t^{j+1}|} \leq \frac{C_1}{\frac{1}{(\mathfrak{C}_1)^2 C_2} e^{-\mathfrak{C} C_1} \mathbf{r}_i} \lesssim_\xi \frac{1}{\mathbf{r}_i},$$

and this complete our claim (143).

Step 7. Whole intermediate groups

We claim that, there exists $C_3 > 0$ such that

$$\prod_{i=1}^{[\frac{|t-s||v|}{L_\xi}]} J^{\ell_{i+1}}_{\ell_{i+1}-1} \times \cdots \times J^{\ell_i+1}_{\ell_i} \leq (C_3)^{|t-s||v|} \mathcal{P}(\mathbf{r}_{[\frac{|t-s||v|}{L_\xi}]}) \mathcal{P}^{-1}(\mathbf{r}_1). \tag{146}$$

From the one group estimate (143),

$$\begin{aligned}
&\prod_{i=1}^{[\frac{|t-s||v|}{L_\xi}]} J^{\ell_{i+1}}_{\ell_{i+1}-1} \times \cdots \times J^{\ell_i+1}_{\ell_i} \\
&\leq \mathcal{P}(\mathbf{r}_{[\frac{|t-s||v|}{L_\xi}]}) (\Lambda(\mathbf{r}_{[\frac{|t-s||v|}{L_\xi}]}))^{\frac{C_\xi}{\mathbf{r}_{[\frac{|t-s||v|}{L_\xi}]}}} \underbrace{\mathcal{P}^{-1}(\mathbf{r}_{[\frac{|t-s||v|}{L_\xi}]}) \times \mathcal{P}(\mathbf{r}_{[\frac{|t-s||v|}{L_\xi}]-1})} \\
&\quad \times (\Lambda(\mathbf{r}_{[\frac{|t-s||v|}{L_\xi}]-1}))^{\frac{C_\xi}{\mathbf{r}_{[\frac{|t-s||v|}{L_\xi}]-1}}} \underbrace{\mathcal{P}^{-1}(\mathbf{r}_{[\frac{|t-s||v|}{L_\xi}]-1}) \times \quad} \cdots \\
&\quad \times \cdots \underbrace{\quad \times \mathcal{P}(\mathbf{r}_{i+1})} (\Lambda(\mathbf{r}_{i+1}))^{\frac{C_\xi}{\mathbf{r}_{i+1}}} \underbrace{\mathcal{P}^{-1}(\mathbf{r}_{i+1}) \times \mathcal{P}(\mathbf{r}_i)} (\Lambda(\mathbf{r}_i))^{\frac{C_\xi}{\mathbf{r}_i}} \underbrace{\mathcal{P}^{-1}(\mathbf{r}_i) \times \mathcal{P}(\mathbf{r}_{i-1})} \\
&\quad \times (\Lambda(\mathbf{r}_{i-1}))^{\frac{C_\xi}{\mathbf{r}_{i-1}}} \underbrace{\mathcal{P}^{-1}(\mathbf{r}_{i-1}) \times \quad} \cdots \\
&\quad \times \cdots \underbrace{\quad \times \mathcal{P}(\mathbf{r}_2)} (\Lambda(\mathbf{r}_2))^{\frac{C_\xi}{\mathbf{r}_2}} \underbrace{\mathcal{P}^{-1}(\mathbf{r}_2) \times \mathcal{P}(\mathbf{r}_1)} (\Lambda(\mathbf{r}_1))^{\frac{C_\xi}{\mathbf{r}_1}} \mathcal{P}^{-1}(\mathbf{r}_1).
\end{aligned}$$

Now we focus on the underbraced terms.

Note that

$$\mathcal{P}^{-1}(\mathbf{r}_{i+1})\mathcal{P}(\mathbf{r}_i) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & \frac{-1+\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}}{5|v|} & 0 & 0 & \frac{1-\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}}{5} \\
0 & 0 & 1 & \frac{1-\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}}{5} & 0 & 0 & |v|\frac{-1+\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}}{5} \\
0 & 0 & 0 & \frac{1+4\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}}{5} & 0 & 0 & 4|v|\frac{1-\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}}{5} \\
0 & 0 & 0 & \frac{1-\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}}{5} & 1 & 0 & |v|\frac{-1+\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}}{5} \\
0 & 0 & 0 & \frac{1-\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}}{5} & 0 & 1 & |v|\frac{-1+\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}}{5} \\
0 & 0 & 0 & \frac{1-\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}}{5|v|} & 0 & 0 & \frac{4+\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}}{5}
\end{bmatrix}.$$

Due to the choice of $\mathbf{r}_i \equiv \mathfrak{C}_1 e^{\frac{\mathfrak{C}}{2}C_1} \mathbf{r}^{\ell_i}$ in (144) we have $\left|\frac{\mathbf{r}_i}{\mathbf{r}_{i+1}}\right| = \left|\frac{\mathbf{r}^{\ell_i}}{\mathbf{r}^{\ell_{i+1}}}\right| \leq C_\xi$, where we have used the Velocity lemma and (65) and (66): $\frac{1}{\mathfrak{C}_1} e^{-\frac{\mathfrak{C}}{2}C_1} \mathbf{r}^{\ell_{i+1}} \leq \frac{1}{\mathfrak{C}_1} e^{-\frac{\mathfrak{C}}{2}|t^{\ell_i} - t^{\ell_{i+1}}|} \mathbf{r}^{\ell_{i+1}} \leq \mathbf{r}^{\ell_i} \leq \mathfrak{C}_1 e^{\frac{\mathfrak{C}}{2}|t^{\ell_i} - t^{\ell_{i+1}}|} \mathbf{r}^{\ell_{i+1}} \leq \mathfrak{C}_1 e^{\frac{\mathfrak{C}}{2}C_1} \mathbf{r}^{\ell_{i+1}}$.

Therefore for sufficiently large $C_\xi > 0$, for all i

$$\mathcal{Q}_{i+1,i} \le \mathcal{Q} := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{C_\xi}{|v|} & 0 & 0 & C_\xi \\ 0 & 0 & 1 & C_\xi & 0 & 0 & C_\xi|v| \\ 0 & 0 & 0 & C_\xi & 0 & 0 & C_\xi|v| \\ 0 & 0 & 0 & C_\xi & 1 & 0 & C_\xi|v| \\ 0 & 0 & 0 & C_\xi & 0 & 1 & C_\xi|v| \\ 0 & 0 & 0 & \frac{C_\xi}{|v|} & 0 & 0 & C_\xi \end{bmatrix}. \tag{147}$$

Again we diagonalize $\mathcal{Q}$ as

$$\mathcal{Q} = \mathcal{F}\mathcal{A}\mathcal{F}^{-1}$$

$$:= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{2C_\xi}{2C_\xi-1} \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{2C_\xi|v|}{2C_\xi-1} \\ 0 & 0 & 0 & 0 & 0 & -|v| & |v| \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{2C_\xi|v|}{2C_\xi-1} \\ 0 & 0 & 0 & 0 & 1 & 0 & \frac{2C_\xi|v|}{2C_\xi-1} \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & 0 & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & 0 & & & & 0 & \\ & & & & & & 2C_\xi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{-C_\xi}{2C_\xi-1}\frac{1}{|v|} & 0 & 0 & \frac{-C_\xi}{2C_\xi-1} \\ 0 & 0 & 1 & \frac{-C_\xi}{2C_\xi-1} & 0 & 0 & \frac{-C_\xi|v|}{2C_\xi-1} \\ 0 & 0 & 0 & \frac{-C_\xi}{2C_\xi-1} & 1 & 0 & \frac{-C_\xi|v|}{2C_\xi-1} \\ 0 & 0 & 0 & \frac{-C_\xi}{2C_\xi-1} & 0 & 1 & \frac{-C_\xi|v|}{2C_\xi-1} \\ 0 & 0 & 0 & \frac{-1}{2|v|} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2|v|} & 0 & 0 & \frac{1}{2} \end{bmatrix},$$

and directly

$$\begin{aligned} &\mathcal{Q}^{[\frac{|t-s||v|}{L_\xi}]} = \mathcal{F}\mathcal{A}^{[\frac{|t-s||v|}{L_\xi}]}\mathcal{F}^{-1} \\ &= \mathcal{F}\ \mathrm{diag}\Big[1,1,1,1,1,0,(2C_\xi)^{[\frac{|t-s||v|}{L_\xi}]}\Big]\ \mathcal{F}^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{|v|}\frac{C_\xi}{2C_\xi-1}((2C_\xi)^{[\frac{|t-s||v|}{L_\xi}]}-1) & 0 & 0 & \frac{C_\xi}{2C_\xi-1}((2C_\xi)^{[\frac{|t-s||v|}{L_\xi}]}-1) \\ 0 & 0 & 1 & \frac{C_\xi}{2C_\xi-1}((2C_\xi)^{[\frac{|t-s||v|}{L_\xi}]}-1) & 0 & 0 & |v|\frac{C_\xi}{2C_\xi-1}((2C_\xi)^{[\frac{|t-s||v|}{L_\xi}]}-1) \\ 0 & 0 & 0 & \frac{(2C_\xi)^{[\frac{|t-s||v|}{L_\xi}]}}{2} & 0 & 0 & |v|\frac{(2C_\xi)^{[\frac{|t-s||v|}{L_\xi}]}}{2} \\ 0 & 0 & 0 & \frac{C_\xi}{2C_\xi-1}((2C_\xi)^{[\frac{|t-s||v|}{L_\xi}]}-1) & 1 & 0 & |v|\frac{C_\xi}{2C_\xi-1}((2C_\xi)^{[\frac{|t-s||v|}{L_\xi}]}-1) \\ 0 & 0 & 0 & \frac{C_\xi}{2C_\xi-1}((2C_\xi)^{[\frac{|t-s||v|}{L_\xi}]}-1) & 0 & 1 & |v|\frac{C_\xi}{2C_\xi-1}((2C_\xi)^{[\frac{|t-s||v|}{L_\xi}]}-1) \\ 0 & 0 & 0 & \frac{1}{|v|}\frac{(2C_\xi)^{[\frac{|t-s||v|}{L_\xi}]}}{2} & 0 & 0 & \frac{(2C_\xi)^{[\frac{|t-s||v|}{L_\xi}]}}{2} \end{bmatrix}. \end{aligned} \tag{148}$$

Finally we can perform the matrix multiplication with special grouping as, with notation $\tilde{t} := t-s$,

$$\begin{aligned} &\prod_{i=1}^{[\frac{|t-s||v|}{L_\xi}]} J_{\ell_{i+1}-1}^{\ell_{i+1}} \times \cdots \times J_{\ell_i}^{\ell_i+1} \\ &\le \mathcal{P}(\mathbf{r}_{[\frac{\tilde{t}|v|}{L_\xi}]})\Big[\Lambda(\mathbf{r}_{[\frac{\tilde{t}|v|}{L_\xi}]})\Big]^{\frac{C_\xi}{\mathbf{r}_{[\frac{\tilde{t}|v|}{L_\xi}]}}} \mathcal{Q}_{[\frac{\tilde{t}|v|}{L_\xi}],[\frac{\tilde{t}|v|}{L_\xi}]-1}\Big[\Lambda(\mathbf{r}_{[\frac{\tilde{t}|v|}{L_\xi}]-1})\Big]^{\frac{C_\xi}{\mathbf{r}_{[\frac{\tilde{t}|v|}{L_\xi}]-1}}} \mathcal{Q}_{[\frac{\tilde{t}|v|}{L_\xi}]-1,[\frac{\tilde{t}|v|}{L_\xi}]-2} \times \cdots \\ &\quad \times \cdots \times \mathcal{Q}_{i+2,i+1}\Big[\Lambda(\mathbf{r}_{i+1})\Big]^{\frac{C_\xi}{\mathbf{r}_{i+1}}} \mathcal{Q}_{i+1,i}(\Lambda(\mathbf{r}_i))^{\frac{C_\xi}{\mathbf{r}_i}} \mathcal{Q}_{i,i-1}\Big[\Lambda(\mathbf{r}_{i-1})\Big]^{\frac{C_\xi}{\mathbf{r}_{i-1}}} \mathcal{Q}_{i-1,i-2} \times \cdots \\ &\quad \times \cdots \times \mathcal{Q}_{3,2}\Big[\Lambda(\mathbf{r}_2)\Big]^{\frac{C_\xi}{\mathbf{r}_2}} \mathcal{Q}_{2,1}\Big[\Lambda(\mathbf{r}_1)\Big]^{\frac{C_\xi}{\mathbf{r}_1}} \mathcal{P}^{-1}(\mathbf{r}_1). \end{aligned}$$

Now we use (143) and take the absolute value of the entries and then use (147) and (148)

$$
\begin{aligned}
&\leq \widetilde{\mathcal{P}(\mathbf{r}_{[\frac{\tilde{t}|v|}{L_\xi}]})}(1+5M\mathbf{r}_{[\frac{\tilde{t}|v|}{L_\xi}]})^{\frac{C_\xi}{\mathbf{r}_{[\frac{\tilde{t}|v|}{L_\xi}]}}}\mathcal{Q}(1+5M\mathbf{r}_{[\frac{\tilde{t}|v|}{L_\xi}]-1})^{\frac{C_\xi}{\mathbf{r}_{[\frac{\tilde{t}|v|}{L_\xi}]-1}}}\mathcal{Q}\times\cdots\\
&\quad\times\cdots\times\mathcal{Q}(1+5M\mathbf{r}_{i+1})^{\frac{C_\xi}{\mathbf{r}_{i+1}}}\mathcal{Q}(1+5M\mathbf{r}_i)^{\frac{C_\xi}{\mathbf{r}_i}}\mathcal{Q}(1+5M\mathbf{r}_{i-1})^{\frac{C_\xi}{\mathbf{r}_{i-1}}}\mathcal{Q}\times\cdots\\
&\quad\times\cdots\times\mathcal{Q}(1+5M\mathbf{r}_2)^{\frac{C_\xi}{\mathbf{r}_2}}\mathcal{Q}(1+5M\mathbf{r}_1)^{\frac{C_\xi}{\mathbf{r}_1}}\widetilde{\mathcal{P}^{-1}(\mathbf{r}_1)}\\
&\leq\prod_{i=1}^{[\frac{\tilde{t}|v|}{L_\xi}]}(1+5M\mathbf{r}_i)^{\frac{C_\xi}{\mathbf{r}_i}}\times\widetilde{\mathcal{P}(\mathbf{r}_{[\frac{\tilde{t}|v|}{L_\xi}]})}\mathcal{Q}^{[\frac{\tilde{t}|v|}{L_\xi}]-1}\widetilde{\mathcal{P}^{-1}(\mathbf{r}_1)}\\
&\leq\left[\sup_{0\leq\mathbf{r}\leq1}(1+5M\mathbf{r})^{\frac{C_\xi}{\mathbf{r}}}\right]^{[\frac{\tilde{t}|v|}{L_\xi}]}\times\widetilde{\mathcal{P}(\mathbf{r}_{[\frac{\tilde{t}|v|}{L_\xi}]})}\mathcal{F}\mathcal{A}^{[\frac{\tilde{t}|v|}{L_\xi}]}\mathcal{F}^{-1}\widetilde{\mathcal{P}^{-1}(\mathbf{r}_1)},
\end{aligned}
$$

where adopted a notation: For a matrix A, the entries of a matrix $\widetilde{A}$ is an absolute value of the entries of A, i.e. $(\widetilde{A})_{ij}=|(A)_{ij}|$.

Now we use the explicit form of (148) to bound

$$
\leq\quad C^{C\tilde{t}|v|}\left[\begin{array}{cc|cc|c|cc}
1 & 0 & \frac{(C_\xi)^{\tilde{t}|v|}}{|v|} & \frac{(C_\xi)^{\tilde{t}|v|}}{|v|} & \frac{(C_\xi)^{\tilde{t}|v|}}{|v|^2}\frac{1}{|\mathbf{r}_1|} & \frac{(C_\xi)^{\tilde{t}|v|}}{|v|^2} & \frac{(C_\xi)^{\tilde{t}|v|}}{|v|^2}\\
0 & 1 & 0 & 0 & 0 & 0 & 0\\ \hline
0 & 0 & (C_\xi)^{\tilde{t}|v|} & (C_\xi)^{\tilde{t}|v|} & \frac{(C_\xi)^{\tilde{t}|v|}}{|v|}\frac{1}{|\mathbf{r}_1|} & \frac{(C_\xi)^{\tilde{t}|v|}}{|v|} & \frac{(C_\xi)^{\tilde{t}|v|}}{|v|}\\
0 & 0 & (C_\xi)^{\tilde{t}|v|} & (C_\xi)^{\tilde{t}|v|} & \frac{(C_\xi)^{\tilde{t}|v|}}{|v|}\frac{1}{|\mathbf{r}_1|} & \frac{(C_\xi)^{\tilde{t}|v|}}{|v|} & \frac{(C_\xi)^{\tilde{t}|v|}}{|v|}\\ \hline
0 & 0 & |v|(C_\xi)^{\tilde{t}|v|}|\mathbf{r}_{[\frac{\tilde{t}|v|}{L_\xi}]}| & |v|(C_\xi)^{\tilde{t}|v|}|\mathbf{r}_{[\frac{\tilde{t}|v|}{L_\xi}]}| & (C_\xi)^{\tilde{t}|v|}\frac{|\mathbf{r}_{[\frac{\tilde{t}|v|}{L_\xi}]}|}{|\mathbf{r}_1|} & (C_\xi)^{\tilde{t}|v|}|\mathbf{r}_{[\frac{\tilde{t}|v|}{L_\xi}]}| & (C_\xi)^{\tilde{t}|v|}|\mathbf{r}_{[\frac{\tilde{t}|v|}{L_\xi}]}|\\ \hline
0 & 0 & |v|(C_\xi)^{\tilde{t}|v|} & |v|(C_\xi)^{\tilde{t}|v|} & (C_\xi)^{\tilde{t}|v|}\frac{1}{|\mathbf{r}_1|} & (C_\xi)^{\tilde{t}|v|} & (C_\xi)^{\tilde{t}|v|}\\
0 & 0 & |v|(C_\xi)^{\tilde{t}|v|} & |v|(C_\xi)^{\tilde{t}|v|} & (C_\xi)^{\tilde{t}|v|}\frac{1}{|\mathbf{r}_1|} & (C_\xi)^{\tilde{t}|v|} & (C_\xi)^{\tilde{t}|v|}
\end{array}\right]
$$

$$
\lesssim\quad C^{C|t-s||v|}\left[\begin{array}{cc|c|c|c}
1 & 0 & \frac{1}{|v|} & \frac{1}{|v||\mathbf{v}_\perp^1|} & \frac{1}{|v|^2}\\
0 & 1 & \mathbf{0}_{1,2} & 0 & \mathbf{0}_{1,2}\\ \hline
\mathbf{0}_{2,1} & \mathbf{0}_{2,1} & O_\xi(1) & \frac{1}{|\mathbf{v}_\perp^1|} & \frac{1}{|v|}\\ \hline
0 & 0 & |\mathbf{v}_\perp^1| & O_\xi(1) & \frac{|\mathbf{v}_\perp^1|}{|v|}\\ \hline
\mathbf{0}_{2,1} & \mathbf{0}_{2,1} & |v| & \frac{|v|}{|\mathbf{v}_\perp^1|} & O_\xi(1)
\end{array}\right]_{7\times7}, \tag{149}
$$

where we have used (145) and the Velocity lemma(Lemma 1) and (65), (66) and $\mathbf{r}_i=\mathfrak{C}_1e^{\frac{\mathfrak{C}}{2}C_1\mathbf{r}^i}\lesssim e^{C|t-s||v|}\frac{|\mathbf{v}_\perp^1|}{|v|}$ and $\frac{\mathbf{r}_{[\frac{|t-s||v|}{L_\xi}]}}{\mathbf{r}_1}=\frac{\mathbf{r}^{[\frac{|t-s||v|}{L_\xi}]}}{\mathbf{r}^1}=\frac{|\mathbf{v}_\perp^{[\frac{|t-s||v|}{L_\xi}]}|}{|\mathbf{v}_\perp^1|}\leq\mathfrak{C}_1e^{\frac{\mathfrak{C}}{2}|v||t-s|}$.

Step 8. Intermediate summary for the matrix method and the final estimate for Type II

Recall from (130) and (132), (149), (134),

$$\begin{aligned}
&\frac{\partial(s^{\ell_*},\mathbf{X}_{\ell_*}(s^{\ell_*}),\mathbf{V}_{\ell_*}(s^{\ell_*}))}{\partial(s^1,\mathbf{X}_1(s^1),\mathbf{V}_1(s^1))} \equiv \frac{\partial(s^{\ell_*},\mathbf{x}_{\perp_{\ell_*}}(s^{\ell_*}),\mathbf{x}_{\parallel_{\ell_*}}(s^{\ell_*}),\mathbf{v}_{\perp_{\ell_*}}(s^{\ell_*}),\mathbf{v}_{\parallel_{\ell_*}}(s^{\ell_*}))}{\partial(s^1,\mathbf{x}_{\perp_1}(s^1),\mathbf{x}_{\parallel_1}(s^1),\mathbf{v}_{\perp_1}(s^1),\mathbf{v}_{\parallel_1}(s^1))}\\
&= \frac{\partial(s^{\ell_*},\mathbf{x}_{\perp_{\ell_*}}(s^{\ell_*}),\mathbf{x}_{\parallel_{\ell_*}}(s^{\ell_*}),\mathbf{v}_{\perp_{\ell_*}}(s^{\ell_*}),\mathbf{v}_{\parallel_{\ell_*}}(s^{\ell_*}))}{\partial(t^{\ell_*},0,\mathbf{x}^{\ell_*}_{\parallel_{\ell_*}},\mathbf{v}^{\ell_*}_{\perp_{\ell_*}},\mathbf{v}^{\ell_*}_{\parallel_{\ell_*}})}\\
&\quad\times\prod_{i=1}^{[\frac{|t-s||v|}{L_\xi}]}\frac{\partial(t^{\ell_{i+1}},0,\mathbf{x}^{\ell_{i+1}}_{\parallel_{\ell_{i+1}}},\mathbf{v}^{\ell_{i+1}}_{\perp_{\ell_{i+1}}},\mathbf{v}^{\ell_{i+1}}_{\parallel_{\ell_{i+1}}})}{\partial(t^{\ell_{i+1}-1},0,\mathbf{x}^{\ell_{i+1}-1}_{\parallel_{\ell_{i+1}-1}},\mathbf{v}^{\ell_{i+1}-1}_{\perp_{\ell_{i+1}-1}},\mathbf{v}^{\ell_{i+1}-1}_{\parallel_{\ell_{i+1}-1}})}\times\cdots\times\frac{\partial(t^{\ell_i+1},0,\mathbf{x}^{\ell_i+1}_{\parallel_{\ell_{i+1}}},\mathbf{v}^{\ell_i+1}_{\perp_{\ell_{i+1}}},\mathbf{v}^{\ell_i+1}_{\parallel_{\ell_{i+1}}})}{\partial(t^{\ell_i},0,\mathbf{x}^{\ell_i}_{\parallel_{\ell_i}},\mathbf{v}^{\ell_i}_{\perp_{\ell_i}},\mathbf{v}^{\ell_i}_{\parallel_{\ell_i}})}\\
&\quad\times\frac{\partial(t^1,0,\mathbf{x}^1_{\parallel_1},\mathbf{v}^1_{\perp_1},\mathbf{v}^1_{\parallel_1})}{\partial(s^1,\mathbf{x}_{\perp_1}(s^1),\mathbf{x}_{\parallel_1}(s^1),\mathbf{v}_{\perp_1}(s^1),\mathbf{v}_{\parallel_1}(s^1))}\\
&\le (132)\times(149)\times(134).
\end{aligned}$$

Then directly we bound

$$\le (132)\times C^{C|t-s||v|}$$
$$\times\left[\begin{array}{c|cc|cc}
1 & \frac{1}{|\mathbf{v}^1_\perp|}+\frac{|v|}{|\mathbf{v}^1_\perp|^2}+|t^1-s^1| & \frac{1}{|v|}+\frac{|v|}{|\mathbf{v}^1_\perp|^2}+|s^1-t^1| & \frac{1}{|v||\mathbf{v}^1_\perp|}+|s^1-t^1|^2 & \frac{1}{|v|^2}+\frac{|s^1-t^1|}{|v|}\\ \hline
0 & 0 & \mathbf{0}_{1,2} & 0 & \mathbf{0}_{1,2}\\
\mathbf{0}_{2,1} & \frac{|v|^2}{|\mathbf{v}^1_\perp|^2}+\frac{|v|}{|\mathbf{v}^1_\perp|}+|v||s^1-t^1| & 1+\frac{|v|^2}{|\mathbf{v}^1_\perp|^2} & \frac{1}{|\mathbf{v}^1_\perp|}+|s^1-t^1| & \frac{1}{|v|}\\ \hline
0 & \frac{|v|^2}{|\mathbf{v}^1_\perp|}+|v| & |\mathbf{v}^1_\perp|+\frac{|v|^2}{|\mathbf{v}^1_\perp|} & 1 & \frac{|\mathbf{v}^1_\perp|}{|v|}\\
\mathbf{0}_{2,1} & \frac{|v|^3}{|\mathbf{v}^1_\perp|^2}+\frac{|v|^2}{|\mathbf{v}^1_\perp|}+|v|^2|s_1-t^1| & |v|+\frac{|v|^3}{|\mathbf{v}^1_\perp|^2} & \frac{|v|}{|\mathbf{v}^1_\perp|}+|v||s_1-t^1| & 1
\end{array}\right], \tag{150}$$

where we have used the Velocity lemma(Lemma 1) and (145), (65), (66) and

$$|v||t^1-s^1|\le\min\{|v|(t_{\mathbf{b}}(x,v)+t_{\mathbf{b}}(x,-v)),(t-s)|v|\}\lesssim_\Omega\min\{\frac{|\mathbf{v}^1_\perp|}{|v|},(t-s)|v|\}\lesssim_\Omega C^{C|t-s||v|}\min\{\frac{|\mathbf{v}^1_\perp|}{|v|},1\}.$$

Again we use the Velocity lemma (Lemma 1) and (145) , (65), (66) and

$$|v||t^{\ell_*}-s^{\ell_*}|\le\min\{|v||t^{\ell_*}-t^{\ell_*+1}|,|t-s||v|\}\lesssim_\Omega\min\{\frac{|\mathbf{v}^{\ell_*}_\perp|}{|v|},|t-s||v|\}\lesssim_\Omega C^{C|t-s||v|}\min\{\frac{|\mathbf{v}^1_\perp|}{|v|},1\},$$

and $|\mathbf{v}_\perp(s^{\ell_*})|\lesssim_\Omega C^{C|v|(t-s)}|\mathbf{v}^1_\perp|$ to have, from (150)

$$\frac{\partial(s^{\ell_*},\mathbf{X}_{\ell_*}(s^{\ell_*}),\mathbf{V}_{\ell_*}(s^{\ell_*}))}{\partial(s^1,\mathbf{X}_1(s^1),\mathbf{V}_1(s^1))}\lesssim C^{C|t-s||v|}\left[\begin{array}{c|c|cc}
0 & \mathbf{0}_{1,3} & 0 & \mathbf{0}_{1,2}\\ \hline
|\mathbf{v}^1_\perp| & O_\xi(1)+\frac{|v|}{|\mathbf{v}^1_\perp|} & \frac{1}{|v|} & \frac{1}{|v|}\\
|v| & \frac{|v|^2}{|\mathbf{v}^1_\perp|^2}+\frac{|v|}{|\mathbf{v}^1_\perp|}+1 & |s^1-t^1|+|t^{\ell_*}-s^{\ell_*}|+\frac{1}{|\mathbf{v}^1_\perp|} & \frac{1}{|v|}\\ \hline
|v|^2 & \frac{|v|^3}{|\mathbf{v}^1_\perp|^2}+\frac{|v|^2}{|\mathbf{v}^1_\perp|}+|v| & O_\xi(1)+\frac{|v|}{|\mathbf{v}^1_\perp|} & O_\xi(1)
\end{array}\right]_{7\times7}. \tag{151}$$

Note that $|s^1-t^1|\lesssim_\xi\frac{1}{|v|}$ and $|t^{\ell_*}-t^{\ell_*}|\lesssim_\xi\frac{1}{|v|}$.

We consider the following case:

$$\text{There exists } \ell\in[\ell_*(s;t,x,v),0] \text{ such that } \mathbf{r}^\ell\ge\sqrt{\delta}. \tag{152}$$

Therefore ℓ is *Type II* in (125). Equivalently $\tau\in[t^{\ell+1},t^\ell]$ for some $\ell_*\le\ell\le0$ and $|\xi(X_{\mathbf{cl}}(\tau;t,x,v))|\ge C\delta$. By the Velocity lemma(Lemma 1), for all $1\le i\le\ell_*(s;t,x,v)$,

$$|\mathbf{r}^i| \;=\; \frac{|\mathbf{v}^i_\perp|}{|v|} \;\gtrsim_\xi\; e^{-C_\xi|v||t^i-t^\ell|}|\mathbf{r}^\ell| \;\gtrsim_\xi\; e^{-C_\xi|v|(t-s)}\sqrt{\delta}.$$

Especially, for all $1 \leq i \leq \ell_*(s;t,x,v)$,

$$|\mathbf{r}^1| \gtrsim_\xi e^{-C_\xi |v|(t-s)}\sqrt{\delta}, \quad \frac{1}{|\mathbf{r}^i|} = \frac{|v|}{|\mathbf{v}^i_\perp|} \lesssim_\xi \frac{e^{C_\xi |v|(t-s)}}{\sqrt{\delta}}.$$

Note that

$$\ell_*(s;t,x,v) \lesssim \max_i \frac{|v||t-s|}{\mathbf{r}^i} \lesssim_\delta C^{C|v||t-s|}.$$

Therefore in the case of (152), from (151),

$$\begin{aligned}
&\frac{\partial(s^{\ell_*}, \mathbf{X}_{\ell_*}(s^{\ell_*}), \mathbf{V}_{\ell_*}(s^{\ell_*}))}{\partial(s^1, \mathbf{X}_1(s^1), \mathbf{V}_1(s^1))} \\
&\lesssim C^{C(t-s)|v|} \left[\begin{array}{c|cc|cc}
0 & 0 & \mathbf{0}_{1,2} & 0 & \mathbf{0}_{1,2} \\ \hline
|\mathbf{v}^1_\perp| & 1+\frac{1}{\sqrt{\delta}} & 1+\frac{1}{\sqrt{\delta}} & \frac{1}{|v|} & \frac{1}{|v|} \\
|v| & 1+\frac{1}{\delta}+\frac{1}{\sqrt{\delta}} & 1+\frac{1}{\delta}+\frac{1}{\sqrt{\delta}} & \frac{1}{|v|}\frac{1}{\sqrt{\delta}} & \frac{1}{|v|} \\ \hline
|v|^2 & |v|(1+\frac{1}{\delta}+\frac{1}{\sqrt{\delta}}) & |v|(1+\frac{1}{\delta}+\frac{1}{\sqrt{\delta}}) & 1+\frac{1}{\sqrt{\delta}} & 1
\end{array}\right] \\
&\lesssim_\delta C^{C|v|(t-s)} \left[\begin{array}{c|c|c}
0 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ \hline
|v| & 1 & \frac{1}{|v|} \\ \hline
|v|^2 & |v| & 1
\end{array}\right],
\end{aligned}$$

and using (131) and (136)

$$\begin{aligned}
&\frac{\partial(s, X_{\mathbf{cl}}(s;t,x,v), V_{\mathbf{cl}}(s;t,x,v))}{\partial(t,x,v)} \\
&\lesssim_{\delta,\xi} C^{C|v|(t-s)} \frac{\partial(s, X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))}{\partial(s^{\ell_*}, \mathbf{X}_{\mathbf{cl}}(s^{\ell_*}), \mathbf{V}_{\mathbf{cl}}(s^{\ell_*}))} \left[\begin{array}{c|c|c}
0 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ \hline
|v| & 1 & \frac{1}{|v|} \\ \hline
|v|^2 & |v| & 1
\end{array}\right] \frac{\partial(s^1, \mathbf{x}_\perp(s^1), \mathbf{x}_\parallel(s^1), \mathbf{v}_\perp(s^1), \mathbf{v}_\parallel(s^1))}{\partial(t,x,v)} \\
&\lesssim_{\delta,\xi} C^{C|v|(t-s)} \begin{bmatrix}
0 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\
|v| & 1 & |s^{\ell_*}-s| \\
\mathbf{0}_{3,1} & |v| & 1
\end{bmatrix} \begin{bmatrix}
0 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\
|v| & 1 & \frac{1}{|v|} \\
|v|^2 & |v| & 1
\end{bmatrix} \begin{bmatrix}
1 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\
\mathbf{0}_{3,1} & 1 & |t-s^1| \\
\mathbf{0}_{3,1} & |v| & 1
\end{bmatrix} \\
&\lesssim_{\delta,\xi} C^{C|v|(t-s)} \begin{bmatrix}
0 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\
|v| & 1 & \frac{1}{|v|} \\
|v|^2 & |v| & 1
\end{bmatrix}.
\end{aligned} \tag{153}$$

Now we only need to consider the remainder case of (152), i.e.

$$\text{For all } \ell \in [\ell_*(s;t,x,v), 0], \text{ we have } \mathbf{r}_\ell \leq \sqrt{\delta}. \tag{154}$$

Note that in this case the moving frame($\mathbf{p}^\ell-$spherical coordinate) is well-defined for all $\tau \in [s,t]$. In next two step we use the ODE method to refine the submatrix of (151):

$$\frac{\partial(\mathbf{x}_{\parallel_{\ell_*}}(s^{\ell_*}), \mathbf{v}_{\parallel_{\ell_*}}(s^{\ell_*}))}{\partial(\mathbf{x}_{\perp_1}(s^1), \mathbf{x}_{\parallel_1}(s^1), \mathbf{v}_{\perp_1}(s^1), \mathbf{v}_{\parallel_1}(s^1))}.$$

Step 8. ODE method within the time scale $|t-s||v| \sim L_\xi$

Recall the end points(time) of intermediate groups from (128):

$$s < \underbrace{t^{\ell_*} < t^{\ell_{[\frac{|t-s||v|}{L_\xi}]+1}}}_{[\frac{|t-s||v|}{L_\xi}]+1} < \underbrace{t^{\ell_{[\frac{|t-s||v|}{L_\xi}]}} < t^{\ell_{[\frac{|t-s||v|}{L_\xi}]-1}+1}}_{[\frac{|t-s||v|}{L_\xi}]} < \cdots < \underbrace{t^{\ell_i} < t^{\ell_{i-1}+1}}_{i} < \cdots < \underbrace{t^{\ell_2} < t^{\ell_1+1}}_{2} < \underbrace{t^{\ell_1} < t^1}_{1} < t,$$

where the underbraced numbering indicates the index of the intermediate group. We further choose points independently on (t,x,v) for all $i=1,2,\cdots,[\frac{|t-s||v|}{L_\xi}]$:

$$
\begin{aligned}
t^{\ell_1+1} < s^2 < t^{\ell_1},\\
t^{\ell_2+1} < s^3 < t^{\ell_2},\\
\vdots\\
t^{\ell_i+1} < s^{i+1} < \underbrace{t^{\ell_i} < \cdots\cdots < t^{\ell_{i-1}+1}}_{i-\text{intermediate group}} < s^i < t^{\ell_{i-1}},\\
\vdots\\
t^{\ell_{[\frac{|t-s||v|}{L_\xi}]}+1} < s^{\ell_{[\frac{|t-s||v|}{L_\xi}]}+1} < t^{\ell_{[\frac{|t-s||v|}{L_\xi}]}}.
\end{aligned}
$$

We claim the following estimate at s^{i+1} via s^i:

$$
\begin{aligned}
&\begin{bmatrix} |\frac{\partial \mathbf{x}_{\parallel_{\ell_i}}(s^{i+1})}{\partial \mathbf{x}_{\perp_1}(s^1)}| & |\frac{\partial \mathbf{x}_{\parallel_{\ell_i}}(s^{i+1})}{\partial \mathbf{x}_{\parallel_1}(s^1)}| \\ |\frac{\partial \mathbf{v}_{\parallel_{\ell_i}}(s^{i+1})}{\partial \mathbf{x}_{\perp_1}(s^1)}| & |\frac{\partial \mathbf{v}_{\parallel_{\ell_i}}(s^{i+1})}{\partial \mathbf{x}_{\parallel_1}(s^1)}| \end{bmatrix}\\
&\lesssim_{\delta,\xi} \begin{bmatrix} 1 & \frac{1}{|v|} \\ |v| & 1 \end{bmatrix} \begin{bmatrix} |\frac{\partial \mathbf{x}_{\parallel_{\ell_i}}(s^{i})}{\partial \mathbf{x}_{\perp_1}(s^1)}| & |\frac{\partial \mathbf{x}_{\parallel_{\ell_i}}(s^{i})}{\partial \mathbf{x}_{\parallel_1}(s^1)}| \\ |\frac{\partial \mathbf{v}_{\parallel_{\ell_i}}(s^{i})}{\partial \mathbf{x}_{\perp_1}(s^1)}| & |\frac{\partial \mathbf{v}_{\parallel_{\ell_i}}(s^{i})}{\partial \mathbf{x}_{\parallel_1}(s^1)}| \end{bmatrix} + e^{C|v||t-s^i|} \begin{bmatrix} 1 & \frac{1}{|v|} \\ |v| & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ |v|\Big(1+\frac{|v|}{|\mathbf{v}_\perp^1|}\Big) & |v|\Big(1+\frac{|v|}{|\mathbf{v}_\perp^1|}\Big) \end{bmatrix},\\
&\begin{bmatrix} |\frac{\partial \mathbf{x}_{\parallel_{\ell_i}}(s^{i+1})}{\partial \mathbf{v}_{\perp_1}(s^1)}| & |\frac{\partial \mathbf{x}_{\parallel_{\ell_i}}(s^{i+1})}{\partial \mathbf{v}_{\parallel_1}(s^1)}| \\ |\frac{\partial \mathbf{v}_{\parallel_{\ell_i}}(s^{i+1})}{\partial \mathbf{v}_{\perp_1}(s^1)}| & |\frac{\partial \mathbf{v}_{\parallel_{\ell_i}}(s^{i+1})}{\partial \mathbf{v}_{\parallel_1}(s^1)}| \end{bmatrix}\\
&\lesssim_{\delta,\xi} \begin{bmatrix} 1 & \frac{1}{|v|} \\ |v| & 1 \end{bmatrix} \begin{bmatrix} |\frac{\partial \mathbf{x}_{\parallel_{\ell_i}}(s^{i})}{\partial \mathbf{v}_{\perp_1}(s^1)}| & |\frac{\partial \mathbf{x}_{\parallel_{\ell_i}}(s^{i})}{\partial \mathbf{v}_{\parallel_1}(s^1)}| \\ |\frac{\partial \mathbf{v}_{\parallel_{\ell_i}}(s^{i})}{\partial \mathbf{v}_{\perp_1}(s^1)}| & |\frac{\partial \mathbf{v}_{\parallel_{\ell_i}}(s^{i})}{\partial \mathbf{v}_{\parallel_1}(s^1)}| \end{bmatrix} + e^{C|v||t-s^i|} \begin{bmatrix} 1 & \frac{1}{|v|} \\ |v| & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.
\end{aligned}
\tag{155}
$$

Within the $i-$th intermediate group, we fix $\mathbf{p}^{\ell_i}-$spherical coordinate in *Step 8*. For the sake of simplicity we drop the index ℓ_i.

Denote, from (116),

$$
F_\parallel(\mathbf{x}_\perp,\mathbf{x}_\parallel,\mathbf{v}_\perp,\mathbf{v}_\parallel) := D(\mathbf{x}_\perp,\mathbf{x}_\parallel,\mathbf{v}_\parallel) + E(\mathbf{x}_\perp,\mathbf{x}_\parallel,\mathbf{v}_\parallel)\mathbf{v}_\perp, \tag{156}
$$

where D is a $\mathbf{r}^3$-vector-valued function and E is a 3×3 matrix-valued function:

$$
\begin{aligned}
D(\mathbf{x}_\perp,\mathbf{x}_\parallel,\mathbf{v}_\parallel) =& \sum_i G_{ij}(\mathbf{x}_\perp,\mathbf{x}_\parallel)\frac{(-1)^{i+1}}{-\mathbf{n}(\mathbf{x}_\parallel)\cdot(\partial_1\eta(\mathbf{x}_\parallel)\times\partial_2\eta(\mathbf{x}_\parallel))}\\
&\times\Big\{\mathbf{v}_\parallel\cdot\nabla^2\eta(\mathbf{x}_\parallel)\cdot\mathbf{v}_\parallel - \mathbf{x}_\perp\mathbf{v}_\parallel\cdot\nabla^2\mathbf{n}(\mathbf{x}_\parallel)\cdot\mathbf{v}_\parallel\Big\}\cdot(-\mathbf{n}(\mathbf{x}_\parallel)\times\partial_{i+1}\eta(\mathbf{x}_\parallel)),
\end{aligned}
$$

and

$$
E(\mathbf{x}_\perp,\mathbf{x}_\parallel,\mathbf{v}_\parallel) = \sum_i G_{ij}(\mathbf{x}_\perp,\mathbf{x}_\parallel)\frac{(-1)^{i+1}}{-\mathbf{n}(\mathbf{x}_\parallel)\cdot(\partial_1\eta(\mathbf{x}_\parallel)\times\partial_2\eta(\mathbf{x}_\parallel))}2\mathbf{v}_\perp\mathbf{v}_\parallel\cdot\nabla\mathbf{n}(\mathbf{x}_\parallel)\cdot(-\mathbf{n}(\mathbf{x}_\parallel)\times\partial_{i+1}\eta(\mathbf{x}_\parallel)).
$$

where $G(\cdot,\cdot)$ is smooth bounded function defined in (122) and we used the notational convention $i\equiv i \bmod 2$.

From Lemma 13 we take the time integration of (114) along the characteristics to have

$$
\begin{aligned}
\mathbf{x}_\parallel(s^{i+1}) &= \mathbf{x}_\parallel(s^i) - \int_{s^{i+1}}^{s^i}\mathbf{v}_\parallel(\tau)\mathrm{d}\tau,\\
\mathbf{v}_\parallel(s^{i+1}) &= \mathbf{v}_\parallel(s^i) - \int_{s^{i+1}}^{s^i}\big\{E(\mathbf{x}_\perp(\tau),\mathbf{x}_\parallel(\tau),\mathbf{v}_\parallel(\tau))\mathbf{v}_\perp(\tau) + D(\mathbf{x}_\perp(\tau),\mathbf{x}_\parallel(\tau),\mathbf{v}_\parallel(\tau))\big\}\,\mathrm{d}\tau.
\end{aligned}
$$

Note that $\mathbf{v}_\perp(\tau)$ is not continuous with respect to the time τ. Using (114) we rewrite this time integration as

$$\int_{s^{i+1}}^{s^i} E(\mathbf{x}_\perp(\tau),\mathbf{x}_\parallel(\tau),\mathbf{v}_\parallel(\tau))\mathbf{v}_\perp(\tau)\mathrm{d}\tau = \int_{t^{\ell_{i-1}+1}}^{s^i} + \sum_{\ell=\ell_i-1}^{\ell_{i-1}+1}\int_{t^{\ell+1}}^{t^\ell} + \int_{s^{i+1}}^{t^{\ell_i}},$$

then we use $\mathbf{v}_\perp(\tau)=\dot{\mathbf{x}}_\perp(\tau)$ and the integration by parts to have

$$\begin{aligned}
&\int_{t^{\ell_{i-1}+1}}^{s^i} E(\mathbf{x}_\perp(\tau),\mathbf{x}_\parallel(\tau),\mathbf{v}_\parallel(\tau))\dot{\mathbf{x}}_\perp(\tau)\mathrm{d}\tau + \sum_{\ell=\ell_i-1}^{\ell_{i-1}+1}\int_{t^{\ell+1}}^{t^\ell} E(\mathbf{x}_\perp(\tau),\mathbf{x}_\parallel(\tau),\mathbf{v}_\parallel(\tau))\dot{\mathbf{x}}_\perp(\tau)\mathrm{d}\tau \\
&+\int_{s^{i+1}}^{t^{\ell_i}} E(\mathbf{x}_\perp(\tau),\mathbf{x}_\parallel(\tau),\mathbf{v}_\parallel(\tau))\dot{\mathbf{x}}_\perp(\tau)\mathrm{d}\tau \\
&= E(s^i)\mathbf{x}_\perp(s^i) - E(t^{\ell_{i-1}+1})\underbrace{\mathbf{x}_\perp(t^{\ell_{i-1}+1})}_{=0} - \int_{t^{\ell_{i-1}+1}}^{s^i}\big[\mathbf{v}_\perp(\tau),\mathbf{v}_\parallel(\tau),F_\parallel(\tau)\big]\cdot\nabla E(\tau)\mathbf{x}_\perp(\tau)\mathrm{d}\tau \\
&+\sum_{\ell=\ell_i-1}^{\ell_{i-1}+1}\Big\{E(t^\ell)\underbrace{\mathbf{x}_\perp(t^\ell)}_{=0} - E(t^{\ell+1})\underbrace{\mathbf{x}_\perp(t^{\ell+1})}_{=0} - \int_{t^{\ell+1}}^{t^\ell}\big[\mathbf{v}_\perp(\tau),\mathbf{v}_\parallel(\tau),F_\parallel(\tau)\big]\cdot\nabla E(\tau)\mathbf{x}_\perp(\tau)\mathrm{d}\tau\Big\} \\
&+E(t^{\ell_i})\underbrace{\mathbf{x}_\perp(t^{\ell_i})}_{=0} - E(s^{i+1})\mathbf{x}_\perp(s^{i+1}) - \int_{s^{i+1}}^{t^{\ell_i}}\big[\mathbf{v}_\perp(\tau),\mathbf{v}_\parallel(\tau),F_\parallel(\tau)\big]\cdot\nabla E(\tau)\mathbf{x}_\perp(\tau)\mathrm{d}\tau \\
&= E(\mathbf{x}_\perp,\mathbf{x}_\parallel,\mathbf{v}_\parallel)(s^i)\mathbf{x}_\perp(s^i) - E(s^{i+1})\mathbf{x}_\perp(s^{i+1}) - \int_{s^i}^{s^{i+1}}\big[\mathbf{v}_\perp(\tau),\mathbf{v}_\parallel(\tau),F_\parallel(\tau)\big]\cdot\nabla E(\tau)\mathbf{x}_\perp(\tau)\mathrm{d}\tau,
\end{aligned}$$

where we have used the fact $X_{\mathbf{cl}}(t^\ell)\in\partial\Omega$(therefore $\mathbf{x}_\perp(t^\ell)=0$) and the notation $E(\tau)=E(\mathbf{x}_\perp(\tau),\mathbf{x}_\parallel(\tau),\mathbf{v}_\parallel(\tau))$, $D(\tau)=D(\mathbf{x}_\perp(\tau),\mathbf{x}_\parallel(\tau),\mathbf{v}_\parallel(\tau))$, $F_\parallel(\tau)=F_\parallel(\mathbf{x}_\perp(\tau),\mathbf{x}_\parallel(\tau),\mathbf{v}_\perp(\tau),\mathbf{v}_\parallel(\tau))$.

Overall we have

$$\begin{aligned}
\mathbf{x}_\parallel(s^{i+1}) &= \mathbf{x}_\parallel(s^i) - \int_{s^{i+1}}^{s^i}\mathbf{v}_\parallel(\tau)\mathrm{d}\tau,\\
\mathbf{v}_\parallel(s^{i+1}) &= \mathbf{v}_\parallel(s^i) - E(s^i)\mathbf{x}_\perp(s^i) + E(s^{i+1})\mathbf{x}_\perp(s^{i+1})\\
&\quad+\int_{s^{i+1}}^{s^i}\big[\mathbf{v}_\perp(\tau),\mathbf{v}_\parallel(\tau),F_\parallel(\tau)\big]\cdot\nabla E(\tau)\mathbf{x}_\perp(\tau)\mathrm{d}\tau - \int_{s^{i+1}}^{s^i}D(\tau)\mathrm{d}\tau.
\end{aligned}\tag{157}$$

Denote

$$\partial = [\partial_{\mathbf{x}_\perp(s^1)},\partial_{\mathbf{x}_\parallel(s^1)},\partial_{\mathbf{v}_\perp(s^1)},\partial_{\mathbf{v}_\parallel(s^1)}] = [\frac{\partial}{\partial\mathbf{x}_\perp(s^1)},\frac{\partial}{\partial\mathbf{x}_\parallel(s^1)},\frac{\partial}{\partial\mathbf{v}_\perp(s^1)},\frac{\partial}{\partial\mathbf{v}_\parallel(s^1)}].$$

We claim that, in a sense of distribution on $(s^1,\mathbf{x}_\perp(s^1),\mathbf{x}_\parallel(s^1),\mathbf{v}_\perp(s^1),\mathbf{v}_\parallel(s^1))\in[0,\infty)\times(0,C_\xi)\times(0,2\pi]\times(\delta,\pi-\delta)\times\mathbb{R}\times\mathbb{R}^2$,

$$\begin{aligned}
&\big[\partial\mathbf{x}_\perp(s^{i+1};s^1,\mathbf{x}(s^1),\mathbf{v}(s^1)),\partial\mathbf{x}_\parallel(s^{i+1};s^1,\mathbf{x}(s^1),\mathbf{v}(s^1)),\ \partial\mathbf{v}_\parallel(s^{i+1};s^1,\mathbf{x}(s^1),\mathbf{v}(s^1))\big]\\
&=\sum_\ell \mathbf{1}_{[t^{\ell+1},t^\ell)}(s^{i+1})[\partial\mathbf{x}_\perp,\partial\mathbf{x}_\parallel,\partial\mathbf{v}_\parallel],\\
&\partial\big[\mathbf{v}_\perp(s^{i+1};s^1,\mathbf{x}(s^1),\mathbf{v}(s^1))\mathbf{x}_\perp(s^{i+1};s^1,\mathbf{x}(s^1),\mathbf{v}(s^1))\big] = \sum_\ell\mathbf{1}_{[t^{\ell+1},t^\ell)}(s^{i+1})\big\{\partial\mathbf{v}_\perp\mathbf{x}_\perp+\mathbf{v}_\perp\partial\mathbf{x}_\perp\big\},
\end{aligned}\tag{158}$$

i.e. the distributional derivatives of $[\mathbf{x}_\perp,\mathbf{x}_\parallel,\mathbf{v}_\parallel]$ and $\mathbf{v}_\perp\mathbf{x}_\perp$ equal the piecewise derivatives. Let $\phi(\tau',\mathbf{x}_\perp,\mathbf{x}_\parallel,\mathbf{v}_\perp,\mathbf{v}_\parallel)\in C_c^\infty([0,\infty)\times(0,C_\xi)\times\mathbb{S}^2\times\mathbb{R}\times\mathbb{R}^2)$. Therefore $\phi\equiv0$ when $\mathbf{x}_\perp<\delta$. For $\mathbf{x}_\perp\geq\delta$ we use the proof of Lemma 13: For $x=\eta(\mathbf{x}_\parallel)+\mathbf{x}_\perp[-\mathbf{n}(\mathbf{x}_\parallel)]$,

$$|\mathbf{x}_\perp| \ \lesssim_\xi\ \xi(x)=\xi(\eta(\mathbf{x}_\parallel)+\mathbf{x}_\perp[-\mathbf{n}(\mathbf{x}_\parallel)])\ \lesssim_\xi\ |\mathbf{x}_\perp|,$$

and therefore $\xi(x) \gtrsim_\xi \delta$ and $\alpha(x,v) \gtrsim_\xi |\xi(x)||v|^2 \gtrsim_\xi |v|^2\delta$. Since we are considering the case $t-s > t_{\mathbf{b}}(x,v)$, from $|v|t_{\mathbf{b}}(x,v) \gtrsim \mathbf{x}_\perp \geq \delta$ we have $|v| \gtrsim_\xi \frac{\delta}{t-s}$ and finally we obtain the lower bound $\alpha(x,v) \gtrsim_\xi \frac{\delta^3}{|t-s|^2} > 0$. By the Velocity lemma, for $(x,v) \in \text{supp}(\phi)$

$$\alpha(x^\ell, v^\ell) \gtrsim_\xi e^{-C|v||t^1-t^\ell|}\alpha(x,v) \gtrsim_\xi e^{-C|v|(t-s)}\frac{\delta^3}{|t-s|^2} \gtrsim_{\xi,|t-s|,\delta,\phi} 1 > 0,$$

where we used the fact that ϕ vanishes away from a compact subset $\text{supp}(\phi)$. Therefore $t^\ell(t,x,v) = t^\ell(t,\mathbf{x}_\perp,\mathbf{x}_\parallel,\mathbf{v}_\perp,\mathbf{v}_\parallel)$ is smooth with respect to $\mathbf{x}_\perp,\mathbf{x}_\parallel,\mathbf{v}_\perp,\mathbf{v}_\parallel$ locally on $\text{supp}(\phi)$ and therefore $\mathcal{M} = \{(\tau',\mathbf{x},\mathbf{v}) \in \text{supp}(\phi) : \tau' = t^\ell(t,\mathbf{x},\mathbf{v})\}$ is a smooth manifold.

It suffices to consider the case $\tau' \sim t^\ell(t,x,v)$. Denote $\partial_{\mathbf{e}} = [\partial_{\mathbf{x}_\perp},\partial_{\mathbf{x}_{\parallel,1}},\partial_{\mathbf{x}_{\parallel,2}},\partial_{\mathbf{v}_\perp},\partial_{\mathbf{v}_{\parallel,1}},\partial_{\mathbf{v}_{\parallel,2}}]$ and $n_{\mathcal{M}} = \mathbf{e}_1$ to have

$$\begin{aligned}
&\int_{\{(\tau',\mathbf{x},\mathbf{v})\in\text{supp}(\phi)\}} [\partial_{\mathbf{e}}\mathbf{x}_\perp(\tau';t,\mathbf{x},\mathbf{v}),\partial_{\mathbf{e}}\mathbf{x}_\parallel(\tau';t,\mathbf{x},\mathbf{v}),\partial_{\mathbf{e}}\mathbf{v}_\parallel(\tau';t,\mathbf{x},\mathbf{v})]\phi(\tau',\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}\mathrm{d}\tau' \\
&= \int_{\tau'<t^\ell} + \int_{\tau'\geq t^\ell} \\
&= \int_{\mathcal{M}} \Big(\lim_{\tau'\uparrow t^\ell}[\mathbf{x}_\perp(\tau'),\mathbf{x}_\parallel(\tau'),\mathbf{v}_\parallel(\tau')] - \lim_{\tau'\downarrow t^\ell}[\mathbf{x}_\perp(\tau'),\mathbf{x}_\parallel(\tau'),\mathbf{v}_\parallel(\tau')]\Big)\phi(\tau',\mathbf{x},\mathbf{v})\{\mathbf{e}\cdot n_{\mathcal{M}}\}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v} \\
&\quad - \int_{\{\tau'\neq t^\ell(t,\mathbf{x},\mathbf{v})\}} [\mathbf{x}_\perp(\tau'),\mathbf{x}_\parallel(\tau'),\mathbf{v}_\parallel(\tau')]\partial_{\mathbf{e}}\phi(\tau',\mathbf{x},\mathbf{v})\mathrm{d}\tau'\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x} \\
&= -\int_{\{\tau'\neq t^\ell(t,\mathbf{x},\mathbf{v})\}} [\mathbf{x}_\perp(\tau'),\mathbf{x}_\parallel(\tau'),\mathbf{v}_\parallel(\tau')]\partial_{\mathbf{e}}\phi(\tau',\mathbf{x},\mathbf{v})\mathrm{d}\tau'\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x},
\end{aligned}$$

where we used the continuity of $[\mathbf{x}_\perp(\tau';t,\mathbf{x},\mathbf{v}),\mathbf{x}_\parallel(\tau';t,\mathbf{x},\mathbf{v}),\mathbf{v}_\parallel(\tau';t,\mathbf{x},\mathbf{v})]$ in terms of τ' near $t^\ell(t,\mathbf{x},\mathbf{v})$.

Note that $\mathbf{v}_\perp(\tau';t,\mathbf{x},\mathbf{v})$ is discontinuous around $\tau' \sim t^\ell$.($\lim_{\tau'\downarrow t^\ell}\mathbf{v}_\perp(\tau') = -\lim_{\tau'\uparrow t^\ell}\mathbf{v}_\perp(\tau')$) However with crucial $\mathbf{x}_\perp(\tau')-$multiplication we have $\mathbf{x}_\perp(t^\ell)\mathbf{v}_\perp(t^\ell) = 0$ and therefore

$$\begin{aligned}
&\int_{\{(\tau',\mathbf{x},\mathbf{v})\in\text{supp}(\phi)\}} \partial_{\mathbf{e}}[\mathbf{x}_\perp(\tau';t,\mathbf{x},\mathbf{v})\mathbf{v}_\perp(\tau';t,\mathbf{x},\mathbf{v})]\phi(\tau',\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}\mathrm{d}\tau' \\
&= \int_{\tau'<t^\ell} + \int_{\tau'\geq t^\ell} \\
&= \int_{\mathcal{M}} \Big(\lim_{\tau'\uparrow t^\ell}[\mathbf{x}_\perp(\tau')\mathbf{v}_\perp(\tau')] - \lim_{\tau'\downarrow t^\ell}[\mathbf{x}_\perp(\tau')\mathbf{v}_\perp(\tau')]\Big)\phi(\tau',\mathbf{x},\mathbf{v})\{\mathbf{e}\cdot n_{\mathcal{M}}\}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v} \\
&\quad - \int_{\{\tau'\neq t^\ell(t,\mathbf{x},\mathbf{v})\}} [\mathbf{x}_\perp(\tau')\mathbf{v}_\perp(\tau')]\partial_{\mathbf{e}}\phi(\tau',\mathbf{x},\mathbf{v})\mathrm{d}\tau'\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x} \\
&= -\int_{\{\tau'\neq t^\ell(t,\mathbf{x},\mathbf{v})\}} [\mathbf{x}_\perp(\tau';t,\mathbf{x},\mathbf{v})\mathbf{v}_\perp(\tau';t,\mathbf{x},\mathbf{v})]\partial_{\mathbf{e}}\phi(\tau',\mathbf{x},\mathbf{v})\mathrm{d}\tau'\mathrm{d}\mathbf{v}\mathrm{d}\mathbf{x}.
\end{aligned}$$

We apply (158) to (157)

$$
\begin{aligned}
&\partial \mathbf{x}_\parallel(s^{i+1}) = \partial \mathbf{x}_\parallel(s^i) - \int_{s^{i+1}}^{s^i} \partial \mathbf{v}_\parallel(\tau)\mathrm{d}\tau,\\
&\partial \mathbf{v}_\parallel(s^{i+1}) = \partial E(s^{i+1})\mathbf{x}_\perp(s^{i+1}) + E(s^{i+1})\partial \mathbf{x}_\perp(s^{i+1}) + \partial \mathbf{v}_\parallel(s^{i+1}) - \partial[E(\mathbf{x}_\perp,\mathbf{x}_\parallel,\mathbf{v}_\parallel)\mathbf{x}_\perp](s^{i+1})\\
&\quad + \int_{s^{i+1}}^{s^i} \partial \mathbf{v}_\perp(\tau)\partial_{\mathbf{x}_\perp}E(\tau)\mathbf{x}_\perp(\tau) + \partial \mathbf{v}_\parallel(\tau)\cdot\nabla_{\mathbf{x}_\parallel}E(\tau)\mathbf{x}_\perp(\tau)\mathrm{d}\tau\\
&\quad + \int_{s^{i+1}}^{s^i} \Big\{ \Big[\partial \mathbf{x}_\perp(\tau)\partial_{\mathbf{x}_\perp}E(\tau) + \partial \mathbf{x}_\parallel(\tau)\cdot\nabla_{\mathbf{x}_\parallel}E(\tau) + \partial \mathbf{v}_\parallel(\tau)\cdot\nabla_{\mathbf{v}_\parallel}E(\tau)\Big]\mathbf{v}_\perp(\tau)\\
&\quad\ + E(\tau)\partial \mathbf{v}_\perp(\tau) + \partial \mathbf{x}_\perp(\tau)\partial_{\mathbf{x}_\perp}D(\tau) + \partial \mathbf{x}_\parallel(\tau)\cdot\nabla_{\mathbf{x}_\parallel}D(\tau) + \partial \mathbf{v}_\parallel(\tau)\nabla_{\mathbf{v}_\parallel}D(\tau)\Big\}\cdot\nabla_{\mathbf{v}_\parallel}E(\tau)\mathbf{x}_\perp(\tau)\mathrm{d}\tau\\
&\quad + \int_{s^{i+1}}^{s^i} \Big\{\mathbf{v}_\perp(\tau)[\partial \mathbf{x}_\perp(\tau),\partial \mathbf{x}_\parallel(\tau),\partial \mathbf{v}_\parallel(\tau)]\cdot\nabla\partial_{\mathbf{x}_\perp}E(\tau) + \mathbf{v}_\parallel(\tau)\cdot[\partial \mathbf{x}_\perp(\tau),\partial \mathbf{x}_\parallel(\tau),\partial \mathbf{v}_\parallel(\tau)]\cdot\nabla\nabla_{\mathbf{x}_\parallel}E(\tau)\\
&\qquad\qquad + F_\parallel(\tau)\cdot[\partial \mathbf{x}_\perp(\tau),\partial \mathbf{x}_\parallel(\tau),\partial \mathbf{v}_\parallel(\tau)]\cdot\nabla\nabla_{\mathbf{v}_\parallel}E(\tau)\Big\}\mathbf{x}_\perp(\tau)\mathrm{d}\tau\\
&\quad + \int_{s^{i+1}}^{s^i} \big\{\mathbf{v}_\perp(\tau)\partial_{\mathbf{x}_\perp}E(\tau) + \mathbf{v}_\parallel(\tau)\cdot\nabla_{\mathbf{x}_\parallel}E(\tau) + F_\parallel(\tau)\cdot\nabla_{\mathbf{v}_\parallel}E(\tau)\big\}\partial \mathbf{x}_\perp(\tau)\mathrm{d}\tau\\
&\quad - \int_{s^{i+1}}^{s^i} \big[\partial \mathbf{x}_\perp(\tau),\partial \mathbf{x}_\parallel(\tau),\partial \mathbf{v}_\parallel(\tau)\big]\cdot\nabla D(\tau)\mathrm{d}\tau.
\end{aligned}
\tag{159}
$$

Now we use (151) to control $[\partial \mathbf{x}_\perp, \partial \mathbf{v}_\perp]$. Notice that we cannot directly use (151) since now we fix the chart for whole $i-$th intermediate group but the estimate (151) is for the moving frame.(For clarity we write the index for the chart for this part.) Note the time of bounces within the $i-$th intermediate group($|t^{\ell_{i-1}} - t^{\ell_i}||v| \sim L_\xi$) are

$$
t^{\ell_i+1} < s^{i+1} < t^{\ell_i} < t^{\ell_i-1} < \cdots\cdots < t^{\ell_{i-1}+2} < t^{\ell_{i-1}+1} < s^i < t^{\ell_{i-1}}.
$$

Now we apply (117) and (151) to bound

$$
|\partial \mathbf{x}_{\perp_{\ell_i}}(\tau')| \lesssim |\partial \mathbf{x}_{\perp_\ell}(\tau')| \lesssim \frac{|v|}{|\mathbf{v}_\perp^{\ell_{i-1}}|}, \quad |\partial \mathbf{v}_{\perp_{\ell_i}}(\tau')| \lesssim |\partial \mathbf{v}_{\perp_\ell}(\tau')| \lesssim \frac{|v|^3}{|\mathbf{v}_\perp^{\ell_{i-1}}|}.
$$

Together with (151) we have (for clarity, we write estimates for each derivative $\partial = [\partial_{\mathbf{x}_\perp}, \partial_{\mathbf{x}_\parallel}, \partial_{\mathbf{v}_\perp}, \partial_{\mathbf{v}_\parallel}]$):

$$
\begin{aligned}
&|\partial_{\mathbf{x}_\perp}\mathbf{x}_\parallel(s^{i+1})| \lesssim_\xi \int_{s^{i+1}}^{s^i} |\partial_{\mathbf{x}_\perp}\mathbf{v}_\parallel(\tau)|\mathrm{d}\tau,\\
&|\partial_{\mathbf{x}_\perp}\mathbf{v}_\parallel(s^{i+1})| \lesssim_\xi \frac{|v|^2}{|\mathbf{v}_\perp^{\ell_{i-1}}|} + |v||\mathbf{x}_\perp(\tau_i)||\partial_{\mathbf{x}_\perp}\mathbf{x}_\parallel(s^{i+1})| + |\mathbf{x}_\perp(s^{i+1})||\partial_{\mathbf{x}_\perp}\mathbf{v}_\parallel(s^{i+1})| + |v|\\
&\qquad\qquad + \int_{s^{i+1}}^{s^i} \Big\{\frac{|v|^3}{|\mathbf{v}_\perp^{\ell_{i-1}}|} + |v|^2|\partial_{\mathbf{x}_\perp}\mathbf{x}_\parallel(\tau)| + |\mathbf{x}_\perp(\tau)|\frac{|v|^4}{|\mathbf{v}_\perp^{\ell_{i-1}}|^2} + |v||\partial_{\mathbf{x}_\perp}\mathbf{v}_\parallel(\tau)|\Big\}\mathrm{d}\tau.
\end{aligned}
$$

We use (65), (66) and (119) and the condition $|\xi(X_{\mathbf{cl}}(\tau))| < \delta$ for all $\tau \in [s,t]$ to have, $\mathbf{x}_\perp(\tau;t,\mathbf{x},\mathbf{v}) \lesssim_\xi |\xi(X_{\mathbf{cl}}(\tau;t,x,v))|$ for all $\tau \in [s,t]$, and therefore

$$
\begin{aligned}
|v|^2|\mathbf{x}_\perp(\tau;t,\mathbf{x},\mathbf{v})| &\lesssim_\xi 2\xi(X_{\mathbf{cl}}(\tau;t,x,v))\{V_{\mathbf{cl}}(\tau;t,x,v)\cdot\nabla^2\xi(X_{\mathbf{cl}}(\tau;t,x,v))\cdot V_{\mathbf{cl}}(\tau;t,x,v)\}\\
&\lesssim_\xi \alpha(\tau;t,\mathbf{x},\mathbf{v}) \lesssim_\xi e^{C|v||t-\tau|}|\mathbf{v}_\perp^1|^2,
\end{aligned}
$$

where we used the convexity of ξ in (2) and the Velocity lemma(Lemma 1).

Hence we rewrite as, for $0 < \delta \ll 1$,

$$\begin{aligned}
|\frac{\partial \mathbf{x}_{\parallel}(s^{i+1})}{\partial \mathbf{x}_{\perp}}| &\lesssim_{\xi} |\frac{\partial \mathbf{x}_{\parallel}(s^{i})}{\partial \mathbf{x}_{\perp}}| + \int_{s^{i+1}}^{s^i} |\frac{\partial \mathbf{v}_{\parallel}(\tau')}{\partial \mathbf{x}_{\perp}}| \mathrm{d}\tau', \\
|\frac{\partial \mathbf{v}_{\parallel}(s^{i+1})}{\partial \mathbf{x}_{\perp}}| - \delta|v||\frac{\partial \mathbf{x}_{\parallel}(s^{i+1})}{\partial \mathbf{x}_{\perp}}| &\lesssim_{\xi,\delta} |\frac{\partial \mathbf{v}_{\parallel}(s^{i})}{\partial \mathbf{x}_{\perp}}| + \int_{s^{i+1}}^{s^i} \Big\{ |v|^2 |\frac{\partial \mathbf{x}_{\parallel}(\tau')}{\partial \mathbf{x}_{\perp}}| + |v||\frac{\partial \mathbf{v}_{\parallel}(\tau')}{\partial \mathbf{x}_{\perp}}| \Big\} \mathrm{d}\tau' \\
&\quad + |v| e^{C|v||t-s^{i+1}|} (1 + \frac{|v|}{|\mathbf{v}_{\perp}^1|}).
\end{aligned} \tag{160}$$

Similarly

$$\begin{aligned}
|\frac{\partial \mathbf{x}_{\parallel}(s^{i+1})}{\partial \mathbf{x}_{\parallel}}| &\lesssim_{\xi} |\frac{\partial \mathbf{x}_{\parallel}(s^{i})}{\partial \mathbf{x}_{\parallel}}| + \int_{s^{i+1}}^{s^i} |\frac{\partial \mathbf{v}_{\parallel}(\tau')}{\partial \mathbf{x}_{\parallel}}| \mathrm{d}\tau', \\
|\frac{\partial \mathbf{v}_{\parallel}(s^{i+1})}{\partial \mathbf{x}_{\parallel}}| - \delta|v||\frac{\partial \mathbf{x}_{\parallel}(s^{i+1})}{\partial \mathbf{x}_{\parallel}}| &\lesssim_{\xi,\delta} |\frac{\partial \mathbf{v}_{\parallel}(s^{i})}{\partial \mathbf{x}_{\parallel}}| + |v| \Big(1 + \frac{|v|}{|\mathbf{v}_{\perp}^1|}\Big) e^{C|v||t-s^{i+1}|} \\
&\quad + \int_{s^{i+1}}^{s^i} \Big\{ |v|^2 |\frac{\partial \mathbf{x}_{\parallel}(\tau')}{\partial \mathbf{x}_{\parallel}}| + |v||\frac{\partial \mathbf{v}_{\parallel}(\tau')}{\partial \mathbf{x}_{\parallel}}| \Big\} \mathrm{d}\tau', \\
|\frac{\partial \mathbf{x}_{\parallel}(s^{i+1})}{\partial \mathbf{v}_{\perp}}| &\lesssim_{\xi} |\frac{\partial \mathbf{x}_{\parallel}(s^{i})}{\partial \mathbf{v}_{\perp}}| + \int_{s^{i+1}}^{s^i} |\frac{\partial \mathbf{v}_{\parallel}(\tau')}{\partial \mathbf{v}_{\perp}}| \mathrm{d}\tau', \\
|\frac{\partial \mathbf{v}_{\parallel}(s^{i+1})}{\partial \mathbf{v}_{\perp}}| - \delta|v||\frac{\partial \mathbf{x}_{\parallel}(s^{i+1})}{\partial \mathbf{v}_{\perp}}| &\lesssim_{\xi,\delta} |\frac{\partial \mathbf{v}_{\parallel}(s^{i})}{\partial \mathbf{v}_{\perp}}| + e^{C|v||t-s^{i+1}|} + \int_{s^{i+1}}^{s^i} \Big\{ |v|^2 |\frac{\partial \mathbf{x}_{\parallel}(\tau')}{\partial \mathbf{v}_{\perp}}| + |v||\frac{\partial \mathbf{v}_{\parallel}(\tau')}{\partial \mathbf{v}_{\perp}}| \Big\} \mathrm{d}\tau', \\
|\frac{\partial \mathbf{x}_{\parallel}(s^{i+1})}{\partial \mathbf{v}_{\parallel}}| &\lesssim_{\xi} |\frac{\partial \mathbf{x}_{\parallel}(s^{i})}{\partial \mathbf{v}_{\parallel}}| + \int_{s^{i+1}}^{s^i} |\frac{\partial \mathbf{v}_{\parallel}(\tau')}{\partial \mathbf{v}_{\parallel}}| \mathrm{d}\tau', \\
|\frac{\partial \mathbf{v}_{\parallel}(s^{i+1})}{\partial \mathbf{v}_{\parallel}}| - \delta|v||\frac{\partial \mathbf{x}_{\parallel}(s^{i+1})}{\partial \mathbf{v}_{\parallel}}| &\lesssim_{\xi,\delta} |\frac{\partial \mathbf{v}_{\parallel}(s^{i})}{\partial \mathbf{v}_{\parallel}}| + e^{C|v||t-s^{i+1}|} + \int_{s^{i+1}}^{s^i} \Big\{ |v|^2 |\frac{\partial \mathbf{x}_{\parallel}(\tau')}{\partial \mathbf{v}_{\parallel}}| + |v||\frac{\partial \mathbf{v}_{\parallel}(\tau')}{\partial \mathbf{v}_{\parallel}}| \Big\} \mathrm{d}\tau'.
\end{aligned}$$

Now we claim a version of Gronwall's estimate: If $a(\tau), b(\tau), f(\tau), g(\tau) \geq 0$ for all $0 \leq \tau \leq t$, and satisfy, for $0 < \delta \ll 1$

$$\left[\begin{array}{cc} 1 & 0 \\ -\delta|v| & 1 \end{array}\right] \left[\begin{array}{c} a(\tau) \\ b(\tau) \end{array}\right] \lesssim_{\xi} \left[\begin{array}{cc} 0 & 1 \\ |v|^2 & |v| \end{array}\right] \left[\begin{array}{c} \int_{\tau}^{t} a(\tau')\mathrm{d}\tau' \\ \int_{\tau}^{t} b(\tau')\mathrm{d}\tau' \end{array}\right] + \left[\begin{array}{c} g(t-\tau) \\ h(t-\tau) \end{array}\right]$$

then

$$\left[\begin{array}{c} a(\tau) \\ b(\tau) \end{array}\right] \lesssim_{\delta,\xi} \int_{\tau}^{t} e^{|v|(\tau'-\tau)} \{ g(\tau') + \frac{h(\tau')}{|v|} \} \mathrm{d}\tau' \left[\begin{array}{c} |v| \\ |v|^2 \end{array}\right] + \left[\begin{array}{c} g(t-\tau) \\ \delta|v|g(t-\tau) + h(t-\tau) \end{array}\right]. \tag{161}$$

Define $\tilde{a}(\tau) := a(t-\tau), \tilde{b}(\tau) := b(t-\tau)$ and $A(\tau) := \int_0^{\tau} \tilde{a}(\tau')\mathrm{d}\tau', B(\tau) := \int_0^{\tau} \tilde{b}(\tau')\mathrm{d}\tau'$. Then

$$\begin{aligned}
\frac{d}{d\tau} \left[\begin{array}{cc} 1 & 0 \\ -\delta|v| & 1 \end{array}\right] \left[\begin{array}{c} A(\tau) \\ B(\tau) \end{array}\right] = \left[\begin{array}{cc} 1 & 0 \\ -\delta|v| & 1 \end{array}\right] \left[\begin{array}{c} \tilde{a}(\tau) \\ b(\tilde{\tau}) \end{array}\right] &\lesssim_{\xi} \left[\begin{array}{cc} 0 & 1 \\ |v|^2 & |v| \end{array}\right] \left[\begin{array}{c} A(\tau) \\ B(\tau) \end{array}\right] + \left[\begin{array}{c} \tilde{g}(\tau) \\ \tilde{h}(\tau) \end{array}\right] \\
&\lesssim_{\xi} \left[\begin{array}{cc} \delta|v| & 1 \\ (1+\delta)|v|^2 & |v| \end{array}\right] \left[\begin{array}{cc} 1 & 0 \\ -\delta|v| & 1 \end{array}\right] \left[\begin{array}{c} A(\tau) \\ B(\tau) \end{array}\right] + \left[\begin{array}{c} \tilde{g}(\tau) \\ \tilde{h}(\tau) \end{array}\right].
\end{aligned}$$

Using $\left[\begin{array}{cc} 0 & 1 \\ |v|^2 & |v| \end{array}\right] \left[\begin{array}{cc} 1 & 0 \\ \delta|v| & 1 \end{array}\right] = \left[\begin{array}{cc} \delta|v| & 1 \\ (1+\delta)|v|^2 & |v| \end{array}\right]$ and the notation $\left[\begin{array}{c} \tilde{A}(\tau) \\ \tilde{B}(\tau) \end{array}\right] := \left[\begin{array}{cc} 1 & 0 \\ -\delta|v| & 1 \end{array}\right] \left[\begin{array}{c} A(\tau) \\ B(\tau) \end{array}\right]$,

$$\frac{d}{d\tau} \left[\begin{array}{c} \tilde{A}(\tau) \\ \tilde{B}(\tau) \end{array}\right] \lesssim_{\xi} \left[\begin{array}{cc} \delta|v| & 1 \\ (1+\delta)|v|^2 & |v| \end{array}\right] \left[\begin{array}{c} \tilde{A}(\tau) \\ \tilde{B}(\tau) \end{array}\right] + \left[\begin{array}{c} \tilde{g}(\tau) \\ \tilde{h}(\tau) \end{array}\right],$$

We diagonalize $\begin{bmatrix} \delta|v| & 1 \\ (1+\delta)|v|^2 & |v| \end{bmatrix}$ as

$$= \begin{bmatrix} 1 & 1 \\ \frac{(1-\delta)+\sqrt{(1+\delta)^2+4}}{2}|v| & \frac{(1-\delta)-\sqrt{(1+\delta)^2+4}}{2}|v| \end{bmatrix} \begin{bmatrix} \frac{(1+\delta)+\sqrt{(1+\delta)^2+4}}{2}|v| & 0 \\ 0 & \frac{(1+\delta)-\sqrt{(1+\delta)^2+4}}{2}|v| \end{bmatrix}$$

$$\times \begin{bmatrix} \frac{-(1-\delta)+\sqrt{(1+\delta)^2+4}}{2\sqrt{(1+\delta)^2+4}} & \frac{1}{\sqrt{(1+\delta)^2+4}}\frac{1}{|v|} \\ \frac{(1-\delta)+\sqrt{(1+\delta)^2+4}}{2\sqrt{(1+\delta)^2+4}} & \frac{-1}{\sqrt{(1+\delta)^2+4}}\frac{1}{|v|} \end{bmatrix}.$$

Denote $\begin{bmatrix} \mathcal{A}(\tau) \\ \mathcal{B}(\tau) \end{bmatrix} := \begin{bmatrix} \frac{-(1-\delta)+\sqrt{(1+\delta)^2+4}}{2\sqrt{(1+\delta)^2+4}} & \frac{1}{\sqrt{(1+\delta)^2+4}}\frac{1}{|v|} \\ \frac{(1-\delta)+\sqrt{(1+\delta)^2+4}}{2\sqrt{(1+\delta)^2+4}} & \frac{-1}{\sqrt{(1+\delta)^2+4}}\frac{1}{|v|} \end{bmatrix} \begin{bmatrix} \tilde{A}(\tau) \\ \tilde{B}(\tau) \end{bmatrix}$ to rewrite

$$\frac{d}{d\tau}\begin{bmatrix} \mathcal{A}(\tau) \\ \mathcal{B}(\tau) \end{bmatrix} \lesssim_\xi \begin{bmatrix} \frac{(1+\delta)+\sqrt{(1+\delta)^2+4}}{2}|v| & 0 \\ 0 & \frac{(1+\delta)-\sqrt{(1+\delta)^2+4}}{2}|v| \end{bmatrix} \begin{bmatrix} \mathcal{A}(\tau) \\ \mathcal{B}(\tau) \end{bmatrix}$$
$$+ \begin{bmatrix} \frac{-(1-\delta)+\sqrt{(1+\delta)^2+4}}{2\sqrt{(1+\delta)^2+4}} & \frac{1}{\sqrt{(1+\delta)^2+4}}\frac{1}{|v|} \\ \frac{(1-\delta)+\sqrt{(1+\delta)^2+4}}{2\sqrt{(1+\delta)^2+4}} & \frac{-1}{\sqrt{(1+\delta)^2+4}}\frac{1}{|v|} \end{bmatrix} \begin{bmatrix} \tilde{g}(\tau) \\ \tilde{h}(\tau) \end{bmatrix}.$$

Therefore

$$\begin{bmatrix} \mathcal{A}(\tau) \\ \mathcal{B}(\tau) \end{bmatrix} \leq \begin{bmatrix} e^{C_{\xi,\delta}\frac{(1+\delta)+\sqrt{(1+\delta)^2+4}}{2}|v|\tau}\mathcal{A}(0) \\ e^{C_{\xi,\delta}\frac{(1+\delta)-\sqrt{(1+\delta)^2+4}}{2}|v|\tau}\mathcal{B}(0) \end{bmatrix}$$

$$+ \int_0^\tau \begin{bmatrix} e^{\frac{(1+\delta)+\sqrt{(1+\delta)^2+4}}{2}|v|(\tau-\tau')} & 0 \\ 0 & e^{\frac{(1+\delta)-\sqrt{(1+\delta)^2+4}}{2}|v|(\tau-\tau')} \end{bmatrix} \begin{bmatrix} \frac{-(1-\delta)+\sqrt{(1+\delta)^2+4}}{2\sqrt{(1+\delta)^2+4}} & \frac{1}{\sqrt{(1+\delta)^2+4}}\frac{1}{|v|} \\ \frac{(1-\delta)+\sqrt{(1+\delta)^2+4}}{2\sqrt{(1+\delta)^2+4}} & \frac{-1}{\sqrt{(1+\delta)^2+4}}\frac{1}{|v|} \end{bmatrix} \begin{bmatrix} \tilde{g}(\tau') \\ \tilde{h}(\tau') \end{bmatrix} \mathrm{d}\tau',$$

and then

$$\begin{bmatrix} A(\tau) \\ B(\tau) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \delta|v| & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{(1-\delta)+\sqrt{(1+\delta)^2+4}}{2}|v| & \frac{(1-\delta)-\sqrt{(1+\delta)^2+4}}{2}|v| \end{bmatrix} \begin{bmatrix} \mathcal{A}(\tau) \\ \mathcal{B}(\tau) \end{bmatrix}$$
$$\lesssim_{\xi,\delta} \int_0^\tau e^{C_{\xi,\delta}|v|(\tau-\tau')}\{\tilde{g}(\tau') + \frac{\tilde{h}(\tau')}{|v|}\}\mathrm{d}\tau' \begin{bmatrix} 1 \\ |v| \end{bmatrix}.$$

Together with the first inequality(the condition of the claim)

$$\begin{bmatrix} a(\tau) \\ b(\tau) \end{bmatrix} \lesssim_\xi \begin{bmatrix} 0 & 1 \\ |v|^2 & (1+\delta)|v| \end{bmatrix} \begin{bmatrix} A(t-\tau) \\ B(t-\tau) \end{bmatrix} + \begin{bmatrix} g(t-\tau) \\ \delta|v|g(t-\tau) + h(t-\tau) \end{bmatrix}$$
$$\lesssim_{\xi,\delta} \int_\tau^t e^{|v|(\tau'-\tau)}\{g(\tau') + \frac{h(\tau')}{|v|}\}\mathrm{d}\tau' \begin{bmatrix} |v| \\ |v|^2 \end{bmatrix} + \begin{bmatrix} g(t-\tau) \\ \delta|v|g(t-\tau) + h(t-\tau) \end{bmatrix}$$
$$\lesssim_{\xi,\delta} e^{C|v||t-\tau|} \begin{bmatrix} 1 & \frac{1}{|v|} \\ |v| & 1 \end{bmatrix} \begin{bmatrix} \sup|g| \\ \sup|h| \end{bmatrix},$$

and this proves the claim (161). We apply (161) to (160) and we prove the claim (155).

Step 9. ODE method within the time scale $|t-s| \sim 1$: Refinement of the estimate (151) We claim that

$$\begin{bmatrix} |\frac{\partial \mathbf{x}_{\parallel_{\tilde{\ell}}}(s)}{\partial \mathbf{x}_{\perp_1}}| & |\frac{\partial \mathbf{x}_{\parallel_{\tilde{\ell}}}(s)}{\partial \mathbf{x}_{\parallel_1}}| & |\frac{\partial \mathbf{x}_{\parallel_{\tilde{\ell}}}(s)}{\partial \mathbf{v}_{\perp_1}}| & |\frac{\partial \mathbf{x}_{\parallel_{\tilde{\ell}}}(s)}{\partial \mathbf{v}_{\parallel_1}}| \\ |\frac{\partial \mathbf{v}_{\parallel_{\tilde{\ell}}}(s)}{\partial \mathbf{x}_{\perp_1}}| & |\frac{\partial \mathbf{v}_{\parallel_{\tilde{\ell}}}(s)}{\partial \mathbf{x}_{\parallel_1}}| & |\frac{\partial \mathbf{v}_{\parallel_{\tilde{\ell}}}(s)}{\partial \mathbf{v}_{\perp_1}}| & |\frac{\partial \mathbf{v}_{\parallel_{\tilde{\ell}}}(s)}{\partial \mathbf{v}_{\parallel_1}}| \end{bmatrix} \lesssim C^{C|v||t-s|} \begin{bmatrix} (1+\frac{|v|}{|\mathbf{v}_\perp^1|}) & (1+\frac{|v|}{|\mathbf{v}_\perp^1|}) & \frac{1}{|v|} & \frac{1}{|v|} \\ |v|(1+\frac{|v|}{|\mathbf{v}_\perp^1|}) & |v|(1+\frac{|v|}{|\mathbf{v}_\perp^1|}) & 1 & 1 \end{bmatrix}, \tag{162}$$

where $\tilde{\ell} = [\frac{|t-s||v|}{L_\xi}]$. From (117)

$$\begin{bmatrix} D_{\mathbf{x}}\mathbf{x}_{\|_i} & D_{\mathbf{v}}\mathbf{x}_{\|_i} \\ D_{\mathbf{x}}\mathbf{v}_{\|_i} & D_{\mathbf{v}}\mathbf{v}_{\|_i} \end{bmatrix} = \frac{\partial(\mathbf{x}_{\|_i}, \mathbf{v}_{\|_i})}{\partial(\mathbf{x}_{\|_{i-1}}, \mathbf{v}_{\|_{i-1}})} \begin{bmatrix} D_{\mathbf{x}}\mathbf{x}_{\|_{i-1}} & D_{\mathbf{v}}\mathbf{x}_{\|_{i-1}} \\ D_{\mathbf{x}}\mathbf{v}_{\|_{i-1}} & D_{\mathbf{v}}\mathbf{v}_{\|_{i-1}} \end{bmatrix},$$

where

$$\frac{\partial(\mathbf{x}_{\|_i}, \mathbf{v}_{\|_i})}{\partial(\mathbf{x}_{\|_{i-1}}, \mathbf{v}_{\|_{i-1}})} \leq C \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline |v| & 1 \end{array}\right] \leq C \left[\begin{array}{c|c} 1 & \frac{1}{|v|} \\ \hline |v| & 1 \end{array}\right] := C\mathbf{B}.$$

Denote

$$\mathbf{D}_i(s) = \begin{bmatrix} |D_{\mathbf{x}}\mathbf{x}_{\|_i}(s)| & |D_{\mathbf{v}}\mathbf{x}_{\|_i}(s)| \\ |D_{\mathbf{x}}\mathbf{v}_{\|_i}(s)| & |D_{\mathbf{v}}\mathbf{v}_{\|_i}(s)| \end{bmatrix}, \quad \mathbf{G} := \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ |v|(1 + \frac{|v|}{|\mathbf{v}_\perp^1|}) & 1 \end{bmatrix}.$$

Note that from (155)

$$\mathbf{D}_i(s^{i+1}) \leq C\mathbf{B}\mathbf{D}_i(s^i) + C\mathbf{B}\mathbf{G}.$$

Therefore

$$\begin{aligned}
\mathbf{D}_{[\frac{|t-s||v|}{L_\xi}]}(s) &\leq C\mathbf{D}_{[\frac{|t-s||v|}{L_\xi}]}(\tau_{[\frac{|t-s||v|}{L_\xi}]}) + C\mathbf{B}\mathbf{G} \\
&\leq C^2\mathbf{B}\mathbf{D}_{[\frac{|t-s||v|}{L_\xi}]-1}(\tau_{[\frac{|t-s||v|}{L_\xi}]}) + C\mathbf{B}\mathbf{G} \\
&\leq C^2\mathbf{B}\mathbf{D}_{[\frac{|t-s||v|}{L_\xi}]-1}(\tau_{[\frac{|t-s||v|}{L_\xi}]-1}) + C^3\mathbf{B}\mathbf{G} + C\mathbf{B}\mathbf{G} \\
&\leq C^3\mathbf{B}^2\mathbf{D}_{[\frac{|t-s||v|}{L_\xi}]-1}(\tau_{[\frac{|t-s||v|}{L_\xi}]-2}) + \{C^2\mathbf{B} + \mathbf{Id}\}C\mathbf{B}\mathbf{G} \\
&\leq C^4\mathbf{B}^3\mathbf{D}_{[\frac{|t-s||v|}{L_\xi}]-2}(\tau_{[\frac{|t-s||v|}{L_\xi}]-2}) + \{C^3\mathbf{B}^2 + C^2\mathbf{B} + \mathbf{Id}\}C\mathbf{B}\mathbf{G} \\
&\ \ \vdots \\
&\lesssim C^{C|t-s||v|}\mathbf{B}^{C[|t-s||v|]}\mathbf{D}_1(\tau_1) + \sum_{i=0}^{C[|t-s||v|]} C^{i+1}\mathbf{B}^i\mathbf{B}\mathbf{G}.
\end{aligned}$$

But direct computation yields $\mathbf{B}^j \leq C^j\mathbf{B}$. Therefore

$$\mathbf{D}_{[\frac{|t-s||v|}{L_\xi}]}(s) \lesssim C^{C|t-s||v|}\mathbf{B}\{\mathbf{D}_1(\tau_1) + \mathbf{B}\mathbf{G}\}.$$

From (133) we have $\mathbf{D}_1(\tau_1) \lesssim \left[\begin{array}{c|c} 1 & \frac{1}{|v|} \\ \hline |v| & 1 \end{array}\right]$ and we conclude our claim (162).

With this estimates, we refine (151) to give a final estimate for the case of $|\xi(X_{\mathbf{cl}}(\tau; t, x, v))| < \delta$ for all $\tau \in [s, t]$:

$$\begin{aligned}
&\frac{\partial(s^{\ell_*}, \mathbf{x}_\perp(s^{\ell_*}), \mathbf{x}_\|(s^{\ell_*}), \mathbf{v}_\perp(s^{\ell_*}), \mathbf{v}_\|(s^{\ell_*}))}{\partial(s^1, \mathbf{x}_\perp(s^1), \mathbf{x}_\|(s^1), \mathbf{v}_\perp(s^1), \mathbf{v}_\|(s^1))} \\
&\lesssim C^{C|v|(t-s)} \left[\begin{array}{c|cc|cc}
0 & 0 & \mathbf{0}_{1,2} & 0 & \mathbf{0}_{1,2} \\ \hline
|\mathbf{v}_\perp^1| & 1 + \frac{|v|}{|\mathbf{v}_\perp^1|} & 1 + \frac{|v|}{|\mathbf{v}_\perp^1|} & \frac{1}{|v|} & \frac{1}{|v|} \\
|v| & 1 + \frac{|v|}{|\mathbf{v}_\perp^1|} & 1 + \frac{|v|}{|\mathbf{v}_\perp^1|} & |t-s| & |t-s| \\ \hline
|v|^2 & \frac{|v|^3}{|\mathbf{v}_\perp^1|^2} + \frac{|v|^2}{|\mathbf{v}_\perp^1|} + |v| & \frac{|v|^3}{|\mathbf{v}_\perp^1|^2} + \frac{|v|^2}{|\mathbf{v}_\perp^1|} + |v| & 1 + \frac{|v|}{|\mathbf{v}_\perp^1|} & 1 \\
|v|^2 & |v| + \frac{|v|^2}{|\mathbf{v}_\perp^1|} & |v| + \frac{|v|^2}{|\mathbf{v}_\perp^1|} & 1 & 1
\end{array}\right],
\end{aligned} \tag{163}$$

and from (131) and (136)

$$
\begin{aligned}
&\frac{\partial(s, X_{\mathbf{cl}}(s;t,x,v), V_{\mathbf{cl}}(s;t,x,v))}{\partial(t,x,v)} \\
&\lesssim C^{C|v|(t-s)} \ \frac{\partial(s, X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))}{\partial(s^{\ell_*}, \mathbf{X}_{\mathbf{cl}}(s^{\ell_*}), \mathbf{V}_{\mathbf{cl}}(s^{\ell_*}))} \left[\begin{array}{c|c|c} 0 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ \hline |v| & 1+\frac{|v|}{|\mathbf{v}_\perp^1|} & \frac{1}{|v|} \\ \hline |v|^2 & |v| + \frac{|v|^2}{|\mathbf{v}_\perp^1|} + \frac{|v|^3}{|\mathbf{v}_\perp^2|} & 1+\frac{|v|}{|\mathbf{v}_\perp^1|} \end{array}\right] \\
&\qquad \times \frac{\partial(s^1, \mathbf{x}_\perp(s^1), \mathbf{x}_\parallel(s^1), \mathbf{v}_\perp(s^1), \mathbf{v}_\parallel(s^1))}{\partial(t,x,v)} \\
&\lesssim C^{C|v|(t-s)} \begin{bmatrix} 0 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ |v| & 1 & |s^{\ell_*} - s| \\ \mathbf{0}_{3,1} & |v| & 1 \end{bmatrix} \left[\begin{array}{c|c|c} 0 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ \hline |v| & 1+\frac{|v|}{|\mathbf{v}_\perp^1|} & \frac{1}{|v|} \\ \hline |v|^2 & |v| + \frac{|v|^2}{|\mathbf{v}_\perp^1|} + \frac{|v|^3}{|\mathbf{v}_\perp^2|} & 1+\frac{|v|}{|\mathbf{v}_\perp^1|} \end{array}\right] \qquad (164) \\
&\qquad \times \begin{bmatrix} 1 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ \mathbf{0}_{3,1} & 1 & |t-s^1| \\ \mathbf{0}_{3,1} & |v| & 1 \end{bmatrix} \\
&\lesssim C^{C|v|(t-s)} \left[\begin{array}{c|c|c} 0 & \mathbf{0}_{1,3} & \mathbf{0}_{1,3} \\ \hline |v| & 1+\frac{|v|}{|\mathbf{v}_\perp^1|} & \frac{1}{|v|} \\ \hline |v|^2 & |v| + \frac{|v|^2}{|\mathbf{v}_\perp^1|} + \frac{|v|^3}{|\mathbf{v}_\perp^1|^2} & 1+\frac{|v|}{|\mathbf{v}_\perp^1|} \end{array}\right].
\end{aligned}
$$

Finally from (153) and (164) we conclude, for all $\tau \in [s,t]$

$$
\frac{\partial(X_{\mathbf{cl}}(s;t,x,v), V_{\mathbf{cl}}(s;t,x,v))}{\partial(t,x,v)} \leq Ce^{C|v|(t-s)} \left[\begin{array}{c|c|c} |v| & 1+\frac{|v|}{|\mathbf{v}_\perp^1|} & \frac{1}{|v|} \\ \hline |v|^2 & |v| + \frac{|v|^2}{|\mathbf{v}_\perp^1|} + \frac{|v|^3}{|\mathbf{v}_\perp^1|^2} & 1+\frac{|v|}{|\mathbf{v}_\perp^1|} \end{array}\right]_{6\times 7}
$$

From the Velocity lemma(Lemma 1),

$$
\begin{aligned}
|\mathbf{v}_\perp^1| &= |v^1 \cdot [-n(x^1)]| = |V_{\mathbf{cl}}(t^1;t,x,v) \cdot n(X_{\mathbf{cl}}(t^1;t,x,v))| \\
&= \sqrt{\alpha(X_{\mathbf{cl}}(t^1), V_{\mathbf{cl}}(t^1))} \geq e^{\mathfrak{C}|v||t-t^1|}\alpha(x,v) \gtrsim \alpha(x,v),
\end{aligned}
$$

and this completes the proof for the case (152).

□

Proof of Theorem 3. The approximation sequence is $f^0(t,x,v) \equiv f_0(x,v)$ and $f^n \equiv f^0$ for all $n = -1, -2, \cdots$, and for all $m = 1, 2, \cdots$,

$$
\begin{aligned}
\partial_t f^{m+1} + v \cdot \nabla_x f^{m+1} + \nu(F^m) f^{m+1} - K f^m &= \Gamma_{\text{gain}}(f^m, f^m), \\
f^{m+1}(t,x,v)|_{\gamma_-} = f^m(t,x,R_x v), \quad f^m(0,x,v) &= f_0(x,v).
\end{aligned}
$$

where $\nu(F^m)(v) = \int_{\mathbf{r}^3}\int_{\mathbb{S}^2} B(u-v,\omega)F^m(u)\mathrm{d}\omega\mathrm{d}u = \int_{\mathbf{r}^3}\int_{\mathbb{S}^2} B(u-v,\omega)\{\mu(u)+\sqrt{\mu(u)}f^m(u)\}\mathrm{d}\omega\mathrm{d}u$. Note that due to Lemma 6 we have $\sup_m \sup_{0\leq t\leq T} ||\langle v\rangle^\zeta e^{\theta|v|^2} f^m(t)||_\infty \lesssim_{\xi,T} (1+||\langle v\rangle^\zeta e^{\theta|v|^2} f_0||_\infty)||\langle v\rangle^\zeta e^{\theta|v|^2} f_0||_\infty$ and $\partial f^m \in L^\infty([0,\infty)\times\Omega\times\mathbf{r}^3)$ and the distributional derivatives coincide with the piecewise derivatives(Proposition 1 and Proposition 2). Especially the temporal (distributional) derivative $\partial_t f^m$ satisfies the boundary condition and therefore $\sup_{0\leq t\leq T}||\langle v\rangle^\zeta e^{\theta|v|^2}\partial_t f^m(t)||_\infty \lesssim_{T,\Omega} (1+||\langle v\rangle^\zeta e^{\theta|v|^2} f_0||_\infty)||\langle v\rangle^\zeta e^{\theta|v|^2}\partial_t f_0||_\infty$.

In the sense of distributions we have, for $\partial_{\mathbf{e}} = [\partial_x, \partial_v]$ with $\mathbf{e} \in \{x,v\}$,

$$
\partial_{\mathbf{e}} f^m(t,x,v) = \mathrm{I}_{\mathbf{e}} + \mathrm{II}_{\mathbf{e}} + \mathrm{III}_{\mathbf{e}}. \tag{165}
$$

Here

$$
\mathrm{I}_{\mathbf{e}} \ = \ e^{-\int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1},t^\ell)}(s)\nu(F^{m-\ell})(s)\mathrm{d}s} \ [\partial X_{\mathbf{cl}}(0), \partial V_{\mathbf{cl}}(0)] \cdot \nabla_{x,v} f_0(X_{\mathbf{cl}}(0), V_{\mathbf{cl}}(0)),
$$

and

$$\mathrm{II}_{\mathbf{e}} = -\int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1},t^\ell)}(s) e^{-\int_s^t \sum_j \mathbf{1}_{[t^{j+1},t^j)}(\tau)\nu(F^{m-j})(\tau)\mathrm{d}\tau} \int_s^t \sum_{j=0}^{\ell_*(s)} \mathbf{1}_{[t^{j+1},t^j)}(\tau)\partial_{\mathbf{e}}[\nu(F^{m-j})(\tau, X_{\mathbf{cl}}(\tau), V_{\mathbf{cl}}(\tau))]\mathrm{d}\tau$$
$$\times \Big\{ Kf^{m-\ell} + \Gamma_{\mathrm{gain}}(f^{m-\ell}, f^{m-\ell}) \Big\}(s, X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))\mathrm{d}s$$
$$- \int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1},t^\ell)}(s)\partial_{\mathbf{e}}\big[\nu(F^{m-\ell})(s, X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))\big]\mathrm{d}s \times e^{-\int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1},t^\ell)}(s)\nu(F^{m-\ell})(s)\mathrm{d}s}\ f_0(X_{\mathbf{cl}}(0), V_{\mathbf{cl}}(0)),$$

and

$$\mathrm{III}_{\mathbf{e}} = \int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1},t^\ell)}(s) e^{-\int_s^t \sum_{j=0}^{\ell_*(s)} \mathbf{1}_{[t^{j+1},t^j)}(\tau)\nu(F^{m-j})(\tau)\mathrm{d}\tau}$$
$$\times \partial_{\mathbf{e}}\big[Kf^{m-\ell}(s, X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s)) + \Gamma_{\mathrm{gain}}(f^{m-\ell}, f^{m-\ell})(s, X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))\big]\mathrm{d}s.$$

Recall the weights

$$e^{-\varpi\langle v\rangle t}\frac{[\alpha(x,v)]^\beta}{\langle v\rangle^{2\beta}}|\partial_x f(t,x,v)|, \quad e^{-\varpi\langle v\rangle t}\frac{|v|[\alpha(x,v)]^{\beta-\frac12}}{\langle v\rangle^{2\beta-1}}|\partial_v f(t,x,v)|.$$

From Lemma 24 and the Velocity lemma(Lemma 1) and Lemma 6 and $F^m \geq 0$ for all m

$$e^{-\varpi\langle v\rangle t}\frac{1}{\langle v\rangle^{2\beta}}[\alpha(x,v)]^\beta\ \mathrm{I}_{\mathbf{x}}$$
$$\lesssim_{\xi,t} e^{-\varpi\langle v\rangle t}\frac{1}{\langle v\rangle^{2\beta}}[\alpha(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))]^\beta e^{2\mathfrak{C}|v|t}\Big\{\frac{|v|}{\sqrt{\alpha(x,v)}}|\partial_x f_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))| + \frac{|v|^3}{\alpha(x,v)}|\partial_v f_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))|\Big\}$$
$$\lesssim_{\xi,t} ||\ \frac{|v|}{\langle v\rangle^{2\beta}}\alpha^{\beta-\frac12}\partial_x f_0\ ||_\infty + ||\ \frac{|v|^3}{\langle v\rangle^{2\beta}}\alpha^{\beta-1}\partial_v f_0\ ||_\infty,$$

and

$$e^{-\varpi\langle v\rangle t}\frac{|v|}{\langle v\rangle^{2\beta-1}}[\alpha(x,v)]^{\beta-\frac12}\ \mathrm{I}_{\mathbf{v}}$$
$$\lesssim_{\xi,t}\ e^{-\varpi\langle v\rangle t}\frac{|v|}{\langle v\rangle^{2\beta-1}}[\alpha(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))]^{\beta-\frac12}e^{2\mathfrak{C}|v|t}\Big\{\frac{1}{|v|}|\partial_x f_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))| + \frac{|v|}{\sqrt{\alpha(x,v)}}|\partial_v f_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))|\Big\}$$
$$\lesssim_{\xi,t}\ ||\ \frac{\alpha^{\beta-\frac12}}{\langle v\rangle^{2\beta-1}}\partial_x f_0\ ||_\infty + ||\ \frac{|v|^2}{\langle v\rangle^{2\beta-1}}\alpha^{\beta-1}\partial_v f_0\ ||_\infty,$$

where we have used $\alpha(x,v)\lesssim_\xi |v|^2$ and the choice of $\varpi \gg 1$.

From Lemma 4 and Lemma 5 we have

$$\mathrm{II}_{\mathbf{e}} \lesssim_t P(||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty)\int_0^t \mathrm{d}s \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1},t^\ell)}(s)e^{-C|v|^2}$$
$$\times \int_{\mathbb{R}^3}\Big\{|u-V_{\mathbf{cl}}(s)|^{\kappa-1}|\partial_{\mathbf{e}}V_{\mathbf{cl}}(s)|e^{-C|u|^2} + |u-v|^\kappa\sqrt{\mu(u)}|\partial_{\mathbf{e}}X_{\mathbf{cl}}(s)||\partial_x f^{m-j}(s,X_{\mathbf{cl}}(s),u)|\Big\}\mathrm{d}u.$$

Now we use Lemma 24 to have, for $\varpi \gg 1$,

$$e^{-\varpi\langle v\rangle t}\frac{1}{\langle v\rangle^{2\beta}}[\alpha(x,v)]^\beta\ \mathrm{II}_{\mathbf{x}}$$
$$\lesssim_{t,\xi} P(||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty)\Big\{e^{-C|v|^2}e^{-\varpi\langle v\rangle t}\frac{1}{\langle v\rangle}[\alpha(x,v)]^\beta\langle v\rangle^{\kappa-1}\frac{|v|^3}{\alpha(x,v)}e^{C|v|t}$$
$$+ \int_0^t\int_{\mathbb{R}^3} e^{-C|v-u|^2}e^{-\varpi\langle v\rangle t}\frac{1}{\langle v\rangle}[\alpha(x,v)]^\beta|u-v|^\kappa\frac{|v|e^{C|v|(t-s)}}{\sqrt{\alpha(x,v)}}\frac{e^{\varpi\langle u\rangle s}}{\langle u\rangle^{-1}[\alpha(X_{\mathbf{cl}}(s),u)]^\beta}\mathrm{d}u\mathrm{d}s$$
$$\times \sup_m \sup_{0\leq s\leq t}||e^{-\varpi\langle v\rangle s}\frac{1}{\langle v\rangle^{2\beta}}\alpha^\beta\partial_x f^m(s)||_\infty\Big\}.$$

We first claim

$$e^{-\varpi\langle v\rangle t}e^{\varpi\langle u\rangle s}e^{C|v|(t-s)}e^{-C'|v-u|^2}\lesssim e^{-\frac{\varpi\langle v\rangle}{2}(t-s)}e^{C''(s+s^2)}e^{-C''|v-u|^2}. \tag{166}$$

Using $\langle u\rangle = 1+|u|\leq 1+|v|+|u-v|\leq 1+\langle v\rangle+|v-u|$ we bound the first three exponents as

$$-\varpi\langle v\rangle t+\varpi\langle u\rangle s+C|v|(t-s)=-(\varpi-C)\langle v\rangle(t-s)-\varpi(\langle v\rangle-\langle u\rangle)s\leq -(\varpi-C)\langle v\rangle(t-s)+\varpi|v-u|s+\varpi s.$$

Using a complete square trick, if $0<\sigma\ll 1$:

$$\varpi|v-u|s=\frac{\sigma\varpi^2}{2}|v-u|^2+\frac{s^2}{2\sigma}-\frac{1}{2\sigma}\big[s-\sigma\varpi|v-u|\big]^2\leq\frac{\sigma\varpi^2}{2}|v-u|^2+\frac{s^2}{2\sigma},$$

we bound the whole exponents of (166) by

$$\begin{aligned}&-(\varpi-C)\langle v\rangle(t-s)+\varpi|v-u|s-C'|v-u|^2+\varpi s\\ &\leq -(\varpi-C)\langle v\rangle(t-s)-(C-\frac{\sigma\varpi^2}{2})|v-u|^2+\frac{s^2}{2\sigma}+\varpi s\\ &\leq -(\varpi-C)\langle v\rangle(t-s)-C_{\sigma,\varpi}|v-u|^2+C'_{\sigma,\varpi}\big\{s^2+s\big\},\end{aligned}$$

Here we use (166) to bound as $e^{-\varpi\langle v\rangle t}\frac{1}{\langle v\rangle^{2\beta}}[\alpha(x,v)]^\beta\ \mathrm{II}_{\mathbf{x}}\lesssim_{t,\xi}P(||e^{\zeta|v|^2}f_0||_\infty)\times$

$$\Big\{1+\underbrace{\int_0^t\int_{\mathbb{R}^3}e^{-\frac{\varpi\langle v\rangle}{2}(t-s)}e^{-C'|v-u|^2}\frac{\langle v\rangle[\alpha(x,v)]^{\beta-\frac12}}{[\alpha(X_{\mathbf{cl}}(s),u)]^\beta}\mathrm{d}u\mathrm{d}s}_{(\mathbf{A})}\times\sup_m\sup_{0\leq s\leq t}||e^{-\varpi\langle v\rangle s}\frac{\alpha^\beta}{\langle v\rangle^{2\beta}}\partial_x f^m(s)||_\infty\Big\}.$$

For $(\mathbf{A})$ we use (20), (2) of Lemma 2 with $Z=\langle v\rangle\big[\alpha(x,v)\big]^{\beta-\frac12}$ and $\varpi=\frac{\varpi}{2}$ and $r=1$. Then

$$(\mathbf{A})\ \lesssim\ C_\delta O(\tilde\delta)+C_{\tilde\delta,\delta}O(\varpi^{-1}).$$

Similarly, but with different weight $e^{-\varpi\langle v\rangle t}\frac{|v|}{\langle v\rangle^{2\beta-1}}[\alpha(x,v)]^{\beta-\frac12}$, we use Lemma 24 and (166) to have

$$\begin{aligned}&e^{-\varpi\langle v\rangle t}\frac{|v|}{\langle v\rangle^{2\beta-1}}[\alpha(x,v)]^{\beta-\frac12}\ \mathrm{II}_{\mathbf{v}}\\ &\lesssim_{t,\xi}\ P(||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty)\\ &\quad\times\Big\{e^{-C|v|^2}e^{-\varpi\langle v\rangle t}\frac{|v|}{\langle v\rangle^{2\beta-1}}[\alpha(x,v)]^{\beta-\frac12}\langle v\rangle^{\kappa-1}\frac{|v|}{\sqrt{\alpha(x,v)}}e^{C|v|t}\\ &\qquad+\int_0^t\int_{\mathbb{R}^3}e^{-C(|v|^2+|u|^2)}e^{-\varpi\langle v\rangle t}|v|[\alpha(x,v)]^\beta|u-v|^\kappa\frac{1}{|v|}e^{C|v|(t-s)}\frac{e^{\varpi\langle u\rangle s}}{\langle u\rangle^{-1}[\alpha(X_{\mathbf{cl}}(s),u)]^\beta}\mathrm{d}u\mathrm{d}s\\ &\qquad\qquad\times\sup_m\sup_{0\leq s\leq t}||e^{-\varpi\langle v\rangle s}\frac{\alpha^\beta}{\langle v\rangle^{2\beta}}\partial_x f^m(s)||_\infty\Big\}\\ &\lesssim_{t,\xi}\ P(||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty)\\ &\quad\times\Big\{1+\underbrace{\int_0^t\int_{\mathbb{R}^3}e^{-\frac{\varpi}{2}\langle v\rangle(t-s)}e^{-C'|v-u|^2}\frac{\langle v\rangle[\alpha(x,v)]^{\beta-\frac12}}{[\alpha(X_{\mathbf{cl}}(s),u)]^\beta}\mathrm{d}u\mathrm{d}s}_{(\mathbf{A})}\times\sup_m\sup_{0\leq s\leq t}||e^{-\varpi\langle v\rangle s}\frac{\alpha^\beta}{\langle v\rangle^{2\beta}}\partial_x f^m(s)||_\infty\Big\},\end{aligned}$$

where we used $\frac{\langle u\rangle}{\langle v\rangle}e^{-C|v-u|^2}\lesssim(1+|v-u|)e^{-C|v-u|^2}\lesssim e^{-C'|v-u|^2}$. Again by (20),

$$(\mathbf{A})\lesssim C_\delta O(\tilde\delta)+C_{\tilde\delta,\delta}O(\varpi^{-1}).$$

From (31) and (32) of Lemma 5 and Lemma 24 and (166)

$$
\begin{aligned}
& e^{-\varpi\langle v\rangle t}\frac{1}{\langle v\rangle^{2\beta}}[\alpha(x,v)]^{\beta}\mathrm{III}_{\mathbf{x}} \\
& \lesssim\ e^{-\varpi\langle v\rangle t}\frac{1}{\langle v\rangle^{2\beta}}[\alpha(x,v)]^{\beta}\big\{E_{\mathbf{x}}+F_{\mathbf{x}}+G_{\mathbf{x}}+H_{\mathbf{x}}+M_{\mathbf{x}}\big\} \\
& := P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty)\times \\
& \times\Big\{\quad \int_0^t\int_{\mathbb{R}^3}\frac{e^{-C|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{3-\kappa}}e^{-\varpi\langle v\rangle t}\frac{1}{\langle v\rangle^{2\beta}}[\alpha(x,v)]^{\beta}e^{-C_\theta|u|^2}|\partial_x V_{\mathbf{cl}}(s)| \\
& \quad+\int_0^t\int_{\mathbb{R}^3}\frac{e^{-C|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}}e^{-\varpi\langle v\rangle t}\frac{1}{\langle v\rangle^{2\beta}}[\alpha(x,v)]^{\beta}|\partial_x f^{m-\ell}(s,X_{\mathbf{cl}}(s),u)||\partial_x X_{\mathbf{cl}}(s)| \\
& \quad+\int_0^t e^{-\varpi\langle v\rangle t}\frac{1}{\langle v\rangle^{2\beta}}[\alpha(x,v)]^{\beta}e^{-C_\theta|v|^2}|\partial_x V_{\mathbf{cl}}(s)|\mathrm{d}s\times \\
& \quad+\int_0^t e^{-\varpi\langle v\rangle t}\frac{1}{\langle v\rangle^{2\beta}}[\alpha(x,v)]^{\beta}|\partial_x X_{\mathbf{cl}}(s)|\int_{\mathbb{R}^3}\frac{e^{-C|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}}|\partial_x f^{m-\ell}(s,X_{\mathbf{cl}}(s),u)| \\
& \quad+\int_0^t e^{-\varpi\langle v\rangle t}\frac{1}{\langle v\rangle^{2\beta}}[\alpha(x,v)]^{\beta}|\partial_x V_{\mathbf{cl}}(s)|\int_{\mathbb{R}^3}\frac{e^{-C|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}}|\partial_v f^{m-\ell}(s,X_{\mathbf{cl}}(s),u)|\Big\}.
\end{aligned}
$$

By Lemma 24 and (166) and Lemma 4, for $0<\kappa\le 1$,

$$
\begin{aligned}
& e^{-\varpi\langle v\rangle t}\frac{1}{\langle v\rangle^{2\beta}}[\alpha(x,v)]^{\beta}E_{\mathbf{x}} \\
& \lesssim P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty)\int_0^t e^{-\varpi\langle v\rangle t}\frac{1}{\langle v\rangle^{2\beta}}[\alpha(x,v)]^{\beta}e^{-C_\theta|v|^2}\sup_v\Big(\int_u\frac{e^{-C|v-u|^2}}{|v-u|^{3-\kappa}}\frac{e^{C_\theta|v|^2}}{e^{C_\theta|u|^2}}\Big)e^{C|v|(t-s)}\frac{|v|^3}{\alpha(x,v)}\mathrm{d}s \\
& \lesssim P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty)\int_0^t e^{-\varpi\langle v\rangle t}e^{C|v|(t-s)}\frac{1}{\langle v\rangle^{2\beta}}|v|^3[\alpha(x,v)]^{\beta-1}e^{-C|v|^2}\mathrm{d}s\lesssim_{\xi,t}P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
& e^{-\varpi\langle v\rangle t}\frac{1}{\langle v\rangle^{2\beta}}[\alpha(x,v)]^{\beta}\{F_{\mathbf{x}}+H_{\mathbf{x}}\} \\
& \lesssim\int_0^t\int_{\mathbb{R}^3}\frac{e^{-C|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}}e^{\varpi\langle u\rangle s+|v|(t-s)-\varpi\langle v\rangle t}\frac{\langle v\rangle^{-1}[\alpha(x,v)]^{\beta}}{\langle u\rangle^{-1}[\alpha(X_{\mathbf{cl}}(s),u)]^{\beta}}\frac{|v|}{\sqrt{\alpha(x,v)}}\mathrm{d}u\mathrm{d}s \\
& \qquad\qquad\times\sup_{0\le s\le t}||e^{-\varpi\langle v\rangle s}\frac{1}{\langle v\rangle^{2\beta}}\alpha^{\beta}\partial_x f^{m-\ell}(s)||_\infty P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty) \\
& \lesssim P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty)\sup_{0\le\ell\le m}\sup_{0\le s\le t}||e^{-\varpi\langle v\rangle s}\frac{1}{\langle v\rangle^{2\beta}}\alpha^{\beta}\partial_x f^{m}(s)||_\infty \\
& \quad\times\underbrace{\int_0^t\int_{\mathbb{R}^3}e^{-\frac{\varpi}{2}\langle v\rangle(t-s)}\frac{e^{-C|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}}\frac{\langle v\rangle^{-1}\langle v\rangle[\alpha(x,v)]^{\beta-\frac12}}{\langle u\rangle^{-1}[\alpha(X_{\mathbf{cl}}(s),u)]^{\beta}}\mathrm{d}u\mathrm{d}s}_{(\mathbf{A})}.
\end{aligned}
$$

Now using (20), (2) of Lemma 2,

$$
(\mathbf{A})\lesssim C_\delta O(\tilde\delta)+C_{\tilde\delta,\delta}O(\varpi^{-1}).
$$

By Lemma 24 and (166) and Lemma 4, for $\beta>1$

$$
\begin{aligned}
& e^{-\varpi\langle v\rangle t}\frac{1}{\langle v\rangle^{2\beta}}[\alpha(x,v)]^{\beta}G_{\mathbf{x}} \\
& \lesssim\int_0^t e^{-\varpi\langle v\rangle t}\frac{1}{\langle v\rangle^{2\beta}}[\alpha(x,v)]^{\beta}e^{-C'|v|^2}\frac{|v|^3}{\alpha(x,v)}e^{C|v|(t-s)}\mathrm{d}s\times P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty) \\
& \lesssim_{\xi,t}P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty).
\end{aligned}
$$

Similarly, with $\frac{|v|^3}{|u|}\frac{1}{\langle v\rangle^{2\beta}} \leq \frac{|v|^2}{|u|}\frac{1}{\langle v\rangle^{2\beta-1}}$,

$$
\begin{aligned}
& e^{-\varpi\langle v\rangle t}\frac{1}{\langle v\rangle^{2\beta}}[\alpha(x,v)]^{\beta}M_{\mathbf{x}} \\
& \lesssim \int_0^t e^{-\varpi\langle v\rangle t}\langle v\rangle^{-1}[\alpha(x,v)]^{\beta}\frac{|v|^3}{\alpha(x,v)}e^{C|v|(t-s)}P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty) \\
& \quad \times \int_{\mathbb{R}^3}\frac{e^{-C|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}}\frac{e^{\varpi\langle u\rangle s}}{|u|[\alpha(X_{\mathbf{cl}}(s),u)]^{\beta-\frac{1}{2}}}\mathrm{d}u\mathrm{d}s \times ||e^{-\varpi\langle u\rangle s}\frac{|u|}{\langle u\rangle^{2\beta-1}}\alpha^{\beta-\frac{1}{2}}\partial_v f^{m-\ell}(s)||_\infty \\
& \lesssim P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty)\sup_{0\leq\ell\leq m}\sup_{0\leq s\leq t}||e^{-\varpi\langle v\rangle s}\frac{|v|}{\langle v\rangle^{2\beta-1}}\alpha^{\beta-\frac{1}{2}}\partial_v f^m(s)||_\infty \\
& \quad \times \underbrace{\int_0^t\int_{\mathbb{R}^3}e^{-\frac{\varpi}{2}\langle v\rangle(t-s)}\frac{e^{-C|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}}\frac{|v|^2[\alpha(x,v)]^{\beta-1}}{|u|[\alpha(X_{\mathbf{cl}}(s),u)]^{\beta-\frac{1}{2}}}\mathrm{d}u\mathrm{d}s}_{(\mathbf{B})}.
\end{aligned}
$$

For $(\mathbf{B})$, let $l=\frac{\varpi}{2}$ and $Z=|v|\big[\alpha(x,v)\big]^{\beta-1}$. Note that $\frac{1}{2}<\beta-\frac{1}{2}<1$ for $1<\beta<\frac{3}{2}$. We apply (64) to have

$$
(\mathbf{B}) \lesssim C_\delta O(\tilde{\delta}) + C_{\tilde{\delta},\delta}O(\varpi^{-1}).
$$

On the other hand,

$$
\begin{aligned}
& e^{-\varpi\langle v\rangle t}\frac{|v|}{\langle v\rangle^{2\beta-1}}[\alpha(x,v)]^{\beta-\frac{1}{2}}\ \mathrm{III}_{\mathbf{v}} \\
& \lesssim\ e^{-\varpi\langle v\rangle t}\frac{|v|}{\langle v\rangle^{2\beta-1}}[\alpha(x,v)]^{\beta-\frac{1}{2}}\ \{E_{\mathbf{v}}+F_{\mathbf{v}}+G_{\mathbf{v}}+H_{\mathbf{v}}+M_{\mathbf{v}}\} \\
& := P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty) \\
& \times\Big\{\int_0^t e^{-\varpi\langle v\rangle t}\frac{|v|}{\langle v\rangle^{2\beta-1}}[\alpha(x,v)]^{\beta-\frac{1}{2}}\ \int_{\mathbb{R}^3}\frac{e^{-C|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{3-\kappa}}e^{-C_\theta|u|^2}|\partial_v V_{\mathbf{cl}}(s)|\mathrm{d}u\mathrm{d}s \\
& \ + \int_0^t e^{-\varpi\langle v\rangle t}\frac{|v|}{\langle v\rangle^{2\beta-1}}[\alpha(x,v)]^{\beta-\frac{1}{2}}\ \int_{\mathbb{R}^3}\frac{e^{-C|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}}|\partial_x f^{m-\ell}(s,X_{\mathbf{cl}}(s),u)||\partial_v X_{\mathbf{cl}}(s)|\mathrm{d}u\mathrm{d}s \\
& \ + \int_0^t e^{-\varpi\langle v\rangle t}\frac{|v|}{\langle v\rangle^{2\beta-1}}[\alpha(x,v)]^{\beta-\frac{1}{2}}\ e^{-C_\theta|v|^2}|\partial_v V_{\mathbf{cl}}(s)|\mathrm{d}s \\
& \ + \int_0^t e^{-\varpi\langle v\rangle t}\frac{|v|}{\langle v\rangle^{2\beta-1}}[\alpha(x,v)]^{\beta-\frac{1}{2}}|\partial_v X_{\mathbf{cl}}(s)|\int_{\mathbb{R}^3}\frac{e^{-C|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}}|\partial_x f^{m-\ell}(s,X_{\mathbf{cl}}(s),u)|\mathrm{d}u\mathrm{d}s \\
& \ + \int_0^t e^{-\varpi\langle v\rangle t}\frac{|v|}{\langle v\rangle^{2\beta-1}}[\alpha(x,v)]^{\beta-\frac{1}{2}}\ |\partial_v V_{\mathbf{cl}}(s)|\int_{\mathbb{R}^3}\frac{e^{-C|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}}|\partial_v f^{m-\ell}(s,X_{\mathbf{cl}}(s),u)|\mathrm{d}u\mathrm{d}s\Big\}.
\end{aligned}
$$

By Lemma 24 and (166) and Lemma 4, for $0<\kappa\leq 1$,

$$
\begin{aligned}
& e^{-\varpi\langle v\rangle t}\frac{|v|}{\langle v\rangle^{2\beta-1}}[\alpha(x,v)]^{\beta-\frac{1}{2}}\ E_{\mathbf{v}} \\
& \lesssim P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty)\int_0^t e^{-\varpi\langle v\rangle t}\frac{|v|}{\langle v\rangle^{2\beta-1}}[\alpha(x,v)]^{\beta-\frac{1}{2}}\ e^{-C_\theta|v|^2}\sup_v\Big(\int_u\frac{e^{-C|v-u|^2}}{|v-u|^{3-\kappa}}\frac{e^{C_\theta|v|^2}}{e^{C_\theta|u|^2}}\Big)\frac{|v|e^{C|v|(t-s)}}{\sqrt{\alpha(x,v)}}\mathrm{d}s \\
& \lesssim P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty)\int_0^t\frac{|v|^2}{\langle v\rangle^{2\beta-1}}[\alpha(x,v)]^{\beta-1}\ e^{-C|v|^2}e^{-\varpi\langle v\rangle t}e^{C|v|(t-s)}\mathrm{d}s \\
& \lesssim_{\xi,t}\ P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&e^{-\varpi\langle v\rangle t}\frac{|v|}{\langle v\rangle^{2\beta-1}}[\alpha(x,v)]^{\beta-\frac{1}{2}}\ \{F_{\mathbf{v}}+H_{\mathbf{v}}\}\\
&\lesssim \int_0^t e^{-\varpi\langle v\rangle t}|v|[\alpha(x,v)]^{\beta-\frac{1}{2}}\ \int_{\mathbb{R}^3}\frac{e^{-C|V_{\mathbf{cl}}(s)-u|^{\kappa}}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}}\frac{e^{\varpi\langle u\rangle s}e^{C|v|(t-s)}}{\langle u\rangle^{-1}[\alpha(X_{\mathbf{cl}}(s),u)]^{\beta}}\frac{1}{|v|}\mathrm{d}u\mathrm{d}s\\
&\quad\times\sup_{0\le s\le t}||e^{-\varpi\langle u\rangle s}\frac{1}{\langle u\rangle^{2\beta}}\alpha^{\beta}\partial_x f^{m-\ell}(s)||_\infty P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty)\\
&\lesssim P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty)\sup_{0\le\ell\le m}\sup_{0\le s\le t}||e^{-\varpi\langle u\rangle s}\frac{\alpha^{\beta}}{\langle v\rangle^{2\beta-1}}\partial_x f^{m}(s)||_\infty\\
&\quad\times\underbrace{\int_0^t\int_{\mathbb{R}^3}e^{-\frac{\varpi}{2}\langle v\rangle(t-s)}\frac{e^{-C|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}}\frac{\langle v\rangle^{-1}\langle v\rangle[\alpha(x,v)]^{\beta-\frac{1}{2}}}{\langle u\rangle^{-1}[\alpha(X_{\mathbf{cl}}(s),u)]^{\beta}}\mathrm{d}u\mathrm{d}s}_{(\mathbf{A})},
\end{aligned}
$$

where from (20)

$$
(\mathbf{A})\lesssim C_{\delta}O(\tilde{\delta})+C_{\tilde{\delta},\delta}O(\varpi^{-1}).
$$

Similarly

$$
\begin{aligned}
&e^{-\varpi\langle v\rangle t}\frac{|v|}{\langle v\rangle^{2\beta-1}}[\alpha(x,v)]^{\beta-\frac{1}{2}}\ G_{\mathbf{v}}\\
&\lesssim\int_0^t e^{-\varpi\langle v\rangle t}|v|\frac{1}{\langle v\rangle^{2\beta-1}}[\alpha(x,v)]^{\beta-\frac{1}{2}}e^{-C'|v|^2}\frac{|v|}{\sqrt{\alpha(x,v)}}e^{C|v|(t-s)}\mathrm{d}s\times P(||\langle v\rangle^{\eta}e^{\theta|v|^2}f_0||_\infty)\\
&\lesssim_{\xi,t} P(||\langle v\rangle^{\eta}e^{\theta|v|^2}f_0||_\infty).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&e^{-\varpi\langle v\rangle t}\frac{|v|}{\langle v\rangle^{2\beta-1}}[\alpha(x,v)]^{\beta-\frac{1}{2}}M_{\mathbf{v}}\\
&\lesssim\int_0^t e^{-\varpi\langle v\rangle t}|v|[\alpha(x,v)]^{\beta-\frac{1}{2}}\frac{|v|e^{C|v|(t-s)}}{\sqrt{\alpha(x,v)}}\\
&\quad\times\int_{\mathbb{R}^3}\frac{e^{-C|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}}\frac{e^{\varpi\langle u\rangle s}e^{C|v|(t-s)}}{|u|[\alpha(X_{\mathbf{cl}}(s),u)]^{\beta-\frac{1}{2}}}\times P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty)\sup_{0\le s\le t}||e^{-\varpi\langle u\rangle s}\frac{|u|}{\langle u\rangle^{2\beta-1}}\alpha^{\beta-\frac{1}{2}}\partial_v f^m(s)||_\infty\\
&\lesssim P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty)\sup_{0\le\ell\le m}\sup_{0\le s\le t}||e^{-\varpi\langle u\rangle s}\frac{|u|}{\langle u\rangle^{2\beta-1}}\alpha^{\beta-\frac{1}{2}}\partial_v f^m(s)||_\infty\\
&\quad\times\underbrace{\int_0^t\int_{\mathbb{R}^3}e^{-\frac{\varpi}{2}\langle v\rangle(t-s)}\frac{e^{-C|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}}\frac{|v|^2[\alpha(x,v)]^{\beta-1}}{|u|[\alpha(X_{\mathbf{cl}}(s),u)]^{\beta-\frac{1}{2}}}\mathrm{d}u\mathrm{d}s}_{(\mathbf{B})},
\end{aligned}
$$

where, from (64),

$$
(\mathbf{B})\lesssim C_{\delta}O(\tilde{\delta})+C_{\tilde{\delta},\delta}O(\varpi^{-1}).
$$

Overall, for $1 < \beta < \frac{3}{2}$,

$$\sup_m \sup_{0\le s\le t} ||e^{-\varpi\langle v\rangle t}\frac{\big[\alpha(x,v)\big]^{\beta}}{\langle v\rangle^{2\beta}}\partial_x f^m(t,x,v)||_\infty + \sup_m \sup_{0\le s\le t} ||e^{-\varpi\langle v\rangle t}\frac{|v|\big[\alpha(x,v)\big]^{\beta-\frac12}}{\langle v\rangle^{2\beta-1}}\partial_v f^m(t,x,v)||_\infty$$

$$\lesssim ||\frac{|v|}{\langle v\rangle^{2\beta}}\alpha^{\beta-\frac12}\partial_x f_0||_\infty + ||\frac{|v|^3\alpha^{\beta-1}}{\langle v\rangle^{2\beta}}\partial_v f_0||_\infty + ||\frac{\alpha^{\beta-\frac12}}{\langle v\rangle^{2\beta-1}}\partial_x f_0||_\infty + ||\frac{|v|^2\alpha^{\beta-1}}{\langle v\rangle^{2\beta-1}}\partial_v f_0||_\infty + P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty)$$

$$+ P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty)\big[\delta t e^{2\delta t} + C_\delta O(\tilde\delta) + C_{\tilde\delta,\delta,\beta}O(\varpi^{-1})\big]\sup_m\sup_{0\le s\le t}||e^{-\varpi\langle v\rangle t}\frac{\alpha^\beta}{\langle v\rangle^{2\beta}}\partial_x f^m||_\infty$$

$$+ P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty)\big[\delta t e^{2\delta t} + C_\delta O(\tilde\delta) + C_{\tilde\delta,\delta,\beta}O(\varpi^{-1})\big]\sup_m\sup_{0\le s\le t}||e^{-\varpi\langle v\rangle t}\frac{|v|\alpha^{\beta-\frac12}}{\langle v\rangle^{2\beta-1}}\partial_x f^m||_\infty.$$

First we choose small $\delta = \delta(t,\Omega,||e^{\zeta|v|^2}f_0||_\infty) \ll 1$ and then choose small $\tilde\delta = \tilde\delta(\delta,\Omega,||e^{\zeta|v|^2}f_0||_\infty) \ll 1$ and then we choose large $\varpi = \varpi(\tilde\delta,\delta,\Omega,||e^{\zeta|v|^2}f_0||_\infty) \gg 1$ to have

$$\sup_m \sup_{0\le s\le t} ||e^{-\varpi\langle v\rangle t}\frac{\big[\alpha(x,v)\big]^{\beta}}{\langle v\rangle^{2\beta}}\partial_x f^m(t,x,v)||_\infty + \sup_m \sup_{0\le s\le t} ||e^{-\varpi\langle v\rangle t}\frac{|v|\big[\alpha(x,v)\big]^{\beta-\frac12}}{\langle v\rangle^{2\beta-1}}\partial_v f^m(t,x,v)||_\infty$$

$$\lesssim ||\frac{|v|\alpha^{\beta-\frac12}}{\langle v\rangle^{2\beta}}\partial_x f_0||_\infty + ||\frac{|v|^3\alpha^{\beta-1}}{\langle v\rangle^{2\beta}}\partial_v f_0||_\infty + ||\frac{\alpha^{\beta-\frac12}}{\langle v\rangle^{2\beta-1}}\partial_x f_0||_\infty + ||\frac{|v|^2\alpha^{\beta-1}}{\langle v\rangle^{2\beta-1}}\partial_v f_0||_\infty + P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty)$$

$$\lesssim ||\frac{\alpha^{\beta-\frac12}}{\langle v\rangle^{2\beta-1}}\partial_x f_0||_\infty + ||\frac{|v|^2\alpha^{\beta-1}}{\langle v\rangle^{2\beta-1}}\partial_v f_0||_\infty + P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty).$$

We remark that this sequence f^m is Cauchy in $L^\infty([0,T]\times\bar\Omega\times\mathbb{R}^3)$ for $0 < T \ll 1$. Therefore the limit function f is a solution of the Boltzmann equation satisfying the specular reflection BC. On the other hand, due to the weak lower semi-continuity of L^p, $p > 1$, we pass a limit $\partial f^m \rightharpoonup \partial f$ weakly in the weighted L^∞−norm.

Now we consider the continuity of $e^{-\varpi\langle v\rangle t}\langle v\rangle^{-1}\alpha^\beta\partial_x f$ and $e^{-\varpi\langle v\rangle t}|v|\alpha^{\beta-\frac12}\partial_v f$. Remark that $e^{-\varpi\langle v\rangle t}\langle v\rangle^{-1}\alpha^\beta\partial_x f^m$ and $e^{-\varpi\langle v\rangle t}|v|\alpha^{\beta-\frac12}\partial_v f^m$ satisfy all the conditions of Proposition 2. Therefore we conclude

$$e^{-\varpi\langle v\rangle t}\langle v\rangle^{-1}\alpha^\beta\partial_x f^m \in C^0([0,T^*]\times\bar\Omega\times\mathbb{R}^3),\ e^{-\varpi\langle v\rangle t}|v|\alpha^{\beta-\frac12}\partial_v f^m \in C^0([0,T^*]\times\bar\Omega\times\mathbb{R}^3).$$

Now we follow $W^{1,\infty}$ estimate proof for $e^{-\varpi\langle v\rangle t}\langle v\rangle^{-1}\alpha^\beta[\partial_x f^{m+1} - \partial_x f^m]$ and $e^{-\varpi\langle v\rangle t}|v|\alpha^{\beta-\frac12}[\partial_v f^{m+1} - \partial f^m]$ to show that $e^{-\varpi\langle v\rangle t}\langle v\rangle^{-1}\alpha^\beta\partial_x f^m$ and $e^{-\varpi\langle v\rangle t}|v|\alpha^{\beta-\frac12}\partial_v f^m$ are Cauchy in L^∞. Then we pass a limit $e^{-\varpi\langle v\rangle t}\langle v\rangle^{-1}\alpha^\beta\partial_x f^m \to e^{-\varpi\langle v\rangle t}\langle v\rangle^{-1}\alpha^\beta\partial_x f$ and $e^{-\varpi\langle v\rangle t}|v|\alpha^{\beta-\frac12}\partial_v f^m \to e^{-\varpi\langle v\rangle t}|v|\alpha^{\beta-\frac12}\partial_v f$ strongly in L^∞ so that $e^{-\varpi\langle v\rangle t}\langle v\rangle^{-1}\alpha^\beta\partial_x f \in C^0([0,T^*]\times\bar\Omega\times\mathbb{R}^3)$ and $e^{-\varpi\langle v\rangle t}|v|\alpha^{\beta-\frac12}\partial_v f \in C^0([0,T^*]\times\bar\Omega\times\mathbb{R}^3)$.

□

6. Bounce-Back Reflection

We recall the bounce-back cycles from *(4)* of Definition 1: $(t^0,x^0,v^0) = (t,x,v)$ and for $\ell \ge 1$,

$$t^\ell = t^1 - (\ell-1)t_{\mathbf{b}}(x^1,v^1),\quad x^\ell = \frac{1-(-1)^\ell}{2}x^1 + \frac{1+(-1)^\ell}{2}x^2,\quad v^{\ell+1} = (-1)^{\ell+1}v.$$

Lemma 14. *For all* $0 \le s \le t$,

$$\min\{\alpha(x^1,v^1),\alpha(x^2,v^2)\} \lesssim_\Omega \alpha(X_{\mathbf{cl}}(s;t,x,v),V_{\mathbf{cl}}(s;t,x,v)) \lesssim_\Omega \max\{\alpha(x^1,v^1),\alpha(x^2,v^2)\}.$$

For $\ell_*(s;t,x,v) \in \mathbb{N}$ *(therefore* $t^{\ell_*+1}(t,x,v) \le s \le t^{\ell_*}(t,x,v)$*)*

$$\ell_*(s;t,x,v) \le \frac{t-s}{t_{\mathbf{b}}(x_1,v_1)} \lesssim_\Omega \frac{|t-s||v|^2}{\sqrt{\alpha(x,v)}}.$$

For all $0 \leq s \leq t$ *uniformly*

$$\begin{aligned}
|\partial_{x_i} t^\ell(t,x,v)| &= \Big| -\ell \frac{\partial_{x_i}\xi(x^1)}{v\cdot\nabla\xi(x^1)} - (\ell-1)\frac{\partial_{x_i}\xi(x^2)}{-v\cdot\nabla\xi(x^2)}\Big| \lesssim_\Omega \frac{t|v|^2}{\alpha(\tau;t,x,v)},\\
|\partial_{v_i} t^\ell(t,x,v)| &= \Big|\ell t_{\mathbf{b}}(x,v)\frac{\partial_{x_i}\xi(x^1)}{v\cdot\nabla\xi(x^1)} + (\ell-1)t_{\mathbf{b}}(x,-v)\frac{\partial_{x_i}\xi(x^2)}{-v\cdot\nabla\xi(x^2)}\Big| \lesssim_\Omega \frac{t}{\sqrt{\alpha(\tau;t,x,v)}},\\
|\partial_{x_i} x_j^\ell(x,v)| &= \Big|\frac{1-(-1)^\ell}{2}\Big\{\delta_{ij} - \frac{v_j\partial_{x_i}\xi(x_1)}{v\cdot\nabla\xi(x_1)}\Big\} + \frac{1+(-1)^\ell}{2}\Big\{\delta_{ij} - \frac{v_j\partial_{x_i}\xi(x_2)}{v\cdot\nabla\xi(x_2)}\Big\}\Big| \lesssim_\Omega 1 + \frac{|v|}{\sqrt{\alpha(\tau;t,x,v)}},\\
|\partial_{v_i} x_j^\ell(x,v)| &= \Big|\frac{1-(-1)^\ell}{2}(-t_{\mathbf{b}}(x,v))\Big\{\delta_{ij} - \frac{v_j\partial_{x_i}\xi(x_1)}{v\cdot\nabla\xi(x_1)}\Big\} + \frac{1+(-1)^\ell}{2}(-t_{\mathbf{b}}(x,-v))\Big\{\delta_{ij} - \frac{v_j\partial_{x_i}\xi(x_2)}{v\cdot\nabla\xi(x_2)}\Big\}\Big|,\\
&\lesssim_\Omega \frac{1}{|v|} + \frac{\sqrt{\alpha(\tau;t,x,v)}}{|v|^2},\\
\partial_{x_i} v^\ell &= 0, \quad |\partial_{v_i} v_j^\ell| = |(-1)^\ell \delta_{ij}| \lesssim_\Omega 1,\\
|\partial_{x_i}(t^\ell - t^{\ell+1})| &= \Big|\frac{\partial_{x_i}\xi(x_1)}{v\cdot\nabla\xi(x_1)} + \frac{\partial_{x_i}\xi(x_2)}{-v\cdot\nabla\xi(x_2)}\Big| \lesssim_\Omega \frac{1}{\sqrt{\alpha(\tau;t,x,v)}},\\
|\partial_{v_i}(t^\ell - t^{\ell+1})| &= \Big|t_{\mathbf{b}}(x,v)\frac{-\partial_{x_i}\xi(x_1)}{v\cdot\nabla\xi(x_1)} + t_{\mathbf{b}}(x,-v)\frac{\partial_{x_i}\xi(x_2)}{v\cdot\nabla\xi(x_2)}\Big| \lesssim_\Omega \frac{1}{|v|^2}.
\end{aligned}$$

Proof. These are direct consequence of (49) and Velocity lemma(Lemma 1). □

Now we prove the key fact: Assume $\lim_{\tau\downarrow t^{j+1}} A^{m-j}(\tau, x^j-(t^j-\tau)v^j, v^j) = \lim_{\tau\uparrow t^{j+1}} A^{m-(j+1)}(\tau, x^{j+1}-(t^j-\tau)v^j, v^{j+1})$. Then in the sense of distribution,

$$\begin{aligned}
&\partial_{\mathbf{e}}\Big[\sum_{\ell=0}^{\ell_*(s)}\int_{\max\{s,t^{j+1}\}}^{t^j} A^{m-j}(\tau, x^j-(t^j-\tau)v^j, v^j)\mathrm{d}\tau\Big]\\
&=\sum_{j=0}^{\ell_*(s)}\int_{\max\{s,t^{j+1}\}}^{t^j}\big[\partial_{\mathbf{e}}t^j, \partial_{\mathbf{e}}x^j+\tau\partial_{\mathbf{e}}v^j, \partial_{\mathbf{e}}v^j\big]\cdot\nabla_{t,x,v}A^{m-j}(\tau, x^j-(t^j-\tau)v^j, v^j)\mathrm{d}\tau\\
&\quad+\sum_{j=0}^{\ell_*(s)-1}\partial_{\mathbf{e}}[t^j-t^{j+1}]A^{m-j}(t^{j+1}, x^{j+1}, v^j)\\
&\quad+\partial_{\mathbf{e}}t^{\ell_*(s)}A^{m-\ell_*(s)}(s, x^j-(t^j-s)v^j, v^j).
\end{aligned}\tag{167}$$

Note that (167) is more general then Lemma 3.

Proof of Lemma 3 and (167). For each time intervals $[t^{\ell+1}, t^\ell]$ and $[t^{j+1}, t^j]$, we apply the change of variables

$$\begin{aligned}
x^\ell-(t^\ell-s)v^\ell,\ s\in[t^{\ell+1},t^\ell] \quad&\mapsto\quad x^\ell+sv^\ell,\ s\in[-(t^\ell-t^{\ell+1}),0],\\
x^j-(t^j-s)v^m,\ \tau\in[t^{j+1},t^j] \quad&\mapsto\quad x^j+\tau v^j,\ \tau\in[-(t^j-t^{j+1}),0].
\end{aligned}\tag{168}$$

From (54) the piecewise derivatives equal distributional derivatives almost everywhere. Therefore we prove Lemma 3. Moreover

$$\begin{aligned}
&\partial_{\mathbf{e}}\Big[\sum_{j=0}^{\ell_*(s)}\int_{\max\{-(t^j-s),-(t^j-t^{j+1})\}}^{0} A^{m-j}(\tau+t^j, x^j+\tau v^j, v^j)\mathrm{d}\tau\Big]\\
&=\sum_{j=0}^{\ell_*(s)}\int_{\max\{-(t^j-s),-(t^j-t^{j+1})\}}^{0}\partial_{\mathbf{e}}\big[A^{m-j}(\tau+t^j, x^j+\tau v^j, v^j)\big]\mathrm{d}\tau\\
&\quad+\sum_{j=0}^{\ell_*(s)-1}\partial_{\mathbf{e}}[t^j-t^{j+1}]A^{m-j}(t^{j+1}, x^{j+1}, v^j)\\
&\quad+\partial_{\mathbf{e}}t^{\ell_*(s)}A^{m-\ell_*(s)}(s, x^j-(t^j-s)v^j, v^j).
\end{aligned}$$

Then we apply the inverse of the change of variables in (168) to the time integration term:

$$\sum_{j=0}^{\ell_*(s)} \int_{\max\{s,t^{j+1}\}}^{t^j} \big[\partial_{\mathbf{e}} t^j, \partial_{\mathbf{e}} x^j + \tau \partial_{\mathbf{e}} v^j, \partial_{\mathbf{e}} v^j\big] \cdot \nabla_{t,x,v} A^{m-j}(\tau, x^j - (t^j-\tau)v^j, v^j) \mathrm{d}\tau.$$

□

Now we are ready to proof the main theorem:

Proof of Theorem 5. From the iteration (34) and (38), for $\ell_*(0;t,x,v) = \ell_*(\ t^{\ell_*+1} \leq 0 < t^{\ell_*})$

$$\begin{aligned}
& f^{m+1}(t,x,v) \\
&= e^{-\sum_{j=0}^{\ell_*(0)} \int_{\max\{0,t^{j+1}\}}^{t^j} \nu(F^{m-j})(\tau)\mathrm{d}\tau} f_0(x^{\ell_*(0)} - t^{\ell_*(0)} v^{\ell_*(0)}, v^{\ell_*(0)}) \\
&+ \sum_{\ell=0}^{\ell_*(0)} \int_{\max\{0,t^{\ell+1}\}}^{t^\ell} e^{-\sum_{j=0}^{\ell_*(s)} \int_{\max\{0,t^{j+1}\}}^{t^j} \nu(F^{m-j})(\tau)\mathrm{d}\tau} K f^{m-\ell}(s, x^\ell - (t^\ell - \tau)v^\ell, v^\ell)\mathrm{d}s \\
&+ \sum_{\ell=0}^{\ell_*(0)} \int_{\max\{0,t^{\ell+1}\}}^{t^\ell} e^{-\sum_{j=0}^{\ell_*(s)} \int_{\max\{0,t^{j+1}\}}^{t^j} \nu(F^{m-j})(\tau)\mathrm{d}\tau} \Gamma_{\text{gain}}(f^{m-\ell}, f^{m-\ell})(s, x^\ell - (t^\ell - s)v^\ell, v^\ell)\mathrm{d}s.
\end{aligned}$$

where $\nu(F^{m-j})(\tau) = \nu(F^{m-j})(\tau, x^j - (t^j-\tau)v^j, v^j)$.

From Lemma 3 and (167), in the sense of distribution we have, for $\partial_{\mathbf{e}} = [\partial_x, \partial_v]$ with $\mathbf{e} \in \{x, v\}$,

$$\begin{aligned}
& e^{-\int_0^t \sum_j \mathbf{1}_{[t^{j+1},t^j)}(\tau)\nu(F^{m-j})(\tau, X_{\mathbf{cl}}(\tau), V_{\mathbf{cl}}(\tau))\mathrm{d}\tau} f_0(X_{\mathbf{cl}}(0), V_{\mathbf{cl}}(0)) \\
&\times \Big\{ -\underbrace{\sum_{j=0}^{\ell_*(0)} \int_{\max\{0,t^{j+1}\}}^{t^j} \big[\partial_{\mathbf{e}} t^j, \partial_{\mathbf{e}} x^j + \tau\partial_{\mathbf{e}} v^j, \partial_{\mathbf{e}} v^j\big] \cdot \nabla_{t,x,v} \nu(F^{m-j})(\tau, x^j - (t^j-\tau)v^j, v^j)\mathrm{d}\tau}_{\mathbf{III}} \\
&\qquad - \sum_{j=0}^{\ell_*(0)-1} \underbrace{\partial_{\mathbf{e}}[t^j - t^{j+1}]\nu(F^{m-j})(t^{j+1}, x^{j+1}, v^j)}_{\mathbf{II}} - \underbrace{\partial_{\mathbf{e}} t^{\ell_*(0)} \nu(F^{m-\ell_*(0)})(0, x^j - t^j v^j, v^j)}_{\mathbf{I}} \Big\} \\
&+ \underbrace{e^{-\int_0^t \sum_j \mathbf{1}_{[t^{j+1},t^j)}(\tau)\nu(F^{m-j})(\tau, X_{\mathbf{cl}}(\tau), V_{\mathbf{cl}}(\tau))\mathrm{d}\tau} \partial_{\mathbf{e}}\big[x^{\ell_*(0)} - t^{\ell_*(0)} v^{\ell_*(0)}, v^{\ell_*(0)}\big] \cdot \nabla_{x,v} f_0(X_{\mathbf{cl}}(0), V_{\mathbf{cl}}(0))}_{\mathbf{I}} \\
&+ \sum_{\ell=0}^{\ell_*(0)-1} \partial_{\mathbf{e}}[t^\ell - t^{\ell+1}] e^{-\sum_{j=0}^{\ell_*(t^\ell - t^{\ell+1})} \int_{\max\{t^\ell - t^{\ell+1} - t^j, -(t^j - t^{j+1})\}}^{0} \nu(F^{m-j})(\tau + t^j, x^j + \tau v^j, v^j)\mathrm{d}\tau} \\
&\quad \underbrace{\times [K f^{m-\ell} + \Gamma_{\text{gain}}(f^{m-\ell}, f^{m-\ell})](t^{\ell+1}, x^{\ell+1}, v^\ell)}_{\mathbf{II}} \\
&+ \underbrace{\partial_{\mathbf{e}} t^{\ell_*(0)} e^{-\int_0^t \mathbf{1}_{[t^{j+1},t^j)}(s)\nu(F^{m-j})(\tau)\mathrm{d}\tau} [K f^{m-\ell_*(0)} + \Gamma_{\text{gain}}(f^{m-\ell_*(0)}, f^{m-\ell_*(0)})](0, x^{\ell_*(0)} - t^{\ell_*(0)} v^{\ell_*(0)}, v^{\ell_*(0)})}_{\mathbf{I}} \\
&+ \int_0^t \mathbf{1}_{[t^{\ell+1},t^\ell)}(s) e^{-\int_s^t \sum_{j=0}^{\ell_*(s)} \mathbf{1}_{[t^{j+1},t^j)}(s)\nu(F^{m-j})(\tau)\mathrm{d}\tau} \\
&\quad \times \underbrace{[\partial_{\mathbf{e}} t^\ell, \partial_{\mathbf{e}} x^\ell + s\partial_{\mathbf{e}} v^\ell, \partial_{\mathbf{e}} v^\ell] \cdot \nabla_{t,x,v}[K f^{m-\ell} + \Gamma_{\text{gain}}](f^{m-\ell}, f^{m-\ell})](s, x^\ell - (t^\ell - s)v^\ell, v^\ell)\mathrm{d}s}_{\mathbf{III}} \\
&+ \int_0^t \mathbf{1}_{[t^{\ell+1},t^\ell)}(s)[K f^{m-\ell} + \Gamma_{\text{gain}}(f^{m-\ell}, f^{m-\ell})](s, X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))\mathrm{d}s \\
&\quad \times \Big\{ -\sum_{j=0}^{\ell_*(s)-1} \underbrace{\partial_{\mathbf{e}}[t^j - t^{j+1}]\nu(F^{m-j})(t^{j+1}, x^{j+1}, v^j)}_{\mathbf{II}} - \underbrace{\partial_{\mathbf{e}} t^{\ell_*(s)} \nu(F^{m-\ell_*(s)})(s, X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))}_{\mathbf{I}} \\
&\qquad - \underbrace{\sum_{j=0}^{\ell_*(s)} \int_{\max\{s,t^{j+1}\}}^{t^j} [\partial_{\mathbf{e}} t^j, \partial_{\mathbf{e}} x^j + \tau\partial_{\mathbf{e}} v^j, \partial_{\mathbf{e}} v^j] \cdot \nabla_{t,x,v} \nu(F^{m-j})(\tau, x^j - (t^j-\tau)v^j, v^j)\mathrm{d}\tau}_{\mathbf{III}} \Big\},
\end{aligned} \tag{169}$$

where we can rewrite it as

$$\partial_{\mathbf{e}} f^m(t,x,v) = \mathbf{I_e} + \mathbf{II_e} + \mathbf{III_e}.$$

Recall that we want to estimate

$$e^{-\varpi\langle v\rangle t}\frac{\alpha(x,v)}{\langle v\rangle^2}\partial_x f(t,x,v),\qquad e^{-\varpi\langle v\rangle t}\frac{|v|\alpha(x,v)^{1/2}}{\langle v\rangle^2}\partial_v f(t,x,v).$$

Using Lemma 14 and Lemma 5 and $F^m\geq 0$ from (34) and Lemma 6

$$\begin{aligned}
&e^{-\varpi\langle v\rangle t}\phi_{\mathbf{x}}(v)[\alpha(x,v)]^{\beta_{\mathbf{x}}}\mathbf{I_x}\\
&\lesssim e^{-\varpi\langle v\rangle t}\phi_{\mathbf{x}}(v)[\alpha(x,v)]^{\beta_{\mathbf{x}}}\\
&\times\Big\{||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty\langle v\rangle^{-\zeta}e^{-\theta|v|^2}\frac{t|v|^2}{\alpha(x,v)}\langle v\rangle^{\kappa}(1+||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty)\\
&\quad+\big[(1+\frac{|v|}{\alpha(x,v)})+\frac{t|v|^3}{\alpha(x,v)}\big]|\partial_x f_0|\\
&\quad+\frac{t|v|^2}{\alpha(x,v)}\langle v\rangle^{-\zeta}e^{-\theta|v|^2}||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty(1+||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty)\\
&\quad+t\langle v\rangle^{-\zeta}e^{-\theta|v|^2}||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty(1+||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty)\langle v\rangle^{\kappa}\frac{t|v|^2}{\alpha(x,v)}(1+||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty)\Big\}.
\end{aligned}$$

Therefore, with $\phi_{\mathbf{x}}=\langle v\rangle^{-2}$ and $\beta_{\mathbf{x}}=1$,

$$\begin{aligned}
e^{-\varpi\langle v\rangle t}\langle v\rangle^{-2}\alpha\mathbf{I_x}&\lesssim||\langle v\rangle^{-2}\alpha(1+\frac{|v|+|v|^3}{\alpha(x,v)})\partial_x f_0||_\infty+\langle v\rangle^{-2}e^{-C_\theta|v|^2}P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty)\\
&\lesssim\{1+||\langle v\rangle\partial_x f_0||_\infty\}\times P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty).
\end{aligned}$$

Similarly

$$\begin{aligned}
&e^{-\varpi\langle v\rangle t}\phi_{\mathbf{v}}\alpha^{\beta_{\mathbf{v}}}\mathbf{I_v}\\
&\lesssim e^{-\varpi\langle v\rangle t}\phi_{\mathbf{v}}(v)[\alpha(x,v)]^{\beta_{\mathbf{v}}}\\
&\times\Big\{||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty\langle v\rangle^{-\zeta}e^{-\theta|v|^2}\frac{te^{C|v|t}}{\alpha(x,v)^{1/2}}\langle v\rangle^{\kappa}(1+||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty)\\
&\quad+\big[(\frac{1}{|v|}+\frac{e^{C|v|t}\alpha(x,v)^{1/2}}{|v|^2})+\frac{t|v|e^{C|v|t}}{\alpha(x,v)^{1/2}}+t\big]|\partial_x f_0|+|\nabla_v f_0|\\
&\quad+\frac{te^{C|v|t}}{\alpha(x,v)^{1/2}}\langle v\rangle^{-\zeta}e^{-\theta|v|^2}||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty(1+||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty)\\
&\quad+t\langle v\rangle^{-\zeta}e^{-\theta|v|^2}||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty(1+||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty)\langle v\rangle^{\kappa}\frac{te^{C|v|t}}{\alpha(x,v)^{1/2}}(1+||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty)\Big\}.
\end{aligned}$$

Therefore $\phi_{\mathbf{v}}=|v|\langle v\rangle^{-2}$ and $\beta_{\mathbf{v}}=1/2$,

$$\begin{aligned}
e^{-\varpi\langle v\rangle t}|v|\langle v\rangle^{-2}\alpha^{1/2}\mathbf{I_v}&\lesssim|||v|\langle v\rangle^{-1}\partial_x f_0||_\infty+|||v|\langle v\rangle^{-2}\alpha^{1/2}\nabla_v f_0||_\infty+P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty)\\
&\lesssim\{1+||\langle v\rangle\partial_x f_0||_\infty+||\partial_v f_0||_\infty\}P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty).
\end{aligned}$$

For **II** using Lemma 5

$$\begin{aligned}
&e^{-\varpi\langle v\rangle t}\phi_{\mathbf{x}}\alpha^{\beta_{\mathbf{x}}}\mathbf{II_x}\\
&\lesssim e^{-\varpi\langle v\rangle t}\phi_{\mathbf{x}}\alpha^{\beta_{\mathbf{x}}}\\
&\times\Big\{\langle v\rangle^{-\zeta}e^{-\theta|v|^2}||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty\frac{t|v|^2}{\alpha(x,v)^{1/2}}\frac{1}{\alpha(x,v)^{1/2}}\langle v\rangle^{\kappa}(1+||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty)\\
&\quad+\frac{t|v|^2}{\alpha(x,v)^{1/2}}\frac{1}{\alpha(x,v)^{1/2}}\langle v\rangle^{-\zeta}e^{-\theta|v|^2}||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty(1+||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty)\\
&\quad+t\langle v\rangle^{-\zeta}e^{-\theta|v|^2}||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty\frac{t|v|^2}{\alpha(x,v)^{1/2}}\frac{1}{\alpha(x,v)^{1/2}}\langle v\rangle^{\kappa}||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty,
\end{aligned}$$

and hence with $\phi_{\mathbf{x}} = \langle v\rangle^{-2}$ and $\beta_{\mathbf{x}} = 1$,

$$e^{-\varpi\langle v\rangle t}\langle v\rangle^{-2}\alpha \mathbf{II}_{\mathbf{x}} \lesssim P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty).$$

Similarly using Lemma 5 with $\phi_{\mathbf{v}} = |v|\langle v\rangle^{-2}$ and $\beta_{\mathbf{v}} = 1/2$

$$\begin{aligned}
&e^{-\varpi\langle v\rangle t}\phi_{\mathbf{v}}\alpha^{\beta_{\mathbf{v}}}\mathbf{II}_{\mathbf{v}}\\
&\lesssim e^{-\varpi\langle v\rangle t}\phi_{\mathbf{v}}\alpha^{\beta_{\mathbf{v}}} \ \times \Big\{\langle v\rangle^{-\zeta}e^{-\theta|v|^2}||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty \frac{t|v|^2}{\alpha(x,v)^{1/2}}\frac{1}{|v|^2}\langle v\rangle^{\kappa}(1+||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty)\\
&\quad + \frac{t|v|^2}{\alpha(x,v)^{1/2}}\frac{1}{|v|^2}\langle v\rangle^{-\zeta}e^{-\theta|v|^2}||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty(1+||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty)\\
&\quad + t\langle v\rangle^{-\zeta}e^{-\theta|v|^2}||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty \frac{t|v|^2}{\alpha(x,v)^{1/2}}\frac{1}{|v|^2}\langle v\rangle^{\kappa}||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty\\
&\lesssim P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty).
\end{aligned}$$

For $\mathbf{III}$ we have

$$\begin{aligned}
&e^{-\varpi\langle v\rangle t}\phi_{\mathbf{e}}(v)[\alpha(x,v)]^{\beta_{\mathbf{e}}}\mathbf{III}_{\mathbf{e}}\\
&:= e^{-\varpi\langle v\rangle t}\phi_{\mathbf{e}}(v)[\alpha(x,v)]^{\beta_{\mathbf{e}}}\{E'_{\mathbf{e}} + F'_{\mathbf{e}} + G'_{\mathbf{e}} + E_{\mathbf{e}} + F_{\mathbf{e}} + G_{\mathbf{e}}\}\\
&\lesssim e^{-\varpi\langle v\rangle t}\phi_{\mathbf{e}}(v)[\alpha(x,v)]^{\beta_{\mathbf{e}}}\Big\{\langle v\rangle^{-\zeta}e^{-\theta|v|^2}||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty\Big\}\\
&\times\Big\{\int_0^t\sum_{j=0}^{\ell_*(0)}\mathbf{1}_{[t^{j+1},t^j)}(s)|\partial_{\mathbf{e}}t^j|\langle v\rangle^{\kappa}\mathrm{d}s(1+||\langle v\rangle^{\zeta}e^{\theta|v|^2}\partial_t f_0||_\infty)\\
&\quad + \int_0^t\sum_{j=0}^{\ell_*(0)}\mathbf{1}_{[t^{j+1},t^j)}(s)\{|\partial_{\mathbf{e}}x^{\ell}| + t|\partial_{\mathbf{e}}v^{\ell}|\}\nu(\partial_x F^{m-\ell})(s,X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))\mathrm{d}s\\
&\quad + \int_0^t\sum_{j=0}^{\ell_*(0)}\mathbf{1}_{[t^{j+1},t^j)}(s)|\partial_{\mathbf{e}}v^{\ell}|\int_{\mathbb{R}^3}|V_{\mathbf{cl}}(s)-u|^{\kappa-1}\sqrt{\mu(u)}F^{m-\ell}(s,X_{\mathbf{cl}}(s),u)\mathrm{d}u\mathrm{d}s\Big\}\\
&+ e^{-\varpi\langle v\rangle t}\phi_1(v)[\alpha(x,v)]^{\beta_1}\Big\{1+(1+t)\langle v\rangle^{-\zeta}e^{-\theta|v|^2}||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty(1+||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty)\Big\}\\
&\times\Big\{\int_0^t\sum_{\ell=0}^{\ell_*(0)}\mathbf{1}_{[t^{\ell+1},t^{\ell})}(s)|\partial_{\mathbf{e}}t^{\ell}|\big[|K\partial_t f^{m-\ell}| + |\Gamma_{\mathrm{gain}}(\partial_t f^{m-\ell},f^{m-\ell})| + |\Gamma_{\mathrm{gain}}(f^{m-\ell},\partial_t f^{m-\ell})|\big]\mathrm{d}s\\
&\quad + \int_0^t\sum_{\ell=0}^{\ell_*(0)}\mathbf{1}_{[t^{\ell+1},t^{\ell})}(s)\big\{|\partial_{\mathbf{e}}x^{\ell}| + t|\partial_{\mathbf{e}}v^{\ell}|\big\}\big[|K\partial_x f^{m-\ell}| + |\Gamma_{\mathrm{gain}}(\partial_x f^{m-\ell},f^{m-\ell})| + |\Gamma_{\mathrm{gain}}(f^{m-\ell},\partial_x f^{m-\ell})|\big]\mathrm{d}s\\
&\quad + \int_0^t\sum_{\ell=0}^{\ell_*(0)}\mathbf{1}_{[t^{\ell+1},t^{\ell})}(s)|\partial_{\mathbf{e}}v^{\ell}|\big[|K_v f^{m-\ell}| + |\Gamma_{\mathrm{gain},v}(f^{m-\ell},f^{m-\ell})|\\
&\qquad\qquad + |\Gamma_{\mathrm{gain}}(f^{m-\ell},\partial_v f^{m-\ell})| + |\Gamma_{\mathrm{gain}}(\partial_v f^{m-\ell},f^{m-\ell})|\big]\mathrm{d}s\Big\}.
\end{aligned}$$

We consider $\partial_{\mathbf{e}}t^j-$contribution, $E'_{\mathbf{e}}$ and $E_{\mathbf{e}}$, first. Note that from Lemma 6

$$\sup_{0\le t\le T}||\langle v\rangle^{\zeta}e^{\theta|v|^2}\partial_t f^m(t)||_\infty \lesssim ||\langle v\rangle^{\zeta}e^{\theta|v|^2}\partial_t f_0||_\infty + P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty).$$

Then from Lemma 14 and *(2)* of Lemma 5 for $\beta_{\mathbf{x}} \geq 1$ and $\phi_{\mathbf{x}} e^{-\theta|v|^2} \lesssim 1$

$$
\begin{aligned}
& e^{-\varpi\langle v\rangle t}\phi_{\mathbf{x}}(v)[\alpha(x,v)]^{\beta_{\mathbf{x}}}\{E'_{\mathbf{x}} + E_{\mathbf{x}}\} \\
& \lesssim e^{-\varpi\langle v\rangle t}\phi_{\mathbf{x}}(v)\alpha(x,v)^{\beta_{\mathbf{x}}}\langle v\rangle^{-\zeta}e^{-\theta|v|^2}t\frac{t|v|^2}{\alpha(x,v)}\langle v\rangle||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty(1+||\langle v\rangle^{\zeta}e^{\theta|v|^2}\partial_t f_0||_\infty) \\
& \ + e^{-\varpi\langle v\rangle t}\phi_{\mathbf{x}}(v)\alpha(x,v)^{\beta_{\mathbf{x}}}(1+t)||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty(1+||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty) \\
& \quad \times t\frac{t|v|^2}{\alpha(x,v)}\langle v\rangle^{-\zeta}e^{-\theta|v|^2}||\langle v\rangle^{\zeta}e^{\theta|v|^2}\partial_t f_0||_\infty(1+||\langle v\rangle^{\zeta}e^{\theta|v|^2}\partial_t f_0||_\infty) \\
& \lesssim_t \phi_{\mathbf{x}}(v)[\alpha(x,v)]^{\beta_{\mathbf{x}}-1}e^{-C_\theta|v|^2}\big[||\langle v\rangle^{\zeta}e^{\theta|v|^2}\partial_t f_0||_\infty + ||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty\big]^3 \\
& \lesssim_t 1 + \big[||\langle v\rangle^{\zeta}e^{\theta|v|^2}\partial_t f_0||_\infty + ||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty\big]^3.
\end{aligned}
$$

Similarly for $\beta_{\mathbf{v}} \geq 1/2$ and a bounded $\phi_{\mathbf{v}}$

$$
\begin{aligned}
& e^{-\varpi\langle v\rangle t}\phi_{\mathbf{v}}(v)[\alpha(x,v)]^{\beta_{\mathbf{v}}}\{E'_{\mathbf{v}} + E_{\mathbf{v}}\} \\
& \lesssim_t \phi_{\mathbf{v}}(v)\alpha(x,v)^{\beta_{\mathbf{v}}-\frac{1}{2}}e^{-C_\theta|v|^2}\big[||\langle v\rangle^{\zeta}e^{\theta|v|^2}\partial_t f_0||_\infty + ||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty\big]^3 \\
& \lesssim_t 1 + \big[||\langle v\rangle^{\zeta}e^{\theta|v|^2}\partial_t f_0||_\infty + ||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty\big]^3.
\end{aligned}
$$

Now we consider $F'_{\mathbf{e}}$ and $F_{\mathbf{e}}$ which is the most complicated terms in bounce-back proof. We use *(2)* of Lemma 5 and Lemma 14 and (166)

$$
\begin{aligned}
& e^{-\varpi\langle v\rangle t}\phi_{\mathbf{x}}[\alpha(x,v)]^{\beta_{\mathbf{x}}}\{F'_{\mathbf{x}} + F_{\mathbf{x}}\} \\
& \lesssim_t \big(1 + \big[||\langle v\rangle^{\zeta}e^{\theta|v|^2}\partial_t f_0||_\infty + ||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty\big]^3\big) \\
& \quad \times \sum_{\ell=0}^{\ell_*(0;t,x,v)}\int_{t^{\ell+1}}^{t^\ell}\int_{\mathbb{R}^3} e^{-\varpi\langle v\rangle t}\phi_{\mathbf{x}}(v)|v|\alpha(x,v)^{\beta_{\mathbf{x}}-\frac{1}{2}}\mathbf{k}_{\kappa,\frac{\theta}{4}}(u,v^\ell)|\partial_x f^{m-\ell}(s,X_{\mathbf{cl}}(s),u)|\mathrm{d}u\mathrm{d}s, \\
& \lesssim_t \big(1 + \big[||\langle v\rangle^{\zeta}e^{\theta|v|^2}\partial_t f_0||_\infty + ||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty\big]^3\big)\max_{0\leq\ell\leq m}\sup_{0\leq s\leq t}||e^{-\varpi\langle v\rangle s}\phi_{\mathbf{x}}\alpha^{\beta_{\mathbf{x}}}\partial_x f^{m-\ell}(s)||_\infty \\
& \quad \times \int_0^t\sum_{\ell=0}^{\ell_*(0;t,x,v)}\mathbf{1}_{[t^{\ell+1},t^\ell)}(s)\int_{\mathbb{R}^3} e^{-\frac{\varpi}{2}\langle v\rangle(t-s)}\frac{\phi_{\mathbf{x}}(v)}{\phi_{\mathbf{x}}(u)}\frac{|v|\alpha(x,v)^{\beta_{\mathbf{x}}-\frac{1}{2}}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}\alpha(X_{\mathbf{cl}}(s),u)^{\beta_{\mathbf{x}}}}e^{-C_\theta|v-u|^2}\mathrm{d}u\mathrm{d}s.
\end{aligned}
$$

Note $\phi_{\mathbf{x}}(v)/\phi_{\mathbf{x}}(u) = \frac{|v|}{|u|}\langle v-u\rangle^2$. By (64),

$$
\begin{aligned}
& \lesssim_t \big(1 + \big[||\langle v\rangle^{\zeta}e^{\theta|v|^2}\partial_t f_0||_\infty + ||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty\big]^3\big)\max_{0\leq\ell\leq m}\sup_{0\leq s\leq t}||e^{-\varpi\langle v\rangle s}\phi_{\mathbf{x}}\alpha^{\beta_{\mathbf{x}}}\partial_x f^{m-\ell}(s)||_\infty \\
& \quad \times \frac{C_\delta O(\tilde{\delta}) + C_{\tilde{\delta},\delta}O(\varpi^{-1})}{\langle v\rangle\alpha(x,v)^{\beta_{\mathbf{x}}-1/2}}|v|\alpha(x,v)^{\beta_{\mathbf{x}}-1/2} \\
& \lesssim [C_\delta O(\tilde{\delta}) + C_{\tilde{\delta},\delta}O(\varpi^{-1})]\big(1 + \big[||\langle v\rangle^{\zeta}e^{\theta|v|^2}\partial_t f_0||_\infty + ||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty\big]^3\big) \\
& \quad \times \max_{0\leq\ell\leq m}\sup_{0\leq s\leq t}||e^{-\varpi\langle v\rangle s}\alpha\langle v\rangle^{-2}\partial_x f^{m-\ell}(s)||_\infty.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
& e^{-\varpi\langle v\rangle t}\phi_{\mathbf{v}}[\alpha(x,v)]^{\beta_{\mathbf{v}}}\{F'_{\mathbf{v}} + F_{\mathbf{v}}\} \\
& \lesssim_t \big(1 + \big[||\langle v\rangle^{\zeta}e^{\theta|v|^2}\partial_t f_0||_\infty + ||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty\big]^3\big) \\
& \quad \times \sum_{\ell=0}^{\ell_*(0;t,x,v)}\int_{t^{\ell+1}}^{t^\ell}\int_{\mathbb{R}^3} e^{-\varpi\langle v\rangle t}\phi_{\mathbf{v}}(v)(\frac{1}{|v|}+1)\alpha(x,v)^{\beta_{\mathbf{v}}}\mathbf{k}_{\kappa,\frac{\theta}{4}}(u,v^\ell)|\partial_x f^{m-\ell}(s,X_{\mathbf{cl}}(s),u)|\mathrm{d}u\mathrm{d}s, \\
& \lesssim_t \big(1 + \big[||\langle v\rangle^{\zeta}e^{\theta|v|^2}\partial_t f_0||_\infty + ||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty\big]^3\big)\max_{0\leq\ell\leq m}\sup_{0\leq s\leq t}||e^{-\varpi\langle v\rangle t}\phi_{\mathbf{x}}\alpha^{\beta_{\mathbf{x}}}\partial_x f^{m-\ell}(s,X_{\mathbf{cl}}(s),u)||_\infty \\
& \quad \times \sum_{\ell=0}^{\ell_*(0;t,x,v)}\int_{t^{\ell+1}}^{t^\ell}\int_{\mathbb{R}^3} e^{-\frac{\varpi}{2}\langle v\rangle(t-s)}\frac{\phi_{\mathbf{v}}(v)}{\phi_{\mathbf{x}}(u)}\frac{(\frac{1}{|v|}+1)\alpha(x,v)^{\beta_{\mathbf{v}}}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}\alpha(x,u)^{\beta_{\mathbf{x}}}}\mathrm{d}u\mathrm{d}s.
\end{aligned}
$$

Due to our choice of β and ϕ, the integration is bounded by

$$\sum_{\ell=0}^{\ell_*(0;t,x,v)} \int_{t^{\ell+1}}^{t^\ell} \int_{\mathbb{R}^3} e^{-\frac{\varpi}{2}\langle v\rangle(t-s)} \langle v-u\rangle^2 \frac{\langle v\rangle \alpha(x,v)^{1/2}}{\alpha(x,u)} \frac{e^{-C|V_{\mathbf{cl}}(s)-u|^2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}} \mathrm{d}u\mathrm{d}s.$$

By (20) this is bounded

$$\lesssim_t \big(1+\big[||\langle v\rangle^\zeta e^{\theta|v|^2}\partial_t \tilde{f}_0||_\infty + ||\langle v\rangle^\zeta e^{\theta|v|^2} f_0||_\infty\big]^3\big) \max_{0\le\ell\le m}\sup_{0\le s\le t} ||e^{-\varpi\langle v\rangle s}\phi_{\mathbf{x}}\alpha^{\beta_{\mathbf{x}}}\partial_x f^{m-\ell}(s)||_\infty$$
$$\times [C_\delta O(\tilde{\delta}) + C_{\tilde{\delta},\delta} O(\varpi^{-1})].$$

Consider $\partial_{\mathbf{e}} v^\ell-$contribution, $G'_{\mathbf{e}}$ and $G_{\mathbf{e}}$. Note that $G'_{\mathbf{x}} = 0 = G_{\mathbf{x}}$ since $\partial_x v^j \equiv 0$. From Lemma 14 and *(3)* of Lemma 5

$$e^{-\varpi\langle v\rangle t}\phi_{\mathbf{v}}(v)\alpha(x,v)^{\beta_{\mathbf{v}}}\{G'_{\mathbf{v}}+G_{\mathbf{v}}\} \lesssim (1+||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty)^4 + ||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty$$
$$\times \int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1},t^\ell)}(s) e^{-\varpi\langle v\rangle t}\phi_{\mathbf{v}}(v)\alpha(x,v)^{\beta_{\mathbf{v}}} \int_{\mathbb{R}^3} \mathbf{k}_{\kappa,\theta/4}(V_{\mathbf{cl}}(s),u)|\partial_v f^{m-\ell}(s,X_{\mathbf{cl}}(s),u)|\mathrm{d}u\mathrm{d}s$$
$$\lesssim (1+||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty)^4 + ||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty$$
$$\times \int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1},t^\ell)}(s) \int_{\mathbb{R}^3} e^{-\varpi\langle v\rangle t} e^{-\varpi\langle u\rangle s} \frac{\phi_{\mathbf{v}}(v)\alpha(x,v)^{\beta_{\mathbf{v}}}}{\phi_{\mathbf{v}}(u)\alpha(X_{\mathbf{cl}}(s),u)^{\beta_{\mathbf{v}}}} e^{-\theta|v|^2}\mathbf{k}_{\kappa,\theta/4}(V_{\mathbf{cl}}(s),u)\mathrm{d}u\mathrm{d}s$$
$$\times \sup_{0\le s\le t}\max_{0\le\ell\le m} ||e^{-\varpi\langle v\rangle s}\phi_{\mathbf{v}}(v)\alpha(x,u)^{\beta_{\mathbf{v}}}\partial_v f^{m-\ell}(s,x,u)||_\infty.$$

Now for $\beta_{\mathbf{v}} \ge 1/2$ we choose any $\beta_{\mathbf{v}} + 1/2 > \beta' > \beta_{\mathbf{v}}$

$$\frac{1}{[\alpha(X_{\mathbf{cl}}(s),u)]^{\beta_{\mathbf{v}}}} \lesssim \frac{|u|^{2(\beta'-\beta_{\mathbf{v}})}}{[\alpha(X_{\mathbf{cl}}(s),u)]^{\beta'}}.$$

Now we use (166) and $\phi_{\mathbf{v}} = |v|\langle v\rangle^{-2}$ to bound the integration by

$$\int_0^t \sum_{\ell=0}^{\ell_*(0)} \mathbf{1}_{[t^{\ell+1},t^\ell)}(s) \int_{\mathbb{R}^3} e^{-\varpi\langle v\rangle(t-s)} \frac{|v|}{|u|} \frac{|u|^{2\beta'-1}\alpha(x,v)^{1/2}}{|V_{\mathbf{cl}}(s)-u|^{2-\kappa}\alpha(X_{\mathbf{cl}}(s),u)^{\beta'}} e^{-\theta|v|^2} e^{-C_\theta|V_{\mathbf{cl}}(s)-u|^2}\mathrm{d}u\mathrm{d}s$$

Now we use $|u|^{2\beta'-1} \le \langle v\rangle^{2\beta'-1}\langle u-v\rangle^{2\beta'-1}$ and we apply (64) to bound this integration by

$$[C_\delta O(\tilde{\delta}) + C_{\tilde{\delta},\delta}O(-\varpi)]\langle v\rangle^{-2+2\beta'}\alpha^{1-\beta'} \lesssim [C_\delta O(\tilde{\delta}) + C_{\tilde{\delta},\delta}O(-\varpi)],$$

Hence,

$$e^{-\varpi\langle v\rangle t}\phi_{\mathbf{v}}(v)\alpha(x,v)^{\beta_{\mathbf{v}}}\{G'_{\mathbf{v}}+G_{\mathbf{v}}\}$$
$$\lesssim (1+||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty)^4 + ||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty \sup_{0\le s\le t}\max_{0\le\ell\le m} ||e^{-\varpi\langle v\rangle s}\phi_{\mathbf{v}}(v)\alpha(x,u)^{\beta_{\mathbf{v}}}\partial_v f^{m-\ell}(s,x,u)||_\infty$$
$$\times \Big\{\frac{C_\delta O(\tilde{\delta}) + C_{\tilde{\delta},\delta}O(-\varpi)}{\langle v\rangle[\alpha(x,v)]^{\beta'-1/2}}\alpha(x,v)^{\beta_{\mathbf{v}}}e^{-C_\theta|v|^2} + O(\varpi^{-1})\Big\}$$
$$\lesssim (1+||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty)^4$$
$$+ [C_\delta O(\tilde{\delta}) + C_{\tilde{\delta},\delta}O(-\varpi)]||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty \sup_{0\le s\le t}\max_{0\le\ell\le m} ||e^{-\varpi\langle v\rangle s}\phi_{\mathbf{v}}(v)\alpha(x,u)^{\beta_{\mathbf{v}}}\partial_v f^{m-\ell}(s,x,u)||_\infty.$$

Now we gather all the estimates and choose small $\delta > 0$ and $\tilde{\delta} > 0$ and large $\varpi > 0$ to close the estimate. Then we follow the exactly same argument as the specular case and this complete the proof of Theorem 5.

□

Appendix. Non-Existence of Second Derivatives

In the previous theorem, we consider the *first-order derivative* of the Boltzmann solution with several boundary conditions. Now we show that the second order normal spatial derivative $\partial_n^2 f$ does not exist up to the boundary in general so that our result is quite optimal.

Assume that $D^2 f$ exist away from the grazing set $\gamma_0 = \{(x,v) \in \partial\Omega \times \mathbb{R}^3 : n(x)\cdot v = 0\}$ but up to the boundary $\partial\Omega \times \mathbb{R}^3$. Taking the normal derivative ∂_n of the Boltzmann equation directly yields

$$v_n\partial_n\partial_n f = -\partial_t\partial_n f - (\partial_n v_n)\partial_n f - \sum_{i=1}^2 \partial_n(v_{\tau_i})\partial_{\tau_i} f - \sum_{i=1}^2 v_{\tau_i}\partial_n\partial_{\tau_i} f - \nu(F)\partial_n f + \underbrace{K(\partial_n f)} + \partial_n\Gamma_{\text{gain}}(f,f),$$

where we used $v\cdot\nabla_x\partial_n f = v_n\partial_n\partial_n f + \sum_{i=1}^2 v_{\tau^i}\partial_{\tau^i}\partial_n f$.

Inside $K(\partial_n f)$, one expects

$$\partial_n f \backsim \frac{1}{n\cdot v}, \tag{168}$$

hence at the boundary $\partial\Omega$, $\partial_n f$ has a non-integrable behavior in v so that $K(\partial_n f)$ can not exist at the boundary. Consequently, $\partial_n^2 f$ does not exist at any boundary point so that $\alpha^\beta D^2 f$ can not exist up to the boundary in any L^p and any β, due to Ukai's trace theorem.

First consider the diffuse reflection boundary condition. We prove (168) will become valid for a general class of initial data after some $t>0$, so that $D^2 f$ does not exist. Theorem 2 plays an important role in our proof.

Proposition 3 (Diffuse BC)**.** *Assume Ω is convex (2). Assume the initial datum is given by*

$$f_0(x,v) = \delta e^{-\eta|v|^2}\chi_\varepsilon(x)h_N(v),$$

where χ_ε, h_N are non-negative smooth functions and $\chi_\varepsilon(x)\equiv 0$ for $\mathrm{dist}(x,\partial\Omega)<\varepsilon$ and $\chi_\varepsilon(x)\equiv 1$ for $\mathrm{dist}(x,\partial\Omega)\geq 2\varepsilon$ and $h_N(v)\equiv 0$ for $|v|\in[0,N-2]\cup[N+2,\infty)$ and $h_N(v)\equiv 1$ for $|v|\in[N-1,N+1]$. For some $\delta,\eta\in(0,\infty)$ chosen later, and $t=\frac{\varepsilon}{N}>0$, $N\gg 1$, the solution of the nonlinear Boltzmann equation with the diffuse BC satisfies

$$K(\partial_n f) = \infty, \qquad (x,v)\in\partial\Omega\times\mathbb{R}^3. \tag{169}$$

Remark that for $0<\delta\ll 1$, $0<\eta<\frac14$ we have $\sup_t ||e^{\eta|v|^2}f(t)||_\infty \lesssim ||e^{\eta|v|^2}f_0||_\infty$ due to [8, 2] and $||\alpha^{1/2}\partial f(t)||_\infty \lesssim 1$ due to Theorem 2. Also remark that f_0, $\partial^k f_0 \equiv 0$ for all $k\in\mathbb{N}$ near $\partial\Omega\times\mathbb{R}^3$ and $\nabla_x f_0(x,v) \sim \delta e^{-\eta|v|^2}\varepsilon^{-1}h_N(v)$ for $\mathrm{dist}(x,\partial\Omega)>\varepsilon$. In particular, $\partial_n^2 f_0\equiv 0$ at the boundary. The convexity of the domain is crucial in the proof especially at (173).

Proof. We first study the behavior of $\partial_n f$ for $n\cdot v\backsim 0$ at $\partial\Omega$. Recall the expression of $\nabla_x f$ from (50) and its boundary identity (16). On $\gamma_- = \{(x,v)\in\partial\Omega\times\mathbb{R}^3 : n(x)\cdot v<0\}$, for $n(x)\cdot v<0$

$$\begin{aligned}
[\partial_n f(t,x,v)]_- &= \frac{-1}{n(x)\cdot v}\Big\{\partial_t f + \sum_{i=1}^2 (v\cdot\tau_i)\partial_{\tau_i} f + \nu(F)f - Kf - \Gamma_{\text{gain}}(f,f)\Big\}(t,x,v) \qquad (170)\\
&= \frac{-1}{n(x)\cdot v}\Big\{\sqrt{\mu(v)}\int_{n(x)\cdot u>0}\partial_t f(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u\\
&\quad + \sum_{i=1}^2 (v\cdot\tau_i)\sqrt{\mu(v)}\int_{n(x)\cdot u>0}\partial_{\tau_i} f(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u\\
&\quad + \sum_{i=1}^2 (v\cdot\tau_i)\sqrt{\mu(v)}\int_{n(x)\cdot u>0}\nabla_v f(t,x,u)\frac{\partial\mathcal{T}^t}{\partial\tau_i}\mathcal{T}(x)u\sqrt{\mu(u)}\{n(x)\cdot u\}\mathrm{d}u\\
&\quad + \nu(F)f(t,x,v) - Kf(t,x,v) + \Gamma_{\text{gain}}(f,f)(t,x,v)\Big\}\\
:= &\ \frac{-1}{n(x)\cdot v}\times G(t,x,v) := \frac{-1}{n(x)\cdot v}\times\sum_{j=1}^6 G_j(t,x,v).
\end{aligned}$$

On the other hand, on the outgoing part $\gamma_+ = \{(x,v) \in \partial\Omega \times \mathbb{R}^3 : n(x)\cdot v > 0\}$, for any $t > 0$, by the Velocity Lemma, $t_{\mathbf{b}} > 0$ for $n(x)\cdot v > 0$, so that

$$\begin{aligned}
&[\partial_n f(t,x,v)]_+ \\
= \ & \frac{-n(x)\cdot n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})}\Big\{\partial_t f + \sum_{i=1}^2 (v\cdot\tau_i)\partial_{\tau_i} f + \nu(F)f - Kf - \Gamma_{\text{gain}}(f,f)\Big\}(t-t_{\mathbf{b}}, x_{\mathbf{b}}, v) \qquad (171)\\
&+ \int_0^{t_{\mathbf{b}}} e^{-\int_s^t \nu} n(x)\cdot\nabla_x\{Kf + \Gamma(f,f)\}(t-s, x-sv, v)\mathrm{d}s \\
= \ & \frac{-n(x)\cdot n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})}\Big\{\sqrt{\mu(v)}\int_{n(x_{\mathbf{b}})\cdot u>0} \partial_t f(t-t_{\mathbf{b}}, x_{\mathbf{b}}, u)\sqrt{\mu(u)}\{n(x_{\mathbf{b}})\cdot u\}\mathrm{d}u \\
&\qquad + \sum_{i=1}^2 (v\cdot\tau_i(x_{\mathbf{b}}))\sqrt{\mu(v)}\int_{n(x_{\mathbf{b}})\cdot u>0} \partial_{\tau_i} f(t-t_{\mathbf{b}}, x_{\mathbf{b}}, u)\sqrt{\mu(u)}\{n(x_{\mathbf{b}})\cdot u\}\mathrm{d}u \\
&\qquad + \sum_{i=1}^2 (v\cdot\tau_i(x_{\mathbf{b}}))\sqrt{\mu(v)}\int_{n(x_{\mathbf{b}})\cdot u>0} \nabla_v f(t-t_{\mathbf{b}}, x_{\mathbf{b}}, u)\frac{\partial \mathcal{T}^t}{\partial \tau_i}\mathcal{T}(x_{\mathbf{b}})u\sqrt{\mu(u)}\{n(x_{\mathbf{b}})\cdot u\}\mathrm{d}u \\
&\qquad + \nu(v)f(t-t_{\mathbf{b}}, x_{\mathbf{b}}, v) - Kf(t-t_{\mathbf{b}}, x_{\mathbf{b}}, v) + \Gamma(f,f)(t-t_{\mathbf{b}}, x_{\mathbf{b}}, v)\Big\} \\
&+ \int_0^{t_{\mathbf{b}}} e^{-\int_s^t \nu} n(x)\cdot\nabla_x\{Kf + \Gamma(f,f)\}(t-s, x-sv, v)\mathrm{d}s \\
= \ & \frac{-n(x)\cdot n(x_{\mathbf{b}})}{v\cdot n(x_{\mathbf{b}})} \times \sum_{j=1}^6 G_j(t-t_{\mathbf{b}}, x_{\mathbf{b}}, v) + \int_{t_{\mathbf{b}}}^t e^{-\int_s^t \nu} n(x)\cdot\nabla_x\{\nu(F)f + Kf + \Gamma_{\text{gain}}(f,f)\}(t-s, x-sv, v)\mathrm{d}s.
\end{aligned}$$

where G is defined in (170).

Step 1. We first claim that up to a constant

$$[\partial_n f]_+ + [\partial_n f]_- \backsim \frac{G}{n(x)\cdot v}. \qquad (172)$$

To establish (172), we first note that from the proof of Theorem 2, the last integral $\int_{t_{\mathbf{b}}}^t$ in (171) is clearly bounded for $n\cdot v \backsim 0$. We now show as $n(x)\cdot v \downarrow 0$,

$$G(t-t_{\mathbf{b}}, x_{\mathbf{b}}, v) \to G(t,x,v). \qquad (173)$$

From our choice of f_0, $\alpha^{1/2}\partial f \in C^0([0,\infty)\times\bar{\Omega}\times\mathbb{R}^3)$ from our Theorem 2. Therefore the first three terms of G converge correspondingly, as $0 < n(x)\cdot v < \delta$,

$$\begin{aligned}
&\sum_{j=1}^3 G_j(t-t_{\mathbf{b}}, x_{\mathbf{b}}, v) \\
&= \sum_{j=1}^3 G_j(t,x,v) + O(\delta + e^{-\frac{1}{4\delta^2}})||e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f_0||_\infty \\
&+ \sqrt{\mu(v)}\int_{n(x_{\mathbf{b}})\cdot u>\delta,\ |u|\le\delta^{-1}} \mathrm{d}u\sqrt{\mu(u)}\Big\{|\nabla\xi(x_{\mathbf{b}})|^{-1}e^{-\frac{l\langle u\rangle}{2}(t-t_{\mathbf{b}})}e^{-\varpi\langle u\rangle(t-t_{\mathbf{b}})}\alpha^{1/2}\partial_t f(t-t_{\mathbf{b}}, x_{\mathbf{b}}, u) \\
&\quad - |\nabla\xi(x)|^{-1}e^{-\frac{l\langle u\rangle}{2}t}e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial_t f(t,x,u) \\
&\quad + |\nabla\xi(x_{\mathbf{b}})|^{-1}e^{-\frac{l\langle u\rangle}{2}(t-t_{\mathbf{b}})}\sum_{i=1}^2 (v\cdot\tau_i(x_{\mathbf{b}}))e^{-\varpi\langle u\rangle(t-t_{\mathbf{b}})}\alpha^{1/2}\partial_{\tau_i} f(t-t_{\mathbf{b}}, x_{\mathbf{b}}, u) \\
&\quad - |\nabla\xi(x)|^{-1}e^{-\frac{l\langle u\rangle}{2}t}\sum_{i=1}^2 (v\cdot\tau_i(x))e^{-\varpi\langle u\rangle t}\alpha^{1/2}\partial_{\tau_i} f(t,x,u)
\end{aligned}$$

$$+|\nabla\xi(x_{\mathbf{b}})|^{-1}e^{-\frac{l\langle u\rangle}{2}(t-t_{\mathbf{b}})}\sum_{i=1}^{2}(v\cdot\tau_i(x_{\mathbf{b}}))\frac{\partial\mathcal{T}^t}{\partial\tau_i}(x_{\mathbf{b}})\mathcal{T}(x_{\mathbf{b}})ue^{-\varpi\langle u\rangle(t-t_{\mathbf{b}})}\alpha^{1/2}\nabla_v f(t-t_{\mathbf{b}},x_{\mathbf{b}},u)$$

$$-|\nabla\xi(x)|^{-1}e^{-\frac{l\langle u\rangle}{2}t}\sum_{i=1}^{2}(v\cdot\tau_i(x))\frac{\partial\mathcal{T}^t}{\partial\tau_i}(x)\mathcal{T}(x)u\mathbf{d}\partial_{\tau_i}f(t,x,u)\Big\}$$

$$=\sum_{j=1}^{3}G_j(t,x,v)+O(\delta+e^{-\frac{1}{4\delta^2}})||\alpha^{1/2}\partial f_0||_\infty+O(\delta)\to\sum_{j=1}^{3}G_j(t,x,v).$$

For G_j, for $j=4,5,6$, we use the continuity of f (see [8, 2, 11]). As $0<n(x)\cdot v<\delta$,

$$\sum_{j=4}^{5}G_j(t-t_{\mathbf{b}},x_{\mathbf{b}},v)$$

$$=\sum_{j=4}^{5}G_j(t,x,v)+O(\delta+e^{-\frac{1}{4\delta^2}})||f_0||_\infty$$

$$+\nu(v)c_\mu\sqrt{\mu(v)}\int_{|n(x_{\mathbf{b}})\cdot u|>\delta,\ |u|\le\delta^{-1}}\big[f(t-t_{\mathbf{b}},x_{\mathbf{b}},u)\{n(x_{\mathbf{b}})\cdot u\}-f(t,x,u)\{n(x)\cdot u\}\big]\sqrt{\mu(u)}\mathrm{d}u$$

$$+\int_{|n(x_{\mathbf{b}})\cdot u|>\delta,\ |u|\le\delta^{-1}}\mathbf{k}(v,u)\big[f(t-t_{\mathbf{b}},x_{\mathbf{b}},u)-f(t,x,u)\big]\mathrm{d}u$$

$$=\sum_{j=4}^{5}G_j(t,x,v)+O(\delta+e^{-\frac{1}{4\delta^2}})||f_0||_\infty+O(\delta)\to\sum_{j=4}^{5}G_j(t,x,v).$$

For G_6, recall $\Gamma_{\text{gain}}(f,f),\ \nu(F)f$. For $\nu(F)f$ we use the boundary condition to have

$$\begin{aligned}
&\nu(F)f(t-t_{\mathbf{b}},x_{\mathbf{b}},v)\\
=\ &\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}q_0(\theta)|v-u|^\kappa F(t-t_{\mathbf{b}},x_{\mathbf{b}},u)\mathrm{d}\omega\mathrm{d}u\\
&\times c_\mu\sqrt{\mu(v)}\int_{n(x_{\mathbf{b}})\cdot u>0}f(t-t_{\mathbf{b}},x_{\mathbf{b}},u)\sqrt{\mu(u)}\{n(x_{\mathbf{b}})\cdot u\}\mathrm{d}u\\
=\ &\nu(F)f(t,x,v)+O(\delta+e^{-\frac{1}{4\delta^2}})||f_0||_\infty^2\\
&+\int_{|n(x_{\mathbf{b}})\cdot u|\ge\delta,\ |u|\le\delta^{-1}}C|v-u|^\kappa\sqrt{\mu(u)}\big[f(t-t_{\mathbf{b}},x_{\mathbf{b}},u)-f(t,x,u)\big]\mathrm{d}u\times O(||e^{\eta|v|^2}f_0||_\infty)\\
&+O(||e^{\eta|v|^2}f_0||_\infty)\\
&\quad\times c_\mu\sqrt{\mu(v)}\int_{n(x_{\mathbf{b}})\cdot u>\delta,\ |u|\le\delta^{-1}}\big[f(t-t_{\mathbf{b}},x_{\mathbf{b}},u)\{n(x_{\mathbf{b}})\cdot u\}-f(t,x,u)\{n(x)\cdot u\}\big]\sqrt{\mu(u)}\mathrm{d}u\\
=\ &\nu(F)f(t,x,v)+O(\delta+e^{-\frac{1}{4\delta^2}})||e^{\eta|v|^2}f_0||_\infty+O(\delta)O(||e^{\eta|v|^2}f_0||_\infty)\to\nu(F)f(t,x,v).
\end{aligned}$$

For the gain term we consider the sets, for $|n(x_{\mathbf{b}})\cdot v|<\delta$,

$$\begin{aligned}
&\{(u,\omega)\in\mathbb{R}^3\times\mathbb{S}^2:|n(x_{\mathbf{b}})\cdot u'|<2\delta\}\cup\{(u,\omega)\in\mathbb{R}^3\times\mathbb{S}^2:|n(x_{\mathbf{b}})\cdot v'|<2\delta\}\\
\subset\ &\Big\{\Big|n(x_{\mathbf{b}})\cdot\sum_{i=1}^{2}(\omega_\perp)_i[(u-v)\cdot(\omega_\perp)_i]\Big|<\delta\Big\}\cup\Big\{\big|(n(x_{\mathbf{b}})\cdot\omega)[(u-v)\cdot\omega]\big|<\delta\Big\}\\
\subset\ &\Big\{\sum_{i=1}^{2}|n(x_{\mathbf{b}})\cdot(\omega_\perp)_i|<\sqrt{\delta}\Big\}\cup\bigcup_{i=1}^{2}\big\{|(u-v)\cdot(\omega_\perp)_i|<2\sqrt{\delta}\big\}\cup\big\{|(u-v)\cdot\omega|<\delta\big\}\\
:=\ &S_\delta,
\end{aligned}$$

where $(\omega_\perp)_i\perp\omega$ for $i=1,2$ and $|\omega_\perp|=1$ and, we have used $n(x_{\mathbf{b}})\cdot(u'+v')=n(x_{\mathbf{b}})\cdot(u+v)$, and the fact that if $\sum_{i=1}^{2}|n(x_{\mathbf{b}})\cdot(\omega_\perp)_i|\ge\sqrt{\delta}$ then either for $i=1$ or $i=2$ we should have $|n(x_{\mathbf{b}})\cdot(\omega_\perp)_i|\ge\frac{\sqrt{\delta}}{2}$ and therefore $|(u-v)\cdot(\omega_\perp)_i|<\frac{\delta}{|n(x_{\mathbf{b}})\cdot(\omega_\perp)_i|}\le 2\sqrt{\delta}$, if $\big|n(x_{\mathbf{b}})\cdot\sum_{i=1}^{2}(\omega_\perp)_i[(u-v)\cdot(\omega_\perp)_i]\big|<\delta$.

Therefore, for $n(x_{\mathbf{b}})\cdot v\sim 0$,

$$\begin{aligned}
&\Gamma_{\text{gain}}(f,f)(t-t_{\mathbf{b}},x_{\mathbf{b}},v)\\
=\;&\iint_{(u,\omega)\in S_\delta}+\iint_{(u,\omega)\in\mathbb{R}^3\times\mathbb{S}^2\backslash S_\delta}\\
=\;&\Gamma_{\text{gain}}(f,f)(t,x,v)+O(\sqrt{\delta})||e^{\eta|v|^2}f_0||_\infty^2+\iint_{(u,\omega)\in\mathbb{R}^3\times\mathbb{S}^2\backslash S_\delta}q_0(\theta)|v-u|^\kappa\sqrt{\mu(u)}\\
&\qquad\qquad\times\big\{f(t-t_{\mathbf{b}},x_{\mathbf{b}},v')f(t-t_{\mathbf{b}},x_{\mathbf{b}},u')-f(t,x,v')f(t,x,u')\big\}\mathrm{d}\omega\mathrm{d}u\\
=\;&\Gamma_{\text{gain}}(f,f)(t,x,v)+O(\sqrt{\delta})||e^{\eta|v|^2}f_0||_\infty^2+O(\delta)\to\Gamma_{\text{gain}}(f,f)(t,x,v).
\end{aligned}$$

This completes the proof of (173).

To complete the proof of (172), recalling $x_{\mathbf{b}}=x-t_{\mathbf{b}}(x,v)v$ we identify the main contribution for $(x,v)\in\gamma_+$ from

$$\begin{aligned}
&n(x)\cdot\left\{\frac{n(x_{\mathbf{b}})}{n(x_{\mathbf{b}})\cdot v}-\frac{n(x)}{n(x)\cdot v}\right\}\\
=\;&\frac{n(x)\cdot\nabla\xi(x-t_{\mathbf{b}}v)\big(\nabla\xi(x)\cdot v\big)-n(x)\cdot\nabla\xi(x)\big(\nabla\xi(x-t_{\mathbf{b}}v)\cdot v\big)}{\big(\nabla\xi(x)\cdot v\big)\big(\nabla\xi(x-t_{\mathbf{b}}v)\cdot v\big)}\\
=\;&\frac{t_{\mathbf{b}}\big\{\big(n(x)\cdot[v\cdot\nabla]\nabla\xi(x-\tilde{\tau}v)\big)\big(\nabla\xi(x)\cdot v\big)-n(x)\cdot\nabla\xi(x)\big(v\cdot\nabla^2\xi(x-\tilde{\tau}v)\cdot v\big)\big\}}{\big(\nabla\xi(x)\cdot v\big)\big(-\nabla\xi(x-t_{\mathbf{b}}v)\cdot v\big)}\qquad(173)\\
=\;&\frac{t_{\mathbf{b}}}{(-\nabla\xi(x_{\mathbf{b}})\cdot v)}\left\{-\frac{\big(v\cdot\nabla^2\xi(x-\tilde{\tau}v)\cdot v\big)}{n(x)\cdot v}+\big(n(x)\cdot[v\cdot\nabla]\nabla\xi(x-\tilde{\tau}v)\big)\right\}\\
:=\;&-\frac{A(x,v)}{n(x)\cdot v}+B(x,v).
\end{aligned}$$

where for some $\tilde{\tau}\in[0,t_{\mathbf{b}}]$. From Velocity lemma(Lemma 1), $A\geq 0$ and from (38) of [8], $t_{\mathbf{b}}\geq C_\xi\frac{-\nabla\xi(x_{\mathbf{b}})\cdot v}{|v|^2}$ so that

$$A(x,v)\geq C_\xi\frac{v}{|v|}\cdot\nabla^2\xi(x-\tilde{\tau}v)\cdot\frac{v}{|v|}\gtrsim_\Omega 1,\quad B(x,v)\sim_\Omega\frac{1}{|v|}.$$

Therefore near the grazing set $n(x)\cdot v\sim 0$

$$\begin{aligned}
&\lim_{n(x)\cdot v\downarrow 0}\{n(x)\cdot v\}\partial_n f(t,x,v)\\
\to\;&-G(t-t_{\mathbf{b}},x_{\mathbf{b}},v)+A(x,v)\times G(t-t_{\mathbf{b}},x_{\mathbf{b}},v)\\
&-\{n(x)\cdot v\}B(x,v)\times G(t-t_{\mathbf{b}},x_{\mathbf{b}},v)\\
&+\{n(x)\cdot v\}\int_0^{t_{\mathbf{b}}}e^{-s\nu}n(x)\cdot\nabla_x\{Kf+\Gamma(f,f)\}(t-s,x-sv,v)\mathrm{d}s\\
\to\;&-G(t,x,v)+A(x,v)\times G(t,x,v)\\
=\;&\lim_{n(x)\cdot v\uparrow 0}\{n(x)\cdot v\}\partial_n f(t,x,v)+A(x,v)G(t,x,v).
\end{aligned}$$

Since $n(x)\cdot v$ has the opposite signs in these limits, we deduce step 1.

Step 2. We deduce from our choice of f_0, $G(\frac{\varepsilon}{N},x,v)\backsim K_2(f)(\frac{\varepsilon}{N},x,v)\neq 0$.

Recall the expression of G in (170) and $Kf=K_1f-K_2f$ from (28). From Theorem2,

$$||e^{-\varpi\langle v\rangle t}\alpha^{1/2}\partial f(\frac{\varepsilon}{N})||_\infty\lesssim C(\frac{\varepsilon}{N})\{||\alpha^{1/2}\partial f_0||_\infty+||e^{\eta|v|^2}f_0||_\infty\}.$$

The first three terms G_1,G_2,G_3 are therefore bounded by,

$$C(\frac{\varepsilon}{N})\langle v\rangle\sqrt{\mu(v)}\big\{||\alpha^{1/2}\partial f_0||_\infty+||e^{\eta|v|^2}f_0||_\infty\big\}\sim e^{-\frac{|v|^2}{5}}\delta(1+\varepsilon^{-1}).$$

The terms $\nu(v)f, K_1 f, \Gamma(f,f)$ are bounded by, from $\sup_t ||e^{\eta|v|^2}f(t)||_\infty \lesssim ||e^{\eta|v|^2}f_0||_\infty$,

$$\Big\{ \langle v\rangle \sqrt{\mu(v)}||e^{\eta|v|^2}f_0||_\infty + \langle v\rangle e^{-\eta|v|^2}||e^{\eta|v|^2}f_0||_\infty^2 \Big\} \ \sim \ e^{-\frac{|v|^2}{5}}\delta + \langle v\rangle e^{-\eta|v|^2}\delta^2.$$

The main contribution is given by $K_2 f$. Choose $0 < C_\Omega \ll 1$, which is independent on $\varepsilon, N>0$ such that

$$\mathrm{dist}\Big(x - \frac{\varepsilon}{N}u, \partial\Omega\Big) \geq 2\varepsilon, \quad \text{for} \ \ |n(x)\cdot u| \leq C_\Omega |u|, \quad u\in[N-1,N+1].$$

Then for $t=\frac{\varepsilon}{N},\ x\in\partial\Omega$,

$$\begin{aligned} K_2 f(\frac{\varepsilon}{N},x,v) \geq & \int_{\left|n(x)\cdot\frac{u}{|u|}\right|\leq C_\Omega, \left||u|-N\right|\leq 2} \mathbf{k}_2(v,u)\Big\{ e^{-\nu(u)t}\delta e^{-\eta|u|^2}\chi_\varepsilon(x-\frac{\varepsilon}{N}u)h_N(u) \\ & \qquad\qquad + \int_0^{\frac{\varepsilon}{N}} e^{-\nu(u)(t-s)}\big[Kf+\Gamma(f,f)\big](s, x-(\frac{\varepsilon}{N}-s)u,u)\mathrm{d}u\Big\} \\ \geq & \ \ \delta\ e^{-\varepsilon} \underbrace{\int_{\frac{u_1}{|u|}\leq C_\Omega,\ \left||u|-N\right|\leq 2} \mathbf{k}_2(v,u)h_N(u)\mathrm{d}u}_{(\star)} - \frac{\varepsilon}{N}\delta(1+\delta), \end{aligned}$$

where we used $\sup_t ||e^{\eta|v|^2}f(t)||_\infty \lesssim ||e^{\eta|v|^2}f_0||_\infty$.

Recall the definition of K_2 and $\mathbf{k}_2$ to rewrite $(\star)$

$$\int_{\frac{u_1}{|u|}\leq C_\Omega,\ \left||u|-N\right|\leq 2}\int_{\mathbb{S}^2} q_0(\theta)|v-u|^\kappa \mu^{\frac12}(u)\big[\mu^{\frac12}(v')e^{-\eta|u'|^2} + \mu^{\frac12}(u')e^{-\eta|v'|^2}\big]\mathrm{d}\omega\mathrm{d}u.$$

Define $u_\parallel := u\cdot\omega,\ u_\perp := u - u_\parallel\omega$. Then by the definitions $u' = u - [(u-v)\cdot\omega]\omega = u_\perp + v_\parallel,\ v' = v + [(u-v)\cdot\omega]\omega = v_\perp + u_\parallel$, for $0<\eta<\frac14$,

$$\begin{aligned} \mu^{\frac12}(v')e^{-\eta|u'|^2} &= e^{-\eta|v|^2}e^{-\eta|u|^2}e^{(-\frac14+\eta)|u'|^2} = e^{-\eta|v|^2}e^{-\eta|u|^2}e^{(-\frac14+\eta)|v_\parallel|^2}e^{(-\frac14+\eta)|u_\perp|^2}, \\ \mu^{\frac12}(u')e^{-\eta|v'|^2} &= e^{-\eta|v|^2}e^{-\eta|u|^2}e^{(-\frac14+\eta)|v'|^2} = e^{-\eta|v|^2}e^{-\eta|u|^2}e^{(-\frac14+\eta)|v_\perp|^2}e^{(-\frac14+\eta)|u_\parallel|^2}. \end{aligned}$$

On $C := \Big\{\omega\in\mathbb{S}^2 : \left|\frac{v}{|v|}\cdot\omega\right| \leq \frac14,\ \left|\frac{u}{|u|} - \big(\frac{u}{|u|}\cdot\omega\big)\omega\right| \leq \frac14\Big\}$, we have $\mu^{\frac12}(v')e^{-\eta|u'|^2} \geq e^{(-\frac{1}{16}-\frac34\eta)(|v|^2+|u|^2)}$.
On $D := \Big\{\omega\in\mathbb{S}^2 : \left|\frac{v}{|v|} - \big(\frac{v}{|v|}\cdot\omega\big)\omega\right| \leq \frac14,\ \left|\frac{u}{|u|}\cdot\omega\right| \leq \frac14\Big\}$ we have $\mu^{\frac12}(u')e^{-\eta|v'|^2} \geq e^{(-\frac{1}{16}-\frac34\eta)(|v|^2+|u|^2)}$.
Therefore $K_2 f(\frac{\varepsilon}{N},x,v)$ is bounded below by

$$\begin{aligned} &\delta\ e^{-\varepsilon}e^{(-\frac{1}{16}-\frac34\eta)|v|^2}\int_{\frac{u_1}{|u|}\leq C_\Omega, \left||u|-N\right|\leq 2}\int_{C\cup D} q_0(\theta)|v-u|^\kappa e^{(-\frac{1}{16}-\frac34\eta)|u|^2}\mathrm{d}\omega\mathrm{d}u \\ &\gtrsim \delta\ e^{-\varepsilon}N^2 e^{(-\frac{1}{16}-\frac34\eta)|v|^2}e^{(-\frac{1}{16}-\frac34\eta)N^2}. \end{aligned}$$

Collecting terms, we finally conclude, at $(x,v)\in\gamma_0$,

$$G(t,x,v)\mathbf{1}_{\{|v|\sim N\}} \gtrsim \delta\Big\{\ e^{-\varepsilon}N^2 e^{-(\frac18+\frac32\eta)N^2} - (1+\varepsilon^{-1})e^{-\frac{N^2}{5}} - \delta N e^{-\eta N^2} - \frac{\varepsilon}{N}(1+\delta)\Big\}.$$

Choose $\eta=\frac18$ and large $N\gg1$ so that $N^2e^{-(\frac18+\frac32\eta)N^2} \gg e^{-\frac{N^2}{5}}$. Then choose small δ such that $N^2e^{-(\frac18+\frac32\eta)N^2} \gg \delta N e^{-\eta N^2}$. Then choose small $\varepsilon>0$ so that $N^2e^{-(\frac18+\frac32\eta)N^2} \gg \frac{\varepsilon}{N}(1+\delta)$.

Combining Steps 1 and 2, we deduce that $\partial_n f$ has a non-integrable singularity $\frac{1}{n\cdot v}$ with no possible cancellation. Hence (169) follows. $\square$

For the bounce-back case we identify the condition inducing non-existence of $\nabla^2 f$ up to the boundary:

Proposition 4 (Bounce-Back BC)**.** *Assume $f_0\in C^1$ and $|\nabla_{x,v}f_0| \lesssim 1$ and $||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty \ll 1$ and for $x\in\partial\Omega$ and $|v|\sim 1$ with $n(x)\cdot v=0$,*

$$v\cdot\nabla_v f_0(y,u) \neq 0, \tag{174}$$

for $(y,u)\sim(x,v)$. Then there exists $t>0$ such that $K(\partial_n f)(x)$ satisfies (169).

Proof. Step 1. We claim (172). On $(x,v)\in\gamma_-$

$$\partial_n f(t,x,v)=\frac{e^{\varpi\langle v\rangle t}}{n(x)\cdot v}e^{-\varpi\langle v\rangle t}\sqrt{\alpha(x,v)}\partial_n f(t,x,v).$$

On $(x',v')\in\gamma_+$, $\partial_n f(t',x',v')$ equals

$$
\begin{aligned}
=&n(x')\cdot\nabla_x f(t',x',v')\\
=&n(x')\cdot\Big\{\nabla_x f(t'-t_{\mathbf{b}}(x',v'),x_{\mathbf{b}}(x',v'),v')e^{-\int_{t'}^t\nu(F)(\tau)\mathrm{d}\tau}\\
&+\int_{t'-t_{\mathbf{b}}(x',v')}^{t'}e^{-\int_s^t\nu(F)(\tau)\mathrm{d}\tau}\big\{K\nabla_x f+\Gamma_{\text{gain}}(\nabla_x f,f)+\Gamma_{\text{gain}}(f,\nabla_x f)-\nu(\nabla_x F)f\big\}(s,x'-(t'-s)v',v')\mathrm{d}s\Big\}.
\end{aligned}
$$

Note that the time integration term is bounded by

$$P(||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty)\int_{t'-t_{\mathbf{b}}(x',v')}^{t'}\int_{\mathbb{R}^3}\frac{e^{-|v'-u|^2}}{|v'-u|^{2-\gamma}}|\nabla_x f(s,x'-(t'-s)v',v')|\mathrm{d}u\mathrm{d}s.$$

If we use Theorem 5 and the non-local to local estimate(Lemma 2) then this term is bounded term in terms of α.

On the other hand the main term can be decomposed as

$$
\begin{aligned}
&n(x')\cdot\nabla_x f(t'-t_{\mathbf{b}}(x',v'),x_{\mathbf{b}}(x',v'),v')\\
=&\frac{-n(x')\cdot n(x_{\mathbf{b}}(x',v'))e^{\varpi\langle v\rangle(t'-t_{\mathbf{b}}(x',v'))}}{n(x_{\mathbf{b}}(x',v'))\cdot v'}e^{-\varpi\langle v'\rangle(t'-t_{\mathbf{b}}(x',v'))}\sqrt{\alpha(x',v')}\partial_n f(t'-t_{\mathbf{b}}(x',v'),x_{\mathbf{b}}(x',v'),v')\\
&+\sum_{i=1}^2 n(x')\cdot\tau(x_{\mathbf{b}}(x',v'))\partial_{\tau_i}f(t'-t_{\mathbf{b}}(x',v'),x_{\mathbf{b}}(x',v'),v').
\end{aligned}
$$

Note that $n(x')\cdot\tau(x_{\mathbf{b}}(x',v'))\to 0$ as $|n(x')\cdot v'|\to 1$. Therefore by (173) we have for $|n(x)\cdot v|\ll 1$

$$[\partial_n f]_+-[\partial_n f]_-\sim\frac{e^{-\varpi\langle v\rangle t}\sqrt{\alpha(x,v)}\partial_n f(t,x,v)}{n(x)\cdot v}.$$

Step 2. We use (169) and apply Theorem 5 and Lemma 14 to have

$$
\begin{aligned}
&\frac{\partial}{\partial x_n}f(t,x,v)\\
&=e^{-\int_0^t\nu(F)(\tau)\mathrm{d}\tau}\Big\{[\partial_n x^{\ell_*(0)}-\partial_n t^{\ell_*(0)}v^{\ell_*(0)}]\cdot\nabla_x f_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))+v^{\ell_*(0)}\cdot\nabla_v f_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))\Big\}\\
&+P(||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty)\frac{t^2|v|^2}{\alpha(x,v)}e^{-\frac{\theta}{4}|v|^2}||\langle v\rangle^\zeta e^{\theta|v|^2}\partial_t f_0||_\infty\\
&+||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty P(||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty)\frac{t(1+t)|v|^2}{\alpha(x,v)}e^{-\frac{\theta}{4}|v|^2}P(||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty)\\
&+P(||\langle v\rangle^\zeta e^{\theta|v|^2}f_0||_\infty)\sup_{0\le s\le t}||e^{-\varpi\langle v\rangle t}\alpha\partial_x f(t)||_\infty\\
&\quad\times\underbrace{\int_0^t\int_{\mathbb{R}^3}\sum_\ell\mathbf{1}_{[t^{\ell+1},t^\ell)}(s)\mathbf{k}_{\kappa,\theta/4}(v^\ell,u)\frac{\partial_n x^\ell}{e^{-\varpi\langle u\rangle s}\alpha(X_{\mathbf{cl}}(s),u)}\mathrm{d}u\mathrm{d}s}_{(\mathbf{BB})}.
\end{aligned}
$$

Here we use Lemma 20 and Lemma 14 and (166), for $\varepsilon\ll 1$

$$
\begin{aligned}
(\mathbf{BB})&\lesssim e^{\varpi\langle v\rangle t}\frac{|v|}{\sqrt{\alpha(x,v)}}\int_0^t\int_{\mathbb{R}^3}e^{-\varpi\langle v\rangle(t-s)}\mathbf{k}_{\kappa,\theta/8}(V_{\mathbf{cl}}(s),u)\frac{1}{\alpha(X_{\mathbf{cl}}(s),u)}\mathrm{d}u\mathrm{d}s\\
&\lesssim e^{\varpi\langle v\rangle t}\frac{|v|}{\sqrt{\alpha(x,v)}}\frac{C_\delta O(\tilde\delta)+C_{\tilde\delta,\delta}O(\varpi^{-1})}{\langle v\rangle[\alpha(x,v)]^{1/2}}\\
&\lesssim\varepsilon e^{\varpi\langle v\rangle t}\frac{|v|}{\langle v\rangle}\frac{1}{\alpha(x,v)}.
\end{aligned}
$$

Now by the explicit computations in Lemma 14

$$
\begin{aligned}
&e^{-\int_0^t \nu(F)(\tau)\mathrm{d}\tau}\Big\{[\partial_n x^{\ell_*(0)} - \partial_n t^{\ell_*(0)} v^{\ell_*(0)}]\cdot\nabla_x f_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0)) + v^{\ell_*(0)}\cdot\nabla_v f_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))\Big\}\\
&\geq e^{-t\langle v\rangle ||\langle v\rangle^\zeta e^{\theta|v|^2} f_0||_\infty}\Big\{\ell_*(0)\frac{n(x)\cdot\nabla\xi(x^1)}{v\cdot\nabla\xi(x^1)} + (\ell_*(0)-1)\frac{n(x)\cdot\nabla\xi(x^2)}{-v\cdot\nabla\xi(x^2)}\Big\}V_{\mathbf{cl}}(0)\cdot\nabla_x f_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))\\
&\quad - C_\xi\frac{|v|}{\sqrt{\alpha(x,v)}}|\nabla_x f_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))| - C_\xi|v||\nabla_v f_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))|\\
&\geq e^{-t\langle v\rangle ||\langle v\rangle^\zeta e^{\theta|v|^2} f_0||_\infty} O_\xi(1)\frac{t|v|^2}{\alpha(x,v)}V_{\mathbf{cl}}(0)\cdot\nabla_x f_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))\\
&\quad - \frac{O_\xi(1+t|v|)}{\sqrt{\alpha(x,v)}}|v||\nabla_x f_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))| - C_\xi|v||\nabla_v f_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))|,
\end{aligned}
$$

where we used (173)

$$
\begin{aligned}
&\ell_*(0)\Big[\frac{n(x)\cdot\nabla\xi(x^1)}{v\cdot\nabla\xi(x^1)} - \frac{n(x)\cdot\nabla\xi(x^2)}{v\cdot\nabla\xi(x^2)}\Big] + \frac{n(x)\cdot\nabla\xi(x^2)}{v\cdot\nabla\xi(x^2)}\\
&= \ell_*(0)\frac{A(x,v)}{n(x^1)\cdot v} + \ell_*(0)B(x,v) + O\Big(\frac{1}{n(x^1)\cdot v}\Big)\\
&= O_\xi(1)\frac{t|v|^2}{\alpha(x,v)} + \frac{O_\xi(1+t|v|)}{\sqrt{\alpha(x,v)}}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{\partial}{\partial x_n}f(t,x,v) &\geq e^{-t\langle v\rangle ||\langle v\rangle^\zeta e^{\theta|v|^2} f_0||_\infty} O_\xi(1)\frac{t|v|^2}{\alpha(x,v)}V_{\mathbf{cl}}(0)\cdot\nabla_x f_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))\\
&\quad - \frac{O_\xi(1+t|v|)}{\sqrt{\alpha(x,v)}}|v||\nabla_x f_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))| - C_\xi|v||\nabla_v f_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))|\\
&\quad - P(||\langle v\rangle^\zeta e^{\theta|v|^2} f_0||_\infty)\frac{t^2|v|^2}{\alpha(x,v)}e^{-\frac{\theta}{4}|v|^2}||\langle v\rangle^\zeta e^{\theta|v|^2}\partial_t f_0||_\infty\\
&\quad - ||\langle v\rangle^\zeta e^{\theta|v|^2} f_0||_\infty P(||\langle v\rangle^\zeta e^{\theta|v|^2} f_0||_\infty)\frac{t(1+t)|v|^2}{\alpha(x,v)}e^{-\frac{\theta}{4}|v|^2}P(||\langle v\rangle^\zeta e^{\theta|v|^2} f_0||_\infty)\\
&\quad - P(||\langle v\rangle^\zeta e^{\theta|v|^2} f_0||_\infty)\sup_{0\leq s\leq t}||e^{-\varpi\langle v\rangle t}\alpha\partial_x f(t)||_\infty \times \varepsilon e^{\varpi\langle v\rangle t}\frac{|v|}{\langle v\rangle}\frac{1}{\alpha(x,v)}.
\end{aligned}
$$

Now we choose $||\langle v\rangle^\zeta e^{\theta|v|^2} f_0||_\infty \ll 1$ and $0 < t \ll 1$ and choose small $\varepsilon > 0$ and then finally we choose small $\frac{\sqrt{\alpha(x,v)}}{|v|} \ll 1$ to have lower bound

$$
\frac{\partial}{\partial x_n}f(t,x,v) \geq e^{-C_1}\frac{t|v|^2}{\alpha(x,v)}V_{\mathbf{cl}}(0)\cdot\nabla_x f_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0)) \geq C_2\frac{|v|^2}{\alpha(x,v)}.
$$

Combining Step 1 and Step 2 we deduce that $\partial_n f$ has a non-integrable singularity $\frac{1}{n\cdot v}$ with no possible cancellation. Hence (169) follows. □

In order to show the non-existence of $\nabla^2 f$ up to the boundary for the specular reflection case(Proposition 5) we find the explicit lower bound of (24) with a simple domain, 2D disk.

Example 1. *Let* $\Omega = \{\bar{x} = (x_1,x_2)\in\mathbb{R}^2 : |x_1|^2+|x_2|^2 < 1\}$. *Define*

$$
\begin{aligned}
&r := \sqrt{x_1^2+x_2^2}\in[0,1], \quad \theta\in[0,2\pi) \quad \textit{such that} \quad (\cos\theta,\sin\theta) = \frac{1}{\sqrt{x_1^2+x_2^2}}(x_1,x_2),\\
&v_r := v_1\cos\theta + v_2\sin\theta, \quad v_\theta := -v_1\sin\theta + v_2\cos\theta.
\end{aligned}
$$

We claim that as $\alpha \to 0$(therefore $r \sim 1, v_r \sim 0$) asymptotically

$$\begin{aligned} |\partial_r \bar{X}_{\mathbf{cl}}(s;t,x,v)\cdot n^{\perp}(\bar{X}_{\mathbf{cl}}(s;t,x,v))| &\sim \frac{|t-s||v_\theta|^2}{\sqrt{v_r^2+(1-r^2)v_\theta^2}} \sim \frac{|t-s||\bar{v}|^2}{\sqrt{\alpha(x,v)}},\\ |\partial_r \bar{V}_{\mathbf{cl}}(s;t,x,v)\cdot n(\bar{X}_{\mathbf{cl}}(s;t,x,v))| &\sim \frac{|t-s||\bar{v}|^4}{v_r^2+(1-r^2)v_\theta^2} \sim \frac{|t-s||\bar{v}|^4}{\alpha(x,v)}, \end{aligned} \tag{175}$$

where $n^{\perp} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} n$.

Explicitly $x = (r\cos\theta, r\sin\theta, x_3)$, *and* $v = (v_r\cos\theta - v_\theta\sin\theta, v_r\sin\theta + v_\theta\cos\theta, v_3)$, *and*

$$\begin{aligned} x^\ell &= (\cos\theta^\ell, \sin\theta^\ell, x_3-(t-t^\ell)v_3),\\ v^\ell &= (\sqrt{v_r^2+v_\theta^2}\cos\psi^\ell, \sqrt{v_r^2+v_\theta^2}\sin\psi^\ell, v_3),\\ t^1 &= t - \frac{r|v_r| + \sqrt{(1-r^2)v_\theta^2+v_r^2}}{v_r^2+v_\theta^2},\\ t^\ell &= t - \frac{r|v_r| + (2\ell-1)\sqrt{(1-r^2)v_\theta^2+v_r^2}}{v_r^2+v_\theta^2}, \end{aligned}$$

and

$$\ell_*(s;t,x,v) \le \frac{(t-s)|\bar{v}|^2}{2\sqrt{(1-r^2)v_\theta^2+v_r^2}} - \frac{r|v_r|}{2\sqrt{(1-r^2)v_\theta^2+v_r^2}} + \frac{1}{2} < \ell_*(s;t,x,v)+1,$$

where, for $\ell \ge 1$

$$\begin{aligned} \theta^0 &= \theta, \quad \theta^\ell = \theta - \cos^{-1}\Big(\frac{v_\theta}{\sqrt{v_\theta^2+v_r^2}}\Big) - (2\ell-1)\cos^{-1}\Big(\frac{rv_\theta}{\sqrt{v_r^2+v_\theta^2}}\Big),\\ \psi^0 &= \cos^{-1}\Big(\frac{v_r\cos\theta - v_\theta\sin\theta}{\sqrt{v_r^2+v_\theta^2}}\Big), \quad \psi^\ell = \psi^0 - 2\ell\cos^{-1}\Big(\frac{rv_\theta}{\sqrt{v_r^2+v_\theta^2}}\Big). \end{aligned}$$

Therefore, if $t^{\ell+1} < s < t^\ell$,

$$X_{\mathbf{cl}}(s) = x^\ell - (t^\ell - s)v^\ell, \quad V_{\mathbf{cl}}(s) = v^\ell,$$

and

$$\begin{aligned} r(s) &= |\bar{X}_{\mathbf{cl}}(s)| = |\bar{x}^\ell - (t^\ell-s)\bar{v}^\ell|,\\ v_r(s) &= \bar{V}_{\mathbf{cl}}(s)\cdot\frac{\bar{X}_{\mathbf{cl}}(s)}{|\bar{X}_{\mathbf{cl}}(s)|}, \quad v_\theta(s) = \bar{V}_{\mathbf{cl}}(s)\cdot\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{\bar{X}_{\mathbf{cl}}(s)}{|\bar{X}_{\mathbf{cl}}(s)|},\\ v_3(s) &= v_3. \end{aligned}$$

Directly

$$\begin{aligned} &\partial_\theta v_r = v_\theta, \quad \partial_\theta v_\theta = -v_r,\\ &\partial_r \cos^{-1}\Big(\frac{rv_\theta}{\sqrt{v_r^2+v_\theta^2}}\Big) = \frac{-v_\theta}{\sqrt{v_r^2+(1-r^2)v_\theta^2}},\\ &\partial_\theta \cos^{-1}\Big(\frac{v_\theta}{\sqrt{v_r^2+v_\theta^2}}\Big) = 1, \quad \partial_\theta \cos^{-1}\Big(\frac{rv_\theta}{\sqrt{v_r^2+v_\theta^2}}\Big) = \frac{rv_r}{\sqrt{v_r^2+(1-r^2)v_\theta^2}},\\ &\partial_{v_r} \cos^{-1}\Big(\frac{v_\theta}{\sqrt{v_r^2+v_\theta^2}}\Big) = \frac{v_\theta}{v_r^2+v_\theta^2}, \quad \partial_{v_r} \cos^{-1}\Big(\frac{rv_\theta}{\sqrt{v_r^2+v_\theta^2}}\Big) = \frac{rv_\theta}{v_r^2+v_\theta^2},\\ &\partial_{v_\theta} \cos^{-1}\Big(\frac{v_\theta}{\sqrt{v_r^2+v_\theta^2}}\Big) = \frac{-v_r}{v_r^2+v_\theta^2}, \quad \partial_{v_\theta} \cos^{-1}\Big(\frac{rv_\theta}{\sqrt{v_r^2+v_\theta^2}}\Big) = \frac{-rv_r}{v_r^2+v_\theta^2},\\ &\partial_{v_r} \cos^{-1}\Big(\frac{v_r\cos\theta - v_\theta\sin\theta}{\sqrt{v_r^2+v_\theta^2}}\Big) = \frac{v_\theta}{v_r^2+v_\theta^2}, \quad \partial_{v_\theta} \cos^{-1}\Big(\frac{v_r\cos\theta - v_\theta\sin\theta}{\sqrt{v_r^2+v_\theta^2}}\Big) = \frac{v_r}{v_r^2+v_\theta^2},\\ &\partial_\theta \cos^{-1}\Big(\frac{v_r\cos\theta - v_\theta\sin\theta}{\sqrt{v_r^2+v_\theta^2}}\Big) = 0 = \partial_r \cos^{-1}\Big(\frac{v_r\cos\theta - v_\theta\sin\theta}{\sqrt{v_r^2+v_\theta^2}}\Big), \end{aligned}$$

and

$$\partial_{v_\theta}\theta^\ell = \frac{|v_r|}{|\bar{v}|^2} + |t-s|, \quad \partial_{v_r}\theta^\ell = -\frac{v_\theta}{|\bar{v}|^2} - (2\ell-1)\frac{rv_\theta}{|\bar{v}|^2},$$
$$\partial_\theta\theta^\ell \lesssim \frac{|t-s||\bar{v}|^2|v_r|}{v_r^2+(1-r^2)v_\theta^2}, \ \partial_r\theta^\ell = \frac{(2\ell-1)v_\theta}{\sqrt{v_r^2+(1-r^2)v_\theta^2}},$$

and

$$\partial_\theta\psi^\ell \lesssim \frac{|t-s||\bar{v}|^2|v_r|}{v_r^2+(1-r^2)v_\theta^2}, \quad \partial_r\psi^\ell = \frac{2\ell v_\theta}{\sqrt{v_r^2+(1-r^2)v_\theta^2}},$$
$$\partial_{v_\theta}\psi^\ell \lesssim \frac{|t-s||v_r|}{\sqrt{v_r^2+(1-r^2)v_\theta^2}} \lesssim |t-s|, \quad \partial_{v_r}\psi^\ell = -2\ell\frac{rv_\theta}{v_r^2+v_\theta^2} + O_\xi(1)\frac{1}{|\bar{v}|},$$

and

$$t^\ell - t^{\ell+1} \leq \frac{2\sqrt{(1-r^2)v_\theta^2+v_r^2}}{v_r^2+v_\theta^2},$$
$$\ell_*(s) \leq \frac{|t-s||\bar{v}|^2}{2\sqrt{v_r^2+(1-r^2)v_\theta^2}} - \frac{r|v_r|}{2\sqrt{v_r^2+(1-r^2)v_\theta^2}} + \frac{1}{2} \leq \ell_*(s)+1,$$

and

$$\partial_r t^\ell = \frac{-|v_r|}{v_r^2+v_\theta^2} + (2\ell-1)\frac{rv_\theta^2}{|\bar{v}|^2\sqrt{v_r^2+(1-r^2)v_\theta^2}},$$
$$\partial_\theta t^\ell = \frac{-(2\ell-1)v_rv_\theta r^2}{|\bar{v}|^2\sqrt{v_r^2+(1-r^2)v_\theta^2}} \lesssim \frac{|t-s||v_\theta|r^2}{\sqrt{v_r^2+(1-r^2)v_\theta^2}},$$

$$\partial_{v_r} t^\ell = -(2\ell-1)\frac{v_r}{|\bar{v}|^2\sqrt{v_r^2+(1-r^2)v_\theta^2}} + O_\xi(1)\frac{1+|v||t-s|}{|\bar{v}|^2},$$
$$\partial_{v_\theta} t^\ell \leq (2\ell-1)\frac{(1-r^2)|v_\theta|}{|\bar{v}|^2\sqrt{v_r^2+(1-r^2)v_\theta^2}} + 2(2\ell-1)\frac{|v_\theta|\sqrt{v_r^2+(1-r^2)v_\theta^2}}{|\bar{v}|^4}$$
$$\lesssim |t-s|\frac{(1-r^2)|v_\theta|}{v_r^2+(1-r^2)v_\theta^2} + |t-s|\frac{|v_\theta|}{|\bar{v}|^2}.$$

If $r < \frac{1}{2}$ *then* $(1-r^2)v_\theta^2 + v_r^2 \geq \frac{3}{4}|\bar{v}|^2$ *and* $\partial_{v_\theta}t^\ell \lesssim |t-s|\frac{4|v_\theta|}{3|\bar{v}|^2} + |t-s|\frac{|v_\theta|}{|\bar{v}|^2} \lesssim |t-s|\frac{|v_\theta|}{|\bar{v}|^2}$. *If* $r \geq \frac{1}{2}$ *and* $|v_\theta| \leq |v_r|$ *then* $\partial_{v_\theta}t^\ell \lesssim \frac{|t-s||v_\theta|}{\frac{v_r^2}{2}+\frac{v_r^2}{2}} + |t-s|\frac{|v_\theta|}{|\bar{v}|^2} \lesssim |t-s|\frac{|v_\theta|}{|\bar{v}|^2}$. *If* $r \geq \frac{1}{2}$ *and* $|v_\theta| \geq |v_r|$ *then* $\partial_{v_\theta}t^\ell \lesssim |t-s|\frac{(1-r^2)|v_\theta|}{(1-r^2)|v_\theta|(\frac{|v_\theta|}{2}+\frac{|v_\theta|}{2})} + |t-s|\frac{|v_\theta|}{|\bar{v}|^2} \lesssim \frac{|t-s|}{|\bar{v}|}$.

Therefore

$$\partial_{v_\theta}t^\ell \lesssim \frac{|t-s|}{|\bar{v}|}.$$

Directly

$$\partial_r \bar{X}_{\mathbf{cl}}(s) = \partial_r\theta^\ell \begin{pmatrix} -\sin\theta^\ell \\ \cos\theta^\ell \end{pmatrix} - \frac{\partial t^\ell}{\partial r}|\bar{v}| \begin{pmatrix} \cos\psi^\ell \\ \sin\psi^\ell \end{pmatrix} - (t^\ell - s)|\bar{v}|\frac{\partial\psi^\ell}{\partial r}\begin{pmatrix} -\sin\psi^\ell \\ \cos\psi^\ell \end{pmatrix}$$
$$= \frac{(2\ell-1)v_\theta^2}{|\bar{v}|\sqrt{v_r^2+(1-r^2)v_\theta^2}}\begin{pmatrix} -\sin\theta^\ell - \cos\psi^\ell \\ \cos\theta^\ell - \sin\psi^\ell \end{pmatrix}$$
$$+ O_\xi(1)\Big\{\frac{(2\ell-1)|v_\theta||v_r|}{|\bar{v}|\sqrt{v_r^2+(1-r^2)v_\theta^2}} + \frac{(2\ell-1)(1-r)|v_\theta|^2}{|\bar{v}|\sqrt{v_r^2+(1-r^2)v_\theta^2}} + \frac{|v_r|}{|\bar{v}|} + \ell|t^\ell - t^{\ell+1}|\Big\},$$

where $O_\xi(1)$ remainder is bounded by

$$
\begin{aligned}
&\lesssim \frac{|t-s||\bar{v}|^2}{v_r^2+(1-r^2)v_\theta^2}\Big[\frac{|v_\theta||v_r|}{|\bar{v}|}+\frac{|1-r||v_\theta^2|}{|\bar{v}|}+\frac{\sqrt{v_r^2+(1-r^2)v_\theta^2}}{|\bar{v}|}\Big]+\frac{|v_r|}{|\bar{v}|}\\
&\lesssim \frac{|t-s||\bar{v}|(1+|\bar{v}|)}{\sqrt{v_r^2+(1-r^2)v_\theta^2}}+|t-s||\bar{v}|+\frac{|v_r|}{|\bar{v}|}.
\end{aligned}
$$

Now we use some trigonometric identities to have

$$
\begin{aligned}
&\sin\theta^\ell+\cos\psi^\ell\\
=&\sin\theta\cos\big(\cos^{-1}(\frac{v_\theta}{|\bar{v}|})+(2\ell-1)\cos^{-1}(\frac{rv_\theta}{|\bar{v}|})\big)-\cos\theta\sin\big(\cos^{-1}(\frac{v_\theta}{|\bar{v}|})+(2\ell-1)\cos^{-1}(\frac{rv_\theta}{|\bar{v}|})\big)\\
&+\frac{v_r\cos\theta-v_\theta\sin\theta}{|\bar{v}|}\cos(2\ell\cos^{-1}(\frac{rv_\theta}{|\bar{v}|}))+\sin\big(\cos^{-1}(\frac{v_r\cos\theta-v_\theta\sin\theta}{|\bar{v}|})\big)\sin\big(-2\ell\cos^{-1}(\frac{rv_\theta}{|\bar{v}|})\big).
\end{aligned}
$$

Here

$$
\begin{aligned}
&\cos\Big(\cos^{-1}(\frac{v_\theta}{|\bar{v}|})-\cos^{-1}(\frac{rv_\theta}{|\bar{v}|})+2\ell\cos^{-1}(\frac{rv_\theta}{|\bar{v}|})\Big)\\
=&\cos\Big(\cos^{-1}(\frac{v_\theta}{|\bar{v}|})-\cos^{-1}(\frac{rv_\theta}{|\bar{v}|})\Big)\cos\Big(2\ell\cos^{-1}(\frac{rv_\theta}{|\bar{v}|})\Big)-\sin\Big(\cos^{-1}(\frac{v_\theta}{|\bar{v}|})-\cos^{-1}(\frac{rv_\theta}{|\bar{v}|})\Big)\sin\Big(2\ell\cos^{-1}(\frac{rv_\theta}{|\bar{v}|})\Big)\\
=&\cos\Big(2\ell\cos^{-1}(\frac{rv_\theta}{|\bar{v}|})\Big)+O_\xi(1)\Big|1-\cos\Big(\cos^{-1}(\frac{v_\theta}{|\bar{v}|})-\cos^{-1}(\frac{rv_\theta}{|\bar{v}|})\Big)\Big|\\
&+O_\xi(1)\Big|\sin\Big(\cos^{-1}(\frac{v_\theta}{|\bar{v}|})-\cos^{-1}(\frac{rv_\theta}{|\bar{v}|})\Big)\sin\Big(2\ell\cos^{-1}(\frac{rv_\theta}{|\bar{v}|})\Big)\Big|\\
=&\cos\Big(2\ell\cos^{-1}(\frac{rv_\theta}{|\bar{v}|})\Big)+O_\xi(1)\Big\{\Big|1-\frac{rv_\theta^2}{|\bar{v}|^2}-\frac{v_r\sqrt{v_r^2+(1-r^2)v_\theta^2}}{|\bar{v}|^2}\Big|+\Big|\frac{rv_rv_\theta}{|\bar{v}|^2}-\frac{v_\theta\sqrt{v_r^2+(1-r^2)v_\theta^2}}{|\bar{v}|^2}\Big|\Big\}\\
=&\cos\Big(2\ell\cos^{-1}(\frac{rv_\theta}{|\bar{v}|})\Big)+O_\xi(1)\big\{\frac{(1-r)v_\theta^2}{|\bar{v}|^2}+\frac{\sqrt{v_r^2+(1-r^2)v_\theta^2}}{|\bar{v}|}\big\},
\end{aligned}
$$

and

$$
\begin{aligned}
&\sin\Big(\cos^{-1}(\frac{v_\theta}{|\bar{v}|})-\cos^{-1}(\frac{rv_\theta}{|\bar{v}|})+2\ell\cos^{-1}(\frac{rv_\theta}{|\bar{v}|})\Big)\\
=&\sin\Big(\cos^{-1}(\frac{v_\theta}{|\bar{v}|})-\cos^{-1}(\frac{rv_\theta}{|\bar{v}|})\Big)\cos\Big(2\ell\cos^{-1}(\frac{rv_\theta}{|\bar{v}|})\Big)\Big)+\cos\Big(\cos^{-1}(\frac{v_\theta}{|\bar{v}|})-\cos^{-1}(\frac{rv_\theta}{|\bar{v}|})\Big)\sin\Big(2\ell\cos^{-1}(\frac{rv_\theta}{|\bar{v}|})\Big)\Big)\\
=&\sin\Big(2\ell\cos^{-1}(\frac{rv_\theta}{|\bar{v}|})\Big)+O_\xi(1)\big\{\frac{(1-r)v_\theta^2}{|\bar{v}|^2}+\frac{\sqrt{v_r^2+(1-r^2)v_\theta^2}}{|\bar{v}|}\big\}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\sin\theta^\ell+\cos\psi^\ell=&(1-\frac{|v_\theta|}{|\bar{v}|})\sin\theta\cos\Big((2\ell-1)\cos^{-1}(\frac{rv_\theta}{|\bar{v}|})\Big)-(1-\frac{|v_\theta|}{|\bar{v}|})\cos\theta\sin\Big(2\ell\cos^{-1}(\frac{rv_\theta}{|\bar{v}|})\Big)\\
&+O_\xi(1)\big\{\frac{(1-r)v_\theta^2}{|\bar{v}|^2}+\frac{\sqrt{v_r^2+(1-r^2)v_\theta^2}}{|\bar{v}|}\big\}\\
\sim&\frac{\sqrt{v_r^2+(1-r^2)v_\theta^2}}{|\bar{v}|}.
\end{aligned}
$$

Since $\cos\theta^\ell-\sin\psi^\ell=\sin(\theta^\ell+\frac{\pi}{2})+\cos(\psi^\ell+\frac{\pi}{2})$,

$$
\cos\theta^\ell-\sin\psi^\ell\sim\frac{\sqrt{v_r^2+(1-r^2)v_\theta^2}}{|\bar{v}|}.
$$

Therefore we conclude our claim for $\partial_r\bar{X}_{\mathbf{cl}}$.

Using the same estimates

$$\partial_{v_r}\bar{X}_{\mathbf{cl}}(s) = (2\ell-1)\Big\{\frac{-rv_\theta}{|\bar{v}|^2}\begin{pmatrix} -\sin\theta^\ell \\ \cos\theta^\ell\end{pmatrix} + \frac{v_r}{|\bar{v}|\sqrt{v_r^2+(1-r^2)v_\theta^2}}\begin{pmatrix}\cos\psi^\ell \\ \sin\psi^\ell\end{pmatrix}\Big\} + O_\xi(1)\frac{1+|\bar{v}||t-s|}{|\bar{v}|}$$
$$= \frac{2\ell-1}{|\bar{v}|}\begin{pmatrix}\sin\theta^\ell+\cos\psi^\ell \\ -\cos\theta^\ell+\sin\psi^\ell\end{pmatrix} + O_\xi(1)\frac{1+|\bar{v}||t-s|}{|\bar{v}|} \lesssim \frac{1}{|\bar{v}|}.$$

Let $\ell_*+1<0<\ell_*$ *then*

$$\partial_r v^\ell = \partial_r\psi^\ell(-|\bar{v}|\sin\psi^\ell, |\bar{v}|\cos\psi^\ell, 0) = \frac{|t-s||\bar{v}|^2|v_\theta|}{v_r^2+(1-r^2)v_\theta^2}(-|\bar{v}|\sin\psi^\ell, |\bar{v}|\cos\psi^\ell, 0).$$

Therefore we conclude our claim for $\partial_r\bar{V}_{\mathbf{cl}}$. *Moreover*

$$|\partial_\theta\bar{X}_{\mathbf{cl}}(s)| \lesssim \frac{|\bar{v}|}{\sqrt{v_r^2+(1-r^2)v_\theta^2}}|\bar{v}||t-s|, \quad |\partial_\theta\bar{V}_{\mathbf{cl}}(s)| \lesssim \frac{|\bar{v}|^2}{\sqrt{v_r^2+(1-r^2)v_\theta^2}}|\bar{v}||t-s|,$$
$$|\partial_{v_r}\bar{V}_{\mathbf{cl}}(s)| \lesssim 1+\frac{|\bar{v}|}{\sqrt{v_r^2+(1-r^2)v_\theta^2}}|\bar{v}||t-s|, \quad |\partial_{v_\theta}\bar{X}_{\mathbf{cl}}(s)| \lesssim \frac{1}{|\bar{v}|}, \quad |\partial_{v_\theta}\bar{V}_{\mathbf{cl}}(s)| \lesssim 1+|\bar{v}||t-s|. \tag{176}$$

Based on Example 1, we naturally consider the 2D specular problem. We consider the 2D specular problem for $f(t,x_1,x_2,v_1,v_2,v_3)$ solving

$$\partial_t f + v_1\partial_{x_1}f + v_2\partial_{x_2}f + \nu(F)f = \Gamma_{\text{gain}}(f,f), \tag{177}$$

where v_3 is a paramter. Here $(x_1,x_2)\in\Omega=\{x\in\mathbb{R}^2:\xi(x)>0\}$ and the convexity (2) is valid for all $\zeta\in\mathbb{R}^2$. We study (177) with specular boundary condition (3). Denote $v:=(\bar{v},v_3)=(v_1,v_2;v_3)\in\mathbb{R}^3$. We define

$$\alpha(x,\bar{v}) = |\bar{v}\cdot\nabla\xi(x)|^2 - 2\{\bar{v}\cdot\nabla^2\xi(x)\cdot\bar{v}\}\xi(x).$$

Note that $\nabla\xi(x)=(\partial_{x_1}\xi(x),\partial_{x_2}\xi(x),0)$.

The following estimate is crucial to establish the weighted C^1-estimate(Theorem 6) and non-existence of $\nabla^2 f$ up to the boundary(Proposition 5).

Lemma 15. *For* $\theta>0$ *and for* $i=1,2$,

$$e^{-\varpi\langle v\rangle s}|\partial_{v_i}\Gamma_{\text{gain}}(f,f)| \lesssim ||e^{\theta|v|^2}f||_\infty\Big\{||e^{\theta|v|^2}f||_\infty + ||\partial_{v_3}f||_\infty + \Big|\Big|e^{-\varpi\langle v\rangle s}\frac{|\bar{v}|}{\langle\bar{v}\rangle}\alpha^{1/2}\nabla_{\bar{v}}f\Big|\Big|_\infty\Big\}. \tag{178}$$

Proof. The key is to use my old idea of splitting $u_{||,3}$ with respect to $|\bar{v}+\bar{u}_\perp|\sqrt{\alpha(\bar{v}+\bar{u}_\perp)}$. From [7] or (29) we note

$$\begin{aligned} &\partial_{v_i}\Gamma_{\text{gain}}(f,f) \\ =\ & 2\Gamma_{\text{gain}}(\partial_{v_i}f,f) \\ &+C\int_{\mathbb{R}^3}\mathrm{d}u\int_{u\cdot w=0}\mathrm{d}w f(v+w)f(v+u)q_0^*(\frac{|u|}{|u+w|})\frac{|u+w|^{\kappa-1}}{|u|}e^{-\frac{|u+v+w|^2}{4}}(-\frac{\mathbf{e}_i}{2})\cdot(u+v+w) \\ =\ & 2\Gamma_{\text{gain}}(\partial_{v_i}f,f) + O_\xi(1)e^{-C|v|^2}||\langle v\rangle^\zeta e^{\theta|v|^2}f||_\infty^2. \end{aligned}$$

Denote the standard cutoff function $\chi\geq 0$: $\chi\equiv 1$ on $[0,1]$ and $\chi\equiv 0$ for $[2,\infty)$. We have

$$\Gamma_{\text{gain}}(\partial_{v_i}f,f) = \int_{\mathbb{R}^3}\mathrm{d}u_{||}f(v+u_{||})\int_{\mathbb{R}^2}\mathrm{d}u_\perp\partial_{v_i}f(v+u_\perp)e^{-\frac{|u_\parallel+u_\perp+v|^2}{4}}q_0^*(\frac{|u_\parallel|}{|u_\parallel+u_\perp|})\frac{|u_\parallel+u_\perp|^{\kappa-1}}{|u_{||}|}.$$

We further split it into

$$\int_{\mathbb{R}^3} \mathrm{d}u_{||} f(v+u_{||}) \int_{\mathbb{R}^2} \chi\left(\frac{|\bar{v}+\bar{u}_\perp|^{1-}\alpha^{1/2-}}{u_{||,3}}\right) \partial_{v_i} f(v+u_\perp) e^{-\frac{|u_\| +v+u_\perp|^2}{4}} q_0^*(\frac{|u_\||}{|u_\| + u_\perp|}) \frac{|u_\| + u_\perp|^{\kappa-1}}{|u_{||}|} \mathrm{d}u_\perp$$

$$+ \int_{\mathbb{R}^3} du_{||} f(v+u_{||}) \int_{\mathbb{R}^2} \left\{1 - \chi\left(\frac{|\bar{v}+\bar{u}_\perp|^{1-}\alpha^{1/2-}}{u_{||,3}}\right)\right\}$$

$$\times \partial_{v_i} f(v+u_\perp) e^{-\frac{|u_\| +v+u_\perp|^2}{4}} q_0^*(\frac{|u_\||}{|u_\| + u_\perp|}) \frac{|u_\| + u_\perp|^{\kappa-1}}{|u_{||}|} \mathrm{d}u_\perp.$$

For the first part, $|u_{||,3}| \geq |\bar{v}+\bar{u}_\perp|^{1-}\alpha^{1/2-}$, and we parametrize $u_\perp$ as $u_{\perp,3} = -\frac{\bar{u}_{||}\cdot\bar{u}_\perp}{u_{||,3}}$ so that

$$\mathrm{d}u_\perp = \frac{|u_{||}|}{|u_{||,3}|} \mathrm{d}\bar{u}_\perp,$$

and the first part equals

$$\int_{\mathbb{R}^3} \mathrm{d}u_{||} f(v+u_{||}) \int_{\mathbb{R}^2} \mathrm{d}\bar{u}_\perp \partial_{v_i} f(v_1 + u_{\perp,1}, v_2 + u_{\perp,2}, v_3 - \tfrac{\bar{u}_{||}\cdot\bar{u}_\perp}{u_{||,3}})$$
$$\times \chi\left(\tfrac{|\bar{v}+\bar{u}_\perp|^{1-}\alpha^{1/2-}}{u_{||,3}}\right) e^{-\frac{|u_\| +v+u_\perp|^2}{4}} \frac{q_0^*(\frac{|u_\||}{|u_\| + u_\perp|})}{|u_{||,3}||u_\| + u_\perp|^{1-\kappa}}$$

We now integrate by part in $u_{\perp,i}$ for $i = 1,2$ to get

$$- \int_{\mathbb{R}^3} \mathrm{d}u_{||} f(v+u_{||}) \int_{\mathbb{R}^2} \partial_{v_3} f(\bar{v}+\bar{u}_\perp, v_3 - \frac{\bar{u}_{||}\cdot\bar{u}_\perp}{u_{||,3}}) \chi\left(\frac{|\bar{v}+\bar{u}_\perp|^{1-}\alpha^{1/2-}}{u_{||,3}}\right) e^{-\frac{|u_\| +v+u_\perp|^2}{4}} \frac{q_0^*(\frac{|u_\||}{|u_\| + u_\perp|}) u_{||,i} \mathrm{d}\bar{u}_\perp}{|u_{||,3}|^2 |u_{||} + u_\perp|^{1-\kappa}}$$

$$- \int_{\mathbb{R}^3} \mathrm{d}u_{||} f(v+u_{||}) \int_{\mathbb{R}^2} f(\bar{v}+\bar{u}_\perp, v_3 - \frac{\bar{u}_{||}\cdot\bar{u}_\perp}{u_{||,3}})$$

$$\times \partial_{u_\perp,i} \left\{ \chi\left(\frac{|\bar{v}+\bar{u}_\perp|^{1-}\alpha^{1/2-}}{u_{||,3}}\right) e^{-\frac{|u_\| +v+u_\perp|^2}{4}} \frac{q_0^*(\frac{|u_\||}{|u_\| + u_\perp|}) \bar{u}_{||,i}}{|u_{||,3}||u_{||} + u_\perp|^{1-\kappa}} \right\} \mathrm{d}\bar{u}_\perp.$$

Directly we have $|\partial_{u_{\perp,i}} \alpha(\bar{v}+\bar{u}_\perp)| \lesssim \alpha(\bar{v}+\bar{u}_\perp)^{1/2}$ and $|\frac{du_{\perp,3}}{du_{\perp,i}}| \leq \frac{|\bar{u}_\||}{|u_{\|,3}|}$ to conclude

$$\begin{aligned}
|\partial_{u_\perp,i}\{\ \ \}| \sim\ & \chi' \frac{|\bar{v}+\bar{u}_\perp|^{0-}\alpha^{1/2-} + |\bar{v}+\bar{u}_\perp|^{1-}\alpha^{0-}}{u_{\|,3}} e^{-\frac{|u_\| +v+u_\perp|^2}{4}} \frac{||q_0^*||_\infty |\bar{u}_{||}|}{|u_{||,3}||u_{||} + u_\perp|^{1-\kappa}} \\
& + \chi e^{-C|u_\| + u_\perp + v|^2} \Big\{ \frac{||q_0^*||_\infty |\bar{u}_\||^2}{|u_{\|,3}|^2 |u_\| + u_\perp|^{1-\kappa}} + \frac{||q_0^*||_{C^1} |\bar{u}_\||^2}{|u_{\|,3}|^2 |u_\| + u_\perp|^{3-\kappa}} + \frac{||q_0^*||_\infty |\bar{u}_\||(1 + \frac{|\bar{u}_\||}{|u_{\|,3}|})}{|u_{\|,3}||u_\| + u_\perp|^{2-\kappa}} \Big\} \\
\lesssim_{q_0^*}\ & \mathbf{1}_{\{u_{\|,3} \sim |\bar{v}+\bar{u}_\perp|^{1-}\alpha^{1/2-}\}} \Big\{ \frac{1}{|\bar{v}+\bar{u}_\perp|} + \frac{|\bar{v}+\bar{u}_\perp|^{1-}\alpha^{0-}}{|u_{\|,3}|} \Big\} \frac{e^{-\frac{|u_\| +v+u_\perp|^2}{4}} |\bar{u}_{||}|}{|u_{||,3}||u_{||} + u_\perp|^{1-\kappa}} \\
& + \mathbf{1}_{\{|\bar{v}+\bar{u}_\perp|^{1-}\alpha^{1/2-} \leq u_{\|,3}\}} \frac{|\bar{u}_\||(1 + |\bar{u}_\||)}{|u_{\|,3}|^2 |u_\| + u_\perp|^{1-\kappa}} (1 + \frac{1}{|u_\| + u_\perp|^2}) e^{-C|u_\| + v + u_\perp|^2}.
\end{aligned}$$

Note that $|f(v+u_\|)| \lesssim e^{-C|v+u_\||^2} ||e^{\theta|v|^2} f||_\infty$ and

$$|f(\bar{v}+\bar{u}_\perp, v_3 - \frac{\bar{u}_\| \cdot \bar{u}_\perp}{u_{\|,3}})| \lesssim e^{-C|\bar{v}+\bar{u}_\perp|^2 - C|v_3 + u_{\perp,3}(u_\|,\bar{u}_\perp)|^2} ||e^{\theta|v|^2} f||_\infty,$$

and

$$e^{-|u_\| + u_\perp + v|^2} e^{-C|v+u_\||^2} e^{-C|\bar{v}+\bar{u}_\perp|^2 - C|v_3 + u_{\perp,3}(u_\|,\bar{u}_\perp)|^2} \lesssim e^{-C'|v|^2} e^{-C'|u_\perp|^2} e^{-C|v+u_\||^2},$$

where $v := v_\| + v_\perp$ with $v_\| := v \cdot \frac{u_\|}{|u_\||}$ and

$$|v+u_\||^2 + |v+u_\perp|^2 = |v_\| + u_\||^2 + |v_\perp|^2 + |v_\perp + u_\perp|^2 + |v_\||^2 \geq |v|^2.$$

The $\partial_{u_{\perp,i}}\{\ \}-$contribution are bounded by following three estimates: For the first term

$$e^{-|v|^2}||e^{\theta|v|^2}f||_\infty^2\int_{\mathbb{R}^2}\frac{e^{-|u_\perp|^2}\mathrm{d}\bar{u}_\perp}{|\bar{v}+\bar{u}_\perp|}\int_{\mathbb{R}^2}|\bar{u}_\parallel|^\kappa e^{-|\bar{v}+\bar{u}_\parallel|^2}\mathrm{d}\bar{u}_\parallel\int_{u_{\parallel,3}\sim|\bar{v}+\bar{u}_\perp|^{1-}\alpha^{\frac{1}{2}-}}\frac{\mathrm{d}u_{\parallel,3}}{|u_{\parallel,3}|}\lesssim e^{-C'|v|^2}.$$

For the second term we use $f(\bar{v}+\bar{u}_\perp,v_3-\frac{\bar{u}_\parallel\cdot\bar{u}_\perp}{u_{\parallel,3}})\lesssim e^{-C|v+u_\perp|^2}||e^{\theta|v|^2}f||_\infty\frac{1}{1+(v_3-\frac{\bar{u}_\parallel\cdot\bar{u}_\perp}{|u_{\parallel,3}|})^\varepsilon}$ such that

$$\begin{aligned}
&f(v+u_\parallel)\frac{|\bar{v}+\bar{u}_\perp|^{1-}\alpha^{0-}}{|u_{\parallel,3}|^2}f(v_3-\frac{\bar{u}_\parallel\cdot\bar{u}_\perp}{u_{\parallel,3}})\frac{e^{-\frac{|u_\parallel+u_\perp+v|^2}{4}}|\bar{u}_\parallel|}{|u_\parallel+u_\perp|^{1-\kappa}}\\
\lesssim&e^{-|v|^2}e^{-|v+u_\parallel|^2}e^{-|u_\perp|^2}||e^{\theta|v|^2}f||_\infty^2\frac{|\bar{v}+\bar{u}_\perp|^{1-}\alpha^{0-}|\bar{u}_\parallel|}{|u_\parallel+u_\perp|^{1-\kappa}}\frac{1}{|u_{\parallel,3}|^{2-\varepsilon}}\frac{1}{|u_{\parallel,3}|^\varepsilon+[v_3u_{\parallel,3}-\bar{u}_\parallel\cdot\bar{u}_\perp]^\varepsilon}\\
\lesssim&e^{-|v|^2}e^{-|v+u_\parallel|^2}e^{-|u_\perp|^2}||e^{\theta|v|^2}f||_\infty^2\frac{|\bar{v}+\bar{u}_\perp|^{1-}\alpha^{0-}\langle v\rangle}{|u_\parallel+u_\perp|^{1-\kappa}}\frac{1}{|u_{\parallel,3}|^{2-\varepsilon}|\bar{u}_\perp|^\varepsilon}\frac{1}{[\frac{v_3u_{\parallel,3}}{|\bar{u}_\perp|}-\bar{u}_\parallel\cdot\frac{\bar{u}_\perp}{|\bar{u}_\perp|}]^\varepsilon},
\end{aligned}$$

where we have used $e^{-|v+u_\parallel|^2}|u_\parallel|\lesssim\{|u_\parallel+v|+|v|\}e^{-|v+u_\parallel|^2}\lesssim(1+|v|)e^{-C|v+u_\parallel|^2}$. Now we decompose $\bar{u}_\parallel=\bar{u}_{\parallel,a}+\bar{u}_{\parallel,b}:=\bar{u}_\parallel\cdot\frac{\bar{u}_\perp}{|\bar{u}_\perp|}+(\bar{u}_\parallel-\bar{u}_\parallel\cdot\frac{\bar{u}_\perp}{|\bar{u}_\perp|})$ and bound as $e^{-|v|^2}||e^{\theta|v|^2}f||_\infty^2\times$

$$\begin{aligned}
&\int_{u_{\parallel,3}\sim|\bar{v}+\bar{u}_\perp|^{1-}\alpha^{\frac{1}{2}-}}\mathrm{d}u_{\parallel,3}\int_{\mathbb{R}^2}\mathrm{d}\bar{u}_\perp e^{-|\bar{u}_\perp|^2}\frac{|\bar{v}+\bar{u}_\perp|^{1-}\alpha^{0-}}{|\bar{u}_\perp|^\varepsilon|u_{\parallel,3}|^{2-\varepsilon}}\\
&\times\int_{\mathbb{R}}\frac{e^{-|\bar{u}_{\parallel,b}+(v-v\cdot\frac{\bar{u}_\perp}{|\bar{u}_\perp|})|^2}\mathrm{d}\bar{u}_{\parallel,b}}{|\bar{u}_{\parallel,b}|^{1-\kappa}}\int_{\mathbb{R}}\frac{\mathrm{d}\bar{u}_{\parallel,a}e^{-|\bar{u}_{\parallel,a}|^2}}{[\bar{u}_{\parallel,a}-\frac{v_3u_{\parallel,3}}{|\bar{u}_\perp|}-v\cdot\frac{\bar{u}_\perp}{|\bar{u}_\perp|}]^\varepsilon}\\
\lesssim&\int_{\mathbb{R}^2}\mathrm{d}\bar{u}_\perp e^{-|\bar{u}_\perp|^2}\int_{u_{\parallel,3}\sim|\bar{v}+\bar{u}_\perp|^{1-}\alpha^{\frac{1}{2}-}}\mathrm{d}u_{\parallel,3}\frac{|\bar{v}+\bar{u}_\perp|^{1-}\alpha(\bar{v}+\bar{u}_\perp)^{0-}}{|\bar{u}_\perp|^\varepsilon|u_{\parallel,3}|^{2-\varepsilon}}
\end{aligned}$$

where we have used $\bar{u}_{\parallel,a}\mapsto\bar{u}_{\parallel,a}-v\cdot\frac{\bar{u}_\perp}{|\bar{u}_\perp|}$. The $u_{\parallel,3}-$integration yields

$$\begin{aligned}
\lesssim&\int_{\mathbb{R}^2}\mathrm{d}\bar{u}_\perp e^{-|\bar{u}_\perp|^2}\frac{|\bar{v}+\bar{u}_\perp|^{1-}\alpha(\bar{v}+\bar{u}_\perp)^{0-}}{|\bar{u}_\perp|^\varepsilon}\int_{u_{\parallel,3}\sim|\bar{v}+\bar{u}_\perp|^{1-}\alpha^{\frac{1}{2}-}}\frac{\mathrm{d}u_{\parallel,3}}{|u_{\parallel,3}|^{2-\varepsilon}}\\
\lesssim&\int_{\mathbb{R}^2}\mathrm{d}\bar{u}_\perp e^{-|\bar{u}_\perp|^2}\frac{|\bar{v}+\bar{u}_\perp|^{1-\delta}\alpha(\bar{v}+\bar{u}_\perp)^{-\delta}}{|\bar{u}_\perp|^\varepsilon}\frac{1}{|\bar{v}+\bar{u}_\perp|^{(1-\varepsilon)(1-\delta)}}\frac{1}{\alpha(\bar{v}+\bar{u}_\perp)^{(\frac{1}{2}-\delta)(1-\varepsilon)}}\\
\lesssim&\int_{\mathbb{R}^2}\mathrm{d}\bar{u}_\perp e^{-|\bar{u}_\perp|^2}\frac{|\bar{v}+\bar{u}_\perp|^{\varepsilon(1-\delta)}}{|\bar{u}_\perp|^\varepsilon}\frac{1}{\alpha(\bar{v}+\bar{u}_\perp)^{\frac{1}{2}-(1-\delta)\varepsilon}}.
\end{aligned}$$

Note that $\alpha(\bar{v}+\bar{u}_\perp)^{\frac{1}{2}-}\gtrsim[n(x)\cdot(\bar{v}+\bar{u}_\perp)]^{1-}$ and $|\bar{u}_\perp|^\varepsilon\gtrsim[n^\perp\cdot\bar{u}_\perp]^\varepsilon$ to bound

$$\lesssim\langle v\rangle\int_{\mathbb{R}}\frac{e^{-|n\cdot\bar{u}_\perp|^2}}{[n\cdot\bar{u}_\perp+n\cdot\bar{v}]^{1-}}\mathrm{d}[n\cdot\bar{u}_\perp]\int_{\mathbb{R}}\frac{e^{-|n^\perp\cdot\bar{u}_\perp|^2}}{|n^\perp\cdot\bar{u}_\perp|^\varepsilon}\mathrm{d}[n^\perp\cdot\bar{u}_\perp]\lesssim\langle v\rangle.$$

For the third term is bounded by $||\langle v\rangle^\zeta e^{\theta|v|^2}f||_\infty^2\times$

$$\begin{aligned}
&\int_{\mathbb{R}^3}d\bar{u}_{||}\int_{\mathbb{R}^2}d\bar{u}_\perp\left\{\int_{|u_{||,3}|\geq|\bar{v}+\bar{u}_\perp|^{1-}\alpha^{1/2-}}\frac{e^{-|u_{||,3}-v_3|^2}}{|u_{||,3}|^2}du_{||,3}\right\}\frac{\langle u_{||}\rangle^2e^{-C\{|u_{||}+v|^2-|u_\perp|^2\}}}{|u+w|^{1-\kappa}}(1+\frac{1}{|u_{||}+u_\perp|^2})\\
\lesssim&\int_{\mathbb{R}^3}d\bar{u}_{||}\left\{\int_{\mathbb{R}^2}\frac{e^{-C|u_\perp|^2}d\bar{u}_\perp}{|\bar{v}+\bar{u}_\perp|^{1-}\alpha^{1/2-}}\right\}\frac{\langle\bar{u}_{||}\rangle^2e^{-C\{|u_{||}+v|^2\}}}{|u_\parallel+u_\perp|^{1-\kappa}}(1+\frac{1}{|u_{||}+u_\perp|^2}),
\end{aligned}$$

where we have used

$$\begin{aligned}
&\int_{|u_{\parallel,3}|\geq|\bar{v}+\bar{u}_\perp|^{1-\delta}\alpha^{\frac{1}{2}-\delta}}\frac{e^{-|u_{\parallel,3}-v_3|^2}}{|u_{\parallel,3}|^{1-\varepsilon}}\frac{1}{|u_{\parallel,3}|^{1+\varepsilon}}\mathrm{d}u_{\parallel,3}\lesssim\frac{1}{|\bar{v}+\bar{u}_\perp|^{(1-\varepsilon)(1+\delta)}\alpha^{(\frac{1}{2}-\varepsilon)(1+\delta)}}\int_{\mathbb{R}}\frac{e^{-|u_{\parallel,3}-v_3|^2}}{|u_{\parallel,3}|^{1-\varepsilon}}\mathrm{d}u_{\parallel,3}\\
&\lesssim\frac{1}{|\bar{v}+\bar{u}_\perp|^{1-}\alpha^{\frac{1}{2}-}}.
\end{aligned}$$

We note that, by separating $|\xi| \geq \delta$ or $|\xi| \leq \delta$, we can write $\alpha^{1/2-} \geq \{n\cdot[\bar{v}+\bar{u}_\perp]\}^{1-}$ and $|\bar{v}+\bar{u}_\perp|^{1-} \geq \{n^\perp\cdot[\bar{v}+\bar{u}_\perp]\}^{1-}$, where $n^\perp = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} n, x$ so that the inner 2D integral are two convergent 1D one

$$\begin{aligned}
&\int_{|\bar{v}+\bar{u}_\perp|\geq 1} \frac{e^{-C|u_\perp|^2} d\bar{u}_\perp}{\alpha^{1/2-}} + \int_{|\bar{v}+\bar{u}_\perp|\leq 1, |n^\perp\{\bar{v}+\bar{u}_\perp\}|\leq 1} \frac{d\bar{u}_\perp}{|\bar{v}+\bar{u}_\perp|^{1-}\alpha^{1/2-}} + \int_{|n^\perp\{\bar{v}+\bar{u}_\perp\}|\geq 1} \frac{e^{-C|u_\perp|^2} d\bar{u}_\perp}{|\bar{v}+\bar{u}_\perp|^{1-}} \\
\leq\ & 1 + \int_{|n^\perp\{\bar{v}+\bar{u}_\perp\}|\leq 1} \frac{e^{-C|u_\perp|^2} d\bar{u}_\perp}{\alpha^{1/2-}} + \int_{|\bar{v}+\bar{u}_\perp|\leq 1, |n^\perp\{\bar{v}+\bar{u}_\perp\}|\leq 1} \frac{d\bar{u}_\perp}{|\bar{v}+\bar{u}_\perp|^{1-}\alpha^{1/2-}} \\
&+ \int_{|n^\perp\{\bar{v}+\bar{u}_\perp\}|\leq 1} \frac{e^{-C|u_\perp|^2} d\bar{u}_\perp}{|\bar{v}+\bar{u}_\perp|^{1-}} \\
<\ & +\infty.
\end{aligned}$$

Similarly, the first term is bounded by

$$||\langle v\rangle^\zeta e^{\theta|v|^2} f||_\infty ||\partial_{v_3} f|| \int_{\mathbb{R}^3} du_{||} \left\{ \int_{|u_{||,3}|\geq|\bar{v}+\bar{u}_\perp|^{1-}\alpha^{1/2-}} \frac{e^{-|v_3-u_{||,3}|^2}}{|u_{||,3}|^2} \right\} \frac{q_0^*(\frac{|u|}{|v+w|}) u_{||,i} d\bar{u}_\perp}{|u_{||}+u_\perp|^{\kappa-1}} e^{-\frac{|u+v+w|^2}{4}}$$

and the same argument yields the same bound.

We now turn to

$$e^{-\varpi\langle v\rangle s} \int_{\mathbb{R}^3} du_{||} f(v+u_{||}) \int_{\mathbb{R}^2} \left\{ 1-\chi\left(\frac{|\bar{v}+\bar{u}_\perp|^{1-}\alpha^{1/2-}}{u_{||,3}} \right) \right\} du_\perp \partial_{v_i} f(v+u_\perp) e^{-\frac{|u+v+w|^2}{4}} \frac{q_0^*(\frac{|u|}{|v+w|})}{|u_{||}||u+w|^{1-\kappa}}$$

In this case,

$$|\bar{v}+\bar{u}_\perp|^{1-}\alpha^{1/2-} \geq |u_{||,3}|$$

We now parametrize $du_\perp$ in two different ways. We decompose

$$\bar{u}_\parallel = \bar{u}_{\parallel,n} + \bar{u}_{\parallel,n^\perp} := \bar{u}_\parallel\cdot n + \bar{u}_\parallel\cdot n^\perp.$$

If $|u_{||,3}| \geq |\bar{u}_{\parallel,n^\perp}|$, then we use the same parametrization to get

$$\begin{aligned}
& e^{-\varpi\langle v\rangle s} e^{\varpi\langle v+u_\perp\rangle s} \int_{\mathbb{R}^3} du_{||} f(v+u_{||}) \int_{\mathbb{R}^2} d\bar{u}_\perp e^{-\varpi\langle v+u_\perp\rangle s} \partial_{v_i} f(\bar{v}+\bar{u}_\perp, v_3 - \frac{\bar{u}_{||}\cdot\bar{u}_\perp}{u_{||,3}}) \\
& \times[1-\chi\left(\frac{|\bar{v}+\bar{u}_\perp|^{1-}\alpha^{1/2-}}{u_{||,3}} \right)] e^{-\frac{|u_\parallel+v+u_\perp|^2}{4}} \frac{q_0^*(\frac{|u_\parallel|}{|u_\parallel+u_\perp|})}{|u_{||,3}||u_\parallel+u_\perp|^{1-\kappa}} \\
\lesssim_s\ & ||e^{\theta|v|^2} f||_\infty ||e^{-\varpi\langle v\rangle s} \frac{\bar{v}\alpha^{1/2}}{\langle\bar{v}\rangle} \partial_{v_i} f(s)||_\infty \int_{\mathbb{R}^3} du_{||} \int_{|\bar{v}+\bar{u}_\perp|^{1-}\alpha^{1/2-}\geq u_{||,3}} \frac{d\bar{u}_\perp}{|\bar{v}+\bar{u}_\perp|\alpha^{1/2}} \frac{e^{-C|u_\perp|^2} e^{-C|v+u_\parallel|^2}}{|u_{||,3}||u_\parallel+u_\perp|^{1-\kappa}}.
\end{aligned}$$

First we integrate $u_{\parallel,n}$ to drop $\frac{1}{|u_\parallel+u_\perp|^{1-\kappa}}$ singular term for $0<\kappa\leq 1$

$$\int \frac{e^{-|v_n+u_{\parallel,n}|^2}}{|u_\parallel-u_\perp|^{1-\kappa}} du_{\parallel,n} \leq \int \frac{e^{-|v_n+u_{\parallel,n}|^2}}{|u_{\parallel,n}-u_{\perp,n}|^{1-\kappa}} du_{\parallel,n} < \infty,$$

so that we only need to bound

$$\begin{aligned}
& \int du_{||} \int \frac{e^{-|\bar{u}_\perp|^2-|v+u_{||}|^2} d\bar{u}_\perp}{|\bar{v}+\bar{u}_\perp|^\varepsilon \alpha^\varepsilon |\bar{v}+\bar{u}_\perp|^{1-\varepsilon}\alpha^{1/2-\varepsilon}} \frac{1}{|u_{||,3}||u_\parallel+u_\perp|^{1-\kappa}} \\
\leq\ & \int du_{||} \left\{ \int \frac{e^{-|\bar{u}_\perp|^2} d\bar{u}_\perp}{|\bar{v}+\bar{u}_\perp|^\varepsilon\alpha^\varepsilon} \right\} \frac{e^{|v+u_\parallel+u_\perp|^2} e^{-|v+u_{||}|^2}}{|u_{||,3}|^{2-}|u_\parallel+u_\perp|^{1-\kappa}}.
\end{aligned}$$

The inner integral is finite, since $\alpha \geq |n\cdot\{\bar{v}+\bar{u}_\perp\}| = |\bar{v}\cdot n+\bar{u}_{\perp,n}|$, and the integral is a 1D integral:

$$\int_{\mathbb{R}} \frac{e^{-|\bar{u}_{\perp,n}|^2} d\bar{u}_{\perp,n}}{|n\cdot\bar{v}+\bar{u}_{\perp,n}|^{3\varepsilon}} < +\infty.$$

Moreover, from $|u_{||,3}| \geq |\bar{u}_{||,n^\perp}|$, the outer integral takes the form

$$\int_{\mathbb{R}} \frac{e^{-|u_{\parallel,3}+v_3|^2} e^{-|v_{\parallel,n^\perp}+u_{\parallel,n^\perp}|^2} \mathrm{d}u_{||,3}\mathrm{d}u_{\parallel,n^\perp}}{|u_{||,3}|^{2-}} \leq \int \frac{e^{-|u_{\parallel,3}+v_3|^2} e^{-|v_{\parallel,n^\perp}+u_{\parallel,n^\perp}|^2} \mathrm{d}u_{||,n^\perp}\, \mathrm{d}u_{||,3}}{\{|u_{||,n^\perp}| + |u_{||,3}|\}^{2-}} < \infty.$$

We are done in this case.

We now consider the case $|u_{||,3}| \leq |u_{\parallel,n^\perp}|$. We now choose a different parametrization. We define

$$u_{\perp,n} := u_\perp \cdot n, \quad u_{\perp,n^\perp} := u_\perp \cdot n^\perp.$$

Now we choose $u_{\perp,n}$ and $u_{\perp,3}$ as parameters so that $u_{\perp,n^\perp} = -\frac{u_{\perp,n}u_{||,n}+u_{\perp,3}u_{||,3}}{u_{||,n^\perp}}$ and

$$\mathrm{d}u_\perp = \frac{|u_{||}|}{|u_{||,n^\perp}|}\mathrm{d}u_{\perp,n}\mathrm{d}u_{\perp,3},$$

so that we need to bound

$$e^{-\varpi\langle v\rangle s}\int_{\mathbb{R}^3}\mathrm{d}u_{||} f(v+u_{||})\int_{\mathbb{R}^2}\mathrm{d}u_{\perp,n}\mathrm{d}u_{\perp,3}\partial_{v_i} f(v_n+u_{\perp,n}, v_{n^\perp} - \frac{u_{\perp,n}u_{||,n}+u_{\perp,3}u_{||,3}}{u_{||,n^\perp}}, v_3+u_{\perp,3})\frac{|u_{||}|}{|u_{||,n^\perp}|}$$
$$\times[1-\chi\left(\frac{|\bar{v}+\bar{u}_\perp|^{1-}\alpha^{1/2-}}{u_{||,3}}\right)]e^{-\frac{|u_\parallel+v+u_\perp|^2}{4}}\frac{q_0^*(\frac{|u_\parallel|}{|u_\parallel+u_\perp|})}{|u_{||}||u_\parallel+u_\perp|^{1-\kappa}}.$$

Directly this is bounded by $||\langle v\rangle^\zeta e^{\theta|v|^2} f||_\infty ||e^{-\varpi\langle v\rangle s}\frac{|\bar{v}|\alpha^{1/2}\partial_{v_i} f}{\langle\bar{v}\rangle}||_\infty\times$

$$\int_{\mathbb{R}^3}\mathrm{d}u_{||}\int_{|\bar{v}+\bar{u}_\perp|^{1-}\alpha^{1/2-}\geq|u_{||,3}|}\frac{\langle\bar{v}+\bar{u}_\perp\rangle e^{-|\bar{u}_\perp|^2-|v+u_{||}|^2}\mathrm{d}u_{\perp,n}\mathrm{d}u_{\perp,3}}{|\bar{v}+\bar{u}_\perp|\alpha^{1/2}}\frac{\mathrm{d}u_\parallel}{|u_{||,n^\perp}||u_\parallel+u_\perp|^{1-\kappa}}$$
$$\lesssim_s \int_{\mathbb{R}^3}\mathrm{d}u_{||}\int_{|\bar{v}+\bar{u}_\perp|^{1-}\alpha^{1/2-}\geq|u_{||,3}|}\frac{\langle\bar{v}+\bar{u}_\perp\rangle e^{-|\bar{u}_{\perp,n}|^2-|v+u_{||}|^2}}{|\bar{v}+\bar{u}_\perp|^\varepsilon\alpha^\varepsilon|\bar{v}+\bar{u}_\perp|^{1-\varepsilon}\alpha^{1/2-\varepsilon}}\frac{1}{|u_{||,n^\perp}|}\mathrm{d}\bar{u}_{\perp,n}.$$

where we integrate $u_{\perp,3}$ first to drop $\int_{\mathbb{R}}\frac{e^{-C|u_{\perp,3}|^2}\mathrm{d}u_{\perp,3}}{|u_\parallel+u_\perp|^{1-\kappa}}\lesssim\int_{\mathbb{R}}\frac{e^{-C|u_{\perp,3}|^2}\mathrm{d}u_{\perp,3}}{|u_{\parallel,3}+u_{\perp,3}|^{1-\kappa}}<+\infty$.

In the case of $|\bar{v}+\bar{u}_\perp|\leq 1$, this is bounded

$$\lesssim_s \int_{\mathbb{R}^3}\mathrm{d}u_{||}\int\frac{e^{-|u_\perp|^2-|v+u_{||}|^2}\mathrm{d}u_{\perp,n}}{|\bar{v}+\bar{u}_\perp|^\varepsilon\alpha^\varepsilon}\frac{1}{|u_{||,n^\perp}||u_{||,3}|^{1-}}$$
$$\lesssim_s \int_{\mathbb{R}^3}du_{||}\int\frac{e^{-|u_\perp|^2-|v+u_{||}|^2}\mathrm{d}u_{\perp,n}}{|\bar{u}_{\perp,n}+\bar{v}_n|^{3\varepsilon}}\frac{1}{|u_{||,n^\perp}||u_{||,3}|^{1-}}$$
$$\lesssim_s \int_{\mathbb{R}^3}\mathrm{d}u_{||}\Big\{\int\frac{e^{-|\bar{u}_{\perp,n}|^2}\mathrm{d}u_{\perp,n}}{|\bar{u}_{\perp,n}+\bar{v}_n|^{3\varepsilon}}\Big\}\frac{e^{-|v+u_{||}|^2}}{|u_{||,n^\perp}||u_{||,3}|^{1-}},$$

where the inner integral is 1D which is finite and bounded. On the other hand, from the assumption $|u_{||,3}|\leq|\bar{u}_{||,n^\perp}|$, the outer integral is

$$\int\frac{e^{-|v+u_{||}|^2}\mathrm{d}\bar{u}_{||,n^\perp}\mathrm{d}\bar{u}_{\parallel,n}\mathrm{d}u_{||,3}}{|\bar{u}_{||,n^\perp}||u_{||,3}|^{1-}}$$
$$\leq \int\left\{\int_0^{|\bar{u}_{||,n^\perp}|}\frac{\mathrm{d}u_{||,3}}{|u_{||,3}|^{1-}}\right\}\frac{e^{-|\bar{v}_n+\bar{u}_{||,n}|^2}e^{-|\bar{v}_{n^\perp}+\bar{u}_{||,n^\perp}|^2}}{|\bar{u}_{||,n^\perp}|}\mathrm{d}\bar{u}_{||,n^\perp}\mathrm{d}\bar{u}_{||,n}$$
$$\leq \iint_{\mathbb{R}^2}\frac{e^{-|\bar{v}_n+\bar{u}_{||,n}|^2}e^{-|\bar{v}_{n^\perp}+\bar{u}_{||,n^\perp}|^2}}{|\bar{u}_{||,n^\perp}|}\mathrm{d}\bar{u}_{||,n^\perp}\mathrm{d}\bar{u}_{||,n}<\infty.$$

In the case of $|\bar v + \bar u_\perp| \geq 1$ we bound the integration as

$$\begin{aligned}&\int_{\mathbb{R}^3}\int_{\mathbb{R}} \frac{\langle \bar v + \bar u_\perp\rangle^{\varepsilon}\langle \bar v + \bar u_\perp\rangle^{1-\varepsilon}}{|u_{\parallel,3}|^{\frac{\varepsilon}{1-\delta}}|\bar v + \bar u_\perp|^{1-\varepsilon}\alpha^{\frac12-\varepsilon}} \frac{e^{-|\bar u_{\perp,n}|^2}e^{-|v+u_\parallel|^2}}{|u_{\parallel,n^\perp}||u_\parallel + u_\perp|^{1-\kappa}} \mathrm{d}\bar u_{\perp,n}\mathrm{d}u_\parallel \\ \lesssim &\int_{\mathbb{R}^3}\mathrm{d}u_\parallel \int \frac{\langle \bar v_n + \bar u_{\perp,n}\rangle^{\varepsilon}\langle \bar v_{n^\perp}\rangle^{\varepsilon} e^{-|\bar u_{\perp,n}|^2}e^{-|v+u_\parallel|^2}}{|u_{\parallel,3}|^{\frac{\varepsilon}{1-\delta}}[\bar u_{\perp,n}+\bar v_n]^{2(\frac12-\varepsilon)}|\bar u_{\parallel,n^\perp}|}\mathrm{d}\bar u_{\perp,n}.\end{aligned}$$

Again $\int_0^{|\bar u_{\parallel,n^\perp}|}\frac{\mathrm{d}u_{\parallel,3}}{|u_{\parallel,3}|^{\frac{\varepsilon}{1-\delta}}} \lesssim |\bar u_{\parallel,n^\perp}|^{1-\frac{\varepsilon}{1-\delta}}$ and hence the integration is bounded by

$$\langle \bar v_n\rangle^{\varepsilon}\langle \bar v_{n^\perp}\rangle^{\varepsilon}\iint \frac{e^{-|\bar u_{\perp,n}|^2}e^{-|\bar v+\bar u_\parallel|^2}}{|\bar u_{\parallel,n^\perp}|^{\frac{\varepsilon}{1-\delta}}|\bar u_{\perp,n}+\bar v_n|^{1-2\varepsilon}} \leq \langle \bar v_n\rangle^{\varepsilon}\langle \bar v_{n^\perp}\rangle^{\varepsilon}\langle \bar v_{n^\perp}\rangle^{-\frac{\varepsilon}{1-\delta}}\langle \bar v_n\rangle^{-(1-2\varepsilon)} \lesssim 1.$$

□

Our main result for 2D specular case is

Theorem 6. *Assume that $f_0 \in W^{1,\infty}$ and satisfies (3). Assume that*

$$\sup_{0<t\leq T}\{||\langle v\rangle^{\zeta}e^{\theta|v|^2}f(t)||_\infty + ||\partial_{v_3}f(t)||_\infty\} \leq c_{T,\zeta,f_0} < +\infty,$$

then

$$\begin{aligned}&\sup_{0\leq t\leq T}\{||e^{-\varpi\langle \bar v\rangle t}\frac{\alpha}{1+|\bar v|^2}\nabla_x f(t)||_\infty + ||e^{-\varpi\langle \bar v\rangle t}\frac{|\bar v|}{\langle \bar v\rangle}\sqrt{\alpha}\nabla_v f(t)||_\infty\} \\ &\lesssim_{T,\Omega,L} ||\frac{\alpha^{1/2}}{\langle \bar v\rangle}\nabla_{\bar x}f_0||_\infty + ||\frac{|\bar v|^2}{\langle v\rangle}\nabla_{\bar v}f_0||_\infty + ||\partial_{v_3}f||_\infty + P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty),\end{aligned}$$

where P is some polynomial. If $f_0 \in C^1$ and the compatibility conditions (21) and (27) are satisfied, then $f\in C^1$ away from the grazing set γ_0. Furthermore, if $||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty \ll 1$, and $\partial\Omega$(therefore ξ) is real analytic, then T can be arbitrarily large.

We remark that powers of singularity α and $\sqrt{\alpha}$ are barely missed in 3D case.

Proof. We repeat our program in 3D for the simpler 2D case, and we only point out the differences. Lemma 6 is valid with easy adaptations. The new $\partial_{v_3}f(t)$ estimate follows from taking the v_3 derivative

$$\{\partial_t+v_1\partial_{x_1}+v_2\partial_{x_2}\}\partial_{v_3}f+K\partial_{v_3}f+K_{v_3}f+\nu(F)\partial_{v_3}f = -\nu_{v_3}(F)f+\Gamma_{\mathrm{gain},v_3}(f,f)+\Gamma_{\mathrm{gain}}(\partial_{v_3}f,f)+\Gamma_{\mathrm{gain}}(f,\partial_{v_3}f).$$

Since

$$\begin{aligned}|K_{v_3}f|+|\nu_{v_3}(F)f|+|\Gamma_{\mathrm{gain},v_3}(f,f)| &\lesssim P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f||_\infty),\\ |K\partial_{v_3}f|+|\Gamma_{\mathrm{gain}}(\partial_{v_3}f,f)| &\lesssim P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f||_\infty)\int\frac{e^{-C|v-u|^2}}{|v-u|^{2-\kappa}}|\partial_{v_3}f(u)|\mathrm{d}u\end{aligned}$$

and for $(x,v)\in\gamma_-$

$$\partial_{v_3}f(t,x,v) = \partial_{v_3}f(t,x,R_x v),$$

then we follow the proof of Lemma 6(similar to $\partial_t f$ proof) to conclude

$$||\partial_{v_3}f(t)||_\infty \lesssim ||\partial_{v_3}f_0||_\infty + P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f||_\infty).$$

The Velocity lemma (Lemma 1) is valid with changing v to $\bar v$. The non-local to local estimates (19) and (20) are valid for $0<\kappa\leq 1$ for $\bar v = (v_1,v_2)$: In the proof of (19) in Lemma 2, Step 1, the claim (68) is valid. *Step 2*, (69) and (70)is valid with $\alpha(x,\bar v)$. In Step 3 we define $\tilde\sigma_1$ and $\tilde\sigma_2$ with changing v to $\bar v$. Then (72), and (74) hold with changing v to $\bar v$. We follow the same proof of *Step 4* to bound $\int_0^{t_{\mathbf{b}}(x,\bar v)}\frac{e^{-l\langle \bar v\rangle(t-s)}}{|v|^{2\beta-1}|\xi|^{\beta-\frac12}}Z(s,v)\mathrm{d}s$. We use $\frac{1}{|v|}\leq\frac{1}{|\bar v|}$ to conclude (19). For the proof of (20) in Lemma 2, we use the same time splitting of (76) with changing $|v|$ to $|\bar v|$. Then all the proofs are followed and we conclude the proof using $\frac{1}{|v|}\leq\frac{1}{|\bar v|}$.

The fundamental Theorem 4 is valid with simpler proof with changing all v to $\bar{v}$. In fact, due to topological advantage, we can use a global chart $x_{||} = \theta$ in $\mathbb{R}^1$ (such as the polar co-ordinates) for the boundary as

$$\eta(x_{||}) = [R(x_{||})\cos x_{||}, R(x_{||})\sin x_{||}]$$

(vector-valued function) with a global ODE for in the polar co-ordinate system near the boundary! The proof of Theorem 4 follows step by step of the 3D case but with simpler argument without changes of charts. The estimate of $e^{-\varpi\langle v\rangle t}\frac{\alpha}{1+|v|^2}\nabla_x f(t)$ exactly as in 3D case, valid for α. The most delicate part is to estimate $\partial_{v_3}\Gamma_{\text{gain}}(f,f)$, where a weight stronger than $\sqrt{\alpha}$, due to $\beta > 1/2$ in (20). It is important to know, that we are unable to establish (20) in the 2D case with $\beta = 1/2$. However, we are able to close the estimate by using additional bounds on $\partial_{v_3} f$.

More precisely we follow the *Proof of Theorem 3* only except M_x and M_v contributions which contains the term (**B**). Again we note that we cannot use the non-local to local estimate for (**B**) with $\beta = 1$.(For 3D, $\beta > 1$) Here we use (178), Lemma 15 crucially to control those $M-$contributions: Using (24)

$$\begin{aligned}
&M_x - \text{contribution}\\
&\lesssim \int_0^t e^{-\varpi\langle v\rangle t}\frac{\alpha(x,\bar{v})}{\langle\bar{v}\rangle^2}|\partial_{\bar{x}}\bar{V}_{\mathbf{cl}}(s)||\partial_{\bar{v}}\Gamma_{\text{gain}}(f^{m-\ell}, f^{m-\ell})|\mathrm{d}s\\
&\lesssim \int_0^t e^{-\varpi\langle v\rangle(t-s)}\frac{|\bar{v}|^3 e^{C|\bar{v}||t-s|}}{\langle\bar{v}\rangle^2}|e^{\varpi\langle v\rangle s}\partial_{\bar{v}}\Gamma_{\text{gain}}(f^{m-\ell}, f^{m-\ell})|\mathrm{d}s\\
&\lesssim \int_0^t e^{-\varpi\langle\bar{v}\rangle(t-s)}|\bar{v}|\mathrm{d}s \times \{\text{ RHS of (178)}\}\\
&\lesssim \frac{1}{\varpi}P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty)\{1 + ||\partial_{v_3}f_0||_\infty + \Big|\Big|e^{-\varpi\langle v\rangle s}\frac{|\bar{v}|}{\langle\bar{v}\rangle}\alpha^{1/2}\partial_{\bar{v}}f\Big|\Big|_\infty\},
\end{aligned}$$

where we used (24).

Similarly

$$\begin{aligned}
&M_v - \text{contribution}\\
&\lesssim \int_0^t e^{-\varpi\langle v\rangle t}\frac{\alpha(x,\bar{v})}{\langle\bar{v}\rangle^2}|\partial_{\bar{v}}\bar{V}_{\mathbf{cl}}(s)||\partial_{\bar{v}}\Gamma_{\text{gain}}(f^{m-\ell}, f^{m-\ell})|\mathrm{d}s\\
&\lesssim \int_0^t e^{-\varpi\langle v\rangle(t-s)}\frac{|\bar{v}|^2 e^{C|\bar{v}||t-s|}}{\langle\bar{v}\rangle}|e^{\varpi\langle v\rangle s}\partial_{\bar{v}}\Gamma_{\text{gain}}(f^{m-\ell}, f^{m-\ell})|\mathrm{d}s\\
&\lesssim \int_0^t e^{-\varpi\langle\bar{v}\rangle(t-s)}|\bar{v}|\mathrm{d}s \times \{\text{ RHS of (178)}\}\\
&\lesssim \frac{1}{\varpi}P(||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty)\{1 + ||\partial_{v_3}f_0||_\infty + \Big|\Big|e^{-\varpi\langle v\rangle s}\frac{|\bar{v}|}{\langle\bar{v}\rangle}\alpha^{1/2}\partial_{\bar{v}}f\Big|\Big|_\infty\}.
\end{aligned}$$

With these estimate for $M-$contribution we are able to close the estimate. □

Proposition 5 (Specular BC)**.** *Let Ω be 2D disk. Assume $f_0 \in C^1$ and $|\nabla_{x,v}f_0| \lesssim 1$ and $||\langle v\rangle^{\zeta}e^{\theta|v|^2}f_0||_\infty \ll 1$ and for all $x \in \partial\Omega$ and $|\bar{v}| \sim 1$ with $n(x)\cdot\bar{v} = 0$*

$$n(x)\cdot\nabla_v f_0(x,v) \gtrsim 1. \tag{179}$$

Then there exists $t > 0$ such that $\partial_n f(t)$ satisfies (169).

Proof. Using the fact that weighted ∂f is continuous we follow *Step 1* of Proposition 4 to conclude (172).

Now we claim (169) We take the spatial normal derivation

$$\begin{aligned}
f(t,x,v) = {}& e^{-\int_0^t \nu(F)(\tau)\mathrm{d}\tau}f_0(X_{\mathbf{cl}}(0), V_{\mathbf{cl}}(0))\\
&+ \int_0^t e^{-\int_s^t \nu(F)(\tau)}\{Kf + \Gamma_{\text{gain}}(f,f)\}(s, X_{\mathbf{cl}}(s), V_{\mathbf{cl}}(s))\mathrm{d}s,
\end{aligned}$$

to have $\partial_n f(t,x,v) = \mathbf{I}+\mathbf{II}+\mathbf{III}$ as (165). Note that

$$\begin{aligned}\mathbf{I} = &\ \partial_n \bar{V}_{\mathbf{cl}}(0)\cdot\nabla_{\bar{v}} f_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0)) e^{-\int_0^t \nu(F)(\tau)\mathrm{d}\tau}\\ &+\partial_n \bar{X}_{\mathbf{cl}}(0)\cdot\nabla_{\bar{x}} f_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0)) e^{-\int_0^t \nu(F)(\tau)\mathrm{d}\tau}.\end{aligned}$$

From the condition (179), we use (175) to have for $\alpha(x,\bar{v})\ll 1$

$$\begin{aligned}|\mathbf{I}| &\gtrsim \frac{t|\bar{v}|^4}{\alpha(x,\bar{v})}|n(X_{\mathbf{cl}}(s))\cdot\nabla_{\bar{v}} f_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))| - \frac{t|\bar{v}|}{\sqrt{\alpha(x,\bar{v})}}||\nabla_{\bar{x}} f_0||_\infty\\ &\gtrsim \frac{t|\bar{v}|^4}{\alpha(x,\bar{v})}|n(X_{\mathbf{cl}}(s))\cdot\nabla_{\bar{v}} f_0(X_{\mathbf{cl}}(0),V_{\mathbf{cl}}(0))|.\end{aligned}$$

Except $M-$contribution

$$\int_0^t e^{-\int_s^t \nu(F)(\tau)\mathrm{d}\tau}\partial_n \bar{V}_{\mathbf{cl}}\cdot\partial_{\bar{v}}\Gamma_{\text{gain}}(f,f)(s,X_{\mathbf{cl}}(s),V_{\mathbf{cl}}(s))\mathrm{d}s,$$

we use the non-local to local estimates(Lemma 2) and follow the proof of Theorem 3 to bound all the other terms by

$$O(\varepsilon)\frac{t\langle v\rangle^{10}}{\alpha(x,\bar{v})} + O(||\langle v\rangle^\zeta e^{\theta|v|^2} f_0||_\infty).$$

Note that the smallness $O(\varepsilon)$ is from the non-local to local estimate.

Now we use Lemma 15 and (24) to bound the $M-$contribution by

$$\int_0^t \frac{|v|^3 e^{C|v|(t-s)}}{\alpha(x,\bar{v})} e^{\varpi\langle v\rangle s}\big\{\text{RHS of } (178)\big\} \lesssim e^{\varpi\langle v\rangle t}\frac{t|v|^3}{\alpha(x,\bar{v})} O(||\langle v\rangle^\zeta e^{\theta|v^2|} f_0||_\infty).$$

Therefore we choose sufficiently small $\varepsilon>0$ and $||\langle v\rangle^\zeta e^{\theta|v|^2} f_0||_\infty>0$ we can let $\mathbf{I}$ dominate $\mathbf{II}$ and $\mathbf{III}$. Hence for $\alpha\sim 0$

$$\partial_n f \neq 0,$$

and combining with (172) we prove (169). □

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