GLOBAL WEAK SOLUTIONS FOR KOLMOGOROV-VICSEK TYPE EQUATIONS WITH ORIENTATIONAL INTERACTION

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Abstract. We prove the global existence of weak solutions to kinetic Kolmogorov-Vicsek models that can be considered a non-local non-linear Fokker-Planck type equation describing the dynamics of individuals with orientational interaction. This model is derived from the discrete Couzin-Vicsek algorithm as mean-field limit [2, 9], which governs the interactions of stochastic agents moving with a velocity of constant magnitude. Therefore, the velocity variable of kinetic Kolmogorov-Vicsek models lies on the unit sphere. For our analysis, we take advantage of the boundedness of velocity space to get L^p estimates and compactness property.

1. Introduction

Recently, a variety of mathematical models capturing the emergent phenomena of self-driven agents have received lots of attention extensively. In particular, the discrete Couzin-Vicsek algorithm (CVA) has been proposed a model describing the interactions of agents moving with a velocity of constant magnitude, and with angles measured from a reference direction (See [1, 3, 15, 21]).

In this manuscript, we look into analytical issues for the kinetic (mesoscopic) description associated to the discrete Couzin-Vicsek algorithm with stochastic dynamics corresponding to Brownian motion on a sphere. More precisely, we consider the corresponding kinetic Kolmogorov-Vicsek model describing stochastic particles with orientational interaction:

$$
\begin{aligned}
&\partial_t f + \omega \cdot \nabla_x f = -\nabla_\omega \cdot (f F_o) + \Delta_\omega f, \\
&F_o(x,\omega,t) = \nu(\omega \cdot \Omega(f))(Id - \omega \otimes \omega)\Omega(f), \\
&\Omega(f)(x,t) = \frac{J(f)(x,t)}{|J(f)(x,t)|}, \quad J(f)(x,t) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} K(|x-y|)\omega f(y,\omega,t)\,dy\,d\omega, \\
&f(x,\omega,0) = f_0(x,\omega),\ x \in \mathbb{R}^d,\ \omega \in \mathbb{S}^{d-1},\ t > 0.
\end{aligned} \tag{1.1}
$$

where $f = f(x,\xi,t)$ is the one-particle distribution function at position $x \in \mathbb{R}^d$, velocity direction $\omega \in \mathbb{S}^{d-1}$ and time t. The operators ∇_ω and Δ_ω denote the gradient and the Laplace-Beltrami operator on the sphere $\mathbb{S}^{d-1}$. The term $F_o(x,\omega,t)$ is the mean-field force

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that governs the orientational interaction of self-driven particles by aligning them with the direction $\Omega(x,t) \in \mathbb{S}^{d-1}$ that depends on the flux $J(x,t)$.

This mean-field force is also proportional to the interaction frequency ν. Its algebraic inverse ν^{-1} represents the typical time-interval between two successive changes in the trajectory of the orientational swarm particle to accommodate the presence of other particle in the neighborhood. The function K is an isotropic observation kernel around each particle and it is assumed to be integrable in $\mathbb{R}$.

Following Degond and Motsch in [9], the interaction frequency function ν is taken to be a positive function of $\cos\theta$, where θ is the angle between ω and Ω. Such dependence of ν with respect to the angle θ represents different turning transition rates at different angles. Hence, the constitutive form of such interaction frequency $\nu(\theta)$ is inherent to species being modeled by orientational interactions. As in [9], we assume that $\nu(\theta)$ is a smooth and bounded function of its argument.

The kinetic Kolmogorov-Fokker-Planck type model with orientational interactions (1.1) was formally derived in [9] as a mean-field limit of the discrete Couzin-Vicsek algorithm (CVA) with stochastic dynamics. There, the authors mainly focused on the model with the following interaction term

$$
\begin{aligned}
&\partial_t f + \omega\cdot\nabla_x f = -\nabla_\omega\cdot(fF_o) + \Delta_\omega f,\\
&F_o(x,\omega,t) = \nu(\omega\cdot\Omega(f))(Id - \omega\otimes\omega)\Omega(f),\\
&\Omega(f)(x,t) = \frac{J(f)(x,t)}{|J(f)(x,t)|}, \quad J(f)(x,t) = \int_{\mathbb{S}^{d-1}} \omega f(x,\omega,t)\,d\omega,\\
&f(x,\omega,0) = f_0(x,\omega),\ x\in\mathbb{R}^d,\ \omega\in\mathbb{S}^{d-1},\ t>0.
\end{aligned} \tag{1.2}
$$

This form of $J(f)(x,t)$ was derived in [9] from the corresponding one (1.1) with the kernel K by rescaling time and spatial variables. Such scaling describes dynamics for the model in (1.1) at large time and length scales compared with scales of the individuals:

We discuss more on the specifics of both models, (1.1) and (1.2), in the next section.

The purpose of this article is to present the global existence of weak solutions to models (1.1) and (1.2) in appropriate Sobolev spaces. In fact we will show that the proofs for both models, (1.1) and (1.2), are exactly the same.

The classical Vicsek model have been receiving lots of attention in the last few years concerning the rigor of mathematical studies of its mean-field limit, hydrodynamic limit and phase transition, among the main properties. More specifically, Bolley, Cañizo and J. A. Carrillo have rigorously justified mean-field limit in [2]. When the force acting on the particles is not normalized, i.e., $\nu\Omega(x,t)$ replaced by $J(x,t)$ in force term F_o. This modification leads to the appearance of phase transitions from disordered states at low density to aligned states at high densities. This phase transition problem has been studied in [1, 3, 7, 8, 13, 15]. Also, issues on hydrodynamic descriptions of kinetic Vicsek model have been discussed in [7, 8, 9, 10, 11, 12]. We also refer to [6, 16] concerning related models of Vicsek model.

Although the Vicsek model has been studied via a variety of scales from microscopic level to macroscopic level, there is few results on existence theory of kinetic description. In [13], Frouvelle and Liu have shown the well-posedness of the space-homogeneous case of (1.2), in which $\nu\Omega(x,t)$ is replaced by $J(x,t)$. More precisely, when there is no advection due to spatial homogeneity, the kinetic equation (1.2) becomes Smoluchowski equation defined on sphere, for which Frouvelle and Liu have proved the global solvability and regularity, moreover by making use of Onsager free energy functional, the large time behavior has been shown. On the other hand, Bolley, Cañizo and J. A. Carrillo [2] have presented the existence of weak solution of space-inhomogeneous equation with a different kind of force. More precisely, they considered the difference between spatial convolutions of mass and momentum with Lipschitz and bounded kernel K, namely $\omega K *_x \rho - K *_x J$ instead of $\nu\Omega$. This force has regular effect for spatial variable compared to our case $\nu\Omega$. To the best of our knowledge, the well-posedness of the space-inhomogeneous for both kinetic Kolmogorov-Vicsek models, (1.1) and (1.2), is still open.

This paper is devoted to show the existence of weak solutions to the kinetic Kolmogorov-Vicsek type models with orientational interactions, given by (1.1) and (1.2). In Section 2, we briefly provides some known results for these kinetic models, which give a heuristic justification for a priori non-zero assumption to be stated in our main result. In Section 3, we present a priori estimates and compactness lemma, which play crucial roles in the main proof in Section 4.

2. Preliminaries and Main result

In this section, we briefly review how the kinetic Kolmogorov-Viscek equations, (1.1) and (1.2) can be formally derived from the discrete Couzin-Vicsek algorithm model [9] with stochastic dynamics. Then we provide our main result and useful formulations.

2.1. **Kinetic Kolmogorov-Vicsek models.** Following [9], the kinetic Kolmogorov-Vicsek model considered in (1.1) is derived from the following classical discrete Vicsek model modeling Brownian motion of the sphere $\mathbb{S}^{d-1}$ given by the stochastic differential equations for $1 \le i \le N$,

$$
\begin{aligned}
&dX_i = \omega_i dt,\\
&d\omega_i = (Id - \omega_i \otimes \omega_i)\nu(\omega_i \cdot \bar{\Omega}_i)\bar{\Omega}_i dt + \sqrt{2\mu}(Id - \omega_i \otimes \omega_i) \circ dB_t^i,\\
&\bar{\Omega}_i = \frac{\bar{J}_i}{|\bar{J}_i|}, \quad \bar{J}_i = \sum_{j,\ |X_j - X_i| \le R} \omega_j.
\end{aligned}
\tag{2.3}
$$

Here, the neighborhood of the i-th particle is the ball centered at $X_i \in \mathbb{R}^d$ with radius $R > 0$. The velocity director $\omega_i \in \mathbb{S}^{d-1}$ of the i-th particle tends to be aligned to the director Ω_i of the average velocity of the neighboring particles with noise B_t^i that stand for N independent standard Brownian motions on $\mathbb{R}^d$ with intensity $\sqrt{2\mu}$. Then, its projection $(Id - \omega_i \otimes \omega_i) \circ dB_t^i$ represents the contribution of a Brownian motion on the sphere $\mathbb{S}^{d-1}$. We refer to [17] for more details on Brownian motions on Riemannian manifolds. The first interaction term of $(2.3)_2$ is the sum of smooth binary interactions with same speed, whereas there is no constraint on the velocity in the Cucker-Smale model [4]. In addition the interaction frequency (weight) function $\nu(\omega_i \cdot \Omega_i)$ depends on the angle between ω_i and Ω_i, parametrized by $\cos\theta_i = \omega_i \cdot \Omega_i$.

From the individual-based model (2.3), the corresponding kinetic mean-field limit as the number of particles N tends to infinity was proposed in [2, 9], results in the phenomenological model for $f = f(x, \omega, t)$, the one-point probability density function of finding a particle at position $x \in \mathbb{R}^d$, with velocity $\omega \in \mathbb{S}^{d-1}$ and time t, evolving according

$$\begin{aligned}
&\partial_t f + \omega \cdot \nabla_x f = -\nabla_\omega \cdot (fF) + \mu \Delta_\omega f,\\
&F(x,\omega,t) = \nu(\omega \cdot \Omega(f))(Id - \omega \otimes \omega)\Omega(f),\\
&\Omega(f)(x,t) = \frac{J(f)(x,t)}{|J(f)(x,t)|}, \quad J(f)(x,t) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} K(|x-y|)\omega f(y,\omega,t)dyd\omega,
\end{aligned} \tag{2.4}$$

where K is an isotropic observation kernel around each particle, which is assumed to be of L^1 class in our framework. Notice that μ corresponds to the diffusive coefficient associated to the Brownian motion on the sphere $\mathbb{S}^{d-1}$.

On the other hand, if we observe the dynamics of the system at large time and length scales compared with the scales of the individuals by introducing new dimensionless variables $\tilde{x} = \varepsilon x, \tilde{t} = \varepsilon t$ with $\varepsilon \ll 1$, this makes the interaction local and to be aligned the particle velocity to the direction of the local particle flux. This interaction term is balanced at leading order ε by the diffusion term. Under this consideration, (2.4) can be considered as the following equation (see [9])

$$\begin{aligned}
&\partial_t f + \omega \cdot \nabla_x f = Q(f),\\
&Q(f) = -\nabla_\omega \cdot (f\nu(\omega \cdot \Omega(f))(Id - \omega \otimes \omega)\Omega(f)) + \mu \Delta_\omega f,\\
&\Omega(f)(x,t) = \frac{J(f)(x,t)}{|J(f)(x,t)|}, \quad J(f)(x,t) = \int_{\mathbb{S}^{d-1}} \omega f(x,\omega,t)d\omega.
\end{aligned} \tag{2.5}$$

Notice that $\Omega(f)$ in (2.4) and (2.5) have a singularity when $J(f)$ becomes 0. To avoid this singularity issue, we are going to present the existence of weak solutions to (2.4) and (2.5) in a subclass of solutions with the non-zero local momentum, i.e. $J(f) \neq 0$. As shown in [9], since ω is not a collisional invariant of operator Q, the momentum is not conserved. Thus, it is not straightforward to get $J(f)(x,t) \neq 0$ for all (x,t) from imposing non-zero initial momentum, i.e. $J(f)(x,0) \neq 0$ for all x. Moreover, there is no canonical entropy for a type of the kinetic equations (2.4) and (2.5). Due to these analytical difficulties, we are going to heuristically justify our constraint $J(f) \neq 0$ by observing equilibria only for (2.5), which has been studied in [9] as follows: For classification of equilibria in the case of dimension $d = 3$, we define the Fisher-von Mises distribution by

$$M_\Omega(\omega) = \frac{1}{\int_{\mathbb{S}^2} \exp(\frac{\sigma(\omega\cdot\Omega)}{\mu})d\omega} \exp\Big(\frac{\sigma(\omega \cdot \Omega)}{\mu}\Big)$$

for a given unit vector $\Omega \in \mathbb{S}^2$. Here, σ denotes an antiderivative of ν, i.e. $\frac{d\sigma}{d\tau}(\tau) = \nu(\tau)$. Since ν is positive, σ is an increasing function, M_Ω is maximal for $\omega \cdot \Omega = 1$, i.e. for ω pointing in the direction of Ω. Therefore, Ω plays the same role as the averaged velocity of the classical Maxwellian of gas dynamics. Moreover, the diffusion constant μ corresponds to the strength of temperature, which measures the spreading of the equilibrium about the average direction Ω. Here, the value of the diffusion constant is fixed, in contrast with the classical gas dynamics where the temperature is a thermodynamical variable whose evolution is determined by the energy balance equation.

By using Fisher-von Mises distribution, the operator Q and equilibria of (2.5) are expressed as follows:

Lemma 2.1. [9] *(i) The operator $Q(f)$ can be written as*

$$Q(f) = \mu \nabla_\omega \cdot \Big[M_{\Omega(f)} \nabla_\omega \Big(\frac{f}{M_{\Omega(f)}}\Big)\Big].$$

(ii) The equilibria, i.e. the solutions $f(\omega)$ satisfying $Q(f) = 0$ form a three dimensional manifold $\mathcal{E}$ given by

$$\mathcal{E} = \{\rho M_\Omega(\omega) \mid \rho > 0,\ \Omega \in \mathbb{S}^2\},$$

where ρ is the total mass and Ω is the director of the flux of $\rho M_\Omega(\omega)$, i.e.

$$\rho = \int_{\mathbb{S}^2} \rho M_\Omega(\omega) d\omega, \quad \Omega = \frac{J(\rho M_\Omega)}{\Big|J(\rho M_\Omega)\Big|},$$

$$J(\rho M_\Omega) := \int_{\mathbb{S}^2} \rho M_\Omega(\omega)\omega d\omega = \rho c(\mu)\Omega,$$

where

$$c(\mu) = \frac{\int_0^\pi \cos\theta \exp\left(\frac{\sigma(\omega\cdot\Omega)}{\mu}\right)\sin\theta d\theta}{\int_0^\pi \exp\left(\frac{\sigma(\omega\cdot\Omega)}{\mu}\right)\sin\theta d\theta}.$$

We note that $c(\mu) \to 1$ as $\mu \to 0$, and $c(\mu) \to 0$ as $\mu \to \infty$. This means that the local momentum $J(\rho M_\Omega)$ of equilibrium solution $f = \rho M_\Omega$ is not zero as long as the strength μ of diffusion is not sufficiently large compared to orientational interaction. Therefore, if the diffusion constant μ is not large, it would make sense that the local momentum $J(f)$ of our solution f is assumed to be nonzero at least near the equilibrium.

2.2. **Main result.** In this part, we present the main results for global existence of weak solutions to equations (1.1) and (1.2).
Before giving the main theorem, we introduce the following notations for simplification.

• **Notation** : From now on, D denotes $\mathbb{R}^d \times \mathbb{S}^{d-1}$, and $\mathbb{P}_{\omega^\perp}$ denotes $Id - \omega\otimes\omega$. Note that the mapping $v \mapsto (Id - \omega\otimes\omega)v$ is the projection of the vector v onto the normal plane to ω.

• **Framework** : In the following sections, it turns out that the proof for (1.1) is the same as that for (1.2). Thus, for convenience, we rewrite (1.1) and (1.2) as one form:

$$\begin{aligned} &\partial_t f + \omega\cdot\nabla_x f = -\nabla_\omega\cdot(fF_o) + \mu\Delta_\omega f, \\ &F_o(x,\omega,t) = \nu(\omega\cdot\Omega(f))\mathbb{P}_{\omega^\perp}\Omega(f), \quad \Omega(f)(x,t) = \frac{J(f)(x,t)}{|J(f)(x,t)|}, \\ &f(x,\omega,0) = f_0(x,\omega),\ (x,\omega)\in D,\ t>0. \end{aligned} \tag{2.6}$$

where $J(f)(x,t)$ denotes either

$$\bar{J}(f)(x,t) := \int_D K(|x-y|)\omega f(y,\omega,t) dy d\omega,$$

or

$$\tilde{J}(f)(x,t) := \int_{\mathbb{S}^{d-1}} \omega f(x,\omega,t) d\omega.$$

As already mentioned, we assume that $\nu(\cdot)$ is a smooth and bounded function of its argument and $K(|\cdot|) \in L^1(\mathbb{R}^d)$. Moreover, we go around the singularity issue for $\Omega(f)$ by imposing a priori assumptions that the weak solution f of (1.1) satisfies

$$\bar{J}(f)(x,t) \neq 0 \quad \text{for all } x\in\mathbb{R}^d,\ t>0, \tag{2.7}$$

and the weak solution f of (1.2) satisfies

$$\tilde{J}(f)(x,t) \neq 0 \quad \text{for all } x \in \mathbb{R}^d,\ t > 0. \tag{2.8}$$

Theorem 2.1. *Assume that f_0 satisfies*

$$f_0 \in L^1 \cap L^\infty(D) \quad \textit{and} \quad f_0 \geq 0. \tag{2.9}$$

Then, for given $T > 0$, under each assumption (2.7) and (2.8), the equations (1.1) and (1.2) respectively have a weak solution f, which satisfies

$$\begin{aligned}
&f \geq 0,\\
&f \in C(0,T;L^1(D)) \cap L^\infty((0,T)\times D),\\
&\nabla_\omega f \in L^2((0,T)\times D).
\end{aligned}$$

and the following weak formulation: for any $\phi \in C_c^\infty([0,T)\times D)$,

$$\begin{aligned}
&\int_0^t \int_D f\partial_t\phi + f\omega\cdot\nabla_x\phi + fF_o\cdot\nabla_\omega\phi - \mu\nabla_\omega f\cdot\nabla_\omega\phi\, dxd\omega ds\\
&\qquad\qquad + \int_D f_0\phi(0,\cdot)dxd\omega = 0,\\
&F_o(x,\omega,t) = \nu(\omega\cdot\Omega(f))\mathbb{P}_{\omega^\perp}\Omega(f).
\end{aligned} \tag{2.10}$$

Moreover, f satisfies that for any $1 \leq p < \infty$,

$$\|f\|_{L^\infty(0,T;L^p(D))} + \frac{2\mu(p-1)}{p}\|\nabla_\omega f^{\frac{p}{2}}\|^{\frac{2}{p}}_{L^2((0,T)\times D)} \leq e^{CT\frac{p}{p-1}}\|f_0\|_{L^p(D)}, \tag{2.11}$$

and

$$\|f\|_{L^\infty((0,T)\times D)} \leq e^{CT}\|f_0\|_{L^\infty(D)}. \tag{2.12}$$

Remark 2.1. *In the following sections, it turns out that the proof of Theorem 2.1 is based on energy method, in which the diffusion term $\mu\Delta_\omega f$ plays a crucial role, but the strength $\mu > 0$ does not essentially affect the proof of existence. Thus from now on, we set $\mu = 1$ without loss of generality.*

2.3. **Formulas for Calculus on sphere.** We here present some useful formulas on sphere $\mathbb{S}^{d-1}$, which are extensively used in this paper.

Let F be a vector-valued function and f be scalar-valued function. Then we have a formula related to the integration by parts:

$$\int_{\mathbb{S}^{d-1}} f\nabla_\omega\cdot F d\omega = -\int_{\mathbb{S}^{d-1}} F\cdot(\nabla_\omega f - 2\omega f)d\omega. \tag{2.13}$$

By the definition of the projection $\mathbb{P}_{\omega^\perp}$, it is obvious that

$$\begin{aligned}
&\mathbb{P}_{\omega^\perp}\omega = 0, \quad \mathbb{P}_{\omega^\perp}\nabla_\omega f = \nabla_\omega f,\\
&\mathbb{P}_{\omega^\perp}u\cdot v = \mathbb{P}_{\omega^\perp}v\cdot u,
\end{aligned} \tag{2.14}$$

for any scalar-valued function f and vectors u, v.

On the other hand, for any constant vector $v \in \mathbb{R}^d$, we have

$$\begin{aligned}
&\nabla_\omega(\omega\cdot v) = \mathbb{P}_{\omega^\perp}v,\\
&\nabla_\omega\cdot(\mathbb{P}_{\omega^\perp}v) = -(d-1)\omega\cdot v.
\end{aligned} \tag{2.15}$$

In order to easily derive the formulas above, one can begin by rewriting them as spherical coordinates. We refer to [13, 20] for the derivations of formulas above.

3. A PRIORI ESTIMATES AND COMPACTNESS LEMMA

In this section, we first derive a priori estimates, then compactness lemma, which play an important role in the proof of Theorem 2.1.

Lemma 3.1. *Assume that f is a smooth solution to* (2.6) *with f_0 satisfying* (2.9)*. Then, for any $1 \leq p < \infty$, we have*

$$\|f\|_{L^\infty(0,T;L^p(D))} + \frac{2(p-1)}{p}\|\nabla_\omega f^{\frac{p}{2}}\|^{\frac{2}{p}}_{L^2((0,T)\times D)} \leq e^{CT\frac{p}{p-1}}\|f_0\|_{L^p(D)}, \tag{3.16}$$

in particular, if $p = \infty$, we have

$$\|f\|_{L^\infty((0,T)\times D)} \leq e^{CT}\|f_0\|_{L^\infty(D)}. \tag{3.17}$$

Proof. For any $1 \leq p < \infty$, it follows from (2.6) that

$$\begin{aligned}\frac{d}{dt}\int_D f^p dxd\omega &= -p\int_D f^{p-1}\nabla_\omega\cdot(f\nu(\omega\cdot\Omega)\mathbb{P}_{\omega^\perp}\Omega)dxd\omega + p\int_D f^{p-1}\Delta_\omega f dxd\omega\\ &=: I_1 + I_2.\end{aligned}$$

By integration by parts as (2.13) and using $\omega\cdot\nabla_\omega f = 0$, we get

$$\begin{aligned}I_2 &= -p(p-1)\int_D f^{p-2}\nabla_\omega f\cdot\nabla_\omega f dxd\omega + 2p\int_D f^{p-1}\omega\cdot\nabla_\omega f dxd\omega\\ &= -\frac{4(p-1)}{p}\int_D|\nabla_\omega f^{\frac{p}{2}}|^2 dxd\omega.\end{aligned}$$

We use the formula (2.15) to have

$$\begin{aligned}I_1 &= -p\int_D f^{p-1}\Big(\nu(\omega\cdot\Omega)\nabla_\omega f\cdot\mathbb{P}_{\omega^\perp}\Omega + f\nu'(\omega\cdot\Omega)|\mathbb{P}_{\omega^\perp}\Omega|^2 - (d-1)f\nu(\omega\cdot\Omega)\omega\cdot\Omega\Big)dxd\omega\\ &\leq p\|\nu(\omega\cdot\Omega)\|_{L^\infty}\int_D f^{p-1}|\nabla_\omega f|dxd\omega + p\|\nu'(\omega\cdot\Omega)\|_{L^\infty}\int_D f^p dxd\omega\\ &\quad + p(d-1)\|\nu(\omega\cdot\Omega)\|_{L^\infty}\int_D f^p dxd\omega.\end{aligned}$$

Using Hölder's inequality, the first integral in right hand side above can be estimates as

$$\begin{aligned}\int_D f^{p-1}|\nabla_\omega f|dxd\omega &\leq \Big(\int_D f^p dxd\omega\Big)^{1/2}\Big(\int_D f^{p-2}|\nabla_\omega f|^2 dxd\omega\Big)^{1/2}\\ &= \frac{2}{p}\Big(\int_D f^p dxd\omega\Big)^{1/2}\Big(\int_D|\nabla_\omega f^{\frac{p}{2}}|^2 dxd\omega\Big)^{1/2}.\end{aligned}$$

Then we have

$$I_1 \leq \frac{2(p-1)}{p}\int_D|\nabla_\omega f^{\frac{p}{2}}|^2 dxd\omega + C(\frac{p}{p-1}+p)\int_D f^p dxd\omega.$$

Combining the estimates above, we get

$$\frac{d}{dt}\int_D f^p dxd\omega + \frac{2(p-1)}{p}\int_D|\nabla_\omega f^{\frac{p}{2}}|^2 dxd\omega \leq C(\frac{p}{p-1}+p)\int_D f^p dxd\omega,$$

which gives the Gronwall type inequality

$$\frac{d}{dt}\|f\|_{L^p(D)} \leq C\frac{p}{p-1}\|f\|_{L^p(D)}.$$

Therefore, we have

$$\|f\|_{L^\infty(0,T;L^p(D))} \le e^{CT\frac{p}{p-1}}\|f_0\|_{L^p(D)},$$

which implies the L^p estimate (3.16). Moreover, taking $p \to \infty$, we have L^∞ bound (3.17). □

Remark 3.1. *In the proof of Lemma 3.1, we only needed the boundedness of Ω rather than specific feature of J itself. In the following proofs, each term corresponding to Ω has to be bounded, thus Lemma 3.1 can be applied.*

The following lemma provides the compactness that ensure the strong L^p convergence of solutions to the following equation (3.18) with bounded force term as a generalized form of our main equations. This strong compactness property is based on the boundedness of force term and velocity space. On the other hand, the celebrated velocity averaging lemma plays a important role on the analysis for kinetic equations with unbounded velocity variable (See for instance [18]). Our compactness property crucially underlies the proof of Theorem 2.1.

Lemma 3.2. *Assume that f_0 satisfies* (2.9)*, and f_n is a smooth solution to*

$$\begin{aligned} &\partial_t f_n + \omega\cdot\nabla_x f_n = -\nabla_\omega\cdot\Big(f_n\nu(\omega\cdot F_n)\mathbb{P}_{\omega^\perp}F_n\Big) + \Delta_\omega f_n, \\ &f_n(x,\omega,0) = f_0(x,\omega), \end{aligned} \tag{3.18}$$

where $F_n : \mathbb{R}^n\times\mathbb{R}_+ \to \mathbb{R}^n$ is a given function of (x,t).
If the sequence (F_n) is bounded in $L^\infty((0,T)\times\mathbb{R}^d)$, for any $1\le p<\infty$, then there exists a limit function f such that up to a subsequence,

$$f_n \to f \quad \text{as } n\to\infty \text{ in } L^p((0,T)\times\mathbb{R}^d)\cap L^2((0,T)\times\mathbb{R}^d;H^1(\mathbb{S}^{d-1})).$$

Moreover, the associated sequences

$$(\bar{J}_n) := \Big(\int_D K(|x-y|)\omega f_n(y,\omega,t)dyd\omega\Big) \quad \text{for a given kernel } K(|\cdot|)\in L^1(\mathbb{R}^d),$$

and

$$(\tilde{J}_n) := \Big(\int_{\mathbb{S}^{d-1}} \omega f_n(x,\omega,t)d\omega\Big)$$

strongly converge to the corresponding limits $\bar{J}$ and $\tilde{J}$ in $L^p((0,T)\times\mathbb{R}^d)$ respectively, where

$$\bar{J} := \Big(\int_D K(|x-y|)\omega f(y,\omega,t)dyd\omega\Big)$$

and

$$\tilde{J} := \Big(\int_{\mathbb{S}^{d-1}} \omega f(x,\omega,t)d\omega\Big).$$

Proof. Since the sequence (F_n) is bounded in $L^\infty((0,T)\times\mathbb{R}^d)$, there exists $F\in L^\infty((0,T)\times\mathbb{R}^d)$ such that up to a subsequence,

$$F_n \rightharpoonup F \quad \text{weakly}-* \text{ in } L^\infty((0,T)\times\mathbb{R}^d). \tag{3.19}$$

Let f be a solution of (3.18) corresponding to F. Then, we subtract the equation (3.18) corresponding to F from (3.18) to obtain

$$\begin{aligned}\partial_t(f_n - f) + \omega\cdot\nabla_x(f_n - f) = &-\nabla_\omega\cdot\Big((f_n-f)\nu(\omega\cdot F_n)\mathbb{P}_{\omega^\perp}F_n\Big)\\ &-\nabla_\omega\cdot\Big(f(\nu(\omega\cdot F_n)-\nu(\omega\cdot F))\mathbb{P}_{\omega^\perp}F_n)\Big)\\ &-\nabla_\omega\cdot\Big(f\nu(\omega\cdot F)\mathbb{P}_{\omega^\perp}(F_n-F)\Big)+\Delta_\omega(f_n-f).\end{aligned}$$

For any fixed $p\in[1,\infty)$, it follows from the above equation that

$$\begin{aligned}&\frac{d}{dt}\int_D (f_n-f)^p dxd\omega\\ &\quad= -p\int_D (f_n-f)^{p-1}\nabla_\omega\cdot\Big((f_n-f)\nu(\omega\cdot F_n)\mathbb{P}_{\omega^\perp}F_n\Big)dxd\omega\\ &\quad\quad - p\int_D (f_n-f)^{p-1}\nabla_\omega\cdot\Big(f(\nu(\omega\cdot F_n)-\nu(\omega\cdot F))\mathbb{P}_{\omega^\perp}F_n\Big)dxd\omega\\ &\quad\quad - p\int_D (f_n-f)^{p-1}\nabla_\omega\cdot\Big(f\nu(\omega\cdot F)\mathbb{P}_{\omega^\perp}(F_n-F)\Big)dxd\omega\\ &\quad\quad + p\int_D (f_n-f)^{p-1}\Delta_\omega(f_n-f)dxd\omega\\ &\quad=: \mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3+\mathcal{J}_4.\end{aligned}$$

We follow the same computation as I_1 in the proof of the Lemma 3.1, to estimate

$$\begin{aligned}\mathcal{J}_1 &= -p\int_D (f_n-f)^{p-1}\Big(\nu(\omega\cdot F_n)\nabla_\omega(f_n-f)\cdot\mathbb{P}_{\omega^\perp}F_n + (f_n-f)\nu'(\omega\cdot F_n)|\mathbb{P}_{\omega^\perp}F_n|^2\\ &\qquad -(d-1)(f_n-f)\nu(\omega\cdot F_n)\omega\cdot F_n\Big)dxd\omega\\ &\le p\|\nu(\omega\cdot F_n)\|_{L^\infty}\|F_n\|_{L^\infty}\int_D (f_n-f)^{p-1}|\nabla_\omega(f_n-f)|dxd\omega\\ &\quad + p\|\nu'(\omega\cdot F_n)\|_{L^\infty}\|F_n\|^2_{L^\infty}\int_D (f_n-f)^p dxd\omega\\ &\quad + p(d-1)\|\nu(\omega\cdot F_n)\|_{L^\infty}\|F_n\|_{L^\infty}\int_D (f_n-f)^p dxd\omega.\\ &\le \frac{2(p-1)}{p}\int_D |\nabla_\omega(f_n-f)^{\frac{p}{2}}|^2 dxd\omega + \frac{Cp^2}{p-1}\int_D (f_n-f)^p dxd\omega\end{aligned}$$

By the same computation as I_2 in the proof of the Lemma 3.1, we have

$$\mathcal{J}_4 = -\frac{4(p-1)}{p}\int_D |\nabla_\omega(f_n-f)^{\frac{p}{2}}|^2 dxd\omega$$

Thus, we have

$$\frac{d}{dt}\int_D (f_n-f)^p dxd\omega \le C\int_D (f_n-f)^p dxd\omega - \frac{2(p-1)}{p}\int_D |\nabla_\omega(f_n-f)^{\frac{p}{2}}|^2 dxd\omega + \mathcal{J}_2+\mathcal{J}_3.$$

Since $f_n = f$ at $t = 0$, applying the Gronwall's inequality to the above inequality, we have that for any $0 < t \le T$,

$$\int_D (f_n - f)^p dx d\omega + \frac{2(p-1)}{p} \int_0^t \int_D |\nabla_\omega (f_n - f)^{\frac{p}{2}}|^2 dx d\omega ds \le e^{CT} \int_0^t (\mathcal{J}_2 + \mathcal{J}_3)(s) ds.$$

We use the formula (2.15) and the mean-value theorem to rewrite $\mathcal{J}_2$ as

$$\begin{aligned}
\mathcal{J}_2 = -p \int_D (f_n - f)^{p-1} \Big[&(\nu(\omega \cdot F_n) - \nu(\omega \cdot F)) \nabla_\omega f \cdot \mathbb{P}_{\omega^\perp} F_n \\
&+ f(\nu'(\omega \cdot F_n) F_n - \nu'(\omega \cdot F) F) \cdot \mathbb{P}_{\omega^\perp} F_n \\
&- (d-1) f (\nu(\omega \cdot F_n) - \nu(\omega \cdot F)) \omega \cdot F_n \Big] dx d\omega \\
= -p \int_D (f_n - f)^{p-1} \Big[&\nu'(\omega \cdot F_n^*) \omega \cdot (F_n - F) \nabla_\omega f \cdot \mathbb{P}_{\omega^\perp} F_n \\
&+ f \Big(\nu'(\omega \cdot F_n)(F_n - F) + \nu''(\omega \cdot F_n^{**}) \omega \cdot (F_n - F) F \Big) \cdot \mathbb{P}_{\omega^\perp} F_n \\
&- (d-1) f \nu'(\omega \cdot F_n^*) \omega \cdot (F_n - F) \omega \cdot F_n \Big] dx d\omega \\
= -p \int_D (f_n - f)^{p-1} \Big[&\nu'(\omega \cdot F_n^*) \nabla_\omega f \cdot \mathbb{P}_{\omega^\perp} F_n \omega \\
&+ f \nu'(\omega \cdot F_n) \mathbb{P}_{\omega^\perp} F_n + f \nu''(\omega \cdot F_n^{**}) F \cdot \mathbb{P}_{\omega^\perp} F_n \omega \\
&- (d-1) f \nu'(\omega \cdot F_n^*) \omega \cdot F_n \omega \Big] \cdot (F_n - F) dx d\omega,
\end{aligned}$$

where F_n^* and F_n^{**} are some bounded functions due to MVT.
Similarly, using (2.15), we rewrite $\mathcal{J}_3$ as

$$\begin{aligned}
\mathcal{J}_3 = -p \int_D (f_n - f)^{p-1} \Big[&\nu(\omega \cdot F) \nabla_\omega f \cdot \mathbb{P}_{\omega^\perp}(F_n - F) + f \nu'(\omega \cdot F) \mathbb{P}_{\omega^\perp} F \cdot \mathbb{P}_{\omega^\perp}(F_n - F) \\
&- (d-1) f \nu(\omega \cdot F) \omega \cdot (F_n - F) \Big] dx d\omega \\
= -p \int_D (f_n - f)^{p-1} \Big[&\nu(\omega \cdot F) \nabla_\omega f + f \nu'(\omega \cdot F) \mathbb{P}_{\omega^\perp} F - (d-1) f \nu(\omega \cdot F) \omega \Big] \cdot (F_n - F) dx d\omega.
\end{aligned}$$

Thus we get

$$\begin{aligned}
&\|f_n - f\|^p_{L^p(D)} + \frac{4(p-1)}{p} \int_0^T \int_D |\nabla_\omega (f_n - f)^{\frac{p}{2}}|^2 dx d\omega ds \\
&\le e^{CT} \int_0^T \int_D \Phi \cdot (F_n - F) dx d\omega ds,
\end{aligned} \tag{3.20}$$

where

$$\begin{aligned}
\Phi = -p(f_n - f)^{p-1} \Big[&\nu'(\omega \cdot F_n^*) \nabla_\omega f \cdot \mathbb{P}_{\omega^\perp} F_n \omega + f \nu'(\omega \cdot F_n) \mathbb{P}_{\omega^\perp} F_n + f \nu''(\omega \cdot F_n^{**}) F \cdot \mathbb{P}_{\omega^\perp} F_n \omega \\
&- (d-1) f \nu'(\omega \cdot F_n^*) \omega \cdot F_n \omega + \nu(\omega \cdot F) \nabla_\omega f + f \nu'(\omega \cdot F) \mathbb{P}_{\omega^\perp} F - (d-1) f \nu(\omega \cdot F) \omega \Big]
\end{aligned}$$

Once we can show $\Phi \in L^1((0,T) \times D)$, we can conclude the proof by the weak convergence F_n as (3.19). Thus, it remains to show the L^1 boundedness of ϕ.

To this end, using the uniform boundedness of (F_n), we have the same estimates as in Lemma 3.1, i.e., for each $g = f_n, f$,

$$\|g\|_{L^\infty(0,T;L^p(D))} \le C\|f_0\|_{L^p(D)}, \quad 1 \le p \le \infty,$$
$$\|\nabla_\omega g^{\frac{p}{2}}\|_{L^2((0,T)\times D)} \le C\|f_0\|^{p/2}_{L^p(D)}, \quad 1 \le p < \infty,$$

where the positive constant C only depends on p and T. Then using those estimates and Hölder's inequality, we have

$$\begin{aligned}\int_0^T \int_D (f_n - f)^{p-1}\nabla_\omega f dx d\omega ds &\le \Big(\int_D (f_n - f)^p dx d\omega\Big)^{1/2}\Big(\int_D (f_n - f)^{p-2}|\nabla_\omega f|^2 dx d\omega\Big)^{1/2}\\ &\le C\Big(\int_D (f_n^p + f^p) dx d\omega\Big)^{1/2}\Big(\int_D |\nabla_\omega f^{\frac{p}{2}}|^2 dx d\omega\Big)^{1/2}\\ &\le C_0,\end{aligned}$$

and

$$\begin{aligned}\int_0^T \int_D (f_n - f)^{p-1} f dx d\omega ds &\le \Big(\int_D (f_n - f)^p dx d\omega\Big)^{\frac{p-1}{p}}\Big(\int_D f^p dx d\omega\Big)^{\frac{1}{p}}\\ &\le C\Big(\int_D (f_n^p + f^p) dx d\omega\Big)^{\frac{p-1}{p}}\Big(\int_D f^p dx d\omega\Big)^{\frac{1}{p}}\\ &\le C_0,\end{aligned}$$

where the positive constant C_0 depends on $\|f_0\|_{L^p(D)}$. Thus we have

$$\begin{aligned}\|\Phi\|_{L^1((0,T)\times D)} &\le C_*(\|(f_n - f)^{p-1}\nabla_\omega f\|_{L^1((0,T)\times D)} + \|(f_n - f)^{p-1} f\|_{L^1((0,T)\times D)})\\ &\le C_* C_0,\end{aligned}$$

where C_* is a positive constant as

$$\begin{aligned}C_* = pd\Big[&\Big((\|\nu'(\omega\cdot F_n^*)\|_{L^\infty} + \|\nu'(\omega\cdot F_n)\|_{L^\infty}) + \|\nu''(\omega\cdot F_n^{**})\|_{L^\infty}\|F\|_{L^\infty}\Big)\|F_n\|_{L^\infty}\\ &+ \|\nu(\omega\cdot F)\|_{L^\infty} + \|\nu'(\omega\cdot F)\|_{L^\infty}\|F\|_{L^\infty}\Big],\end{aligned}$$

which does not depend on n thanks to the uniform boundedness of (F_n).
Therefore, by applying (3.19), it follows from (3.20) that

$$\begin{aligned}& f_n \to f \quad \text{in } L^p((0,T)\times D),\\ & \nabla_\omega f_n \to \nabla_\omega f \quad \text{in } L^2((0,T)\times D).\end{aligned} \tag{3.21}$$

It remains to show that (3.21) implies the strong convergence of the associated sequences $(\bar{J}_n)$ and $(\tilde{J}_n)$. For $(\bar{J}_n)$, using the Minkowski inequality, Hölder's inequality and Young's inequality, we estimate

$$\begin{aligned}\|\bar{J}_n - \bar{J}\|_{L^p((0,T)\times\mathbb{R}^n)} &= \Big(\int_0^T \int_{\mathbb{R}^d} \Big|\int_{\mathbb{S}^{d-1}} K *_x (f_n - f)\omega d\omega\Big|^p dx ds\Big)^{\frac{1}{p}}\\ &\le \int_{\mathbb{S}^{d-1}} \Big(\int_0^T \int_{\mathbb{R}^d} |K *_x (f_n - f)|^p dx ds\Big)^{\frac{1}{p}} d\omega\\ &\le C\Big(\int_0^T \int_{\mathbb{S}^{d-1}} \|K *_x (f_n - f)\|^p_{L^p(\mathbb{R}^d)} d\omega ds\Big)^{\frac{1}{p}}\\ &\le C\|K\|_{L^1(\mathbb{R}^d)}\|f_n - f\|_{L^p((0,T)\times D)}.\end{aligned}$$

Similarly we have

$$\begin{aligned}\|\tilde{J}_n - \tilde{J}\|_{L^p((0,T)\times\mathbb{R}^n)} &= \Big(\int_0^T \int_{\mathbb{R}^d} \Big| \int_{\mathbb{S}^{d-1}} \omega(f_n - f) d\omega \Big|^p dxds\Big)^{\frac{1}{p}} \\ &\le \int_{\mathbb{S}^{d-1}} \Big(\int_0^T \int_{\mathbb{R}^d} |f_n - f|^p dxds\Big)^{\frac{1}{p}} d\omega \\ &\le C\|f_n - f\|_{L^p((0,T)\times D)}.\end{aligned}$$

□

4. Proof of Theorem 2.1

In this section, we use the a priori estimates and compactness property in previous lemmas to prove Theorem 2.1. For this end, we intend to use iteration scheme to construct a sequence of solutions (f_n) to linear approximate equation of (2.6), for which at n-th step, $\Omega(f_{n-1})$ in the force term will be given from previous $(n-1)$-th step. In this iteration, even if we suppose (2.7) or (2.8), since the linear approximate equation is different from the original equation (2.6), the corresponding momentum $J(f_n)$ of the solution f_n of approximate equation at n-th step can be zero, which fails in defining $\Omega(f_n)$ at next step. In order to remove this issue of singularity, we need to first regularize the equation (2.6).

4.1. **Regularized equation.** We first regularize (1.1) and (1.2) by adding $\varepsilon > 0$ to the denominator of Ω as follows.

$$\begin{aligned}&\partial_t f_\varepsilon + \omega\cdot\nabla_x f_\varepsilon = -\nabla_\omega \cdot \Big(f_\varepsilon \nu(\omega\cdot\Omega_\varepsilon)\mathbb{P}_{\omega^\perp}\Omega_\varepsilon\Big) + \Delta_\omega f_\varepsilon, \\ &\Omega_\varepsilon(x,t) = \frac{J_\varepsilon(x,t)}{|J_\varepsilon(x,t)| + \varepsilon}, \\ &f_\varepsilon(x,\omega,0) = f_0(x,\omega),\ x\in\mathbb{R}^d,\ \omega\in\mathbb{S}^{d-1},\ t>0.\end{aligned} \tag{4.22}$$

where $J_\varepsilon(x,t)$ denotes one of either $\bar{J}(f_\varepsilon)(x,t)$ or $\tilde{J}(f_\varepsilon)(x,t)$.

4.2. **Construction of approximate solutions.** We now construct a sequence of approximate solutions to the regularized equation (4.22) by using the following iteration scheme. We define a initial function f_1 by

$$f_1(x,\omega,t) = f_0(x,\omega), \quad (x,\omega)\in D, \quad t\ge 0.$$

Then, we find f_2 solving

$$\begin{aligned}&\partial_t f_2 + \omega\cdot\nabla_x f_2 = -\nabla_\omega\cdot\Big(f_2\nu(\omega\cdot\Omega_1)\mathbb{P}_{\omega^\perp}\Omega_1\Big) + \Delta_\omega f_2, \\ &\Omega_1(x,t) = \frac{J_1(x,t)}{|J_1(x,t)|+\varepsilon}, \quad J_1(x,t) = \text{either } \bar{J}(f_1)(x,t) \text{ or } \tilde{J}(f_1)(x,t), \\ &f_2(x,\omega,0) = f_0(x,\omega).\end{aligned}$$

Inductively, we define f_{n+1} as a solution of

$$\begin{aligned}&\partial_t f_{n+1} + \omega\cdot\nabla_x f_{n+1} = -\nabla_\omega\cdot\Big(f_{n+1}\nu(\omega\cdot\Omega_n)\mathbb{P}_{\omega^\perp}\Omega_n\Big) + \Delta_\omega f_{n+1}, \\ &\Omega_n(x,t) = \frac{J_n(x,t)}{|J_n(x,t)|+\varepsilon}, \quad J_n(x,t) = \text{either } \bar{J}(f_n)(x,t) \text{ or } \tilde{J}(f_n)(x,t), \\ &f_{n+1}(x,\omega,0) = f_0(x,\omega),\end{aligned} \tag{4.23}$$

We have omitted ε-dependence in $f_{\varepsilon,n}$ above for the notational simplicity. First of all, we need to justify the solvability of the approximate equations (4.23) for $n \geq 1$ as follows.

Lemma 4.1. *For any $T > 0$ and fixed $n \geq 1$, assume that f_n is a given integrable function and f_0 satisfies* (2.9)*. Then, there exists a unique solution $f_{n+1} \geq 0$ to the linear equation* (4.23) *satisfying the L^p-estimates:*

$$\|f_{n+1}\|_{L^\infty(0,T;L^p(D))} + \frac{2(p-1)}{p}\|\nabla_\omega f_{n+1}^{\frac{p}{2}}\|_{L^2((0,T)\times D)}^{\frac{2}{p}} \leq e^{CT\frac{p}{p-1}}\|f_0\|_{L^p(D)}, \tag{4.24}$$

and

$$\|f_{n+1}\|_{L^\infty((0,T)\times D)} \leq e^{CT}\|f_0\|_{L^\infty(D)}. \tag{4.25}$$

The proof of Lemma 4.1 follows the same argument as Degond's proof in [5]. We postpone its proof in Appendix for the reader's convenience.

4.3. **Passing to the limit as $n \to \infty$.** We now show that f_n converges to the solution of regularized equation (4.22), which provides the existence of weak solution to (4.22) as follows.

Proposition 4.1. *For a given $T > 0$ and $\varepsilon > 0$, if f_0 satisfies* (2.9)*, then there exists a weak solution $f_\varepsilon \geq 0$ to the* (4.22) *satisfying the L^p-estimates: for $1 \leq p < \infty$,*

$$\|f_\varepsilon\|_{L^\infty(0,T;L^p(D))} + \frac{2(p-1)}{p}\|\nabla_\omega f_\varepsilon^{\frac{p}{2}}\|_{L^2((0,T)\times D)}^{\frac{2}{p}} \leq e^{CT\frac{p}{p-1}}\|f_0\|_{L^p(D)}, \tag{4.26}$$

and

$$\|f_\varepsilon\|_{L^\infty((0,T)\times D)} \leq e^{CT}\|f_0\|_{L^\infty(D)}. \tag{4.27}$$

Proof. Since the sequence (Ω_n) defined in (4.23) is bounded in $L^\infty((0,T)\times\mathbb{R}^d)$, we can apply Lemma 3.2 with $F_n = \Omega_n$. Then, there exists a limit function f_ε such that up to a subsequence,

$$\begin{aligned} &f_n \to f_\varepsilon \quad \text{as } n\to\infty \text{ in } L^p((0,T)\times\mathbb{R}^d)\cap L^2((0,T)\times\mathbb{R}^d;H^1(\mathbb{S}^{d-1})),\\ &J_n \to J_\varepsilon \quad \text{as } n\to\infty \text{ in } L^p((0,T)\times\mathbb{R}^d), \end{aligned}$$

where the pair (J_n, J_ε) is either $(\bar{J}(f_n), \bar{J}(f_\varepsilon))$ or $(\tilde{J}(f_n), \tilde{J}(f_\varepsilon))$.
Moreover, this yields

$$\Omega_n \;\to\; \Omega_\varepsilon := \frac{J_\varepsilon}{|J_\varepsilon|+\varepsilon} \quad \text{as } n\to\infty \;\text{ in } L^\infty(0,T;L^p(D)).$$

Indeed, this is derived from

$$\begin{aligned} &\int_{\mathbb{R}^d}|\Omega_n-\Omega_\varepsilon|^p dx\\ &\quad= \int_{\mathbb{R}^d}\Big|\frac{\varepsilon(J_n-J_\varepsilon)+|J_\varepsilon|(J_n-J_\varepsilon)+J_\varepsilon(|J_\varepsilon|-|J_n|)}{(|J_n|+\varepsilon)(|J_\varepsilon|+\varepsilon)}\Big|^p dx\\ &\quad\leq \frac{1}{\varepsilon^p}\int_{\mathbb{R}^d}\Big|\frac{\varepsilon(J_n-J_\varepsilon)+|J_\varepsilon|(J_n-J_\varepsilon)+J_\varepsilon(|J_\varepsilon|-|J_n|)}{|J_\varepsilon|+\varepsilon}\Big|^p dx\\ &\quad\leq C(\varepsilon)\int_{\mathbb{R}^d}\Big(|J_n-J_\varepsilon|^p+|J_n-J_\varepsilon|^p+||J_n|-|J_\varepsilon||^p\Big)dx\\ &\quad\leq C(\varepsilon)\int_{\mathbb{R}^d}|J_n-J_\varepsilon|^p dx, \end{aligned}$$

Therefore, those imply that the limit f_ε satisfies the following weak formulation of (4.22): for all $\phi \in C_c^\infty([0,T)\times\mathbb{R}^d\times\mathbb{S}^{d-1})$,

$$\begin{aligned}\int_0^t\int_D f_\varepsilon\partial_t\phi + f_\varepsilon\omega\cdot\nabla_x\phi + f_\varepsilon F_\varepsilon\cdot\nabla_\omega\phi - \nabla_\omega f_\varepsilon\cdot\nabla_\omega\phi dxd\omega ds\\ + \int_D f_0\phi(0,\cdot)dxd\omega = 0,\end{aligned}$$

$$F_\varepsilon = \nu(\omega\cdot\Omega_\varepsilon)\mathbb{P}_{\omega^\perp}\Omega_\varepsilon, \quad \Omega_\varepsilon(x,t) = \frac{J_\varepsilon(x,t)}{|J_\varepsilon(x,t)|+\varepsilon},$$

$$J_\varepsilon(x,t) = \text{either } \bar{J}(f_\varepsilon) \text{ or } \tilde{J}(f_\varepsilon).$$

In addition, we use the proof of Lemma 3.1 and the boundedness of Ω_ε above, to complete the L^p estimates (4.26) and (4.27). □

4.4. **Passing to the limit as $\varepsilon \to 0$.** In this part, we complete the proof of Theorem 2.1 by showing the convergence of (4.22) to (2.6) in the weak sense, as $\varepsilon\to 0$. For the convenience, we denote a sequence $f_n := f_{\varepsilon_n}$ for a convergent sequence $\varepsilon_n \to 0$, then consider a sequence

$$F_n := \frac{J_n}{|J_n|+\varepsilon_n}, \quad J_n = \int_{\mathbb{S}^{d-1}} \omega f_n d\omega.$$

Since the sequence (F_n) defined above is bounded in $L^\infty((0,T)\times\mathbb{R}^d)$, we can apply Lemma 3.2. Then, there exists a limit function f such that up to a subsequence,

$$\begin{aligned}&f_n \to f \quad \text{as } n\to\infty \text{ in } L^p((0,T)\times\mathbb{R}^d)\cap L^2((0,T)\times\mathbb{R}^d; H^1(\mathbb{S}^{d-1})),\\ &J_n \to J \quad \text{as } n\to\infty \text{ in } L^p((0,T)\times\mathbb{R}^d),\end{aligned} \tag{4.28}$$

where the pair (J_n, J) is either $(\bar{J}(f_n), \bar{J}(f))$ or $(\tilde{J}(f_n), \tilde{J}(f))$.
We may show that f is the weak solution to (2.6), that is, f satisfies the weak formulation (2.10) as the limit of the following formulation for (4.22):

$$\begin{aligned}\int_0^t\int_D f_n\partial_t\phi + f_n\omega\cdot\nabla_x\phi + f_n\nu\Big(\frac{\omega\cdot J_n}{|J_n|+\varepsilon_n}\Big)\mathbb{P}_{\omega^\perp}\frac{J_n}{|J_n|+\varepsilon_n}\cdot\nabla_\omega\phi - \nabla_\omega f_n\cdot\nabla_\omega\phi dxd\omega ds\\ + \int_D f_0\phi(0,\cdot)dxd\omega = 0,\end{aligned}$$

for any $\phi\in C_c^\infty([0,T)\times\mathbb{R}^d\times\mathbb{S}^{d-1})$.
By the convergence of f_n in (4.28), we can easily show that all terms in the formulation above except for the nonlinear force term converge to the corresponding terms in (2.10). On the other hand, the convergence of the nonlinear term requires further justification as follows.

Lemma 4.2. *As $n\to\infty$,*

$$\int_0^t\int_D f_n\nu\Big(\frac{\omega\cdot J_n}{|J_n|+\varepsilon_n}\Big)\mathbb{P}_{\omega^\perp}\frac{J_n}{|J_n|+\varepsilon_n}\cdot\nabla_\omega\phi dxd\omega ds \to \int_0^t\int_D f\nu\Big(\frac{\omega\cdot J}{|J|}\Big)\mathbb{P}_{\omega^\perp}\frac{J}{|J|}\cdot\nabla_\omega\phi dxd\omega ds,$$

when $|J(x,t)|>0$ by the assumption (2.7) *and* (2.8).

Proof. We here omit the projection operator $\mathbb{P}_{\omega^\perp}$ thanks to (2.14) for convenience, i.e., we show that

$$\int_0^t\int_D f_n\nu\Big(\frac{\omega\cdot J_n}{|J_n|+\varepsilon_n}\Big)\frac{J_n}{|J_n|+\varepsilon_n}\cdot\nabla_\omega\phi dxd\omega ds \to \int_0^t\int_D f\nu\Big(\frac{\omega\cdot J}{|J|}\Big)\frac{J}{|J|}\cdot\nabla_\omega\phi dxd\omega ds,$$

First of all, since (4.27) and ν is bounded, there is a uniform constant C such that

$$\Big\|f_n\nu\Big(\frac{\omega\cdot J_n}{|J_n|+\varepsilon_n}\Big)\frac{J_n}{|J_n|+\varepsilon_n}\Big\|_{L^\infty((0,T)\times D)}\le\|f_n\|_{L^\infty((0,T)\times D)}\|\nu\|_{L^\infty}\le C.$$

which implies that for some F,

$$f_n\nu\Big(\frac{\omega\cdot J_n}{|J_n|+\varepsilon_n}\Big)\frac{J_n}{|J_n|+\varepsilon_n}\rightharpoonup F\quad\text{weakly}-*\text{ in }L^\infty((0,T)\times D).$$

Thus it remains to show

$$F=f\nu\Big(\frac{\omega\cdot J}{|J|}\Big)\frac{J}{|J|},$$

on $\{(t,x,\omega)\in(0,T]\times\mathbb{R}^d\times\mathbb{S}^{d-1}\mid |J(x,t)|>0\}$.
For that, we consider a bounded set

$$X_{R,\delta}:=\{(t,x,\omega)\in(0,T]\times B_R(0)\times\mathbb{S}^{d-1}\mid |J(x,t)|>\delta\},$$

where R and δ are any positive constants, and $B_R(0)$ denote the ball with radius R, centered at 0 in $\mathbb{R}^d$.
Since $f_n\to f$ and $J_n\to J$ a.e. on $X_{R,\delta}$ by (4.28), we use Egorov's theorem to have that for any $\eta>0$, there exists $Y_\eta\subset X_{R,\delta}$ such that $|X_{R,\delta}\backslash Y_\eta|<\eta$ and

$$f_n\to f,\quad J_n\to J\quad\text{in }L^\infty(Y_\eta).$$

Thus, for sufficiently large n,

$$|J_n(x,t)|>\frac{\delta}{2}\quad\text{for }(x,t)\in Y_\eta,$$

which allows us to get

$$\begin{aligned}
&\Big\|\frac{J_n}{|J_n|+\varepsilon_n}-\frac{J}{|J|}\Big\|_{L^\infty(Y_\eta)}\\
&\quad=\Big\|\frac{|J|(J_n-J)+J(|J|-|J_n|)-\varepsilon_nJ}{(|J_n|+\varepsilon_n)|J|}\Big\|_{L^\infty(Y_\eta)}\\
&\quad\le\frac{2}{\delta}\Big\|\frac{|J|(J_n-J)+J(|J|-|J_n|)-\varepsilon_nJ}{|J|}\Big\|_{L^\infty(Y_\eta)}\\
&\quad\le\frac{2}{\delta}\Big(\|J_n-J\|_{L^\infty(Y_\eta)}+\||J_n|-|J|\|_{L^\infty(Y_\eta)}-\varepsilon_n\Big)\\
&\quad\to0.
\end{aligned}$$

This yields

$$\begin{aligned}
&\Big\|f_n\nu\Big(\frac{\omega\cdot J_n}{|J_n|+\varepsilon_n}\Big)\frac{J_n}{|J_n|+\varepsilon_n}-f\nu\Big(\frac{\omega\cdot J}{|J|}\Big)\frac{J}{|J|}\Big\|_{L^\infty(Y_\eta)}\\
&\quad=\Big\|f_n\Big[\nu\Big(\frac{\omega\cdot J_n}{|J_n|+\varepsilon_n}\Big)-\nu\Big(\frac{\omega\cdot J}{|J|}\Big)\Big]\frac{J_n}{|J_n|+\varepsilon_n}\Big\|_{L^\infty(Y_\eta)}\\
&\qquad+\Big\|f_n\nu\Big(\frac{\omega\cdot J}{|J|}\Big)\Big(\frac{J_n}{|J_n|+\varepsilon_n}-\frac{J}{|J|}\Big)\Big\|_{L^\infty(Y_\eta)}+\Big\|(f_n-f)\nu\Big(\frac{\omega\cdot J_n}{|J_n|+\varepsilon_n}\Big)\frac{J_n}{|J_n|+\varepsilon_n}\Big\|_{L^\infty(Y_\eta)}\\
&\quad\le C\|f_n\|_{L^\infty}(\|\nu'\|_{L^\infty}+\|\nu\|_{L^\infty})\Big\|\frac{J_n}{|J_n|+\varepsilon_n}-\frac{J}{|J|}\Big\|_{L^\infty(Y_\eta)}+C\|f_n-f\|_{L^\infty(Y_\eta)}\|\nu\|_{L^\infty}\\
&\quad\to0.
\end{aligned}$$

Thus we have

$$F = f\nu\Big(\frac{\omega\cdot J}{|J|}\Big)\frac{J}{|J|} \quad \text{on } Y_\eta.$$

Since η, R and δ are arbitrary, taking $\eta,\delta \to 0$ and $R\to\infty$,

$$F = f\nu\Big(\frac{\omega\cdot J}{|J|}\Big)\frac{J}{|J|} \quad \text{on } \{(t,x,\omega)\in(0,T]\times\mathbb{R}^d\times\mathbb{S}^{d-1} \mid |J(x,t)|>0\}.$$

Therefore, we complete the convergence. □

Thanks to lemma above and (4.28), we conclude that f satisfies the weak formulation (2.10).

On the other hand, the estimates (2.11) and (2.12) follow directly from the estimates (4.26) and (4.27).

Appendix A. Proof of Lemma 4.1

We here prove the existence of solution f to the linear equation

$$\begin{aligned}
&\partial_t f + \omega\cdot\nabla_x f = -\nabla_\omega\cdot\Big(f\nu(\omega\cdot\bar\Omega)\mathbb{P}_{\omega^\perp}\bar\Omega\Big) + \Delta_\omega f,\\
&\bar\Omega = \frac{\bar J(x,t)}{|\bar J(x,t)|+\varepsilon}, \quad \bar J(x,t) = \int_{\mathbb{S}^{d-1}} \omega g(x,\omega,t)d\omega,\\
&f(x,\omega,0) = f_0(x,\omega),
\end{aligned} \tag{A.29}$$

where g is a given integrable function.

We begin by rewriting (A.29) as

$$\begin{aligned}
&\partial_t f + \omega\cdot\nabla_x f + \nu(\omega\cdot\bar\Omega)\mathbb{P}_{\omega^\perp}\bar\Omega\cdot\nabla_\omega f\\
&\qquad + f\nu'(\omega\cdot\bar\Omega)|\mathbb{P}_{\omega^\perp}\bar\Omega|^2 - (d-1)f\nu(\omega\cdot\bar\Omega)\omega\cdot\bar\Omega - \Delta_\omega f = 0,\\
&f(x,\omega,0) = f_0(x,\omega),
\end{aligned} \tag{A.30}$$

where we have used the formula (2.15).

Then we consider a new function $\bar f(x,\omega,t) := e^{-\lambda t}f(x,\omega,t)$ for a given $\lambda>0$, which leads to

$$\begin{aligned}
&\partial_t\bar f + \omega\cdot\nabla_x\bar f + \psi_1\cdot\nabla_\omega\bar f + \Big(\lambda+\psi_2+\psi_3\Big)\bar f - \Delta_\omega\bar f = 0,\\
&\bar f(x,\omega,0) = f_0(x,\omega),
\end{aligned} \tag{A.31}$$

where the functions ψ_1, ψ_2 and ψ_3 are given by

$$\begin{aligned}
\psi_1(x,\omega,t) &= \nu(\omega\cdot\bar\Omega)\mathbb{P}_{\omega^\perp}\bar\Omega,\\
\psi_2(x,\omega,t) &= \nu'(\omega\cdot\bar\Omega)|\mathbb{P}_{\omega^\perp}\bar\Omega|^2,\\
\psi_3(x,\omega,t) &= -(d-1)\nu(\omega\cdot\bar\Omega)\omega\cdot\bar\Omega.
\end{aligned}$$

Since $|\bar\Omega|\le 1$ and the smooth function ν is bounded, ψ_1, ψ_2 and ψ_3 are also all bounded. Therefore, the Lions' theorem [19] guarantees the existence of (A.31) by the same argument as [5]. More precisely, the equation (A.31) has a solution $\bar f$ in the space

$$Y := \{f\in L^2([0,T]\times\mathbb{R}^d;H^1(\mathbb{S}^{d-1})) \mid \partial_t f+\omega\cdot\nabla_x f\in L^2([0,T]\times\mathbb{R}^d;H^{-1}(\mathbb{S}^{d-1}))\}.$$

Furthermore, by the Green formula in [5], we have the fact that for any $f\in Y$,

$$< \partial_t f + \omega\cdot\nabla_x f, f >= \frac{1}{2}\int_D (|f(x,\omega,T)|^2 - |f(x,\omega,0)|^2)dxd\omega, \tag{A.32}$$

where $<,>$ denotes the pairing of $L^2([0,T]\times\mathbb{R}^d;H^{-1}(\mathbb{S}^{d-1}))$ and $L^2([0,T]\times\mathbb{R}^d;H^1(\mathbb{S}^{d-1}))$. We use (A.32) to show the uniqueness of solutions f in Y as follows.
Let us consider a solution $\bar f\in Y$ to (A.31) with initial data $f_0=0$. Then by using (A.32), we have

(A.33)
$$\begin{aligned}
0 &=< \partial_t\bar f+\omega\cdot\nabla_x\bar f+\psi_1\cdot\nabla_\omega\bar f+(\lambda+\psi_2+\psi_3)\bar f-\Delta_\omega\bar f,\bar f>\\
&=\frac12\int_D|\bar f(x,\omega,T)|^2dxd\omega-\frac12\int_D\nabla_\omega\cdot\psi_1|\bar f|^2dxd\omega\\
&\quad+\int_D(\lambda+\psi_2+\psi_3)|\bar f|^2dxd\omega+\int_D|\nabla_\omega\bar f|^2dxd\omega\\
&\ge\Big(\lambda-\frac12\|\nabla_\omega\cdot\psi_1\|_{L^\infty([0,T]\times D)}-\|\psi_2\|_{L^\infty([0,T]\times D)}-\|\psi_3\|_{L^\infty([0,T]\times D)}\Big)\int_D|\bar f|^2dxd\omega.
\end{aligned}$$

Since
$$\begin{aligned}
\nabla_\omega\cdot\psi_1&=\nu'(\omega\cdot\bar\Omega)\nabla_\omega(\omega\cdot\bar\Omega)\cdot\mathbb{P}_{\omega^\perp}\bar\Omega+\nu(\omega\cdot\bar\Omega)\nabla_\omega\cdot\mathbb{P}_{\omega^\perp}\bar\Omega\\
&=\nu'(\omega\cdot\bar\Omega)|\mathbb{P}_{\omega^\perp}\bar\Omega|^2-(d-1)\nu(\omega\cdot\bar\Omega)\omega\cdot\bar\Omega,
\end{aligned}$$
$\nabla_\omega\cdot\psi_1$ is bounded. Thus we choose λ such that

$$\lambda>\frac12\|\nabla_\omega\cdot\psi_1\|_{L^\infty([0,T]\times D)}+\|\psi_2\|_{L^\infty([0,T]\times D)}+\|\psi_3\|_{L^\infty([0,T]\times D)},\tag{A.34}$$

then (A.33) yields $\bar f=0$, which proves the uniqueness of the linear equation (A.31).

Therefore, (A.31) has a unique solution $\bar f\in L^2([0,T]\times\mathbb{R}^d;H^1(\mathbb{S}^{d-1}))$.
Furthermore, since $f_0\ge0$ and $f_0\in L^\infty(D)$, by using the similar argument as (A.33), we can show
$$\bar f\ge0\quad\text{and}\quad\bar f\in L^\infty([0,T]\times D).$$
For that, we use the following fact in [5], for any $f\in Y$,
$$<\partial_t f+\omega\cdot\nabla_x f,f_->=\frac12\int_D(|f_-(x,\omega,0)|^2-|f_-(x,\omega,T)|^2)dxd\omega,$$
where $f_-:=\max(-f,0)$.
Then, since $f_-(x,\omega,0)=0$ by $f_0\ge0$, we have
$$\begin{aligned}
0&=<\partial_t\bar f+\omega\cdot\nabla_x\bar f+\psi_1\cdot\nabla_\omega\bar f+(\lambda+\psi_2+\psi_3)\bar f-\Delta_\omega\bar f,\bar f_->\\
&=-\frac12\int_D|\bar f_-(x,\omega,T)|^2dxd\omega+\frac12\int_D\nabla_\omega\cdot\psi_1|\bar f_-|^2dxd\omega\\
&\quad-\int_D(\lambda+\psi_2+\psi_3)|\bar f_-|^2dxd\omega-\int_D|\nabla_\omega\bar f_-|^2dxd\omega\\
&\le-\Big(\lambda-\frac12\|\nabla_\omega\cdot\psi_1\|_{L^\infty([0,T]\times D)}-\|\psi_2\|_{L^\infty([0,T]\times D)}-\|\psi_3\|_{L^\infty([0,T]\times D)}\Big)\int_D|\bar f_-|^2dxd\omega.
\end{aligned}$$
By the same choice as (A.34), we $\bar f_-=0$, which proves $\bar f\ge0$.
The same argument also deduces
$$\|\bar f\|_{L^\infty([0,T]\times D)}\le\|f_0\|_{L^\infty(D)}.$$
We now go back to (A.30) by using the transformation $f(x,\omega,t)=e^{\lambda t}\bar f(x,\omega,t)$. Since our results for $\bar f$ are invariant under this transformation, we conclude the proof, together with

the estimates (4.24) and (4.25), which is estimated by using the proof of Lemma 3.1 and the boundedness of $\bar{\Omega}$.

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