Quantitative estimates of propagation of chaos for stochastic systems with $W^{-1,\infty}$ kernels

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Abstract

We derive quantitative estimates proving the propagation of chaos for large stochastic systems of interacting particles. We obtain explicit bounds on the relative entropy between the joint law of the particles and the tensorized law at the limit. We have to develop for this new laws of large numbers at the exponential scale. But our result only requires very weak regularity on the interaction kernel in the negative Sobolev space $\dot{W}^{-1,\infty}$, thus including the Biot-Savart law and the point vortices dynamics for the 2d incompressible Navier-Stokes.

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1 Introduction

1.1 Motivation

We consider large systems of N indistinguishable point-particles given by the coupled stochastic differential equations (SDEs)

$$\mathrm{d}X_i = F(X_i)\,\mathrm{d}t + \frac{1}{N}\sum_{j\neq i} K(X_i - X_j)\,\mathrm{d}t + \sqrt{2\sigma_N}\,\mathrm{d}W_t^i, \quad i = 1, \cdots, N \tag{1.1}$$

where for simplicity $X_i \in \Pi^d$, the d-dimensional torus, the W^i are N independent standard Wiener Processes (Brownian motions) in $\mathbb{R}^d$ and the stochastic term in (1.1) should be understood in the Itô sense.

The interaction term is normalized by the factor $1/N$, corresponding to the mean field scaling. For a fixed N our goal is hence to derive explicit, quantitative estimates comparing System (1.1) to the *mean field limit* $\bar{\rho}$ solving

$$\partial_t \bar{\rho} + \operatorname{div}_x(\bar{\rho}\,[F + K \star_x \bar{\rho}]) = \sigma\,\Delta\bar{\rho}. \tag{1.2}$$

Such estimates in particular imply the propagation of chaos in the limit $N \to \infty$. But precisely because they are quantitative, they also characterize the reduction of complexity of System (1.1) for large and finite N.

A guiding motivation of interaction kernel K in our work is given by the Biot-Savart law in dimension 2, namely

$$K(x) = \alpha \, \frac{x^\perp}{|x|^2} + K_0(x), \tag{1.3}$$

where $x^\perp$ denotes the rotation of vector x by $\pi/2$ and where K_0 is a smooth correction to periodize K on the torus represented by $[-1/2,\ 1/2]^d$. If $\omega(x) \in L^p(\Pi^d)$ with $p \geq 1$, then $u = K \star_x \omega$ solves

$$\operatorname{curl} u = \operatorname{curl} K \star_x \omega = \alpha \left(\omega - \int_{\Pi^d} \omega \right), \qquad \operatorname{div} u = \operatorname{div} K \star \omega = 0.$$

If $F = 0$, the limiting equation (1.2) becomes

$$\partial_t \omega + K \star_x \omega \cdot \nabla_x \omega = \sigma \, \Delta \omega, \tag{1.4}$$

where we now write on $\omega(t, x)$, using the classical notation for the vorticity of a fluid. Eq. (1.4) is invariant by the addition of a constant $\omega \to \omega + C$. We may hence assume that $\int_{\Pi^d} \omega = 0$ and Eq. (1.4) is then equivalent to the 2d incompressible Navier-Stokes system on $u(t, x)$ s.t. $\omega = \operatorname{curl} u$,

$$\begin{aligned} &\partial_t u + u \cdot \nabla_x u = \nabla_x p + \sigma \, \Delta u, \\ &\operatorname{div} u = 0. \end{aligned} \tag{1.5}$$

The system of particles (1.1) now corresponds to a system of interacting point vortices with additive noise. Because we present our method in the simplest framework where particles are indistinguishable, all point vortices necessarily have the same vorticity in this setting.

Our main result provides an *explicit estimate quantifying that the system* (1.1) *is within* $O(N^{-1/2})$ *from the limit* (1.2) in an appropriate statistical sense. However the method that we develop is able to handle general kernels K which are not necessarily singular only at the origin or may not even be functions.

The rest of the introduction is organized as follows: We state precisely our main result in the next subsection. We devote subsection 1.3 to a longer discussion of various examples of kernels K that are covered by our main theorem. While our main focus concerns systems with non-vanishing diffusion, the tools that are developed in this article can also be applied to settings with vanishing diffusion. We give an example of such a result in subsection 1.4. The last subsection in the introduction sketches the proof of our basic *a priori* estimates.

Section 2 presents the proof of our main results, assuming that one has two critical estimates, Theorems 3 and 4, which are sort of modified laws of large numbers. We establish some preliminary combinatorics notations in section 3. This enables us to easily prove Theorem 3 in section 4. The proof of Theorem 4 is considerably more difficult; it is performed in section 5 which is the main technical contribution of this article.

1.2 Main results

We start by recalling the precise definition of the space $\dot{W}^{-1,\infty}(\Pi^d)$ which is used both in Prop. 1 and in Theorem 1 and which is critical to our applications.

Definition 1. *A function f with $\int_{\Pi^d} f = 0$ belongs to $\dot{W}^{-1,\infty}(\Pi^d)$ iff there exists a vector field g in $L^\infty(\Pi^d)$ s.t. $f = \operatorname{div} g$. Similarly a vector field K with $\int_{\Pi^d} K = 0$ belongs to $\dot{W}^{-1,\infty}(\Pi^d)$ iff there exists a matrix field V in $L^\infty(\Pi^d)$ s.t. $K = \operatorname{div} V$ or $K_\alpha = \sum_\beta \partial_\beta V_{\alpha\beta}$. We then denote*

$$\|f\|_{\dot{W}^{-1,\infty}} = \inf_g \|g\|_{L^\infty}, \quad \textit{with} \;\; f = \operatorname{div} g,$$

and similarly

$$\|K\|_{\dot{W}^{-1,\infty}} = \inf_V \|V\|_{L^\infty}, \quad \textit{with} \;\; K = \operatorname{div} V.$$

Following the basic approach introduced in [29], our main idea is to use relative entropy methods to compare the *coupled law* $\rho_N(t, x_1, \ldots, x_N)$ of the whole system (1.1) to the tensorized law

$$\bar{\rho}_N(t, x_1, \ldots, x_N) = \bar{\rho}^{\otimes^N} = \Pi_{i=1}^N \bar{\rho}(t, x_i),$$

consisting of N independent copies of a process following the law $\bar{\rho}$, solution to the limiting equation (1.2).

As our estimates carry over ρ_N, we do not consider directly the system of SDEs (1.1) but instead work at the level of the Liouville equation

$$\partial_t \rho_N + \sum_{i=1}^N \operatorname{div}_{x_i} \left(\rho_N \left(F(x_i) + \frac{1}{N} \sum_{j=1}^N K(x_i - x_j) \right) \right) = \sum_{i=1}^N \sigma_N \, \Delta_{x_i} \rho_N, \tag{1.6}$$

where and hereafter we use the convention that $K(0) = 0$. The law ρ_N encompasses all the statistical information about the system. Given that it is set in $\Pi^{d\,N}$ with $N >> 1$, the observable statistical information is typically contained in the marginals

$$\rho_{N,k}(t, x_1, \ldots, x_k) = \int_{\Pi^{d\,(N-k)}} \rho_N(t, x_1, \ldots, x_N)\ \mathrm{d}x_{k+1} \ldots\ \mathrm{d}x_N. \tag{1.7}$$

Our final goal is to obtain explicit bounds on $\rho_{N,k} - \bar{\rho}^{\otimes^k}$, where $\bar{\rho}^{\otimes^k} = \Pi_{i=1}^k \bar{\rho}(t, x_i)$. Those bounds will follow from a relative entropy estimate between $\bar{\rho}_N$ and a solution ρ_N to (1.6). But for this, we cannot use any weak solution to the Liouville (1.6) and instead require

Definition 2 (Entropy solution)**.** *A density $\rho_N \in L^1(\Pi^{d\,N})$, with $\rho_N \geq 0$ and $\int_{\Pi^{d\,N}} \rho_N\ \mathrm{d}X^N = 1$, is an entropy solution to Eq.* (1.6) *on the time interval* $[0,\ T]$*, iff ρ_N solves* (1.6) *in the sense of distributions, and for a.e. $t \leq T$*

$$\begin{aligned}
&\int_{\Pi^{d\,N}} \rho_N(t, X^N)\ \log \rho_N(t, X^N)\ \mathrm{d}X^N + \sigma_N \sum_{i=1}^N \int_0^t \int_{\Pi^{d\,N}} \frac{|\nabla_{x_i} \rho_N|^2}{\rho_N}\ \mathrm{d}X^N\ \mathrm{d}s \\
&\quad \leq \int_{\Pi^{d\,N}} \rho_N^0\ \log \rho_N^0\ \mathrm{d}X^N \\
&\qquad - \frac{1}{N} \sum_{i,\,j=1}^N \int_0^t \int_{\Pi^{d\,N}} (\,\mathrm{div}\, F(x_i) + \mathrm{div}\, K(x_i - x_j))\, \rho_N\ \mathrm{d}X^N\ \mathrm{d}s,
\end{aligned} \tag{1.8}$$

where for convenience we use in the article the notation $X^N = (x_1, \cdots, x_N)$.

In general it can be difficult to obtain the well posedness of an advection-diffusion equation such as (1.6) under very weak regularity of the advection field K, such as is our case here. We refer to [11] for an example of such study.

In our case though, we do not need the well posedness and it is in fact straightforward to check that there exists at least one entropy solution to (1.6).

Proposition 1. *Assume that $\int_{\Pi^{d\,N}} \rho_N^0\ \log \rho_N^0 < 0$, $\sigma_N \geq \underline{\sigma} > 0$, and that $F,\ \mathrm{div}\, F \in L^\infty$. Assume finally that $K \in \dot{W}^{-1,\infty}$ with as well* $\mathrm{div}\, K \in$

$\dot W^{-1,\infty}$. *Then there exists an entropy solution* ρ_N *satisfying*

$$\int_{\Pi^{d\,N}} \rho_N(t,X^N)\,\log\rho_N(t,X^N)\,\mathrm{d}X^N + \frac{\sigma_N}{2}\sum_{i=1}^N \int_0^t\int_{\Pi^{d\,N}} \frac{|\nabla_{x_i}\rho_N|^2}{\rho_N}\,\mathrm{d}X^N\,\mathrm{d}s$$
$$\leq \int_{\Pi^{d\,N}} \rho_N^0\,\log\rho_N^0\,\mathrm{d}X^N + \frac{N\,t\,\|\operatorname{div}K\|^2_{\dot W^{-1,\infty}}}{2\,\underline{\sigma}} + N\,t\,\|\operatorname{div}F\|_{L^\infty}. \tag{1.9}$$

Moreover for any $\phi\in L^2([0,\ T],\ W^{1,\infty}(\Pi^{2d}))$ *with* $\|\phi\|_{L^2_tW^{1,\infty}_x}\leq 1$

$$\frac{1}{\|K\|_{\dot W^{-1,\infty}}}\int_0^t\int_{\Pi^{2d}} \phi(t,x_1,x_2)\,K(x_1-x_2)\,\rho_{N,2}(t,x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}t$$
$$\leq 1+t+\frac{2}{N\,\underline{\sigma}}\int_{\Pi^{d\,N}} \rho_N^0\,\log\rho_N^0\,\mathrm{d}X^N + \frac{t\,\|\operatorname{div}K\|^2_{\dot W^{-1,\infty}}}{\underline{\sigma}^2} + t\,\frac{2\|\operatorname{div}F\|_{L^\infty}}{\underline{\sigma}}, \tag{1.10}$$

so that the product $K\,\rho_N$ *is well defined.*

Our method revolves around the control of the rescaled relative entropy

$$\mathcal{H}_N(\rho_N\mid\bar\rho_N)(t)=\frac{1}{N}\int_{\Pi^{d\,N}} \rho_N(t,X^N)\,\log\frac{\rho_N(t,X^N)}{\bar\rho_N(t,X^N)}\,\mathrm{d}X^N, \tag{1.11}$$

while our main result is the explicit estimate

Theorem 1. *Assume that* $\operatorname{div}F\in L^\infty(\Pi^d)$, *that* $K\in\dot W^{-1,\infty}(\Pi^d)$ *with* $\operatorname{div}K\in\dot W^{-1,\infty}$. *Assume that* $\sigma_N\geq\underline{\sigma}>0$. *Assume moreover that* ρ_N *is an entropy solution to Eq.* (1.6) *as per Def. 2. Assume finally that* $\bar\rho\in L^\infty([0,\ T],\ W^{2,p}(\Pi^d))$ *for any* $p<\infty$ *solves Eq.* (1.2) *with* $\inf\bar\rho>0$ *and* $\int_{\Pi^d}\bar\rho=1$. *Then*

$$\mathcal{H}_N(\rho_N\mid\bar\rho_N)(t)\leq e^{\bar M\,(\|K\|+\|K\|^2)\,t}\Bigg(\mathcal{H}_N(\rho_N^0\mid\bar\rho_N^0)+\frac{1}{N}$$
$$+\bar M(1+t\,(1+\|K\|^2)\,|\sigma-\sigma_N|\Bigg),$$

where we denote $\|K\|=\|K\|_{\dot W^{-1,\infty}}+\|\operatorname{div}K\|_{\dot W^{-1,\infty}}$ *and* $\bar M$ *is a constant which only depends on*

$$\bar M\left(d,\ \underline{\sigma},\ \inf\bar\rho,\ \|\bar\rho\|_{W^{1,\infty}},\ \sup_{p\geq1}\frac{\|\nabla^2\bar\rho\|_{L^p}}{p},\ \frac{1}{N}\int_{\Pi^{d\,N}}\rho_N^0\,\log\rho_N^0,\ \|\operatorname{div}F\|_{L^\infty}\right).$$

Remark 1. There is no explicit regularity assumption on F in the previous theorem. Nevertheless some regularity on F is implicitly required, in particular to obtain $W^{2,p}$ solution $\bar\rho$ to (1.2).

Remark 2. While our results are presented for simplicity in the torus Π^d, they could be extended to any bounded domain Ω with appropriate boundary conditions. The possible extension to unbounded domains however appears highly non-trivial, in particular in view of the assumption $\inf \bar\rho > 0$ which could not hold anymore.

The proof of Theorem 1 strongly relies on the properties of the relative entropy over tensorized spaces such as $\Pi^{d\,N}$. Those properties are also critical to derive appropriate control on the observables or marginals $\rho_{N,k}$. In particular the sub-additivity implies that the relative entropy of the marginals is bounded by the total relative entropy or

$$\mathcal{H}_k(\rho_{N,k}\mid\bar\rho^{\otimes^k})=\frac{1}{k}\int_{\Pi^{d\,k}}\rho_{N,k}\,\log\frac{\rho_{N,k}}{\bar\rho^{\otimes^k}}\;\mathrm{d}x_1\ldots\,\mathrm{d}x_k\leq\mathcal{H}_N(\rho_N\mid\bar\rho_N),\quad(1.12)$$

for which we refer to [24, 38, 39] where estimates quantifying the classical notion of propagation of chaos are thoroughly investigated.

It is then possible to derive from Theorem 1 the strong propagation of chaos as per

Corollary 1. *Under the assumptions of Theorem 1, if* $\mathcal{H}_N(\rho_N^0\mid\bar\rho_N^0)\to 0$ *as* $N\to\infty$*, then over any fixed time interval* $[0,\ T]$

$$\mathcal{H}_N(\rho_N\mid\bar\rho_N)\longrightarrow 0,\quad \text{as } N\to\infty.$$

As a consequence considering any finite marginal at order k*, one has the strong propagation of chaos*

$$\|\rho_{N,k}-\bar\rho^{\otimes^k}\|_{L^\infty([0,\ T],\ L^1(\Pi^{d\,k}))}\longrightarrow 0.$$

Finally in the particular case where $\sup_N N\,\mathcal{H}_N(\rho_N^0\mid\bar\rho_N^0)=H<\infty$*, and where* $\sup_N N\,|\sigma_N-\sigma|=S<\infty$*, then one has that, for some constant* C *depending only on* k*,* H*,* S*,* T *and* $\|K\|$ *and* $\bar M$ *defined in Theorem 1,*

$$\|\rho_{N,k}-\bar\rho^{\otimes^k}\|_{L^\infty([0,\ T],\ L^1(\Pi^{d\,k}))}\leq\frac{C}{\sqrt{N}}.\quad(1.13)$$

Remark 3. The rate of convergence in $1/\sqrt{N}$ in (1.13) is widely considered to be optimal as it corresponds to the size of stochastic fluctuations. We refer for example to [36] where entropy methods are used in this context for smooth interaction kernels; see also the prior [1] and [3, 9].

Proof. Corollary 1 follows directly from Theorem 1 by using inequality (1.12) and the Csiszár-Kullback-Pinsker inequality (see for instance [47]) for any f and g functions on $\Pi^{d\,k}$

$$\|f-g\|_{L^1(\Pi^{d\,k})} \leq \sqrt{2k\,\mathcal{H}_k(f \mid g)}.$$

$\square$

The starting steps in the proof of Theorem 1, such as the relative entropy and the reduction to a modified law of large numbers, had already been exposed in [29]. However the present contribution expands much on the basic ideas and techniques introduced in [29]: First we make better use of the diffusion, which was instead mostly considered as a perturbation in [29]. This is the main reason why we are essentially able to *gain one full derivative* in our assumption on K with respect to the $K \in L^\infty$ in [29].

The main technical contribution in the present article, namely the modified law of large numbers stated in Theorem 4, is considerably more difficult to prove than any equivalent in [29]. This has lead to several new ideas in the combinatorics approach, detailed in the proof of Theorem 4 in section 5. Theorem 4 is also much more general and we believe that it can be of further and wider use.

The importance of law of large numbers for the propagation of chaos or the mean field limit has of course long been recognized, at least since Kac, see [31] or [45]. We also refer to [20] for an example where the classical law of large numbers is used but which is limited to Lipschitz kernels K.

The relative entropy at the level of the Liouville equation does not seem to have been widely used yet with [48], in the context of hydrodynamics of Ginzburg-Landau, being maybe the closest to the approach developed here. We also refer to [15] for a different, trajectorial, view on the role of the entropy in SDEs.

1.3 Applications

We delve in this section into some examples of kernels K that our method can handle and discuss at the same time where our result stands in comparison to the existing literature. In general quantitative estimates of propagation of chaos were previously only available for smooth, Lipschitz, kernels K such as in the classical result [37]; see also [1, 3, 9, 36] for more on the classical Lipschitz case. Gronwall-like estimates with Lipschitz force fields, but a fixed number of SDEs, were also at the basis of [27].

System (1.1) retains simple additive interactions, contrary to the more complex structure found for example in [40, 41]; but it still includes a large range of first order models, such as swarming, opinion dynamics or aggregation equations, see for instance [2, 5, 10, 13] or [32]. The list of examples given below is hence by no means exhaustive and we refer to our recent survey [30] for a more thorough discussion of current important questions.

- The 2d viscous vortex model where K satisfies (1.3). As mentioned in the introduction, the mean field limit (1.2) is then the 2d incompressible Navier-Stokes equation written in vorticity form, Eq. (1.4). We can write

  $$K = \operatorname{div} V, \quad V = \begin{bmatrix} -\phi \arctan\frac{x_1}{x_2} + \psi_1 & 0 \\ 0 & \phi \arctan\frac{x_2}{x_1} + \psi_2 \end{bmatrix},$$

  where one can choose ϕ smooth with compact support in the representative $(-1/2,\ 1/2)^2$ of Π^2 and $(\psi_1,\ \psi_2)$ a corresponding smooth correction to periodize V. Therefore K satisfies the assumptions of Theorem 1.

  The convergence of the systems of point vortices (1.1) to the limit (1.4) had first been established in [43] for a large enough viscosity σ. The well posedness of the point vortices dynamics has been proved globally in [42]; see also [14]. Finally the convergence to the mean field limit has been obtained with any positive viscosity σ in the recent [16].

  However those results rely on a compactness argument based on a control of the singular interaction provided by the dissipation of entropy in the system.

  As far as we know, this article is *the first to provide a quantitative rate of propagation of chaos for the 2d viscous vortex model.*

- Hamiltonian structure. If the dimension d is even then the previous example can be generalized to include any Hamiltonian structure. In that case one has $d = 2n$, $x = (q, p)$ with $q, p \in \Pi^n$ and for some Hamiltonian $H : \Pi^{2n} \longrightarrow \mathbb{R}$,

  $$K = (\nabla_p H,\ -\nabla_q H).$$

  Theorem 1 now applies if $H \in L^\infty(\Pi^{2n})$, though this may not be the optimal condition (see the discussion below). The theorem provides propagation of chaos for such systems with diffusion with much weaker assumptions than any comparable result in the literature.

We are nevertheless somewhat limited by our framework here. One would for example typically want to apply this to the classical Newtonian dynamics where $H = \sum_i p_i^2/2 + \frac{1}{N}\sum_{i,j} V(q_i - q_j)$. This is formally easy by choosing the appropriate function F in the system of particles (1.1).

The first issue is that the momentum should be unbounded instead of having $p \in \Pi^n$; as we mentioned in one of the remarks after Theorem 1, such an extension of our result to $p \in \mathbb{R}^n$ for example would be non-trivial...

The second issue concerns the diffusion which for such models usually applies only to the momentum. This leads to a degenerate diffusion whereas we absolutely require it in every variable.

- Collision-like interactions. We can even handle extremely singular interactions where some sort of collision event occurs at some fixed horizon. Consider for example any function $\phi \in L^1(\Pi^d)$, any smooth field $M(x)$ of matrices and define

$$K = \operatorname{div}(M\,\mathbb{I}_{\phi\leq 0}), \quad \text{or } K_\alpha(x) = \sum_\beta \partial_\beta(M_{\alpha\beta}(x)\,\mathbb{I}_{\phi(x)\leq 0}).$$

  It is straightforward to choose M s.t. $\operatorname{div} K \in \dot W^{-1,\infty}$ or even $\operatorname{div} K = 0$: A simple example is simply to take M anti-symmetric. As $M\,\mathbb{I}_{\phi\leq 0} \in L^\infty$, Theorem 1 applies. This particular choice of K means that two particles i and j will interact exactly when $\phi(X_i - X_j) = 0$. An obvious example is $\phi(x) = |x|^2 - (2R)^2$ in which case the particles can be seen as balls of radius R which interact when touching.

  But in the context of swarming, one could have birds, or other animals, which interact as soon as they can see each other; this is different from the cone of vision type of interaction found for example in [6] where the interaction is much less singular (bounded). Micro-organisms such as bacteria may also have complicated, non-smooth shapes. In all those cases $\{\phi \leq 0\}$ is not a ball in general and may even be a singular set.

  Since $M(x)$ is smooth, one could interpret K as being supported on the measure $\delta_{\phi=0}$. But in fact we do not need any regularity on ϕ, not even $\phi \in BV$ and here K may not even be a measure...

- Gradient flow structure. The dual to the Hamiltonian case is to take $K = \nabla\psi$ for some potential ψ. This lets us see the system of particles

(1.1) as a gradient flow with diffusion and it endows the mean field limit (1.2) with the derived and nonlinear gradient flow structure.

When ψ is convex, but not necessarily smooth, it is possible to strongly use this gradient flow structure. This is in particular the key to obtain the well posedness of Eq. (1.2), even without diffusion, as in [7, 8] and in [2] for the mean field limit.

However it does not seem easy for our approach to fully make use of such gradient flows. This is seen on the assumptions of Theorem 1 where having $K \in \dot{W}^{-1,\infty}$ is not very demanding, $\psi \in L^\infty$ would be enough, while the condition $\operatorname{div} K \in \dot{W}^{-1,\infty}$ actually forces us to consider Lipschitz potentials ψ. Of course any ψ convex is Lipschitz so that Theorem 1 still extends the known theory for general ψ. But it is clearly not performing as well as in the Hamiltonian case.

A very good example of this is the 2d Patlak-Keller-Segel model of chemotaxis where one would like to have $K = \alpha\, x/|x|^2 + K_0(x)$. This choice of K is just a rotation of $\pi/2$ from the 2d Navier-Stokes kernel given by (1.3). Therefore we still have that $K \in \dot{W}^{-1,\infty}$ by using a rotation of the matrix V that we wrote in the Navier-Stokes setting. But unfortunately $\operatorname{div} K$ is now one full derivative away from $\dot{W}^{-1,\infty}$ and Theorem 1 cannot be applied.

By studying the specific properties of the system though, a convergence result to measure-valued solutions was obtained in [26] while the convergence to weak solutions was achieved in [17] (see also [18] for the sub-critical case). We also refer to [35] for general Coulomb interactions. Those results are not quantitative though and a major open problem remains to find an equivalent of Theorem 1 in this case.

We wish to conclude this subsection about kernels K to which Theorem 1 applies, by discussing more in details the assumption $K \in \dot{W}^{-1,\infty}$.

We first come back to the vortex dynamics for 2d Navier-Stokes and the kernel K given by the Biot-Savart law (1.3). Since $\operatorname{div} K = 0$, the classical way to represent K is by $K = \operatorname{curl} \psi$ with

$$\psi(x) = \alpha\, \log|x| + \psi_0(x),$$

with again ψ_0 a smooth correction to periodize ψ. Obviously ψ is not bounded which at first glance suggests that K does not belong to $\dot{W}^{-1,\infty}$. This is incorrect as the "right" choice of V above demonstrates but it means that *knowing whether* $K \in \dot{W}^{-1,\infty}$ *is not as simple as it may seem.*

The distinction is rather technical but it is *critical* for us as it allows us to handle the crucial example of the vortex model. It also turns out to be *connected with a fundamental difficulty* in our proof. Our estimates directly use a representation $K = \operatorname{div} V$ and the most difficult term would vanish if V were anti-symmetric, which is the case if we take $K = \operatorname{curl}\psi$. *The fact that we cannot take $K = curl\,\psi$ with $\psi \in L^\infty$ is responsible for the main technical difficulty in this article* and in particular this is what requires Theorem 4 whose proof takes all of section 5. We refer to the more specific comments that we make in subsection 2.1.

In general the study of the K for which there exists a matrix field $V \in L^\infty$ s.t. $\operatorname{div} V = K$ turns out to be a very complex mathematical question. This can be done coordinate by coordinate obviously so the question is equivalent to finding the scalar field ϕ for which there exists a vector field $u \in L^\infty$ s.t. $\operatorname{div} u = \phi$.

The difficulty is that for a given K, there *does not exist a unique matrix field V s.t.* $\operatorname{div} V = K$. Of course in dimension $d = 2$ if $\operatorname{div} K = 0$, then there exists a unique ψ up to a constant, s.t. $K = \operatorname{curl}\psi$. In dimension $d > 2$, if $\operatorname{div} K = 0$, there exists an anti-symmetric matrix V s.t. $K = \operatorname{div} V$. The anti-symmetric matrix V is not unique in general though with the well known issue of the gauge choice for vector potential if $d = 3$.

But even in dimension 2, there is no reason why $\psi \in L^\infty$ if $K \in \dot W^{-1,\infty}$. This is indeed connected to the fact that the Riesz transforms are unbounded on L^∞ and the kernel K of (1.3) is the classical example of this. Instead one only has in general that $\psi \in BMO$.

However even in this simple case, *it is not known if $\psi \in BMO$ is equivalent to $K \in \dot W^{-1,\infty}$*. This question is connected to the classical representation of BMO functions in [12]. For any $\psi \in BMO$, [12] showed that there exists $\psi_0,\ \psi_1,\ \psi_2 \in L^\infty$ s.t. $\psi = \psi_0 + R_1\,\psi_1 + R_2\,\psi_2$ with R_i, $i = 1,\ 2$, the Riesz transforms. If it were always possible to take $\psi_0 = 0$ then we would have the equivalence but that seems (at best) highly non-trivial.

Instead the positive results that we have are much more recent and limited. This line of investigation was started in the seminal [4] which proved that if $K \in L^d(\Pi^d)$ then $K \in \dot W^{-1,\infty}(\Pi^d)$. If K is known to be a signed measure then this was extended in [44] to find that $K = \operatorname{div} V$ with $V \in L^\infty$ iff there exists C s.t. for any Borel set U

$$\left|\int_U K(dx)\right| \leq C\,|\partial U|. \tag{1.14}$$

This result in [44] hence has the direct consequence

Proposition 2. *If $d > 1$ and K belongs to the Lorentz space $L^{d,\infty}(\Pi^d)$ then $K \in \dot{W}^{-1,\infty}$.*

Proof. Assuming $K \in L^{d,\infty}$ then for a constant C, we have that

$$|\{x \in \Pi^d,\ |K(x)| \geq M\}| \leq \frac{C}{M^d}.$$

Decompose now dyadically

$$\int_U |K(x)|\,dx \leq |U| + \sum_{k\geq 0} 2^{k+1}\,|\{x \in U,\ |K(x)| \geq 2^k\}|.$$

Define k_0 s.t. $2^{-d\,(k_0+1)} \leq |U| \leq 2^{-d\,k_0}$ and bound

$$|\{x \in U,\ |K(x)| \geq 2^k\}| \leq |U| \text{ for } k \leq k_0,$$
$$|\{x \in U,\ |K(x)| \geq 2^k\}| \leq |\{x \in \Pi^d,\ |K(x)| \geq 2^k\}| \leq \frac{C}{2^{d\,k}} \text{ for } k > k_0.$$

This leads to

$$\begin{aligned}\int_U |K(x)|\,dx &\leq |U| + \sum_{k\leq k_0} 2^{k+1}\,|U| + C \sum_{k>k_0} 2^{(1-d)\,k+1}\\ &\leq |U| + 2^{k_0+2}\,|U| + C\,2^{(1-d)\,k_0+1} \leq C'\,|U|^{\frac{d-1}{d}},\end{aligned}$$

by using the definition of k_0. By the isoperimetric inequality, there exists a constant C_d s.t. $|U|^{\frac{d-1}{d}} \leq C_d\,|\partial U|$ so that we verify the condition (1.14) which concludes the proof. $\square$

Prop. 2 not only applies to K given by (1.3) but proves in general that any K with $|K(x)| \leq C/|x|$ belongs to $\dot{W}^{-1,\infty}$.

The original result in [4] is not constructive, and it is even proved that the $V \in L^\infty$ s.t. $K = \operatorname{div} V$ cannot be obtained linearly from K. The development of constructive algorithms to obtain V is a current important field of research, see [46].

1.4 The case with vanishing diffusion

While we are mostly interested in Eq. (1.6) when the viscosity does not asymptotically vanishes, a nice (and essentially free) consequence of the method developed here is to also provide a result with vanishing viscosity.

The result is of course weaker and requires $K \in L^\infty$ with $\operatorname{div} K \in L^\infty$. Obtaining an entropy solution to (1.6) in the sense of (2) is even more straightforward in this case as there is no need for integration by parts. Moreover we also directly obtain the following bound, which replaces in that case the one provided by Prop. 1.

$$\int_{\Pi^{d\,N}} \rho_N(t, X^N) \log \rho_N(t, X^N)\,\mathrm{d}X^N + \sigma_N \sum_{i=1}^{N} \int_0^t \int_{\Pi^{d\,N}} \frac{|\nabla_{x_i} \rho_N|^2}{\rho_N}\,\mathrm{d}X^N\,\mathrm{d}s$$
$$\leq \int_{\Pi^{d\,N}} \rho_N^0 \log \rho_N^0\,\mathrm{d}X^N + N\,t\,(\|\operatorname{div} K\|_{L^\infty} + \|\operatorname{div} F\|_{L^\infty})\,. \tag{1.15}$$

Under those stronger assumptions on K, we have the following result

Theorem 2. *Assume that* $\operatorname{div} F \in L^\infty(\Pi^d)$, *that* $K \in L^\infty(\Pi^d)$ *with also* $\operatorname{div} K \in L^\infty(\Pi^d)$. *Assume moreover that* ρ_N *is an entropy solution to Eq.* (1.6) *as per Def. 2. Assume finally that* $\bar\rho \in L^\infty([0,\ T],\ W^{1,\infty}(\Pi^d))$ *solves Eq.* (1.2) *with* $\int_{\Pi^d} \bar\rho = 1$. *Then*

$$\mathcal{H}_N(\rho_N \mid \bar\rho_N)(t) \leq e^{\bar M_2\,\|K\|_\infty\,t} \Bigg(\mathcal{H}_N(\rho_N^0 \mid \bar\rho_N^0) + \frac{1}{N}$$
$$+ \bar M_2\,(1 + \|K\|_\infty t)\,|\sigma - \sigma_N| \Bigg),$$

where we now denote $\|K\|_\infty = \|K\|_{L^\infty} + \|\operatorname{div} K\|_{L^\infty}$ *and* $\bar M_2$ *is a constant which only depends on*

$$\bar M_2\left(\sigma, \|\log \bar\rho\|_{W^{1,\infty}},\ \sup_{p\geq 1} \frac{\|\nabla \log \bar\rho\|_{L^p(\bar\rho\,\mathrm{d}x)}}{p},\ \frac{1}{N}\int_{\Pi^{d\,N}} \rho_N^0 \log \rho_N^0, \|\operatorname{div} F\|_{L^\infty}\right).$$

Remark 4. The constant $\bar M_2$ is in the above complex form simply because we include all cases $\sigma_N \to \sigma \geq 0$. For instance if $\sigma_N \equiv \sigma$, then $\bar M_2$ only explicitly depend on

$$\bar M_2\left(\sup_{p\geq 1} \frac{\|\nabla \log \bar\rho\|_{L^p(\bar\rho\,\mathrm{d}x)}}{p}\right).$$

For the vanishing viscosity case $\sigma_N \to \sigma = 0$, $\bar M_2$ only explicitly depends on

$$\bar M_2\left(\sup_{p\geq 1} \frac{\|\nabla \log \bar\rho\|_{L^p(\bar\rho\,\mathrm{d}x)}}{p},\ \|\log \bar\rho\|_{W^{1,\infty}}\right).$$

See the proof of Theorem 2 in subsection 2.6 for more details.

Remark 5. To control the error caused by the difference $|\sigma - \sigma_N|$, we need $\nabla \log \bar\rho \in L^\infty(\Pi^d)$. This can be replaced to moment assumptions like $|\nabla \log \bar\rho(x)| \leq C|x|^k$ so that the result can easily be extended to the whole space $\mathbb{R}^d$.

Theorem 2 is obviously mostly only useful in comparison to our main result if $\sigma_N \to \sigma = 0$, including potentially the purely deterministic setting where $\sigma_N = 0$ or cases where the viscosity is degenerate in some directions. But it also requires less regularity on the limit $\bar\rho$ and could also be of use in such a situation. In particular it does not require that $\inf \bar\rho > 0$ and is hence easy to extend to unbounded domains contrary to Theorem 1.

Because of its usefulness for degenerate viscosities, it is rather natural to compare Theorem 2 to results for kinetic mean field limits based on the 2nd order dynamics

$$\mathrm{d}Q_i = P_i \,\mathrm{d}t, \quad \mathrm{d}P_i = \frac{1}{N} \sum_{j=1}^N K(Q_i - Q_j)\,\mathrm{d}t + \sqrt{2\sigma_N}\, dW_t^i. \tag{1.16}$$

We refer to [19, 28] for an introduction to the mean field question in this kinetic setting. The best results so far have been obtained in [23] for a singular kernel K with $|K(x)| \leq C\,|x|^{-\alpha}$, $|\nabla K(x)| \leq C\,|x|^{-1-\alpha}$ with $\alpha < 1$; in [22] for Hölder continuous K. The most classical case is again the Poisson kernel $K(x) = \gamma_d\, x/|x|^d$ which is unfortunately out of reach so far (except in dimension 1 as in [25]). It is possible to treat truncated kernels such as $K(x) = \gamma_d\, x/(|x| + \varepsilon_N)^d$ with the most realistic ε_N obtained in [33, 34]. However none of the techniques in those articles seems, so far, to be able to handle any diffusion and especially vanishing or degenerate diffusion as in (1.16). In the case of (1.16) where the limiting equation is often called Vlasov-Fokker-Planck, we refer for example to [3] which requires more regularity on K.

We remark that in comparison, the theory of mean field limits for purely 1st order systems without viscosity is much more advanced, with the limit of point vortices already obtained in [21]. However as we noticed before, the techniques developed there do not seem to be compatible with any (vanishing) diffusion.

An obvious point of comparison for Theorem 2 is our previous result in [29]. This previous result covered the case of (1.16) with the same assumption $K \in L^\infty$; it also introduced the basic ideas for the method used here, based on the relative entropy and combinatorics estimates.

However [29] was relying strongly on the simplectic structure of the dynamics in (1.16). Extending the method to general kernels K which may not even be Hamiltonian, as is done by Theorem 2, changes the scope of the result. It has also been proved to be quite complex: From a technical point of view, the whole combinatorics estimates of [29] can be summarized in section 3 of the present article while the new estimates are considerably longer, see section 5.

1.5 Sketch of the proof of Proposition 1

The proof follows very classical ideas: Consider a regularized interaction kernel K_ε. Eq. (1.6) with K_ε now has a unique solution $\rho_{N,\varepsilon}$ for any initial measure ρ_N^0. The goal is to take the limit $\varepsilon \to 0$, by extracting weak-* converging sub-sequences of $\rho_{N,\varepsilon}$, and to derive (1.6) for the limiting kernel K and the various estimates such as (1.8) and (1.9).

The only (small) difficulty in this procedure is to obtain adequate uniform bounds. For this reason we only explain here how to derive those bounds for any weak solution ρ_N to (1.6) which also satisfies (1.8).

The first step is to prove from (1.8) that

$$\sum_{i=1}^N \int_0^t \int_{\Pi^{d\,N}} \frac{|\nabla_{x_i}\rho_N|^2}{\rho_N}\, \mathrm{d}X^N\, \mathrm{d}s \le C \int_{\Pi^{d\,N}} \rho_N^0 \log \rho_N^0\, \mathrm{d}X^N.$$

Observe that if $\operatorname{div} K \in \dot W^{-1,\infty}$, that is $\operatorname{div} K = \operatorname{div} \psi$ with $\|\psi\|_{L^\infty} = \|\operatorname{div} K\|_{\dot W^{-1,\infty}}$, then

$$\begin{aligned} -\frac{1}{N}\sum_{i,\,j=1}^N \int_0^t \int_{\Pi^{d\,N}} \operatorname{div} K(x_i - x_j)\, \rho_N\, \mathrm{d}X^N\, \mathrm{d}s \\ \le \|\operatorname{div} K\|_{\dot W^{-1,\infty}} \sum_i \int_0^t \int_{\Pi^{d\,N}} |\nabla_{x_i}\rho_N|\, \mathrm{d}X^N\, \mathrm{d}s \end{aligned}$$

On the other hand

$$\begin{aligned} &\sum_i \int_0^t \int_{\Pi^{d\,N}} |\nabla_{x_i}\rho_N|\, \mathrm{d}X^N\, \mathrm{d}s \\ &\quad\le \frac{\sigma_N}{2\,\|\operatorname{div} K\|_{\dot W^{-1,\infty}}} \sum_i \int_0^t \int_{\Pi^{d\,N}} \frac{|\nabla_{x_i}\rho_N|^2}{\rho_N}\, \mathrm{d}X^N\, \mathrm{d}s \\ &\qquad + \frac{\|\operatorname{div}\, K\|_{\dot W^{-1,\infty}}}{2\,\sigma_N} \sum_i \int_0^t \int_{\Pi^{d\,N}} \rho_N\, \mathrm{d}X^N\, \mathrm{d}s \\ &\quad\le \frac{N\,t\,\|\operatorname{div}\, K\|_{\dot W^{-1,\infty}}}{2\,\sigma_N} + \frac{\sigma_N}{2\|\operatorname{div} K\|_{\dot W^{-1,\infty}}} \sum_i \int_0^t \int_{\Pi^{d\,N}} \frac{|\nabla_{x_i}\rho_N|^2}{\rho_N}\, \mathrm{d}X^N\, \mathrm{d}s. \end{aligned}$$

This implies that

$$
\begin{aligned}
&-\frac{1}{N}\sum_{i,\,j=1}^{N}\int_0^t\int_{\Pi^{d\,N}} \operatorname{div} K(x_i-x_j)\,\rho_N\,\mathrm{d}X^N\,\mathrm{d}s\\
&\qquad\le \frac{N\,t\,\|\operatorname{div} K\|_{\dot W^{-1,\infty}}^2}{2\,\sigma_N}+\frac{\sigma_N}{2}\sum_i\int_0^t\int_{\Pi^{d\,N}}\frac{|\nabla_{x_i}\rho_N|^2}{\rho_N}\,\mathrm{d}X^N\,\mathrm{d}s.
\end{aligned}
$$

Introducing this bound in (1.8) shows that

$$
\begin{aligned}
&\int_{\Pi^{d\,N}}\rho_N(t,X^N)\,\log\rho_N(t,X^N)\,\mathrm{d}X^N+\frac{\sigma_N}{2}\sum_{i=1}^N\int_0^t\int_{\Pi^{d\,N}}\frac{|\nabla_{x_i}\rho_N|^2}{\rho_N}\,\mathrm{d}X^N\,\mathrm{d}s\\
&\le\int_{\Pi^{d\,N}}\rho_N^0\,\log\rho_N^0\,\mathrm{d}X^N+\frac{N\,t\,\|\operatorname{div} K\|_{\dot W^{-1,\infty}}^2}{2\,\sigma_N}+N\,t\,\|\operatorname{div} F\|_{L^\infty},
\end{aligned}
$$

which since $\sigma_N\ge\underline{\sigma}$ exactly proves (1.9).

From [38] by convexity, we know that

$$
\int_0^t\int_{\Pi^{2\,d}}\frac{|\nabla_{x_1}\rho_{N,2}|^2}{\rho_{N,2}}\,dx_1\,dx_2\le\frac{1}{N}\sum_{i=1}^N\int_0^t\int_{\Pi^{d\,N}}\frac{|\nabla_{x_i}\rho_N|^2}{\rho_N}\,\mathrm{d}X^N\,\mathrm{d}s.
$$

If $K\in\dot W^{-1,\infty}$, *i.e.* if $K(x)=\operatorname{div} V(x)$ or using coordinates $K_\alpha(x)=\sum_{\beta=1}^d\partial_\beta V_{\alpha\beta}(x)$ with V a matrix-valued field, then for any $\phi\in W^{1,\infty}$

$$
\begin{aligned}
&\int_{\Pi^{2\,d}}K(x_1-x_2)\,\phi(x_1,x_2)\,\rho_{N,2}\,dx_1\,dx_2\\
&\quad=-\int_{\Pi^{2\,d}}V(x_1-x_2)\,(\phi\,\nabla_{x_1}\rho_{N,2}+\nabla_{x_1}\phi\ \rho_{N,2})\,dx_1\,dx_2\\
&\quad\le\|V\|_{L^\infty}\,\|\nabla\phi\|_{L^\infty}+\|V\|_{L^\infty}\,\|\phi\|_{L^\infty}\,\left(\int_{\Pi^{2\,d}}\frac{|\nabla_{x_1}\rho_{N,2}|^2}{\rho_{N,2}}\,dx_1\,dx_2\right)^{1/2},
\end{aligned}
$$

which leads to (1.10) using that $\inf_V\|V\|_{L^\infty}=\|K\|_{\dot W^{-1,\infty}}$.

Finally, we note that

$$
\operatorname{div}\frac{x}{|x|^\gamma}=\frac{d}{|x|^\gamma}-\gamma\sum_\alpha\frac{x_\alpha\,x_\alpha}{|x|^{\gamma+2}}=\frac{d-\gamma}{|x|^\gamma},
$$

so that with the same approach it would be possible to derive the bound

$$
\int_{\Pi^{2\,d}}\frac{\rho_{N,2}}{|x_1-x_2|^\gamma}\,dx_1\,dx_2\le\frac{1}{(d-\gamma)^2}\int_{\Pi^{2\,d}}\frac{|\nabla_{x_1}\rho_{N,2}|^2}{\rho_{N,2}}\,dx_1\,dx_2,
$$

for any $\gamma<2$ if $d=2$ and for $\gamma=2$ if $d>2$, which has proved critical in the previous derivation and studies of the 2d incompressible Navier-Stokes for instance see [14, 16, 43].

2 Proofs of Theorems 1 and 2

2.1 Sketch of the proof of Theorem 1

Our goal in this subsection is to present the main steps of the proof. For this reason, we make several simplifying assumptions that allow us to focus on the main ideas. First of all, we assume that

$$F=0,\quad \operatorname{div} K=0,\quad K_\alpha=\sum_\beta \partial_\beta V_{\alpha\beta}\quad \text{with } \|V\|_{L^\infty(\Pi^d)}\le\delta,$$

for δ small in terms of some norms of $\bar\rho$.

We also assume that $\bar\rho\in C^\infty$ with $\inf\bar\rho>0$ and that ρ_N is a classical solution to (1.6) so that we may easily manipulate this equation.

Finally we assume that $\sigma_N=\sigma=1$.

Following our previous discussion about the criticality of the assumption $K=\operatorname{div} V$ with $V\in L^\infty$, we refer the readers in particular to the end of step 2 after formula (2.3) and to step 5 in the following proof. That step requires the use of Theorem 4 whose proof contains the main technical difficulties of the article.

If instead one would assume that V is anti-symmetric then the term $\tilde B$ in step 5 vanishes and as we mentioned above, we would have a much simpler proof. Unfortunately this would not let us handle our most important kernel $K=x^\perp/|x|^2$ corresponding to the 2d incompressible Navier-Stokes system.

Step 1: Time evolution of the relative entropy. First of all it is straightforward to derive an equation on $\bar\rho_N$ from the limiting equation (1.2)

$$\begin{aligned}\partial_t\bar\rho_N+\sum_{i=1}^N\frac1N\sum_{j=1}^N K(x_i-x_j)\cdot\nabla_{x_i}\bar\rho_N&=\sum_{i=1}^N\sigma\,\Delta_{x_i}\bar\rho_N\\ &+\sum_{i=1}^N\left(\frac1N\sum_{j=1}^N K(x_i-x_j)-K\star_x\bar\rho(x_i)\right)\cdot\nabla_{x_i}\bar\rho_N.\end{aligned}$$

Combining this with the Liouville equation (1.6), one obtains that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_N(\rho_N\,|\,\bar\rho_N)(t)\\ &\le-\frac{1}{N^2}\sum_{i,\,j=1}^N\int_{\Pi^{d\,N}}\rho_N\;(K(x_i-x_j)-K\star_x\bar\rho(x_i))\cdot\nabla_{x_i}\log\bar\rho_N\;\mathrm{d}X^N\\ &\quad-\frac1N\sum_{i=1}^N\int_{\Pi^{d\,N}}\rho_N\left|\nabla_{x_i}\log\frac{\rho_N}{\bar\rho_N}\right|^2.\end{aligned}\tag{2.1}$$

A full justification of this calculation is given later in the main proof in Lemma 2.2.

Step 2: Using $K = \operatorname{div} V$. As the kernel K is not bounded but we only have that $K = \operatorname{div} V$ with $V \in L^\infty$, the next step is to integrate by parts to make V explicit in our estimates

$$
\begin{aligned}
&-\frac{1}{N^2}\sum_{i,\,j=1}^N \int_{\Pi^{d\,N}} \rho_N\ (K(x_i-x_j) - K\star_x \bar\rho(x_i))\cdot\nabla_{x_i}\log\bar\rho_N\ \mathrm{d}X^N\\
&=\frac{1}{N^2}\sum_{i,\,j=1}^N \int_{\Pi^{d\,N}} \rho_N\ (V(x_i-x_j) - V\star_x \bar\rho(x_i))\ :\ \frac{\nabla^2_{x_i}\bar\rho_N}{\bar\rho_N}\ \mathrm{d}X^N\\
&\quad+\frac{1}{N^2}\sum_{i,\,j=1}^N \int_{\Pi^{d\,N}} (V(x_i-x_j) - V\star_x \bar\rho(x_i)) : \nabla_{x_i}\bar\rho_N\otimes\nabla_{x_i}\frac{\rho_N}{\bar\rho_N}\ \mathrm{d}X^N.
\end{aligned}
$$

Observe that we were careful in writing that

$$\rho_N\,\nabla_{x_i}\log\bar\rho_N = \frac{\rho_N}{\bar\rho_N}\,\nabla_{x_i}\bar\rho_N,$$

so that after integration by parts, the second term involves a derivative of $\rho_N/\bar\rho_N$ which can be controlled thanks to the dissipation term in (2.1). More precisely by Cauchy-Schwartz

$$
\begin{aligned}
&\frac{1}{N^2}\sum_{i,\,j=1}^N \int_{\Pi^{d\,N}} (V(x_i-x_j) - V\star_x \bar\rho(x_i)) : \nabla_{x_i}\bar\rho_N\otimes\nabla_{x_i}\frac{\rho_N}{\bar\rho_N}\,\mathrm{d}X^N\\
&\le\frac{1}{N}\sum_{i=1}^N\int_{\Pi^{d\,N}}\left|\nabla_{x_i}\frac{\rho_N}{\bar\rho_N}\right|^2\frac{\bar\rho_N^2}{\rho_N}\ \mathrm{d}X^N\\
&\quad+\frac{1}{N}\sum_{i=1}^N\int_{\Pi^{d\,N}}\rho_N\,\frac{|\nabla_{x_i}\bar\rho_N|^2}{\bar\rho_N^2}\left|\frac{1}{N}\sum_{j=1}^N (V(x_i-x_j) - V\star_x\bar\rho(x_i))\right|^2\ \mathrm{d}X^N.
\end{aligned}
$$

Of course

$$\left|\nabla_{x_i}\frac{\rho_N}{\bar\rho_N}\right|^2\frac{\bar\rho_N^2}{\rho_N} = \left|\nabla_{x_i}\log\frac{\rho_N}{\bar\rho_N}\right|^2\ \rho_N$$

so that the first term is actually bounded by the dissipation of entropy. On the other hand

$$\frac{|\nabla_{x_i}\bar\rho_N|^2}{\bar\rho_N^2} = \frac{|\nabla_{x_i}\bar\rho(x_i)|^2}{\bar\rho(x_i)^2}.$$

Hence we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_N(\rho_N \mid \bar{\rho}_N)(t) \leq A + B, \tag{2.2}$$

where

$$\begin{aligned} A &= \frac{C_{\bar{\rho}}}{N}\sum_{i=1}^{N}\int_{\Pi^{d\,N}} \rho_N \left|\frac{1}{N}\sum_j \left(V(x_i - x_j) - V \star_x \bar{\rho}(x_i)\right)\right|^2 \mathrm{d}X^N, \\ B &= \frac{1}{N^2}\sum_{i,\,j=1}^{N}\int_{\Pi^{d\,N}} \rho_N \,\left(V(x_i - x_j) - V \star_x \bar{\rho}(x_i)\right) : \frac{\nabla^2_{x_i}\bar{\rho}(x_i)}{\bar{\rho}(x_i)}\,\mathrm{d}X^N, \end{aligned} \tag{2.3}$$

and $C_{\bar{\rho}}$ is a constant depending only on the smoothness of $\bar{\rho}$.

We point out here that $\nabla^2_{x_i}\bar{\rho}$ is a symmetric matrix. Hence, *if V is anti-symmetric, then the term B completely vanishes: $B = 0$.*

Step 3: Change of law from ρ_N to $\bar{\rho}_N$. The two previous terms A and B can be seen as the expectations of the corresponding random variables with respect to the law ρ_N. Obviously we do not know the properties of ρ_N and would much prefer having expectations with respect to the tensorized law $\bar{\rho}_N$. We hence use the following

Lemma 2.1. *For any two probability densities ρ_N and $\bar{\rho}_N$ on $\Pi^{d\,N}$, and any $\Phi \in L^\infty(\Pi^{d\,N})$, one has that $\forall \eta > 0$*

$$\int_{\Pi^{d\,N}} \Phi\,\rho_N\,\mathrm{d}X^N \leq \mathcal{H}_N(\rho_N\,|\bar{\rho}_N) + \frac{1}{N}\,\log\int_{\Pi^{d\,N}} \bar{\rho}_N\,e^{N\,\Phi}\,\mathrm{d}X^N.$$

Proof. We give the (short) proof for the sake of completeness. Define

$$f = \frac{1}{\lambda}\,e^{N\,\Phi}\,\bar{\rho}_N, \quad \lambda = \int_{\Pi^{d\,N}} \bar{\rho}_N\,e^{N\,\Phi}\,\mathrm{d}X^N.$$

Notice that f is a probability density as $f \geq 0$ and $\int f = 1$. Hence by the convexity of the entropy

$$\frac{1}{N}\int_{\Pi^{d\,N}} \rho_N\,\log f\,\mathrm{d}X^N \leq \frac{1}{N}\int_{\Pi^{d\,N}} \rho_N\,\log\rho_N\,\mathrm{d}X^N.$$

On the other hand, one can easily check that

$$\frac{1}{N}\int_{\Pi^{d\,N}}\rho_N\,\log f\,\mathrm{d}X^N = \int_{\Pi^{d\,N}}\rho_N\,\Phi\,\mathrm{d}X^N + \frac{1}{N}\int_{\Pi^{d\,N}}\rho_N\,\log\bar{\rho}_N\,\mathrm{d}X^N - \frac{\log\lambda}{N},$$

which concludes the proof of the lemma. □

To apply Lemma 2.1 to A, we first expand A coordinate by coordinate as

$$A \le \frac{C_{\bar\rho}}{N} \sum_{i=1}^N \sum_{\alpha,\beta=1}^d \int_{\Pi^{d\,N}} \rho_N \left(\frac{1}{N} \sum_{j=1}^N (V_{\alpha,\beta}(x_i - x_j) - V_{\alpha,\beta} \star_x \bar\rho(x_i)) \right)^2 \, \mathrm{d}X^N.$$

Now applying Lemma 2.1 first to each

$$\Phi_{\alpha,\beta} = \left(\frac{1}{N} \sum_{j=1}^N (V_{\alpha,\beta}(x_i - x_j) - V_{\alpha,\beta} \star_x \bar\rho(x_i)) \right)^2,$$

in A and then to

$$\Phi = \frac{1}{N^2} \sum_{i,j=1}^N (V(x_i - x_j) - V \star_x \bar\rho(x_i)) \, : \, \frac{\nabla^2_{x_i} \bar\rho(x_i)}{\bar\rho(x_i)},$$

in B, we obtain that

$$A + B \le 2\,\mathcal{H}_N(\rho_N \mid \bar\rho_N)(t) + \tilde A + \tilde B,$$

with

$$\begin{aligned} \tilde A &= \frac{C_{\bar\rho}}{N^2} \sum_{i=1}^N \sum_{\alpha,\beta=1}^d \log \int_{\Pi^{d\,N}} \exp \left(\frac{1}{\sqrt{N}} \sum_{j=1}^N (V_{\alpha,\beta}(x_i - x_j) - V_{\alpha,\beta} \star_x \bar\rho(x_i)) \right)^2 \\ &\qquad \bar\rho_N \, \mathrm{d}X^N, \\ \tilde B &= \frac{1}{N} \log \int_{\Pi^{d\,N}} \bar\rho_N \, e^{\frac{1}{N} \sum_{i,\,j=1}^N (V(x_i - x_j) - V \star_x \bar\rho(x_i)) : \frac{\nabla^2_{x_i} \bar\rho(x_i)}{\bar\rho(x_i)}} \, \mathrm{d}X^N. \end{aligned} \tag{2.4}$$

Observe that the cost to perform this change of law is, unfortunately, severe as we now have exponential factors in $\tilde A$ and $\tilde B$. That is the reason why we need L^∞ (or almost L^∞) bounds on V.

Step 4: Bounding $\tilde A$ through a law of large number at the exponential scale. By symmetry of permutation, we may take $i = 1$ in $\tilde A$. Define

$$\psi_{\alpha,\beta}(z, x) = V_{\alpha,\beta}(z - x) - V_{\alpha,\beta} \star_x \bar\rho(z),$$

so that

$$\left(\frac{1}{\sqrt{N}}\sum_{j=1}^{N}(V_{\alpha,\beta}(x_1-x_j)-V_{\alpha,\beta}\star_x\bar\rho(x_1))\right)^2$$
$$=\frac{1}{N}\sum_{j_1,j_2=1}^{N}\psi_{\alpha\beta}(x_1,x_{j_1})\psi_{\alpha\beta}(x_1,x_{j_2}).$$

We remark that each ψ has vanishing expectation with respect to $\bar\rho$

$$\int_{\Pi^d}\psi_{\alpha\beta}(z,x)\,\bar\rho(x)\,dx=0.$$

Theorem 3. *Consider any $\bar\rho\in L^1(\Pi^d)$ with $\bar\rho\geq 0$ and $\int_{\Pi^d}\bar\rho(x)\,\mathrm{d}x=1$. Assume that a scalar function $\psi\in L^\infty$ with $\|\psi\|_{L^\infty}<\frac{1}{2e}$, and that for any fixed z, $\int_{\Pi^d}\psi(z,x)\,\bar\rho(x)\,dx=0$ then*

$$\begin{aligned}\int_{\Pi^{d\,N}}\bar\rho_N\exp\left(\frac{1}{N}\sum_{j_1,j_2=1}^{N}\psi(x_1,x_{j_1})\psi(x_1,x_{j_2})\right)\,\mathrm{d}X^N\\ \leq C=3\left(1+\frac{5\alpha}{(1-\alpha)^3}+\frac{\beta}{1-\beta}\right),\end{aligned}\tag{2.5}$$

where $\bar\rho_N(t,X^N)=\Pi_{i=1}^N\bar\rho(t,x_i)$

$$\alpha=(e\,\|\psi\|_{L^\infty})^4<1,\quad \beta=\left(\sqrt{2e}\,\|\psi\|_{L^\infty}\right)^4<1.\tag{2.6}$$

We give a straightforward proof of Theorem 3 in Section 4, using the combinatorics techniques developed in the article. But note that this theorem is essentially a variant of the well known law of large numbers at exponential scales; the main difference being that $\psi(x_1,x_{j_1})\psi(x_1,x_{j_2})$ does not have vanishing expectation if $j_1=j_2$, $j_1=1$ or $j_2=1$. *Technically Theorem 3 is hence rather simple, contrary to Theorem 4 below.*

Using Theorem 3 and by taking $\|V\|_{L^\infty}$ small enough, we deduce that

$$\tilde A\leq\frac{C_{\bar\rho}}{N}.\tag{2.7}$$

Step 5: Bound on $\tilde B$ through a new modified law of large numbers. We now define

$$\phi(x,z)=(V(x-z)-V\star\bar\rho(x))\,:\,\frac{\nabla_x^2\bar\rho(x)}{\bar\rho(x)},$$

and we apply to $\tilde B$ the following result

Theorem 4. *Consider $\bar\rho \in L^1(\Pi^d)$ with $\bar\rho \geq 0$ and $\int_{\Pi^d} \bar\rho \,\mathrm{d}x = 1$. Consider further any $\phi(x,z) \in L^\infty$ with*

$$\gamma := C \left(\sup_{p\geq 1} \frac{\| \sup_z |\phi(.,z)| \|_{L^p(\bar\rho\,\mathrm{d}x)}}{p} \right)^2 < 1,$$

where C is a universal constant. Assume that ϕ satisfies the following cancellations

$$\int_{\Pi^d} \phi(x,z)\,\bar\rho(x)\,dx = 0 \quad \forall z, \qquad \int_{\Pi^d} \phi(x,z)\,\bar\rho(z)\,dz = 0 \quad \forall x.$$

Then

$$\int_{\Pi^{d\,N}} \bar\rho_N \exp\left(\frac{1}{N} \sum_{i,j=1}^N \phi(x_i,x_j) \right) \mathrm{d}X^N \leq \frac{3}{1-\gamma} < \infty, \tag{2.8}$$

where we recall that $\bar\rho_N(t,X^N) = \Pi_{i=1}^N \bar\rho(t,x_i)$.

Theorem 4 is by far the main technical difficulty in this article. Observe that contrary to classical laws of large numbers, it requires two precise cancellations on ϕ, separately in x where

$$\int_{\Pi^d} \phi(x,z)\,\bar\rho(x)\,\mathrm{d}x = \int_{\Pi^d} (\operatorname{div} K(x-z) - \operatorname{div} K \star_x \bar\rho(x))\,\bar\rho(x)\,dx = 0,$$

as $\operatorname{div} K = 0$ and in z where we use the classical cancellation

$$\int_{\Pi^d} (V(x-z) - V \star_x \bar\rho(x))\,\bar\rho(z)\,dz = 0.$$

Choosing δ so that $\|V\|_{L^\infty}$ is small enough, Theorem 4 again implies that

$$\tilde B \leq \frac{C_{\bar\rho}}{N}. \tag{2.9}$$

While Theorem 4 looks similar to the modified law of large numbers that was at the heart of our previous result [29], it is considerably more difficult to prove. In [29], we relied a lot on the natural simplectic structure of the problem, which is completely absent here. *The proof Theorem 4 is therefore the main technical difficulty and contribution of the article, performed in Section 5.*

As we noticed earlier, if V were anti-symmetric, then one would have $\phi = 0$ and in turn $\tilde B = 0$. The main technical difficulty here is due to the need for a V without symmetries, which is required to handle 2d incompressible Navier-Stokes.

Final step: Conclusion of the proof. Inserting (2.7) and (2.9) in (2.2), we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_N(\rho_N \mid \bar\rho_N) \leq 2\,\mathcal{H}_N(\rho_N \mid \bar\rho_N) + \frac{C_{\bar\rho}}{N},$$

allowing to conclude through Gronwall's lemma.

There are several additional difficulties in the general proof. The fact that $\|V\|_{L^\infty}$ is not small forces us to carefully rescale all our estimates. Similarly since ρ_N is only an entropy solution to the Liouville Eq. (1.6), we have to proceed more carefully in estimating the relative entropy.

2.2 Time evolution of the relative entropy

The first step in the proof is to estimate the time evolution of the relative entropy,

Lemma 2.2. *Assume that ρ_N is an entropy solution to Eq.* (1.6) *as per Def. 2. Assume that $\bar\rho \in W^{1,\infty}([0,\ T]\times\Pi^d)$ solves Eq.* (1.2) *with* $\inf\bar\rho > 0$ *and* $\int_{\Pi^d}\bar\rho = 1$. *Then*

$$\begin{aligned}
\mathcal{H}_N(\rho_N \mid \bar\rho_N)(t) &= \frac{1}{N}\int_{\Pi^{d\,N}} \rho_N(t,X^N)\,\log\frac{\rho_N(t,X^N)}{\bar\rho_N(t,X^N)}\,\mathrm{d}X^N \leq \mathcal{H}_N(\rho_N^0 \mid \bar\rho_N^0)\\
&- \frac{1}{N^2}\sum_{i,\,j=1}^N \int_0^t\int_{\Pi^{d\,N}} \rho_N\,\left(K(x_i-x_j) - K\star_x\bar\rho(x_i)\right)\cdot\nabla_{x_i}\log\bar\rho_N\,\mathrm{d}X^N\,\mathrm{d}s\\
&- \frac{1}{N^2}\sum_{i,\,j=1}^N \int_0^t\int_{\Pi^{d\,N}} \rho_N\,\left(\operatorname{div}\,K(x_i-x_j) - \operatorname{div}K\star_x\bar\rho(x_i)\right)\,\mathrm{d}X^N\,\mathrm{d}s\\
&- \frac{\underline{\sigma}}{N}\sum_{i=1}^N\int_0^t\int_{\Pi^{d\,N}}\rho_N\left|\nabla_{x_i}\log\frac{\rho_N}{\bar\rho_N}\right|^2 + C_1\,t\,|\sigma-\sigma_N|,
\end{aligned}$$

where we recall that $\bar\rho_N(t,X^N) = \Pi_{i=1}^N\bar\rho(t,x_i)$ *and with*

$$C_1 = \frac{1}{N\,t}\,\frac{2}{\underline{\sigma}}\int_{\Pi^{d\,N}}\rho_N^0\,\log\rho_N^0 + 2\,\|\log\bar\rho\|_{W^{1,\infty}}^2 + \frac{\|\operatorname{div}K\|_{\dot W^{-1,\infty}}^2}{\underline{\sigma}^2} + \frac{2\|\operatorname{div}F\|_{L^\infty}}{\underline{\sigma}}.$$

Proof. From the limiting equation (1.2), one can readily check that $\log \bar{\rho}_N$ solves

$$\partial_t \log \bar{\rho}_N + \sum_{i=1}^{N} \frac{1}{N} \sum_{j=1}^{N} (F(x_i) + K(x_i - x_j)) \cdot \nabla_{x_i} \log \bar{\rho}_N = \sum_{i=1}^{N} \sigma \, \frac{\Delta_{x_i} \bar{\rho}_N}{\bar{\rho}_N}$$
$$+ \sum_{i=1}^{N} \left(\frac{1}{N} \sum_{j=1}^{N} K(x_i - x_j) - K \star_x \bar{\rho}(x_i) \right) \cdot \nabla_{x_i} \log \bar{\rho}_N$$
$$- \sum_{i=1}^{N} (\operatorname{div} F(x_i) + \operatorname{div} K \star_x \bar{\rho}(x_i)). \tag{2.10}$$

Remark that $\log \bar{\rho}_N \in W^{1,\infty}([0,\ T] \times \Pi^{d\,N})$ since $\bar{\rho} \in W^{1,\infty}([0,\ T] \times \Pi^{d})$ and $\bar{\rho}$ is bounded from below. Therefore $\log \bar{\rho}_N$ can be used as a test function against ρ_N in Eq. (1.6). This implies that

$$\int_{\Pi^{d\,N}} \rho_N \, \log \bar{\rho}_N \, \mathrm{d}X^N = \int_{\Pi^{d\,N}} \rho_N^0 \, \log \bar{\rho}_N^0 \, \mathrm{d}X^N$$
$$+ \int_0^t \int_{\Pi^{d\,N}} \rho_N \left(\partial_t \log \bar{\rho}_N + \frac{1}{N} \sum_{i,j=1}^{N} (F(x_i) + K(x_i - x_j)) \cdot \nabla_{x_i} \log \bar{\rho}_N \right) \mathrm{d}X^N \, \mathrm{d}s$$
$$- \sigma_N \sum_{i=1}^{N} \int_0^t \int_{\Pi^{d\,N}} \nabla_{x_i} \log \bar{\rho}_N \, \nabla_{x_i} \rho_N \, \mathrm{d}X^N \, \mathrm{d}s.$$

Using the equation (2.10) on $\log \bar{\rho}_N$, we obtain

$$\int_{\Pi^{d\,N}} \rho_N \, \log \bar{\rho}_N \, \mathrm{d}X^N = \int_{\Pi^{d\,N}} \rho_N^0 \, \log \bar{\rho}_N^0 \, \mathrm{d}X^N$$
$$+ \sum_{i=1}^{N} \int_0^t \int_{\Pi^{d\,N}} \rho_N \, \left(\frac{1}{N} \sum_{j=1}^{N} K(x_i - x_j) - K \star_x \bar{\rho}(x_i) \right) \cdot \nabla_{x_i} \log \bar{\rho}_N \, \mathrm{d}X^N \, \mathrm{d}s$$
$$- \int_0^t \int_{\Pi^{d\,N}} \rho_N \, \sum_{i=1}^{N} (\operatorname{div} F(x_i) + \operatorname{div} K \star_x \bar{\rho}(x_i)) \, \mathrm{d}X^N \, \mathrm{d}s$$
$$+ \sum_{i=1}^{N} \int_0^t \int_{\Pi^{d\,N}} \left(\sigma \, \rho_N \, \frac{\Delta_{x_i} \bar{\rho}_N}{\bar{\rho}_N} - \sigma_N \nabla_{x_i} \rho_N \cdot \frac{\nabla_{x_i} \bar{\rho}_N}{\bar{\rho}_N} \right) \, \mathrm{d}X^N \, \mathrm{d}s.$$

Using the entropy dissipation for ρ_N given by (1.8), we have that

$$\begin{aligned}
&\mathcal{H}_N(\rho_N \mid \bar\rho_N)(t) \leq \mathcal{H}_N(\rho_N \mid \bar\rho_N)(0) + \frac{1}{N}\, D_N \\
&- \frac{1}{N^2} \sum_{i,\, j=1}^{N} \int_0^t \int_{\Pi^{d\,N}} \rho_N \, \left(K(x_i - x_j) - K \star_x \bar\rho(x_i)\right) \cdot \nabla_{x_i} \log \bar\rho_N \, \mathrm{d}X^N \, \mathrm{d}s \\
&- \frac{1}{N^2} \sum_{i,\, j=1}^{N} \int_0^t \int_{\Pi^{d\,N}} \rho_N \, \left(\operatorname{div} K(x_i - x_j) - \operatorname{div} K \star_x \bar\rho(x_i)\right) \, \mathrm{d}X^N \, \mathrm{d}s,
\end{aligned} \tag{2.11}$$

with

$$D_N = \sum_{i=1}^{N} \int_0^t \int_{\Pi^{d\,N}} \left(-\sigma\, \rho_N \, \frac{\Delta_{x_i} \bar\rho_N}{\bar\rho_N} + \sigma_N \nabla_{x_i} \rho_N \cdot \frac{\nabla_{x_i} \bar\rho_N}{\bar\rho_N} - \sigma_N \, \frac{|\nabla_{x_i} \rho_N|^2}{\rho_N} \right).$$

By integration by parts

$$\begin{aligned}
&\int_{\Pi^{d\,N}} \left(-\rho_N \, \frac{\Delta_{x_i} \bar\rho_N}{\bar\rho_N} + \nabla_{x_i} \rho_N \cdot \frac{\nabla_{x_i} \bar\rho_N}{\bar\rho_N} - \frac{|\nabla_{x_i} \rho_N|^2}{\rho_N} \right) \\
&\quad = - \int_{\Pi^{d\,N}} \left(\rho_N \frac{|\nabla_{x_i} \bar\rho_N|^2}{\bar\rho_N^2} - 2\, \nabla_{x_i} \rho_N \cdot \frac{\nabla_{x_i} \bar\rho_N}{\bar\rho_N} + \frac{|\nabla_{x_i} \rho_N|^2}{\rho_N} \right) \\
&\quad = - \int_{\Pi^{d\,N}} \rho_N \, \left| \nabla_{x_i} \log \frac{\rho_N}{\bar\rho_N} \right|^2 .
\end{aligned} \tag{2.12}$$

On the other hand,

$$\begin{aligned}
&(\sigma - \sigma_N) \sum_{i=1}^{N} \int_{\Pi^{d\,N}} \rho_N \, \frac{\Delta_{x_i} \bar\rho_N}{\bar\rho_N} \\
&\qquad = (\sigma - \sigma_N) \sum_{i=1}^{N} \int_{\Pi^{d\,N}} \left(-\nabla_{x_i} \rho_N \cdot \frac{\nabla_{x_i} \bar\rho_N}{\bar\rho_N} + \rho_N \, \frac{|\nabla_{x_i} \bar\rho_N|^2}{\bar\rho_N^2} \right).
\end{aligned}$$

Of course

$$\sum_{i=1}^{N} \int_{\Pi^{d\,N}} \rho_N \, \frac{|\nabla_{x_i} \bar\rho_N|^2}{\bar\rho_N^2} = \sum_{i=1}^{N} \int_{\Pi^{d\,N}} \rho_N \, \frac{|\nabla_{x_i} \bar\rho(x_i)|^2}{\bar\rho(x_i)^2} \leq N \, \| \log \bar\rho \|_{W^{1,\infty}}^2,$$

while by Cauchy-Schwartz

$$\sum_{i=1}^N \int_0^t \int_{\Pi^{d\,N}} \nabla_{x_i}\rho_N \cdot \frac{\nabla_{x_i}\bar\rho_N}{\bar\rho_N} \leq N\,t\,\|\log\bar\rho\|_{W^{1,\infty}}^2 + \sum_{i=1}^N \int_0^t \int_{\Pi^{d\,N}} \frac{|\nabla_{x_i}\rho_N|^2}{\rho_N}$$
$$\leq N\,t\,\|\log\bar\rho\|_{W^{1,\infty}}^2 + \frac{2}{\underline{\sigma}} \int_{\Pi^{d\,N}} \rho_N^0\,\log\rho_N^0 + \frac{N\,t\,\|\operatorname{div} K\|_{\dot W^{-1,\infty}}^2}{\underline{\sigma}^2}$$
$$+ N\,t\,\frac{2\|\operatorname{div} F\|_{L^\infty}}{\underline{\sigma}},$$

by Prop. 1 based on the entropy dissipation.

This leads to

$$\begin{aligned}
&(\sigma-\sigma_N)\sum_{i=1}^N \int_0^t \int_{\Pi^{d\,N}} \rho_N\,\frac{\Delta_{x_i}\bar\rho_N}{\bar\rho_N} \\
&\quad \leq |\sigma-\sigma_N|\,\Bigg(N\,t\left[2\,\|\log\bar\rho\|_{W^{1,\infty}}^2 + \frac{\|\operatorname{div} K\|_{\dot W^{-1,\infty}}^2}{\underline{\sigma}^2} + \frac{2\|\operatorname{div} F\|_{L^\infty}}{\underline{\sigma}}\right] \\
&\qquad + \frac{2}{\underline{\sigma}} \int_{\Pi^{d\,N}} \rho_N^0\,\log\rho_N^0\Bigg).
\end{aligned} \tag{2.13}$$

Finally combining (2.13) with (2.12)

$$\begin{aligned}
D_N \leq{}& -\underline{\sigma}\sum_{i=1}^N \int_0^t \int_{\Pi^{d\,N}} \rho_N\,\left|\nabla_{x_i}\log\frac{\rho_N}{\bar\rho_N}\right|^2 + |\sigma-\sigma_N|\,\Bigg(\frac{2}{\underline{\sigma}} \int_{\Pi^{d\,N}} \rho_N^0\,\log\rho_N^0 \\
&+ N\,t\left[2\,\|\log\bar\rho\|_{W^{1,\infty}}^2 + \frac{\|\operatorname{div} K\|_{\dot W^{-1,\infty}}^2}{\underline{\sigma}^2} + \frac{2\|\operatorname{div} F\|_{L^\infty}}{\underline{\sigma}}\right]\Bigg),
\end{aligned}$$

which inserted in (2.11) concludes the proof. □

2.3 Bounding the interaction terms: The bounded divergence term

We now have to obtain the main estimates, starting with the case where the kernel belongs to $\dot W^{-1,\infty}(\Pi^d)$ and has bounded divergence.

Lemma 2.3. *Assume that $\bar\rho \in W^{2,p}(\Pi^d)$ for any $p<\infty$, then for any kernel*

$L \in \dot W^{-1,\infty}(\Pi^d)$ with $\operatorname{div} L \in L^\infty$, one has that

$$
\begin{aligned}
&- \frac{1}{N^2} \sum_{i,\,j=1}^N \int_{\Pi^{d\,N}} \rho_N \, \left(L(x_i - x_j) - L \star_x \bar\rho(x_i)\right) \cdot \nabla_{x_i} \log \bar\rho_N \, \mathrm{d}X^N \\
&- \frac{1}{N^2} \sum_{i,\,j=1}^N \int_{\Pi^{d\,N}} \rho_N \, \left(\operatorname{div} L(x_i - x_j) - \operatorname{div} L \star_x \bar\rho(x_i)\right) \, \mathrm{d}X^N \\
&\qquad \le \frac{\underline{\sigma}}{4\,N} \sum_{i=1}^N \int_{\Pi^{d\,N}} \rho_N \, |\nabla_{x_i} \log \frac{\rho_N}{\bar\rho_N}|^2 \, \mathrm{d}X^N + C\, M_L^1 \left(\mathcal{H}_N(\rho_N \,|\bar\rho_N) + \frac{1}{N}\right),
\end{aligned}
$$

where C is a universal constant and

$$
M_L^1 = d^3 \, \frac{\|\bar\rho\|_{W^{1,\infty}}^2 \, \|L\|_{\dot W^{-1,\infty}}^2}{\underline{\sigma}\,(\inf \bar\rho)^2} + \frac{\|L\|_{\dot W^{-1,\infty}}}{\inf \bar\rho} \, \sup_{p \ge 1} \frac{\|\nabla^2 \bar\rho\|_{L^p}}{p} + \|\operatorname{div} L\|_{L^\infty}.
$$

Proof. Remark that in this estimate, time is now only a fixed parameter and will hence not be specified in this proof.

Denote $V \in L^\infty(\Pi^d)$ s.t. $L = \operatorname{div} V$. By the definition of $\dot W^{-1,\infty}$ we assume that $\|V\|_{L^\infty} \le 2\,\|L\|_{\dot W^{-1,\infty}}$. By integration by parts

$$
\begin{aligned}
&- \frac{1}{N^2} \sum_{i,\,j=1}^N \int_{\Pi^{d\,N}} \rho_N \, \left(L(x_i - x_j) - L \star_x \bar\rho(x_i)\right) \cdot \nabla_{x_i} \log \bar\rho_N \, \mathrm{d}X^N \\
&- \frac{1}{N^2} \sum_{i,\,j=1}^N \int_{\Pi^{d\,N}} \rho_N \, \left(\operatorname{div} L(x_i - x_j) - \operatorname{div} L \star_x \bar\rho(x_i)\right) \, \mathrm{d}X^N = A + B,
\end{aligned}
$$

with

$$
\begin{aligned}
A &= \frac{1}{N^2} \sum_{i,\,j=1}^N \int_{\Pi^{d\,N}} \left(V(x_i - x_j) - V \star_x \bar\rho(x_i)\right) \,:\, \nabla_{x_i} \bar\rho_N \otimes \nabla_{x_i} \frac{\rho_N}{\bar\rho_N} \, \mathrm{d}X^N, \\
B &= \frac{1}{N^2} \sum_{i,\,j=1}^N \int_{\Pi^{d\,N}} \rho_N \Bigg[\left(V(x_i - x_j) - V \star_x \bar\rho(x_i)\right) \,:\, \frac{\nabla_{x_i}^2 \bar\rho_N}{\bar\rho_N} \\
&\qquad\qquad - \operatorname{div} L(x_i - x_j) + \operatorname{div} L \star_x \bar\rho(x_i) \Bigg] \, \mathrm{d}X^N.
\end{aligned}
$$

We treat independently A and B.

The bound on A. First by Cauchy-Schwartz and by using $a\,b \leq a^2/4 + b^2$

$$A \leq \frac{\sigma}{4\,N} \sum_{i=1}^N \int_{\Pi^{d\,N}} \frac{\bar\rho_N^2}{\rho_N} |\nabla_{x_i} \frac{\rho_N}{\bar\rho_N}|^2 \, \mathrm{d}X^N$$
$$+ \frac{d}{N\,\underline{\sigma}} \sum_{i=1}^N \int_{\Pi^{d\,N}} \left(\frac{1}{N}\sum_{j=1}^N (V(x_i-x_j) - V \star_x \bar\rho(x_i))\right)^2 \left|\frac{\nabla_{x_i}\bar\rho_N}{\bar\rho_N}\right|^2 \rho_N \, \mathrm{d}X^N.$$

Remark that

$$\left|\frac{\nabla_{x_i}\bar\rho_N}{\bar\rho_N}\right|^2 = |\nabla_{x_i} \log\bar\rho(x_i)|^2 \leq \frac{\|\bar\rho\|^2_{W^{1,\infty}}}{(\inf\bar\rho)^2}.$$

Hence one has that

$$A \leq \frac{\sigma}{4\,N} \sum_{i=1}^N \int_{\Pi^{d\,N}} \rho_N\, |\nabla_{x_i} \log \frac{\rho_N}{\bar\rho_N}|^2 \, \mathrm{d}X^N$$
$$+ \frac{d\|\bar\rho\|^2_{W^{1,\infty}}}{N\,\underline{\sigma}\,(\inf\bar\rho)^2} \sum_{i=1}^N \sum_{\alpha,\beta=1}^d \int_{\Pi^{d\,N}} \left(\frac{1}{N}\sum_{j=1}^N (V_{\alpha,\beta}(x_i-x_j) - V_{\alpha,\beta} \star_x \bar\rho(x_i))\right)^2$$
$$\rho_N \, \mathrm{d}X^N, \tag{2.14}$$

where $V_{\alpha,\beta}$ is the corresponding coordinate of the matrix field V.

For some $\eta > 0$ to be chosen later, we apply Lemma 2.1 with

$$\Phi = \left(\frac{1}{N}\sum_{j=1}^N \eta\, (V_{\alpha,\beta}(x_i-x_j) - V_{\alpha,\beta} \star_x \bar\rho(x_i))\right)^2,$$

to find

$$\frac{1}{N}\sum_{i=1}^N \sum_{\alpha,\beta=1}^d \int_{\Pi^{d\,N}} \left(\frac{1}{N}\sum_{j=1}^N (V_{\alpha,\beta}(x_i-x_j) - V_{\alpha,\beta} \star_x \bar\rho(x_i))\right)^2 \rho_N \, \mathrm{d}X^N$$
$$\leq \frac{d^2}{\eta^2} \mathcal{H}_N(\rho_N \,|\, \bar\rho_N)$$
$$+ \frac{1}{N^2\,\eta^2} \sum_{i=1}^N \sum_{\alpha,\beta=1}^N \log \int_{\Pi^{d\,N}} \bar\rho_N\, e^{N\left(\frac{1}{N}\sum_j \eta\left(V_{\alpha,\beta}(x_i-x_j) - V_{\alpha,\beta}\star_x\bar\rho(x_i)\right)\right)^2} \, \mathrm{d}X^N. \tag{2.15}$$

By symmetry

$$\frac{1}{N}\sum_{i=1}^{N}\log\int_{\Pi^{d\,N}}\bar\rho_N\, e^{N\left(\frac{1}{N}\sum_j \eta\left(V_{\alpha,\beta}(x_i-x_j)-V_{\alpha,\beta}\star_x\bar\rho(x_i)\right)\right)^2}\,\mathrm{d}X^N$$
$$=\log\int_{\Pi^{d\,N}}\bar\rho_N\, e^{N\left(\frac{1}{N}\sum_j \eta\left(V_{\alpha,\beta}(x_1-x_j)-V_{\alpha,\beta}\star_x\bar\rho(x_1)\right)\right)^2}\,\mathrm{d}X^N.$$

Define $\psi(z,x)=\eta\,V_{\alpha,\beta}(z-x)-\eta\,V_{\alpha,\beta}\star\bar\rho(z)$. Choose $\eta=1/(4\,e\,\|V\|_{L^\infty})$ and note that $\|\psi\|_{L^\infty}\leq\frac{1}{4\,e}$ and that for a fixed z, $\int\bar\rho(x)\,\psi(z,x)\,dx=0$. Since

$$N\left(\frac{1}{N}\sum_{j=1}^{N}\eta\,\left(V_{\alpha,\beta}(x_1-x_j)-V_{\alpha,\beta}\star_x\bar\rho(x_1)\right)\right)^2$$
$$=\frac{1}{N}\sum_{j_1,j_2=1}^{N}\psi(x_1,x_{j_1})\,\psi(x_1,x_{j_2}),$$

we may apply Theorem 3 to obtain that

$$\int_{\Pi^{d\,N}}\bar\rho_N\, e^{N\left(\frac{1}{N}\sum_j \eta\left(V_{\alpha,\beta}(x_1-x_j)-V_{\alpha,\beta}\star_x\bar\rho(x_1)\right)\right)^2}\,\mathrm{d}X^N\leq C,$$

for some explicit universal constant C.

Combining (2.14)-(2.15) with this bound yields the final estimate on A

$$\begin{aligned}A\leq\;&\frac{\sigma}{4\,N}\sum_{i=1}^{N}\int_{\Pi^{d\,N}}\rho_N\,|\nabla_{x_i}\log\frac{\rho_N}{\bar\rho_N}|^2\,\mathrm{d}X^N\\&+C\,d^3\,\frac{\|\bar\rho\|_{W^{1,\infty}}^2\,\|V\|_{L^\infty}^2}{\underline{\sigma}\,(\inf\bar\rho)^2}\,\left(\mathcal{H}_N(\rho_N\,|\bar\rho_N)+\frac{1}{N}\right),\end{aligned}\tag{2.16}$$

again for some universal constant C.

The bound on B. Define

$$\phi(x,z)=(V(x-z)-V\star_x\bar\rho(x))\;:\;\frac{\nabla_x^2\bar\rho(x)}{\bar\rho(x)}-\,\mathrm{div}\,L(x-z)+\,\mathrm{div}\,L\star_x\bar\rho(x),\tag{2.17}$$

so that

$$B = \frac{1}{N^2} \sum_{i,j=1}^{N} \int_{\Pi^{d\,N}} \rho_N \left[(V(x_i - x_j) - V \star_x \bar\rho(x_i)) \,:\, \frac{\nabla^2_{x_i} \bar\rho_N}{\bar\rho_N} \right.$$

$$\left. -\ \mathrm{div}\, L(x_i - x_j) +\ \mathrm{div}\, L \star_x \bar\rho(x_i) \right] \mathrm{d}X^N$$

$$= \frac{1}{N^2} \sum_{i,j=1}^{N} \int_{\Pi^{d\,N}} \rho_N \, \phi(x_i, x_j)\ \mathrm{d}X^N.$$

Apply Lemma 2.1 with

$$\Phi = \frac{1}{N^2} \sum_{i,j=1}^{N} \eta \, \phi(x_i, x_j),$$

so that

$$B \le \frac{1}{\eta}\, \mathcal{H}_N(\rho_N \mid \bar\rho_N) + \frac{1}{N\,\eta} \int_{\Pi^{d\,N}} \bar\rho_N \, e^{\frac{1}{N} \sum_{i,j} \eta\, \phi(x_i, x_j)}\ \mathrm{d}X^N. \tag{2.18}$$

Observe that $\int_{\Pi^d} \phi(x,z)\, \bar\rho(z)\, \mathrm{d}z = 0$. While by integration by parts

$$\int_{\Pi^d} (V(x-z) - V \star_x \bar\rho(x)) \,:\, \frac{\nabla^2_x \bar\rho(x)}{\bar\rho(x)}\, \bar\rho(x)\, \mathrm{d}x$$

$$= \int_{\Pi^d} (\,\mathrm{div}\, L(x-z) -\ \mathrm{div}\, L \star_x \bar\rho(x))\ \bar\rho(x)\, \mathrm{d}x,$$

implying that $\int_{\Pi^d} \phi(x,z)\, \bar\rho(x)\ \mathrm{d}x = 0$. Note as well from (2.17) that

$$\| \sup_z |\phi(.,z)| \|_{L^p(\bar\rho\, dx)} \le 2\, \frac{\|V\|_{L^\infty}}{\inf \bar\rho}\, \|\nabla^2 \bar\rho\|_{L^p} + 2\, \|\,\mathrm{div}\, L\|_{L^\infty}.$$

Hence choosing

$$\eta = \frac{1}{C\, \left(\frac{\|V\|_{L^\infty}}{\inf \bar\rho}\ \sup_p \frac{\|\nabla^2 \bar\rho\|_{L^p}}{p} + \|\,\mathrm{div}\, L\|_{L^\infty} \right)},$$

we may apply Theorem 4 to bound

$$\int_{\Pi^{d\,N}} \bar\rho_N \, e^{\frac{1}{N} \sum_{i,j} \eta\, \phi(x_i, x_j)}\ \mathrm{d}X^N \le C,$$

for some universal constant C. Hence from (2.18), we conclude that

$$B \le C \left(\frac{\|V\|_{L^\infty}}{\inf \bar\rho} \sup_p \frac{\|\nabla^2 \bar\rho\|_{L^p}}{p} + \|\operatorname{div} L\|_{L^\infty} \right) \left(\mathcal{H}_N(\rho_N \mid \bar\rho_N) + \frac{1}{N} \right). \tag{2.19}$$

To finish the proof of the lemma, we simply have to add (2.16) and (2.19), recalling that $\|V\|_{L^\infty} \le 2\|L\|_{\dot W^{-1,\infty}}$. □

2.4 Bounding the interaction terms: The divergence term only in $\dot W^{-1,\infty}$

Lemma 2.4. *Assume that $\bar\rho \in W^{1,p}(\Pi^d)$ for any $p < \infty$, then for any kernel $L \in L^\infty(\Pi^d)$ with $\operatorname{div} L \in \dot W^{-1,\infty}$, one has that*

$$\begin{aligned}
&- \frac{1}{N^2} \sum_{i,\, j=1}^N \int_{\Pi^{d\,N}} \rho_N \, \left(L(x_i - x_j) - L \star_x \bar\rho(x_i)\right) \cdot \nabla_{x_i} \log \bar\rho_N \, \mathrm{d}X^N \\
&- \frac{1}{N^2} \sum_{i,\, j=1}^N \int_{\Pi^{d\,N}} \rho_N \, \left(\operatorname{div} L(x_i - x_j) - \operatorname{div} L \star_x \bar\rho(x_i)\right) \, \mathrm{d}X^N \\
&\qquad \le \frac{\sigma}{4\,N} \sum_i \int_{\Pi^{d\,N}} \rho_N \, |\nabla_{x_i} \log \frac{\rho_N}{\bar\rho_N}|^2 \, \mathrm{d}X^N + C\, M_L^2 \left(\mathcal{H}_N(\rho_N \,|\bar\rho_N) + \frac{1}{N} \right),
\end{aligned}$$

where C is a universal constant and

$$M_L^2 = \left(\|L\|_{L^\infty} + \|\operatorname{div} L\|_{\dot W^{-1,\infty}} \right) \frac{\|\nabla \bar\rho\|_{L^\infty}}{\inf \bar\rho} + \frac{d}{\underline{\sigma}} \|\operatorname{div} L\|^2_{\dot W^{-1,\infty}}.$$

Proof. The proof follows similar ideas to the proof of Lemma 2.3 but now we have to integrate by parts the term with $\operatorname{div} L$ instead of the term with L. Denote $\tilde L \in L^\infty$ s.t. $\operatorname{div} \tilde L = \operatorname{div} L$ and $\|\operatorname{div} L\|_{\dot W^{-1,\infty}} = \|\tilde L\|_{L^\infty}$. Write

$$\begin{aligned}
&- \frac{1}{N^2} \sum_{i,\, j=1}^N \int_{\Pi^{d\,N}} \rho_N \, \left(\operatorname{div} L(x_i - x_j) - \operatorname{div} L \star_x \bar\rho(x_i)\right) \, \mathrm{d}X^N \\
&\quad = \frac{1}{N^2} \sum_{i,\, j=1}^N \int_{\Pi^{d\,N}} \nabla_{x_i} \frac{\rho_N}{\bar\rho_N} \cdot \left(\tilde L(x_i - x_j) - \tilde L \star_x \bar\rho(x_i) \right) \bar\rho_N \, \mathrm{d}X^N \\
&\qquad + \frac{1}{N^2} \sum_{i,\, j=1}^N \int_{\Pi^{d\,N}} \rho_N \, \left(\tilde L(x_i - x_j) - \tilde L \star_x \bar\rho(x_i) \right) \cdot \nabla_{x_i} \log \bar\rho_N \, \mathrm{d}X^N.
\end{aligned}$$

Hence

$$
\begin{aligned}
&-\frac{1}{N^2}\sum_{i,\,j=1}^{N}\int_{\Pi^{d\,N}}\rho_N\ \left(L(x_i-x_j)-L\star_x\bar\rho(x_i)\right)\cdot\nabla_{x_i}\log\bar\rho_N\ \mathrm{d}X^N\\
&-\frac{1}{N^2}\sum_{i,\,j=1}^{N}\int_{\Pi^{d\,N}}\rho_N\ \left(\operatorname{div}L(x_i-x_j)-\operatorname{div}L\star_x\bar\rho(x_i)\right)\ \mathrm{d}X^N=A+B,
\end{aligned}
\tag{2.20}
$$

with

$$
A=\frac{1}{N^2}\sum_{i,\,j=1}^{N}\int_{\Pi^{d\,N}}\nabla_{x_i}\frac{\rho_N}{\bar\rho_N}\cdot\left(\tilde L(x_i-x_j)-\tilde L\star_x\bar\rho(x_i)\right)\ \bar\rho_N\,\mathrm{d}X^N,
$$

and

$$
B=\frac{1}{N^2}\sum_{i,\,j=1}^{N}\int_{\Pi^{d\,N}}\rho_N\ \left(\bar L(x_i-x_j)-\bar L\star_x\bar\rho(x_i)\right)\cdot\nabla_{x_i}\log\bar\rho_N\ \mathrm{d}X^N,
$$

for $\bar L=\tilde L-L$.

Bound for A. We start with Cauchy-Schwartz to bound

$$
\begin{aligned}
A\le&\frac{\underline{\sigma}}{4\,N}\sum_{i=1}^{N}\int_{\Pi^{d\,N}}|\nabla_{x_i}\frac{\rho_N}{\bar\rho_N}|^2\,\frac{\bar\rho_N^2}{\rho_N}\\
&+\frac{1}{N\,\underline{\sigma}}\sum_{i=1}^{N}\sum_{\alpha=1}^{d}\int_{\Pi^{d\,N}}\rho_N\left|\frac{1}{N}\sum_{j=1}^{N}(\tilde L_\alpha(x_i-x_j)-\tilde L_\alpha\star_x\bar\rho(x_i)\right|^2\ \mathrm{d}X^N,
\end{aligned}
$$

where $\tilde L_\alpha$ is the α coordinate of $\tilde L$.

Denote $\psi(z,x)=\eta\,(\tilde L_\alpha(z-x)-\tilde L_\alpha\star\bar\rho(z))$, and use Lemma 2.1 for $\Phi=\left|\frac{1}{N}\sum_{j=1}^{N}\psi(x_i,x_j)\right|^2$ to obtain

$$
\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N}\int_{\Pi^{d\,N}}\rho_N\left|\frac{1}{N}\sum_{j=1}^{N}(\tilde L_\alpha(x_i-x_j)-\tilde L_\alpha\star_x\bar\rho(x_i)\right|^2\ \mathrm{d}X^N\\
&\quad\le\frac{1}{\eta^2}\,\mathcal{H}_N(\rho_N\mid\bar\rho_N)+\frac{1}{N^2\,\eta^2}\sum_{i=1}^{N}\log\int_{\Pi^{d\,N}}\bar\rho_N\,e^{\left|\frac{1}{N}\sum_j\psi(x_i,x_j)\right|^2}\ \mathrm{d}X^N.
\end{aligned}
$$

Of course $\int_{\Pi^d} \psi(z,x)\,\bar\rho(x)\,dx = 0$ so that taking

$$\eta = \frac{1}{4\,e\,\|\tilde L\|_{L^\infty}} = \frac{1}{4\,e\,\|\operatorname{div} L\|_{\dot W^{-1,\infty}}},$$

and applying Theorem 3, we find

$$\begin{aligned} A \leq & \frac{\sigma}{4\,N} \sum_{i=1}^N \int_{\Pi^{d\,N}} \rho_N |\nabla_{x_i} \log \frac{\rho_N}{\bar\rho_N}|^2 \,\mathrm{d}X^N \\ & + C\,d\,\frac{\|\operatorname{div} L\|^2_{\dot W^{-1,\infty}}}{\underline{\sigma}} \left(\mathcal{H}_N(\rho_N \mid \bar\rho_N) + \frac{1}{N}\right). \end{aligned} \tag{2.21}$$

Bound for B. We follow the same steps as before, define

$$\phi(x,z) = \big(\bar L(x-z) - \bar L \star_x \bar\rho(x)\big)\cdot \nabla_x \log \bar\rho(x),$$

and first apply Lemma 2.1 with $\Phi = \frac{\eta}{N^2}\sum_{i,j=1}^N \phi(x_i,x_j)$ to find

$$B \leq \frac{1}{\eta}\mathcal{H}_N(\rho_N \mid \bar\rho_N) + \frac{1}{N\,\eta}\,\log \int_{\Pi^{d\,N}} \bar\rho_N\, e^{\frac{1}{N}\sum_{i,j}\phi(x_i,x_j)}\,\mathrm{d}X^N.$$

Since $\operatorname{div} \bar L = \operatorname{div} \tilde L - \operatorname{div} L = 0$, we have that

$$\int_{\Pi^d} \phi(x,z)\,\bar\rho(z)\,dz = \int_{\Pi^d} \phi(x,z)\,\bar\rho(x)\,dx = 0.$$

Choose

$$\eta = \frac{1}{C\,\|\bar L\|_{L^\infty}\,\sup_p \frac{\|\nabla \log\bar\rho\|_{L^p(\bar\rho\,\mathrm{d}x)}}{p}} = \frac{\inf\bar\rho}{C\,(\|L\|_{L^\infty} + \|\operatorname{div} L\|_{\dot W^{-1,\infty}})\,\|\nabla\bar\rho\|_{L^\infty}},$$

and apply now Theorem 4 to conclude that

$$B \leq C\,(\|L\|_{L^\infty} + \|\operatorname{div} L\|_{\dot W^{-1,\infty}})\,\frac{\|\nabla\bar\rho\|_{L^\infty}}{\inf\bar\rho}\,\left(\mathcal{H}_N(\rho_N \mid \bar\rho_N) + \frac{1}{N}\right). \tag{2.22}$$

Combining (2.21) and (2.22) concludes the proof. $\square$

2.5 Conclusion of the proof of Theorem 1

The proof of Theorem 1 follows from the previous estimates through a careful decomposition of the kernel K.

By the assumption of Theorem 1, we have that $K_\alpha = \partial_\beta V_{\alpha\beta}$ where $V \in L^\infty(\Pi^d)$ is a matrix field, and that there exists $\tilde{K} \in L^\infty$ s.t. $\operatorname{div} K = \operatorname{div} \tilde{K}$ and $\|\tilde{K}\|_{L^\infty(\Pi^d)} \leq 2\, \|\operatorname{div} K\|_{\dot{W}^{-1,\infty}}$. For convenience, we use the notation

$$\|K\| = \|\tilde{K}\|_{L^\infty(\Pi^d)} + \|V\|_{L^\infty(\Pi^d)} \leq 2\, \|\operatorname{div} K\|_{\dot{W}^{-1,\infty}} + 2\, \|K\|_{\dot{W}^{-1,\infty}}.$$

Define $\bar{K} = \operatorname{div} V - \tilde{K}$. Note that $\operatorname{div} \bar{K} = 0$ and obviously since $\tilde{K} \in L^\infty$ and we can choose $\tilde{K}$ s.t. $\int \tilde{K} = 0$, then $\bar{K} \in \dot{W}^{-1,\infty}$ with $\|\bar{K}\|_{\dot{W}^{-1,\infty}} \leq C_d\, \|K\|$.

We combine Lemma 2.2 with Lemma 2.3 for $L = \bar{K}$, and finally with Lemma 2.4 for $L = \tilde{K}$. We obtain

$$\begin{aligned} &\mathcal{H}_N(\rho_N \mid \bar{\rho}_N)(t) \leq \mathcal{H}_N(\rho_N^0 \mid \bar{\rho}_N^0) + C_1\, t\, |\sigma - \sigma_N| \\ &\quad + C \int_0^t \left(M_{\bar{K}}^1 + M_{\tilde{K}}^2\right) \left(\mathcal{H}_N(\rho_N \mid \bar{\rho}_N)(s) + \frac{1}{N}\right) ds. \end{aligned} \tag{2.23}$$

Because of our specific bounds

$$\begin{aligned} M_{\bar{K}}^1 &\leq d^3\, \|\bar{K}\|_{\dot{W}^{-1,\infty}}^2 \frac{\|\bar{\rho}\|_{W^{1,\infty}}^2}{\underline{\sigma}\, (\inf \bar{\rho})^2} + \frac{\|\bar{K}\|_{\dot{W}^{-1,\infty}}}{\inf \bar{\rho}} \sup_p \frac{\|\nabla^2 \bar{\rho}\|_{L^p}}{p}, \\ M_{\tilde{K}}^2 &\leq \left(\|\tilde{K}\|_{L^\infty} + \|\operatorname{div} \tilde{K}\|_{\dot{W}^{-1,\infty}}\right) \frac{\|\nabla \bar{\rho}\|_{L^\infty}}{\inf \bar{\rho}} + \frac{d}{\underline{\sigma}} \|\operatorname{div} \tilde{K}\|_{\dot{W}^{-1,\infty}}^2. \end{aligned}$$

To keep calculations simple, we do not try here to obtain fully explicit bounds (which would still be possible) and simplify (2.23) in

$$\begin{aligned} &\mathcal{H}_N(\rho_N \mid \bar{\rho}_N)(t) \leq \mathcal{H}_N(\rho_N^0 \mid \bar{\rho}_N^0) + \bar{M}\, (1 + t\, (1 + \|K\|^2))\, |\sigma - \sigma_N| \\ &+ \bar{M}\, (\|K\| + \|K\|^2) \int_0^t \left(\mathcal{H}_N(\rho_N \mid \bar{\rho}_N)(s) + \frac{1}{N}\right), \end{aligned} \tag{2.24}$$

where we only kept explicit a simplified dependence on K and where the constant $\bar{M}$ depends only on

$$\bar{M}\left(d,\ \underline{\sigma},\ \inf \bar{\rho},\ \|\bar{\rho}\|_{W^{1,\infty}},\ \sup_p \frac{\|\nabla^2 \bar{\rho}\|_{L^p}}{p},\ \frac{1}{N} \int_{\Pi^{d\,N}} \rho_N^0\, \log \rho_N^0,\ \|\operatorname{div} F\|_{L^\infty}\right).$$

By Gronwall lemma, (2.24) implies that

$$\mathcal{H}_N(\rho_N \mid \bar\rho_N)(t) \leq e^{\bar M\,(\|K\|+\|K\|^2)\,t}\Bigg(\mathcal{H}_N(\rho_N^0 \mid \bar\rho_N^0) + \frac{1}{N} + \bar M(1+t\,(1+\|K\|^2)\,|\sigma-\sigma_N|\Bigg),$$

which concludes the proof of Theorem 1.

2.6 Proof of Theorem 2

The proof of our result for vanishing viscosity is in fact now straightforward as it uses our previous analysis.

First of all, we have an direct equivalent of Lemma 2.2

$$\begin{aligned}
\mathcal{H}_N(\rho_N \mid \bar\rho_N)(t) &= \frac{1}{N}\int_{\Pi^{d\,N}} \rho_N(t,X^N)\,\log\frac{\rho_N(t,X^N)}{\bar\rho_N(t,X^N)}\,\mathrm{d}X^N \leq \mathcal{H}_N(\rho_N^0 \mid \bar\rho_N^0)\\
&- \frac{1}{N^2}\sum_{i,\,j=1}^N \int_0^t\int_{\Pi^{d\,N}} \rho_N\,\left(K(x_i-x_j) - K\star_x\bar\rho(x_i)\right)\cdot\nabla_{x_i}\log\bar\rho_N\,\mathrm{d}X^N\,\mathrm{d}s\\
&- \frac{1}{N^2}\sum_{i,\,j=1}^N \int_0^t\int_{\Pi^{d\,N}} \rho_N\,\left(\operatorname{div}\,K(x_i-x_j) - \operatorname{div} K\star_x\bar\rho(x_i)\right)\,\mathrm{d}X^N\,\mathrm{d}s + \alpha_N
\end{aligned} \tag{2.25}$$

where when $\sigma_N\to\sigma=0$,

$$\alpha_N = \frac{\sigma_N}{4N}\sum_{i=1}^N\int_0^t\int_{\Pi^{d\,N}} \bar\rho_N|\nabla_{x_i}\log\bar\rho(x_i)|^2\,\mathrm{d}X^N\,\mathrm{d}s \leq t|\sigma-\sigma_N|\,\|\log\bar\rho\|_{W^{1,\infty}}^2,$$

while when $\sigma_N\to\sigma>0$ as $N\to\infty$, we can take $\underline{\sigma}=\sigma/2$ as in Lemma 2.2 which gives

$$\alpha_N = C_2\,t\,|\sigma-\sigma_N|$$

with C_2 given by

$$C_2 = \frac{2}{\sigma N\,t}\int_{\Pi^{d\,N}} \rho_N^0\,\log\rho_N^0 + \frac{2}{\sigma}\|\operatorname{div} K\|_{L^\infty} + \frac{2}{\sigma}\|\operatorname{div} F\|_{L^\infty} + 2\|\log\bar\rho\|_{W^{1,\infty}}^2.$$

There is no need for any integration by part on the other terms in (2.25). Denote for some $\eta>0$

$$\frac{1}{\eta}\,\phi(x,z) = (K(x-z) - K\star\bar\rho(x))\cdot\nabla_x\log\bar\rho(x) + \operatorname{div} K(x-z) - \operatorname{div} K\star\bar\rho(x).$$

We directly apply Lemma 2.1 to

$$\Phi = \frac{1}{N^2} \sum_{i,j=1}^{N} \phi(x_i, x_j),$$

and find

$$\begin{aligned}
&- \frac{1}{N^2} \sum_{i,\,j=1}^{N} \int_0^t \int_{\Pi^{d\,N}} \rho_N \,\left(K(x_i - x_j) - K \star_x \bar\rho(x_i)\right) \cdot \nabla_{x_i} \log \bar\rho_N \,\mathrm{d}X^N \,\mathrm{d}s \\
&- \frac{1}{N^2} \sum_{i,\,j=1}^{N} \int_0^t \int_{\Pi^{d\,N}} \rho_N \,\left(\operatorname{div}\, K(x_i - x_j) - \operatorname{div} K \star_x \bar\rho(x_i)\right) \,\mathrm{d}X^N \,\mathrm{d}s \\
&\quad \leq \frac{1}{\eta}\, \mathcal{H}_N(\rho_N \mid \bar\rho_N) + \frac{1}{\eta N}\, \log \int_{\Pi^{d\,N}} \bar\rho_N \,\exp\left(\frac{1}{N} \sum_{i,j=1}^{N} \phi(x_i, x_j)\right) \,\mathrm{d}X^N.
\end{aligned}$$

We use Theorem 4 and observe for this ϕ satisfies the required assumptions and in particular

$$\begin{aligned}
\gamma &= C\, \left(\sup_{p\geq 1} \frac{\|\sup_z |\phi(.,z)|\,\|_{L^p(\bar\rho\, dx)}}{p}\right)^2 \\
&\leq C\, \eta^2\, \|K\|_\infty^2\, \left(\sup_{p\geq 1} \frac{\|\nabla \log \bar\rho\|_{L^p(\bar\rho\, dx)}}{p}\right)^2 < 1,
\end{aligned}$$

provided that

$$\eta < \frac{1}{C\,\|K\|_\infty\, \sup_p\, \frac{\|\nabla \log \bar\rho\|_{L^p(\bar\rho\, dx)}}{p}},$$

and where we recall that

$$\|K\|_\infty = \|K\|_{L^\infty} + \|\operatorname{div} K\|_{L^\infty}.$$

By Theorem 4, we hence have for some universal constant $C > 0$

$$\begin{aligned}
&- \frac{1}{N^2} \sum_{i,\,j=1}^{N} \int_0^t \int_{\Pi^{d\,N}} \rho_N \,\left(K(x_i - x_j) - K \star_x \bar\rho(x_i)\right) \cdot \nabla_{x_i} \log \bar\rho_N \,\mathrm{d}X^N \,\mathrm{d}s \\
&- \frac{1}{N^2} \sum_{i,\,j=1}^{N} \int_0^t \int_{\Pi^{d\,N}} \rho_N \,\left(\operatorname{div}\, K(x_i - x_j) - \operatorname{div} K \star_x \bar\rho(x_i)\right) \,\mathrm{d}X^N \,\mathrm{d}s \\
&\quad \leq \bar M_2\, \|K\|_\infty \int_0^t \left(\mathcal{H}_N(\rho_N \mid \bar\rho_N)(s) + \frac{1}{N}\right) \mathrm{d}s.
\end{aligned}$$

Inserting this in (2.25), we find that

$$\begin{aligned}\mathcal{H}_N(\rho_N \mid \bar\rho_N) \leq & \mathcal{H}_N(\rho_N^0 \mid \bar\rho_N^0) + \bar M_2 \, \|K\|_\infty \int_0^t \left(\mathcal{H}_N(\rho_N \mid \bar\rho_N)(s) + \frac{1}{N}\right) \mathrm{d}s \\ & + \bar M_2 \, (1 + t\|K\|_\infty) \, |\sigma - \sigma_N|\end{aligned}$$

for some constant $\bar M_2$ depending only on

$$\bar M_2 \Big(\sigma, \|\log \bar\rho\|_{W^{1,\infty}}, \sup_{p\geq 1} \frac{\|\nabla \log \bar\rho\|_{L^p(\bar\rho\, dx)}}{p}, \\ \frac{1}{N}\int_{\Pi^{d\,N}} \rho_N^0 \log \rho_N^0, \|\operatorname{div} F\|_{L^\infty}\Big).$$

This concludes the proof by Gronwall lemma.

3 Preliminary of combinatorics

Before the proof of the main estimates Theorem 3 and Theorem 4, we list some useful combinatorics results used throughout this article. We first recall Stirling's formula

$$n! = \lambda_n \sqrt{2\pi n} \left(\frac{n}{e}\right)^n, \tag{3.1}$$

where $1 < \lambda_n < \frac{11}{10}$ and $\lambda_n \to 1$ as $n \to \infty$.

We have the elementary bound following from (3.1)

Lemma 3.1. *For any $1 \leq p \leq q$, one has*

$$\binom{q}{p} \leq e^p q^p p^{-p}.$$

One also has the basic combinatorics on p-tuples

Lemma 3.2. *For any $1 \leq p \leq q$, one has*

$$|\{(b_1, \dots, b_p) \in \mathbb{N}^p \mid \forall l \; 1 \leq b_l \leq q \text{ and } b_1 + b_2 + \cdots + b_p = q\}| = \binom{q-1}{p-1}.$$

Proof of Lemma 3.2. When $p = 1$, the lemma trivially holds true with the convention $\binom{0}{0} = 1$ if $p = q = 1$. We thus assume $p \geq 2$ in the following. Since each $p-$tuple $(b_1, b_2, \cdots, b_p)$ uniquely determines a $(p-1)-$tuple $(c_1, c_2, \cdots, c_{p-1})$ and reciprocally via

$$c_1 = b_1, c_2 = b_1 + b_2, \cdots, c_{p-1} = b_1 + b_2 + \cdots + b_{p-1},$$

it suffices to verify that

$$|\{(c_1, c_2, \cdots, c_{p-1}) \mid 1 \le c_1 < c_2 < \cdots < c_{p-1} \le q-1\}| = \binom{q-1}{p-1}.$$

This is simply obtained by choosing $p-1$ distinct integers from the set $\{1, 2, \cdots, q-1\}$ and assigning the smallest one to c_1, the second smallest to c_2, and so on. □

Much of the combinatorics that we handle is based only on the multiplicity in the multi-indices. It is therefore convenient to know how many multi-indices can have the same multiplicity signature

Lemma 3.3. *For any $a_1, \ldots, a_q \in \mathbb{N}$ s.t. $a_1 + \cdots + a_q = p$, then the set of multi-indices $I_p = (i_1, \ldots, i_p)$ with $1 \le i_k \le q$ and corresponding multiplicities has cardinal*

$$\left|\left\{(i_1, \ldots, i_p) \in \{1, \ldots, q\}^p \mid \forall l \ a_l = |\{k\,,\ i_k = l\}|\right\}\right| = \frac{p!}{a_1! \cdots a_q!}.$$

Proof. This is the basic multinomial relation: We have to choose 1 a_1 times among p positions, 2 a_2 times among the remaining positions and so on... □

Similarly as for the binomial coefficients, $\frac{p!}{a_1! \cdots a_q!}$ is the coefficient of $x_1^{a_1} \ldots x_q^{a_q}$ in the expansion of $(x_1 + \cdots + x_q)^p$ leading to the obvious estimate

$$\sum_{a_1, \ldots, a_q \ge 0,\ a_1 + \cdots + a_q = p} \frac{p!}{a_1! \cdots a_q!} = q^p. \tag{3.2}$$

Let us fix some notations here. We write the **integer valued** $p-$tuple as $I_p = (i_1, \cdots, i_p)$. The **overall set** $\mathcal{T}_{q,p}$ of those indices is defined as

$$\mathcal{T}_{q,p} = \{I_p = (i_1, \cdots, i_p) | 1 \le i_\nu \le q, \text{ for all } 1 \le \nu \le p\}. \tag{3.3}$$

We thus define the **multiplicity function** $\Phi_{q,p} : \mathcal{T}_{q,p} \to \{0, 1, \cdots, p\}^q$ with $\Phi_{q,p}(I_p) = A_q = (a_1, a_2, \cdots, a_q)$, where

$$a_l = |\{1 \le \nu \le p \mid i_\nu = l\}|.$$

In many of our proofs, we use cancellations so that any I_p which has an index of multiplicity exactly 1 leads to a vanishing term.

This leads to the definition of the "**effective set**" $\mathcal{E}_{q,p}$ by

$$\mathcal{E}_{q,p} = \{I_p \in \mathcal{T}_{q,p} \mid \Phi_{q,p}(I_p) = A_q = (a_1, \cdots, a_q) \\ \text{with } \ a_\nu \neq 1 \text{ for any } \ 1 \leq \nu \leq q.\}$$

One has the following combinatorics result

Lemma 3.4. *Assume that* $1 \leq p \leq q$. *Then*

$$|\mathcal{E}_{q,p}| \leq \sum_{l=1}^{\lfloor \frac{p}{2} \rfloor} \binom{q}{l} l^p \leq \lfloor \frac{p}{2} \rfloor \binom{q}{\lfloor \frac{p}{2} \rfloor} \left(\lfloor \frac{p}{2} \rfloor\right)^p \leq \frac{p}{2} e^{\frac{p}{2}} q^{\frac{p}{2}} \left(\frac{p}{2}\right)^{\frac{p}{2}}. \tag{3.4}$$

Proof of Lemma 3.4. Pick any multi-index $I_p = (i_1, \cdots, i_p) \in \mathcal{E}_{q,p}$ and recall that $|I_p| = |\{i_1, \cdots, i_p\}|$. The fact that $I_p \in \mathcal{E}_{q,p}$ implies that the multiplicity of each integer cannot be one. Hence $1 \leq |I_p| \leq \lfloor \frac{p}{2} \rfloor$. Indeed, if $|I_p| \geq \lfloor \frac{p}{2} \rfloor + 1$, then

$$p \geq 2\left(\lfloor \frac{p}{2} \rfloor + 1\right) > 2\left(\frac{p}{2} - 1 + 1\right) = p,$$

which is impossible.

If $p = 1$, then $\mathcal{E}_{q,p} = \emptyset$. The estimate (3.4) holds trivially. In the following we assume that $p \geq 2$.

Denote $l = |I_p|$ which can be $1, 2, \cdots, \lfloor \frac{p}{2} \rfloor$. Consequently, one has by summing all possible choices for $|I_p|$

$$|\mathcal{E}_{q,p}| = \sum_{l=1}^{\lfloor \frac{p}{2} \rfloor} |\{I_p \in \mathcal{E}_{q,p} | \ |I_p| = l\}|.$$

For a fixed $|I_p| = l$, there are $\binom{q}{l}$ many choices of numbers l from $S = \{1, 2, \cdots, q\}$ to compose I_p.

Having already chosen those l numbers from S, without loss of generality we may assume that I_p as a set coincides with $\{1, 2, \cdots, l\}$. The total choices of $p-$tuple I_p can be bounded by l^p trivially since each i_ν has at most l choices.

Remark that $1 \leq l \leq \lfloor \frac{p}{2} \rfloor \leq \lfloor \frac{q}{2} \rfloor$, so that

$$\binom{q}{l} \leq \binom{q}{\lfloor \frac{p}{2} \rfloor}.$$

Hence one has

$$|\mathcal{E}_{q,p}| \leq \sum_{l=1}^{\lfloor \frac{p}{2} \rfloor} \binom{q}{l} l^p \leq \lfloor \frac{p}{2} \rfloor \binom{q}{\lfloor \frac{p}{2} \rfloor} \left(\lfloor \frac{p}{2} \rfloor\right)^p.$$

The last inequality in (3.4) is now ensured by Lemma 3.1, in particular the following inequality

$$\binom{q}{\lfloor \frac{p}{2} \rfloor} \leq e^{\lfloor \frac{p}{2} \rfloor} q^{\lfloor \frac{p}{2} \rfloor} (\lfloor \frac{p}{2} \rfloor)^{-\lfloor \frac{p}{2} \rfloor}.$$

This finishes the proof of Lemma 3.4. □

4 Proof of Theorem 3

The goal here is to bound

$$\int_{\Pi^{d\,N}} \bar{\rho}_N \exp\left(\frac{1}{N} \sum_{j_1,j_2=1}^{N} \psi(x_1, x_{j_1})\, \psi(x_1, x_{j_2}) \right) \mathrm{d}X^N,$$

for any bounded ψ with vanishing average against $\bar{\rho}$.

Since

$$\exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k \leq 3 \sum_{k=0}^{\infty} \frac{1}{(2k)!} A^{2k},$$

it suffices only to bound the series with even terms

$$\begin{aligned} &\int_{\Pi^{d\,N}} \bar{\rho}_N \exp\left(\frac{1}{N} \sum_{j_1,j_2=1}^{N} \psi(x_1, x_{j_1})\, \psi(x_1, x_{j_2}) \right) \mathrm{d}X^N \\ &\quad \leq 3 \sum_{k=0}^{\infty} \frac{1}{(2k)!} \int_{\Pi^{d\,N}} \bar{\rho}_N \left(\frac{1}{N} \sum_{j_1,j_2=1}^{N} \psi(x_1, x_{j_1})\, \psi(x_1, x_{j_2}) \right)^{2k} \mathrm{d}X^N, \end{aligned} \tag{4.1}$$

where in general the $k-$th even term can be expanded as

$$\begin{aligned} &\frac{1}{(2k)!} \int_{\Pi^{d\,N}} \bar{\rho}_N \left(\frac{1}{N} \sum_{j_1,j_2=1}^{N} \psi(x_1, x_{j_1})\, \psi(x_1, x_{j_2}) \right)^{2k} \mathrm{d}X^N \\ &\quad = \frac{1}{(2k)!} \frac{1}{N^{2k}} \sum_{j_1,\cdots,j_{4k}=1}^{N} \int_{\Pi^{d\,N}} \bar{\rho}_N\, \psi(x_1, x_{j_1}) \cdots \psi(x_1, x_{j_{4k}})\, \mathrm{d}X^N. \end{aligned} \tag{4.2}$$

We divide the proof in two different cases: Where k is small compared to N and in the simpler case where k is comparable to or larger than N.

Case: $4 \leq 4k \leq N$ First observe that for any particular choice of indices $j_1, \ldots, j_{4K}$, one has

$$\int_{\Pi^{d\,N}} \bar{\rho}_N \, \psi(x_1, x_{j_1}) \cdots \psi(x_1, x_{j_{4k}}) \, \mathrm{d}X^N \leq \|\psi\|_{L^\infty}^{4k}. \tag{4.3}$$

The whole estimate hence relies on counting how many choices of multi-indices $(j_1, \ldots, j_{4k})$ lead to a non-vanishing term. Denote hence $\mathcal{N}_{N,4k}$ the set of multi-indices $(j_1, \cdots, j_{4k})$ s.t.

$$\int_{\Pi^{d\,N}} \bar{\rho}_N \, \psi(x_1, x_{j_1}) \cdots \psi(x_1, x_{j_{4k}}) \, \mathrm{d}X^N \neq 0.$$

Denote by $(a_1, \ldots, a_N)$ the multiplicity for $(j_1, \ldots, j_{4k})$,

$$a_l = |\{\nu \in \{1, \ldots, 4k\}, \ j_\nu = l\}|.$$

If there exists $l \neq 1$ s.t. $a_l = 1$, then the variable x_l enters exactly once in the integration. Assume for simplicity that $j_1 = l$ then

$$\begin{aligned}
&\int_{\Pi^{d\,N}} \bar{\rho}_N \, \psi(x_1, x_{j_1}) \cdots \psi(x_1, x_{j_{4k}}) \, \mathrm{d}X^N \\
&\quad = \int_{\Pi^{d\,(N-1)}} \psi(x_1, x_{j_2}) \cdots \psi(x_1, x_{j_{4k}}) \, \Pi_{i \neq l} \bar{\rho}(x_i) \; \mathrm{d}x_i \\
&\qquad\qquad \int_{\Pi^d} \bar{\rho}(x_{j_1}) \, \psi(x_1, x_{j_1}) \; \mathrm{d}x_{j_1} = 0,
\end{aligned}$$

by the assumption of vanishing mean average for ψ, *provided* $l = j_1 \neq 1$.

Recall the definitions of the overall set (see (3.3)) and the effective set

$$\begin{aligned}
\mathcal{E}_{q,p} = \{ I_p \in \mathcal{T}_{q,p} \mid \ & (a_1, \cdots, a_q) = \Phi_{q,p}(I_p) \\
& \text{with} \ \ a_\nu \neq 1 \text{ for any } \ 1 \leq \nu \leq q \},
\end{aligned}$$

where $(a_1, \cdots, a_q)$ denotes the multiplicity of the multi-index I_p.

Therefore the integral

$$\int_{\Pi^{d\,N}} \bar{\rho}_N \, \psi(x_1, x_{j_1}) \cdots \psi(x_1, x_{j_{4k}}) \, \mathrm{d}X^N$$

vanishes unless $j_1, \cdots, j_{4k}$ belongs to $\mathcal{E}_{N,4k}$ (all multiplicities are different from 1) or satisfies $a_1 = 1$ and every $a_l \neq 1$ for $l > 1$.

In that last case, we have to choose one index n s.t. $j_n = 1$, with $4k$ possibilities. The rest of the multi-index $(j_1, \cdots, j_{n-1}, j_{n+1}, \cdots, j_{4k})$ must have all multiplicities different from 1. This multi-index hence belongs to $\mathcal{E}_{N-1,4k-1}$.

Consequently,

$$|\mathcal{N}_{N,4k}| \leq |\mathcal{E}_{N,4k}| + 4k\,|\mathcal{E}_{N-1,4k-1}|.$$

We now apply Lemma 3.4

$$\mathcal{N}_{N,4k} \leq (1+4k)\,|\mathcal{E}_{N,4k}| \leq 10\,k^2\,e^{2k}\,N^{2k}\,(2k)^{2k}.$$

Using (4.3) , for $1 \leq k \leq \lfloor \frac{N}{4} \rfloor$, we obtain

$$\begin{aligned}
&\frac{1}{(2k)!}\frac{1}{N^{2k}} \sum_{j_1,\cdots,j_{4k}=1}^{N} \int_{\Pi^{d\,N}} \bar{\rho}_N\,\psi(x_1,x_{j_1})\cdots\psi(x_1,x_{j_{4k}})\,\mathrm{d}X^N \\
&\quad\leq \frac{1}{(2k)!}\frac{10}{N^{2k}}\,k^2\,e^{2k}\,N^{2k}\,(2k)^{2k}\,\|\psi\|_{L^\infty}^{4k} \\
&\quad\leq 5\,e^{4k}\,k^{\frac{3}{2}}\,\|\psi\|_{L^\infty}^{4k},
\end{aligned} \tag{4.4}$$

by the Stirling's formula for $n = 2k$.

Case: $4k > N$**.** In this case, we do not need to use any combinatorics. We simply remark that there can be at most N^{4k} multi-indices. From (4.3), we have for $k > \lfloor \frac{N}{4} \rfloor$

$$\begin{aligned}
&\frac{1}{(2k)!}\frac{1}{N^{2k}} \sum_{j_1,\cdots,j_{4k}=1}^{N} \int_{\Pi^{d\,N}} \bar{\rho}_N\,\psi(x_1,x_{j_1})\cdots\psi(x_1,x_{j_{4k}})\,\mathrm{d}X^N \\
&\qquad\leq \frac{1}{(2k)!}\,\frac{1}{N^{2k}}\,N^{4k}\,\|\psi\|_{L^\infty}^{4k} \leq k^{-\frac{1}{2}}\,2^{2k}\,e^{2k}\,\|\psi\|_{L^\infty}^{4k},
\end{aligned} \tag{4.5}$$

still by the Stirling's formula.

Conclusion of the proof. Combining (4.4), (4.5) and (4.1), we have that

$$\begin{aligned}
&\int_{\Pi^{d\,N}} \bar{\rho}_N \exp\left(\frac{1}{N}\sum_{j_1,j_2=1}^{N}\psi(x_1,x_{j_1})\,\psi(x_1,x_{j_2})\right)\,\mathrm{d}X^N \\
&\qquad\leq 3\left(1 + \sum_{k=1}^{\lfloor \frac{N}{4} \rfloor} 5\,e^{4k}\,k^{\frac{3}{2}}\,\|\psi\|_{L^\infty}^{4k} + \sum_{k=\lfloor \frac{N}{4} \rfloor+1}^{\infty} k^{-\frac{1}{2}}\,2^{2k}\,e^{2k}\,\|\psi\|_{L^\infty}^{4k}\right).
\end{aligned}$$

The proof of Theorem 3 is completed by

$$\sum_{k=1}^{\lfloor \frac{N}{4} \rfloor} 5\, e^{4k}\, k^{\frac{3}{2}}\, \|\psi\|_{L^\infty}^{4k} \le 5\,\alpha \sum_{k=1}^{\infty} k(k+1)\alpha^{k-1}$$
$$= 5\,\alpha\, \frac{\mathrm{d}^2}{\mathrm{d}\alpha^2} \left(\sum_{k=0}^{\infty} \alpha^k \right) = 5\,\alpha \left(\frac{1}{1-\alpha} \right)'' = \frac{10\,\alpha}{(1-\alpha)^3} < \infty,$$

while

$$\sum_{k=\lfloor \frac{N}{4} \rfloor+1}^{\infty} k^{-\frac{1}{2}}\, 2^{2k}\, e^{2k}\, \|\psi\|_{L^\infty}^{4k} \le \sum_{k=1}^{\infty} \beta^k = \frac{1}{1-\beta} - 1$$
$$= \frac{\beta}{1-\beta} < \infty.$$

5 Proof of Theorem 4

We recall that our goal is to bound

$$\int_{\Pi^{d\,N}} \bar\rho_N \exp\left(\frac{1}{N} \sum_{i,j=1}^{N} \phi(x_i, x_j) \right) \mathrm{d}X^N$$

with the assumptions

$$\int_{\Pi^d} \phi(x,z)\, \bar\rho(x)\, dx = 0 \quad \forall z, \qquad \int_{\Pi^d} \phi(x,z)\, \bar\rho(z)\, dz = 0 \quad \forall x. \tag{5.1}$$

As in the proof of Theorem 3, one expands the exponential in series and only needs to bound the even terms

$$\begin{aligned}&\int_{\Pi^{d\,N}} \bar\rho_N\, \exp\left(\frac{1}{N} \sum_{i,j=1}^{N} \phi(x_i, x_j) \right) \mathrm{d}X^N \\ &\qquad \le 3 \sum_{k=0}^{\infty} \frac{1}{(2k)!} \int_{\Pi^{d\,N}} \bar\rho_N \left| \frac{1}{N} \sum_{i,j=1}^{N} \phi(x_i, x_j) \right|^{2k} \mathrm{d}X^N.\end{aligned} \tag{5.2}$$

As in the proof of Theorem 3, we separate the proof into two cases: the case where k is relatively small compared to N which requires a careful combinatorial analysis to take vanishing terms into account and the more straightforward case when k is comparable to or larger than N.

Accordingly Theorem 4 is a consequence of the following two propositions

Proposition 3. *If* $4k > N$*, one has*

$$\frac{1}{(2k)!}\int_{\Pi^{d\,N}} \bar\rho_N \left|\frac{1}{N}\sum_{i,j=1}^N \phi(x_i,x_j)\right|^{2k} \mathrm{d}X^N$$
$$\leq \left(6e^2 \sup_{p\geq 1}\frac{\|\sup_z|\phi(.,z)|\|_{L^p(\bar\rho\,dx)}}{p}\right)^{2k}.$$

Proposition 4. *For* $4 \leq 4k \leq N$*, one has*

$$\frac{1}{(2k)!}\int_{\Pi^{d\,N}} \bar\rho_N \left|\frac{1}{N}\sum_{i,j=1}^N \phi(x_i,x_j)\right|^{2k} \mathrm{d}X^N$$
$$\leq \left(1600 \sup_{p\geq 1}\frac{\|\sup_z|\phi(.,z)|\|_{L^p(\bar\rho\,dx)}}{p}\right)^{2k}.$$

Let us give a quick proof of Theorem 4 assuming Proposition 3 and Proposition 4.

Proof of Theorem 4. By (5.2) and Proposition 3 and Proposition 4, one has

$$\int_{\Pi^{d\,N}} \bar\rho_N \exp\left(\frac{1}{N}\sum_{i,j=1}^N \phi(x_i,x_j)\right) \mathrm{d}X^N$$
$$\leq 3\left(1+\sum_{k=1}^{\lfloor\frac{N}{4}\rfloor}\left(1600 \sup_{p\geq 1}\frac{\|\sup_z|\phi(.,z)|\|_{L^p(\bar\rho\,dx)}}{p}\right)^{2k}\right.$$
$$\left. + \sum_{k=\lfloor\frac{N}{4}\rfloor+1}^{\infty}\left(6e^2 \sup_{p\geq 1}\frac{\|\sup_z|\phi(.,z)|\|_{L^p(\bar\rho\,dx)}}{p}\right)^{2k}\right).$$

We defined $\gamma = C\left(\sup_{p\geq 1}\frac{\|\sup_z|\phi(.,z)|\|_{L^p(\bar\rho\,dx)}}{p}\right)^2 < 1$. One obtains, taking $C = 1600^2 + 36\,e^4$,

$$\int_{\Pi^{d\,N}} \bar\rho_N \exp\left(\frac{1}{N}\sum_{i,j=1}^N \phi(x_i,x_j)\right) \mathrm{d}X^N \leq 3\sum_{k=0}^{\infty}\gamma^k = \frac{3}{1-\gamma} < \infty,$$

completing the proof of Theorem 4. $\square$

5.1 The case $4k > N$: Proof of Proposition 3

For $k > \frac{N}{4}$ the $k-$th even term can be estimated by

$$
\begin{aligned}
&\frac{1}{(2k)!}\int_{\Pi^{d\,N}} \bar\rho_N \left|\frac{1}{N}\sum_{i,j=1}^N \phi(x_i,x_j)\right|^{2k} \,\mathrm{d}X^N \\
&\leq \frac{1}{(2k)!}\frac{1}{N^{2k}}\sum_{i_1,j_1,\cdots,i_{2k},j_{2k}=1}^N \int_{\Pi^{d\,N}} \bar\rho_N \sup_z|\phi(x_{i_1},z)|\cdots\sup_z|\phi(x_{i_{2k}},z)| \,\mathrm{d}X^N \\
&= \frac{1}{(2k)!}\int_{\Pi^{d\,N}} \bar\rho_N \left(\sum_{i=1}^N \sup_z|\phi(x_i,z)|\right)^{2k} \,\mathrm{d}X^N.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\frac{1}{(2k)!}\int_{\Pi^{d\,N}} \bar\rho_N \left|\frac{1}{N}\sum_{i,j=1}^N \phi(x_i,x_j)\right|^{2k} \,\mathrm{d}X^N \\
&= \frac{1}{(2k)!}\sum_{\substack{a_1+\cdots+a_N=2k,\\ a_1\geq 0,\cdots a_N\geq 0}} \frac{(2k)!}{(a_1)!\cdots(a_N)!} M_{a_1}^{a_1}\cdots M_{a_N}^{a_N},
\end{aligned}
\tag{5.3}
$$

where we denote

$$
M_{a_i}^{a_i} = \int_{\Pi^d} \sup_z |\phi(x,z)|^{a_i}\bar\rho(x)\,\mathrm{d}x
$$

with the convention that $M_0^0 = 1$. Remark that

$$
\begin{aligned}
M_{a_i}^{a_i} &\leq a_i^{a_i}\left(\sup_{p\geq 1}\frac{\|\sup_z|\phi(x,z)|\|_{L^p(\bar\rho\,dx)}}{p}\right)^{a_i} \\
&\leq e^{a_i}(a_i)!\left(\sup_{p\geq 1}\frac{\|\sup_z|\phi(x,z)|\|_{L^p(\bar\rho\,dx)}}{p}\right)^{a_i},
\end{aligned}
$$

where the last inequality $n^n \leq e^n n!$ can be easily verified by the Stirling's formula. Inserting it into (5.3), one obtains

$$
\begin{aligned}
&\frac{1}{(2k)!}\int_{\Pi^{d\,N}} \bar\rho_N \left|\frac{1}{N}\sum_{i,j=1}^N \phi(x_i,x_j)\right|^{2k} \,\mathrm{d}X^N \\
&\leq\; e^{2k}\left(\sup_{p\geq 1}\frac{\|\sup_z|\phi(x_{i_1},z)|\|_{L^p(\bar\rho\,dx)}}{p}\right)^{2k} \sum_{\substack{a_1+\cdots+a_N=2k,\\ a_1\geq 0,\cdots a_N\geq 0}} 1.
\end{aligned}
\tag{5.4}
$$

The quantity $\sum_{\substack{a_1+\cdots+a_N=2k\\ a_1\geq 0,\cdots a_N\geq 0}} 1$ is equal to the cardinality of the set

$$\{(a_1, a_2, \cdots, a_N)|a_1 + \cdots + a_N = 2k, a_i \geq 0 \text{ for } \ 1 \leq i \leq N\}$$

or the cardinality of the following equinumerous set

$$\{(b_1, b_2, \cdots, b_N)|b_1 + \cdots + b_N = 2k + N, b_i \geq 1 \text{ for } \ 1 \leq i \leq N\}.$$

Applying Lemma 3.2 in section 3 by taking $p = N$ and $q = 2k + N$, this cardinal is exactly $\binom{2k+N-1}{N-1}$.

This expression can be simplified. Note that if $a \geq b/2$ by Stirling's formula

$$\binom{a+b}{b} \leq \frac{\sqrt{a+b}}{\sqrt{\pi\, a\, b}}\,(a+b)^{a+b}a^{-a}b^{-b} \leq (1+b/a)^a\,(1+a/b)^b \leq 3^a\,(1+a/b)^b.$$

Since $(1+\frac{1}{s})^s < e$ for any $s > 0$, this gives

$$\binom{a+b}{b} \leq (3\,e)^a.$$

Since $4k > N$, $\binom{2k+N-1}{N-1} \leq 3^{2k}\,e^{2k}$ and therefore from (5.4), one obtains that

$$\begin{aligned} &\frac{1}{(2k)!}\int_{\Pi^{d\,N}} \bar\rho_N \left|\frac{1}{N}\sum_{i,j=1}^N \phi(x_i, x_j)\right|^{2k} \,\mathrm{d}X^N \\ &\qquad \leq 3^{2k}\,e^{4k}\left(\sup_{p\geq 1}\frac{\|\sup_z|\phi(x,z)|\|_{L^p(\bar\rho\,dx)}}{p}\right)^{2k}. \end{aligned} \tag{5.5}$$

This proves Proposition 3.

5.2 The case $4 \leq 4k \leq N$: Proof of Proposition 4

In this case, the previous straightforward approach fails, even assuming that $\phi \in L^\infty$ as we would only get

$$\frac{1}{(2k)!}\int_{\Pi^{d\,N}} \bar\rho_N \left|\frac{1}{N}\sum_{i,j=1}^N \phi(x_i, x_j)\right|^{2k} \,\mathrm{d}X^N \leq \frac{N^{2k}}{(2k)!}\,\|\phi\|_{L^\infty}^{2k},$$

which blows up when N goes to infinity. The key here, as is in the proof of Theorem 3, is to identify the right cancellations in the expansion

$$\int_{\Pi^{d\,N}} \bar\rho_N \, \exp\left(\frac{1}{N}\sum_{i,j=1}^N \phi(x_i,x_j)\right)\, \mathrm{d}X^N$$
$$\leq 3\sum_{k=1}^\infty \frac{1}{(2k)!}\frac{1}{N^{2k}} \sum_{i_1,j_1\cdots,i_{2k},j_{2k}=1}^N \int_{\Pi^{d\,N}} \bar\rho_N \, \phi(x_{i_1},x_{j_1})\cdots\phi(x_{i_{2k}},x_{j_{2k}})\, \mathrm{d}X^N. \tag{5.6}$$

5.2.1 Notations and preliminary considerations

We denote by $I_{2k} = (i_1,\cdots,i_{2k})$ the $i-$indices and similarly by $J_{2k} = (j_1,\cdots,j_{2k})$ the $j-$indices, where all i_ν, j_ν are in $\{1,2,\cdots,N\}$ for $1\leq \nu\leq 2k$.

We denote by $(a_1,a_2,\cdots,a_N)$ the multiplicities of I_{2k},

$$a_l = |\{1\leq \nu\leq 2k | i_\nu = l\}|, \quad l=1,2,\cdots,N,$$

and by $(b_1,\cdots,b_N)$ the multiplicities of J_{2k}.

For the study of cancellations, the critical parameter will be the number of multiplicities which are exactly 1 in I_{2k}, so that we denote

$$m_I = |\{l \mid a_l = 1\}|, \quad n_I = |\{l \mid a_l > 1\}|. \tag{5.7}$$

Note that m_I+n_I is exactly the number of integers present in I_{2k}: $m_I+n_I = |\{l \mid a_l\geq 1\}|$.

We start by the following lemma which, for every I_{2k}, identifies the only possible J_{2k} s.t. the integral does not vanish.

First we simplify the possible expression of I_{2k} which makes the counting easier by using the natural symmetry by permutation of the problem. For any $\tau\in\mathcal{S}_N$, we simply define $\tau(I_{2k}) = (\tau(i_1),\ldots,\tau(i_{2k}))$. Thus τ is a one-to-one application on the I_{2k} and moreover

$$\int_{\Pi^{d\,N}} \bar\rho_N \, \phi(x_{i_1},x_{j_1})\cdots\phi(x_{i_{2k}},x_{j_{2k}})\, \mathrm{d}X^N$$
$$= \int_{\Pi^{d\,N}} \bar\rho_N \phi(x_{\tau(i_1)},x_{\tau(j_1)})\cdots\phi(x_{\tau(i_{2k})},x_{\tau(j_{2k})})\, \mathrm{d}X^N.$$

Therefore to identify cancellations, we only need to consider one I_{2k} in each of the equivalence classes $\{\tau(I_{2k}),\ \forall\tau\in\mathcal{S}_N\}$, leading to

Definition 3. *A multi-index I_{2k} belongs to the reduced form set $\mathcal{R}_{N,2k}$ iff $0 < a_1 \leq a_2 \ldots \leq a_n$ and $a_{n+1} = \cdots = a_N = 0$.*

Note that for any I_{2k} there exists only one $\tilde{I}_{2k} \in \mathcal{R}_{N,2k}$ that belongs to the same class, even though there can be several τ s.t. $\tau(I_{2k}) = \tilde{I}_{2k}$ (as any repeated index leaves $\tilde{I}_{2k}$ invariant under the corresponding transposition).

5.2.2 Identifying the "right" indices J_{2k}

Remark that by the definition of m_I and n_I in (5.7), if $I_{2k} \in \mathcal{R}_{N,2k}$ is under its reduced form, one has

$$\begin{aligned} a_l &= 1 \quad \text{for } l = 1, \cdots, m_I, \\ a_l &> 1 \quad \text{for } l = m_I + 1, \cdots, m_I + n_I, \\ a_l &= 0 \quad \text{for } l > m_I + n_I. \end{aligned}$$

Based on this simple structure, we can prove that

Lemma 5.1. *For any m, n, define as $\mathcal{J}_{m,n}$ the set of indices J_{2k} with multiplicities $(b_1, \ldots, b_N)$ satisfying*

- *$b_l \geq 1$ for any $l = 1 \ldots m$;*
- *$b_l \neq 1$ for any $l > m + n$.*

Then for any $I_{2k} \in \mathcal{R}_{N,2k}$ and any $J_{2k} \notin \mathcal{J}_{m_I,n_I}$, one has that

$$\int_{\Pi^{d\,N}} \bar{\rho}_N \, \phi(x_{i_1}, x_{j_1}) \cdots \phi(x_{i_{2k}}, x_{j_{2k}}) \, \mathrm{d}X^N = 0.$$

This lemma identifies, for each $I_{2k} \in \mathcal{R}_{N,2k}$, a relevant subset of indices $\mathcal{J}_{m_I,n_I}$; in the sense that any multi-index J_{2k} out of this set leads to a vanishing integral and hence can be removed from our summation. Lemma 5.1 is not an equivalence though: There can still be indices $J_{2k} \in \mathcal{J}_{m_I,n_I}$ giving a vanishing integral. But the formulation above allows for simpler combinatorics and in particular $\mathcal{J}_{m_I,n_I}$ only depends in a basic manner on I_{2k} through the two integers m_I and n_I.

Proof of Lemma 5.1. Choose any $I_{2k} \in \mathcal{R}_{N,2k}$, up to a permutation, we may freely assume that I_{2k} has the following form

$$I_{2k} = \left(1, 2, \cdots, m_I, \underbrace{m_I + 1, \cdots, m_I + 1}_{a_{m_I+1}}, \cdots, \underbrace{m_I + n_I, \cdots, m_I + n_I}_{a_{m_I+n_I}} \right).$$

Choose any $J_{2k} \notin \mathcal{J}_{m_I,n_I}$. That means that there exists $l \le m_I$ s.t. $b_l = 0$ or that there exists $l > m_I + n_I$ s.t. $b_l = 1$. Each case corresponds to a different cancellation in the integral.

The case $b_l = 0$ for some $l \le m_I$. By the definition of the reduced form, $a_l = 1$ and therefore the index l appears only once in I_{2k} and never in J_{2k} thus being present exactly once in the product inside the integral. Assume that $i_\nu = l$ for some ν so

$$\int_{\Pi^{d\,N}} \bar\rho_N\, \phi(x_{i_1}, x_{j_1}) \cdots \phi(x_{i_{2k}}, x_{j_{2k}})\ \mathrm{d}X^N$$
$$= \int_{\Pi^{d\,(N-1)}} \frac{\bar\rho_N}{\bar\rho(x_{i_\nu})} \left(\int_{\Pi^d} \bar\rho(x_{i_\nu})\, \phi(x_{i_\nu}, x_{j_\nu})\, dx_{i_\nu}\right) \Pi_{\nu'\neq\nu}\phi(x_{i_{\nu'}}, x_{j_{\nu'}})\ \mathrm{d}x_{i_{\nu'}}.$$

Now it is enough to remark that for any i and $j \neq i$, as is the case here since all $j_{\nu'} \neq l$,

$$\int_{\Pi^d} \bar\rho(x_i)\, \psi(x_i, x_j)\ \mathrm{d}x_i = 0,$$

which is exactly the first assumption in (5.1).

The case $b_l = 1$ for some $l > m_I + n_I$. By definition, this means that $a_l = 0$. The index l appears only once in J_{2k} and never in I_{2k}. Again it is present exactly once in the product inside the integral. Assume that $j_\nu = l$ for some ν so

$$\int_{\Pi^{d\,N}} \bar\rho_N\, \phi(x_{i_1}, x_{j_1}) \cdots \phi(x_{i_{2k}}, x_{j_{2k}})\ \mathrm{d}X^N$$
$$= \int_{\Pi^{d\,(N-1)}} \frac{\bar\rho_N}{\bar\rho(x_{j_\nu})} \left(\int_{\Pi^d} \bar\rho(x_{j_\nu})\, \phi(x_{i_\nu}, x_{j_\nu})\, dx_{j_\nu}\right) \Pi_{\nu'\neq\nu}\phi(x_{i_{\nu'}}, x_{j_{\nu'}})\ \mathrm{d}x_{j_{\nu'}}.$$

The results then follows from the fact that for $i \neq j$

$$\int_{\Pi^d} \bar\rho(x_j)\, \phi(x_i, x_j)\ \mathrm{d}x_j = 0,$$

which is the second equality in (5.1). □

5.2.3 The cardinality of $\mathcal{J}_{m,n}$

Our next step is to show that $|\mathcal{J}_{m,n}|$ is much less than the total number of multi-indices J_{2k}, namely N^{2k},

Lemma 5.2. *One has that for some universal constant C*

$$|\mathcal{J}_{m,n}| \leq C^k\, N^{k-m/2}\, k^{k+m/2},$$

where C can be chosen as $512\, e$ or roughly 1400.

Proof of Lemma 5.2. A multi-index J_{2k} belongs to $\mathcal{J}_{m,n}$ iff $b_l \geq 1$ for $l \leq m$ and $b_l = 0,\, 2,\, 3, \ldots$ for $l > m+n$. Let us distinguish further between those $l > m+n$ where $b_l = 0$ and those for which $b_l \geq 2$.

Choose first $p = 0, 1, \ldots, \lfloor \frac{2k-m}{2} \rfloor$ and choose then p indices $l_1, \ldots, l_p$ between $m+n+1$ and N which exactly correspond to $b_l \geq 2$. There are $\binom{N-m-n}{p}$ such possibilities.

Once these $l_1, \ldots, l_p$ have been chosen, the set of possible multiplicities for $J_{2k} \in \mathcal{J}_{m,n}$ is given by

$$\mathcal{B}_{m,n,p,l_1,\ldots,l_p} = \Big\{(b_1, \ldots, b_N) \mid b_1, \ldots, b_m \geq 1,\ b_{l_1}, \ldots, b_{l_p} \geq 2,$$
$$b_l = 0 \text{ if } l > m+n \text{ and } l \neq l_1, \ldots, l_p, \text{ and } b_1 + b_2 + \cdots + b_N = 2k\}.$$

After the multiplicities are known it is straightforward to obtain the number of J_{2k} in $\mathcal{J}_{m,n}$, using Lemma 3.3. Decomposing all the possible J_{2k} according to those possibilities, one hence finds

$$|\mathcal{J}_{m,n}| = \sum_{p=0}^{\lfloor \frac{2k-m}{2} \rfloor} \sum_{l_1,\ldots,l_p = m+n+1,\ldots,N} \sum_{(b_1,\ldots,b_N) \in \mathcal{B}_{m,n,p,l_1,\ldots,l_p}} \frac{(2k)!}{b_1! \cdots b_N!}.$$

Note that since $b_{l_1}, \ldots, b_{l_p} \geq 2$ and $b_1, \ldots, b_m \geq 1$ one has that $m + 2p \leq b_1 + \cdots + b_N = 2k$, leading to the upper bound $p \leq k - m/2$.

Furthermore using the invariance by permutation, one may immediately reduce this expression by assuming that $l_1 = m+n+1$, $l_2 = m+n+2$... Denoting the partial sums $s_m = b_1 + \cdots + b_m$ and $s_n = b_{m+n+1} + \cdot + b_{m+n+p}$, one has

$$|\mathcal{J}_{m,n}| = \sum_{p=0}^{k-m/2} \binom{N-m-n}{p} \sum_{s_m=m}^{2k-2p} \sum_{b_1 \ldots b_m \geq 1,\ b_1+\cdots+b_m = s_m}$$
$$\sum_{s_n=2p}^{2k-s_m} \sum_{b_{m+n+1},\ldots,b_{m+n+p} \geq 2,\ b_{m+n+1}+\cdots+b_{m+n+p} = s_n}$$
$$\sum_{b_{m+1},\ldots,b_{m+n} \geq 0,\ b_{m+1}+\cdots+b_{m+n} = 2k-s_m-s_n} \frac{(2k)!}{b_1! \cdots b_{m+n+p}!}.$$

Using the standard multinomial summation (3.2), one can easily calculate the last sum to obtain

$$|\mathcal{J}_{m,n}| = \sum_{p=0}^{k-m/2} \binom{N-m-n}{p}$$
$$\sum_{s_m=m}^{2k-2p} \sum_{b_1 \ldots b_m \geq 1,\ b_1+\cdots+b_m=s_m} \sum_{s_n=2p}^{2k-s_m} \frac{n^{2k-s_m-s_n}}{(2k-s_m-s_n)!}$$
$$\sum_{b_{m+n+1},\ldots,b_{m+n+p}\geq 2,\ b_{m+n+1}+\cdots+b_{m+n+p}=s_n} \frac{(2k)!}{b_1!\cdots b_m!\, b_{m+n+1}!\cdots b_{m+n+p}!}.$$

Now bound the sum on $b_1 \ldots b_m$ by the sum starting at $b_1, \ldots, b_m = 0$ and similarly for the sum on $b_{m+n+1} \ldots b_{m+n+p}$ to obtain

$$|\mathcal{J}_{m,n}| \leq \sum_{p=0}^{k-m/2} \binom{N-m-n}{p}$$
$$\sum_{s_m=m}^{2k-2p} \sum_{s_n=2p}^{2k-s_m} \frac{(2k)!\, n^{2k-s_m-s_n}\, m^{s_m}\, p^{s_n}}{(2k-s_m-s_n)!\, s_m!\, s_n!}.$$

We recall the obvious bound $\binom{a}{b} \leq 2^a$ so that

$$\frac{(2k)!}{(2k-s_m-s_n)!\, s_m!\, s_n!} = \frac{(2k-s_m)!}{(2k-s_m-s_n)!\, s_n!} \, \frac{(2k)!}{(2k-s_m)!\, s_m!}$$
$$= \binom{2k}{s_m}\binom{2k-s_m}{s_n} \leq 2^{4k}.$$

Furthermore by lemma 3.1 as $m+n \leq N/2$, $\binom{N-m-n}{p} \leq e^p\, N^p\, p^{-p}$. Thus

$$|\mathcal{J}_{m,n}| \leq 2^{4k} \sum_{p=0}^{k-m/2} e^p\, N^p \sum_{s_m=m}^{2k-2p} \sum_{s_n=2p}^{2k-s_m} n^{2k-s_m-s_n} p^{s_n-p}\, m^{s_m}.$$

Note that $2k - s_m - s_n \geq 0$ and $s_n - p \geq 0$ and m, n, $p \leq 2k$ so

$$n^{2k-s_m-s_n}\, p^{s_n-p}\, m^{s_m} \leq (2k)^{2k-p}.$$

Therefore finally

$$|\mathcal{J}_{m,n}| \leq 2^{6k}\, e^k\, (2k)^2 \sum_{p=0}^{k-m/2} N^p\, k^{2k-p}$$
$$\leq 2^{6k}\, e^k\, (2k)^2\, k\, N^{k-m/2}\, k^{k+m/2} < (2^9 e)^k\, N^{k-m/2}\, k^{k+m/2},$$

as since $N \geq k$, the maximum of $N^p\, k^{2k-p}$ is attained for the maximal value of p. $\square$

5.2.4 Conclusion of the proof of the Proposition 4

Observe that for a particular choice of I_{2k} and J_{2k}

$$\begin{aligned}
&\int_{\Pi^{d\,N}} \bar\rho_N\, \phi(x_{i_1}, x_{j_1}) \cdots \phi(x_{i_{2k}}, x_{j_{2k}})\, \mathrm{d}X^N \\
&\quad \leq \int_{\Pi^{d\,N}} \bar\rho_N\, \Pi_{\nu=1}^{2k} \sup_z |\phi(x_{i_\nu}, z)|\, \mathrm{d}X^N \\
&\quad \leq \int_{\Pi^{d\,N}} \bar\rho_N\, (\sup_z |\phi(x_1, z)|)^{a_1} \ldots (\sup_z |\phi(x_N, z)|)^{a_N}\, \mathrm{d}X^N.
\end{aligned} \tag{5.8}$$

As one readily sees this bound only depends on the multiplicity in I_{2k}.

We use the cancellations obtained in Lemma 5.1 to deduce from (5.8),

$$\begin{aligned}
&\int_{\Pi^{d\,N}} \bar\rho_N \left| \frac{1}{N} \sum_{i,j=1}^N \phi(x_i, x_j) \right|^{2k} \mathrm{d}X^N \\
&\quad \leq \frac{1}{N^{2k}} \sum_{\substack{a_1+\cdots+a_N=2k,\\ a_1\geq 0,\cdots,a_N\geq 0.}} |\mathcal{J}_{m_a,n_a}|\, |\{I_{2k} \mid \Phi_{N,2k}(I_{2k}) = (a_1,\ldots,a_N)\}| \\
&\qquad \int_{\Pi^{d\,N}} \bar\rho_N\, (\sup_z |\phi(x_1, z)|)^{a_1} \ldots (\sup_z |\phi(x_N, z)|)^{a_N}\, \mathrm{d}X^N,
\end{aligned}$$

where we denote $m_a = m_{(a_1,\ldots,a_N)} = |\{l \mid a_l = 1\}|$, $n_a = n_{(a_1,\ldots,a_N)} = |\{l \mid a_l > 1\}|$ and we recall that $\Phi_{N,2k}(I_{2k})$ is the multiplicity function associating to each I_{2k} the vector $(a_1, \ldots, a_N)$ of multiplicities.

Remark that

$$\begin{aligned}
&\int_{\Pi^{d\,N}} \bar\rho_N\, (\sup_z |\phi(x_1, z)|)^{a_1} \ldots (\sup_z |\phi(x_N, z)|)^{a_N}\, \mathrm{d}X^N \\
&\quad \leq e^{2k} \left(\sup_{p\geq 1} \frac{\| \sup_z |\phi(., z)| \|_{L^p(\bar\rho\, dx)}}{p} \right)^{2k} a_1! \cdots a_N!,
\end{aligned}$$

since $a^a \leq e^a\, a!$.

On the other hand by Lemma 3.3

$$|\{I_{2k} \mid \Phi_{N,2k}(I_{2k}) = (a_1, \ldots, a_N)\}| \leq \frac{(2k)!}{a_1! \cdots a_N!},$$

which implies that

$$\frac{1}{(2k)!}\int_{\Pi^{d\,N}}\bar\rho_N\left|\frac{1}{N}\sum_{i,j=1}^N\phi(x_i,x_j)\right|^{2k}\,\mathrm{d}X^N$$
$$\leq\frac{e^{2k}}{N^{2k}}\left(\sup_{p\geq 1}\frac{\|\sup_z|\phi(.,z)|\|_{L^p(\bar\rho\,dx)}}{p}\right)^{2k}\sum_{\substack{a_1+\cdots+a_N=2k,\\ a_1\geq 0,\cdots,a_N\geq 0.}}|\mathcal{J}_{m_a,n_a}|.$$

We apply Lemma 5.2

$$\frac{1}{(2k)!}\int_{\Pi^{d\,N}}\bar\rho_N\left|\frac{1}{N}\sum_{i,j=1}^N\phi(x_i,x_j)\right|^{2k}\,\mathrm{d}X^N\leq\frac{e^{2k}}{N^{2k}}$$
$$\cdot\left(\sup_{p\geq 1}\frac{\|\sup_z|\phi(.,z)|\|_{L^p(\bar\rho\,dx)}}{p}\right)^{2k}\sum_{\substack{a_1+\cdots+a_N=2k,\\ a_1\geq 0,\cdots,a_N\geq 0.}}C^k\,N^{k-m_a/2}\,k^{k+m_a/2}.$$

Consider any $(a_1,\ldots,a_N)$ with exactly p coefficients $a_l\geq 1$. Up to $\binom{N}{p}$ permutations, we can actually assume that $a_1,\cdots,a_p\geq 1$. All the other a_l are 0. Since we have $m_a+n_a=p$ and $m_a+2\,n_a\leq 2k$ then $m_a\geq 2\,(p-k)$. As $N\geq k$ then

$$N^{k-m_a/2}\,k^{k+m_a/2}\leq N^{k-(p-k)_+}\,k^{k+(p-k)_+}.$$

Hence

$$\begin{aligned}&\sum_{a_1+\cdots a_N=2k}N^{k-m_a/2}\,k^{k+m_a/2}\\ &\quad=\sum_{p=1}^{2k}\binom{N}{p}\sum_{a_1,\ldots,a_p\geq 1,\;a_1+\cdots+a_p=2k}N^{k-(p-k)_+}\,k^{k+(p-k)_+}\\ &\quad\leq\sum_{p=1}^{2k}\binom{N}{p}\binom{2k-1}{p-1}N^{k-(p-k)_+}\,k^{k+(p-k)_+},\end{aligned}$$

by Lemma 3.2. Bound $\binom{2k-1}{p-1}\leq 2^{2k}$ and for $p\leq k$ since $\binom{N}{p}$ is maximum for $p=k$

$$\sum_{p=1}^{k}\binom{N}{p}\binom{2k-1}{p-1}N^{k-(p-k)_+}\,k^{k+(p-k)_+}\leq 2^{2k}\,k\binom{N}{k}N^k\,k^k\leq(8e)^k\,N^{2k},$$

by Stirling's formula. Still by Stirling's formula for $p > k$,

$$\binom{N}{p} N^{k-(p-k)_+} k^{k+(p-k)_+} \le e^p N^p p^{-p} N^{2k-p} k^p = e^p N^{2k}.$$

Hence again

$$\sum_{p=k+1}^{2k} \binom{N}{p} \binom{2k-1}{p-1} N^{k-(p-k)_+} k^{k+(p-k)_+} \le k 2^{2k} e^{2k} N^{2k} < \frac{1}{2} (8e^2)^k N^{2k}.$$

Finally,

$$\begin{aligned}
\frac{1}{(2k)!} \int_{\Pi^{d\,N}} \bar\rho_N \left| \frac{1}{N} \sum_{i,j=1}^{N} \phi(x_i, x_j) \right|^{2k} \, \mathrm{d}X^N \\
\le (8\, e^4 C)^k \left(\sup_{p\ge 1} \frac{\| \sup_z |\phi(.,z)| \|_{L^p(\bar\rho\, dx)}}{p} \right)^{2k} \\
\le 800^{2k} \left(\sup_{p\ge 1} \frac{\| \sup_z |\phi(.,z)| \|_{L^p(\bar\rho\, dx)}}{p} \right)^{2k},
\end{aligned}$$

concluding the proof of Proposition 4.

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