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A new family of higher order nonlinear degenerate parabolic equations

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Abstract
There has been much investigation of higher order nonlinear degenerate equations of the form $h_t = \left(M(h)\left(\frac{\delta H}{\delta h}\right)_x\right)_x$, where M is a specified function and H is the quadratic first order energy functional $\frac{1}{2}\int h_x^2\,\mathrm{d}x$. The energy functional arises in many physical models, but is not universal among higher order parabolic equations. Recent investigations have motivated the study of other energy functionals, such as $H_p = \int (h_x^2)^{p/2}\,\mathrm{d}x$ for $p \neq 2$. We undertake such a study here, proving the existence of weak solutions for appropriate boundary conditions, nonnegativity and positivity properties of solutions. Moreover, an entropy dissipation–entropy estimate for solutions of this equation is obtained. Support properties and long time behaviour of solutions are also discussed for various cases.

Mathematics Subject Classification: 35B40, 35K25, 35K45, 35K55, 35K65

1. Introduction

Diffusion processes are modelled by a parabolic evolution equation of the form

$$h_t = \left(M(h)\left(\frac{\delta H}{\delta h}\right)_x\right)_x. \tag{1.1}$$

Here $M(h)$ is called the *mobility term* and H is an *energy functional*, so that $\frac{\delta H}{\delta h}$ is the *chemical potential*.

The simplest example arises when $M(h) \equiv 1$ and $H(h) = \frac{1}{2}\int h^2\,\mathrm{d}x$, which gives us the linear heat equation.

More recently a first order energy functional of the form

$$H_2(h) := \frac{1}{2}\int h_x^2\,\mathrm{d}x \tag{1.2}$$

has been employed. For example, the following equation

$$h_t = -(M(h)h_{xxx})_x\,, \qquad -a \leqslant x \leqslant a, \tag{1.3}$$

with either periodic or 'no flux' boundary conditions, i.e. $h_x(t, \pm a) = h_{xxx}(t, \pm a) = 0$, is called the *thin-film equation* and here the mobility term is given by $M(h) = h^n$ and the energy functional is given by (1.2). This is because in the classical derivation of the thin-film equation the local energy density for the given interface profile $h(t, x)$ is given by

$$\tfrac{1}{2}|h_x|^2,$$

for some constant. From this one can easily get the energy functional (sometimes called *the effective interface Hamiltonian*) (1.2).

It is worth noting that the effective interface Hamiltonian taken in the derivations of the thin-film equation is an approximation. In the physical problem we are interested in the two-dimensional cross section Ω of the fluid which is given by the area between the graph of a function $y = h(x) \geqslant 0$ and by $y = 0$. Hence, $\Omega = \{(x, y) : 0 < y < h(x)\}$. Note that in this framework, the surface energy can be written as

$$H_s = \int_{\{h>0\}} \sqrt{1 + h_x^2}\,\mathrm{d}x, \tag{1.4}$$

where we neglected the multiplicative factor. See [16]. From this, the approximate energy functional (1.2) is obtained. This is because in the classical lubrication approximation the basic assumption is that the typical length scale in the vertical direction is negligible compared with the typical horizontal length scale. We also remark that we neglected the constant term as, when we plug the interface energy H into (1.1), the constant term disappears.

In a more general setting of the thin-film equation the mobility is of the form

$$M(h) = h^3 + h^n, \qquad n \in (0, 3).$$

Here n accounts for different forms of the slip condition at the liquid–solid interface. Also it is worth noting that the theory can be generalized to the case

$$M \in C([0, \infty)), \qquad \text{increasing}, \; M(h) \sim h^n \text{ as } h \searrow 0, \; \text{for some } n \in (0, 3).$$

Different mobility terms represent different physical situations. For instance in (1.3), when $n = 1$, or equivalently $M(h) = h$, the equation describes the evolution of the thickness of a thin bridge between two masses of fluid in a Hele–Show cell. $M(h) = h^3$ case is used in the modelling of capillary driven flow. More precisely, here h is the thickness of a fluid film on a substrate where the film is evolving under the influence of the surface tension, but not the gravity. Finally, when $M(h) = h^2$, it is used for the presence of the slip length to allow the contact line to move at the fluid–substrate interface. See the papers [6, 7, 9, 13, 15, 16] for more information and derivations for the thin-film equation.

We also remark here that the thin-film equation is a special case ($p = 2$ case) of the so-called 'doubly nonlinear thin film equation' considered in [1]. The equation reads as follows:

$$h_t + [|h|^n |h_{xxx}|^{p-2} h_{xxx}]_x = 0, \tag{1.5}$$

where $n > 0$ and $p \geqslant 2$ are real constants. Equation (1.5) describes the evolution of the height $h(x, t)$ of a surface-tension driven thin liquid film on a solid surface in a lubrication approximation [1, 15, 21]. The $p = 2$ case in (1.5) corresponds to a Newtonian fluid, and $p \neq 2$ occurs when considering 'power-law' liquids. In [1] the authors prove the existence of solutions to the problem (1.5) and obtain sharp upper bounds for the propagation of their support. They also derive a necessary condition for the occurrence of waiting-time phenomena. The techniques we use in this paper are closely related to the techniques used in [1].

On the other hand, another example is the problem of relaxation of axisymmetric crystal surfaces with a single facet below the roughening transition. In [14] (also the references therein) this problem is analysed via a continuum approach that accounts for step energy g_1 and step–step interaction energy $g_2 > 0$. We point out that the evolution of the surface morphology here is caused by the motion of steps. The energy functional used for this problem is

$$H_3(h) := \int g_0 + g_1|\nabla h| + \frac{1}{3}g_2|\nabla h|^3 \,\mathrm{d}x, \tag{1.6}$$

where the g_0 term represents the surface free energy of the reference plane, g_1 is the step energy and g_2 includes entropic repulsions due to fluctuations at the step edges and pairwise energetic interactions between adjacent steps. We will omit details and moreover we will not analyse the equation obtained closely. We mention this problem to show that there are situations in which different power-law surface energy functionals are used. Readers who are interested in this problem may see [14] and the references therein.

With this background as motivation, we now turn to the study of (1.1) for $H = H_p$, where H_p is given by (1.7) with $p > 0$,

$$H_p(h(t,x)) := \frac{1}{p}\int |h_x(t,x)|^p \,\mathrm{d}x. \tag{1.7}$$

We begin by formally writing down the equation. A simple set of calculations yields that

$$\frac{\delta H_p}{\delta h} = -(p-1)(h_x^2)^{\frac{p}{2}-1}h_{xx},$$

and differentiating this with respect to x yields

$$\left(\frac{\delta H_p}{\delta h}\right)_x = -(p-1)(p-2)(h_x^2)^{\frac{p}{2}-2}h_x h_{xx}^2 - (p-1)(h_x^2)^{\frac{p}{2}-1}h_{xxx}.$$

Plugging this back into (1.1) (choosing $M(h) = h^n$) yields the following equation:

$$h_t = -[h^n((p-1)(p-2)(h_x^2)^{\frac{p}{2}-2}h_x h_{xx}^2 + (p-1)(h_x^2)^{\frac{p}{2}-1}h_{xxx})]_x. \tag{1.8}$$

Therefore, the initial boundary value problem we consider here is

$$h_t = -[h^n[(p-1)(h_x^2)^{\frac{p}{2}-1}h_{xx}]_x]_x, \tag{1.9}$$

in $Q_T := (0,T)\times\Omega$, where $T > 0$ and Ω is the bounded interval

$$\Omega = \{-a < x < a\},$$

with initial conditions

$$h(0,x) = h_0(x), \qquad h_0 \in H^p(\Omega) \tag{1.10}$$

and with *no-flux* boundary conditions

$$h_x = h_{xx} = h_{xxx} = 0 \qquad \text{for } x \in \{-a, a\}. \tag{1.11}$$

Note that in (1.9) we write an alternate form of the equation (1.8) because this is useful in making certain calculations.

Although having some similarities to second order parabolic equations, fourth order parabolic equations do not have a maximum principle. Nonetheless, as in Bernis and Friedman's investigation of the $p = 2$ case [6], we shall prove

$$\text{initial data } \geqslant 0 \Rightarrow \text{the solution} \geqslant 0.$$

Clearly this is wrong for the linear fourth order equation $h_t + h_{xxxx} = 0$. Due to the lack of a maximum principle, one must rely on proving dissipation results for nonlinear entropies.

Singularity formation of the form $h \to 0$ is therefore an interesting question for the fourth order nonlinear degenerate parabolic equations.

A main objective of this paper is to provide a range of dissipated entropy functionals that are useful for nonnegativity, positivity and long time behaviour of solutions. To be more precise and make the reading easier, we briefly describe our main results and their proof techniques concerning the problem (1.9), (1.10) and (1.11).

I. Nonnegativity. Using the ideas of Bernis and Friedman [6] we seek zeroth order Lyapunov functionals which may be useful for proving nonnegativity of solutions to the equation (1.9), combined with the initial and boundary conditions (1.10) and (1.11), respectively. To this end, we define

$$E_0[h(t,x)] := \int_\Omega \Phi(h(t,x))\,\mathrm{d}x, \qquad \text{where } \Phi''(s) = \frac{1}{s^n}, \tag{1.12}$$

and prove that E_0 satisfies

$$E_0[h(t,x)] + (p-1)\int_0^t \int_\Omega (h_x^2)^{p/2-1} h_{xx}^2\,\mathrm{d}x\,\mathrm{d}t = E_0[h_0(x)]\,\mathrm{d}x. \tag{1.13}$$

Conclusion of these calculations is that

$$h(t,x) > 0 \text{ for } n \geqslant 2 + \frac{p}{p-1}.$$

II. Regularization. Analogous to the thin-film equation case, we define

$$P_\epsilon(h) := \frac{h^{(2+\frac{p}{(p-1)})} h^n}{\epsilon h^n + h^{(2+\frac{p}{(p-1)})}} \tag{1.14}$$

and consider the equation

$$h_t = -(p-1)[P_\epsilon(h)[(h_x^2)^{p/2-1}h_x]_{xx}]_x. \tag{1.15}$$

The initial condition of the problem is also modified; indeed we define

$$h_{0\epsilon}(x) = h_0(x) + \epsilon^\theta, \qquad 0 < \theta < 2/5. \tag{1.16}$$

Finally, the boundary conditions (1.11) are kept unchanged. We prove the following theorem, which states the properties of a weak solution obtained by a uniform limit as $\epsilon \to 0$ of solutions h_ϵ of the regularized problems.

Theorem 1 (Properties of positively approximated solution). Any function h obtained by letting $\epsilon \to 0$ so that $h_{\epsilon_k} \to h$ in $C_{\mathrm{loc}}(\bar{Q})$ as $k \to \infty$, where $\{h_\epsilon\}$ is a sequence of solutions to the regularized problem (1.15), (1.16) and (1.11), satisfies

$$h \in C(\bar{Q}_T), \qquad \text{actually } h \in C_{t,x}^{\beta,1/p'}(\bar{Q}_T) \text{ (uniformly in } x), \tag{1.17}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\beta = \frac{p-1}{5p-2}$.

$$h_t, h_x, h_{xx}, h_{xxx}, h_{xxxx} \in C(P), \tag{1.18}$$

where $P = \bar{Q}_T - (\{h = 0\} \cup \{t = 0\})$, and

$$P_\epsilon(h)[(h_x^2)^{p/2-1}h_{xx}]_x \in L^2(P), \tag{1.19}$$

h satisfies (1.15) in the following sense:

$$\int\int_{Q_T} h\phi_t\,\mathrm{d}x\,\mathrm{d}t + (p-1)\int\int_P h^n[(h_x^2)^{p/2-1}h_{xx}]_x\phi_x\,\mathrm{d}x\,\mathrm{d}t = 0, \tag{1.20}$$

for all ϕ that is Lipschitz in $\bar{Q}_T$, and $\phi = 0$ near $t = 0$ and near $t = T$,

$$h(0,x) = h_0(x), \qquad x \in \bar{\Omega}, \tag{1.21}$$

$$h_x(t,.) \to h_{0x} \qquad \text{strongly in } L^p(\Omega) \text{ as } t \to 0, \tag{1.22}$$

and finally h satisfies the boundary conditions (1.11) at all points of the lateral boundary where $h \neq 0$.

This kind of regularization is also useful for improving the result of singularity formation. Indeed we deduce that

for $n \geqslant 2 + p' - \frac{1}{p'}$, where $p' = \frac{p}{p-1}$, singularity formation[1] is not possible.

Remark. There is also another regularization which is somewhat standard in the theory of nonlinear degenerate parabolic equations. This regularization, first introduced by Bernis and Friedman in [6] for the thin-film equation, reads as follows:

$$h_{\epsilon t} = -(p-1)\left((h_\epsilon^n + \epsilon)[(h_{\epsilon x}^2 + \epsilon)^{\frac{p}{2}-1} h_{\epsilon x}]_{xx}\right)_x, \qquad \epsilon > 0. \tag{1.23}$$

Using this kind of a regularization one can show the existence of a weak solution which is a uniform limit as $\epsilon \to 0$ of solutions to (1.23). Since our main results follow from the first regularization considered above we do not give the details of this standard regularization here. The details of this regularization can be found in the author's thesis [19].

III. Entropy dissipation–entropy estimate. We prove that the functional $K_q(h(t,x)) := \int_\Omega \frac{h_x^2}{h^q}\,\mathrm{d}x$ is an *entropy functional* for positive smooth solutions of (1.1) with $p = 3$ and $n = 2$. that is, we prove that we can bound the rate of decrease of K_q in terms of itself along any smooth positive solution of (1.1) with $p = 3$ and $n = 2$. More precisely, we prove that there exists a constant $C > 0$ such that

$$K_q(t) \leqslant \left[\frac{2}{5(Ct + \frac{2}{5}[K_q(0)]^{-5/2})}\right]^{2/5}. \tag{1.24}$$

This clearly gives an initial polynomial decay(like $t^{-2/5}$) of positive smooth solutions to the equilibrium and once $K_q(h)$ is small enough we can then use linearization to obtain an exponential decay.

We also note that numerical calculations suggest that an inequality of the form (1.24) can be proved for a wider range of n values(and also a wider range of p values) but we leave such a study for an upcoming paper.

IV. Support properties. We prove the following result related to the support properties of solutions.

Theorem 2 (Support properties). Let h_0 satisfy

$$n \in (0,\infty), \qquad 0 \leqslant h_0 \in H^p(\Omega), \quad h_0 \not\equiv 0 \text{ in } [-a,a] \tag{1.25}$$

and let h_ϵ be the solution of the equation (1.15) with initial condition

$$h_{0\epsilon}(x) = h_0(x) + \delta(\epsilon), \tag{1.26}$$

[1] As singularity formation we mean $h \to 0$ throughout this paper.

and boundary conditions (1.11), where $P_\epsilon(s)$ is given by (1.14). Let h be a solution of the problems (1.9), (1.10) and (1.11), obtained by

$$h_{\epsilon_k} \to h \qquad \text{in } C_{\mathrm{loc}}(\bar{Q}) \text{ as } \epsilon_k \to 0. \tag{1.27}$$

Then, one has the following conclusions.

(i) If $n \geqslant 1 + \frac{(p-1)}{p}$, then

$$\operatorname{supp} h(t_0, .) \subseteq \operatorname{supp} h(t, .) \qquad \text{for } t > t_0.$$

(ii) If $n > \frac{p}{p-1}$ then

$$h(t_0, x_0) > 0 \Rightarrow h(t, x_0) > 0 \qquad \text{for almost every } t > t_0.$$

(iii) If $n \geqslant 1 + \frac{(p-1)}{p} + \frac{p}{(p-1)}$

$$h(t_0, x_0) > 0 \Rightarrow h(t, x_0) > 0 \qquad \text{for all } t > t_0.$$

V. Asymptotic behaviour of nonnegative solutions. Using the usual energy functional (1.7) we deduce the long time behaviour of both the smooth and the weak solutions. The strategy, same as the one in [17], is to try to control the rate of decrease of the functional (1.7) in terms of itself. The following result, which is a generalization of one of the results of [17], is quite useful for this purpose.

Lemma 7.2 (A useful inequality). For any measurable function $\psi : [0, \infty) \to [0, \infty)$ and for any $0 \leqslant u \in H^3(\Omega)$ with $u_x(\pm a) = 0$, we have that

$$\left(\int_\Omega \frac{u^2}{\psi(u)}\,\mathrm{d}x\right)^{1/2} \left(\int_\Omega \psi(u)[(u_x^2)^{p/2-1}u_{xx}]_x^2\,\mathrm{d}x\right)^{1/2} \geqslant C H_p(u), \tag{1.28}$$

where C is a finite constant depending on a and p. Using this lemma we deduce the following proposition, which is useful for obtaining the long time behaviour result for nonnegative smooth solutions.

Proposition 7.3 (Energy dissipation bound). Suppose that $0 < n < \infty$ and h is a nonnegative smooth solution (i.e. classical solution) of the equation (1.9) with initial and boundary conditions (1.10) and (1.11). Moreover, suppose that the initial condition $h_0 \in H^1(\Omega)$ has finite mass. Then, we have the following.

(i) If $0 < n < 2$, then there exists a constant $0 < C = C(||h_{0x}||_{L^p(\Omega)}, p, a, n)$ such that

$$\int_\Omega h^n(t, x)[(h_x^2)^{p/2-1}h_{xx}]_x\,\mathrm{d}x \geqslant C[H_p(h(t, x))]^2, \qquad \forall t > 0. \tag{1.29}$$

(ii) If $n = 2$ then (1.29) holds with $C = C(p, a)$. that is, C is now independent of $||h_{0x}||_{L^p(\Omega)}$.

(iii) If $n > 2$ and $\int_\Omega h_0^{2-n}(x)\,\mathrm{d}x < \infty$, then there exists a constant $0 < C = C(\int_\Omega h_0^{2-n}(x)\,\mathrm{d}x, p, a)$ such that (1.29) holds.

By the proposition we deduce that

$$H_p[h(t, x)] \leqslant [H_p[h_0]^{-1} + Ct]^{-1}, \qquad t > 0. \tag{1.30}$$

Hence, from this, $H_p(h)$ becomes sufficiently small after some finite time and so $h(t, x)$ becomes uniformly bounded from below away from 0. From this point on we can then deduce from linearization that there is an exponential decay.

Note that we could not deduce the long time behaviour of weak solutions using entropy dissipation–entropy estimate section. However, using the usual energy it is possible to prove the following result.

Proposition 7.4 (Energy dissipation for weak solutions). Assume that $n \in (0,1) \cap (2,\infty)$ and $h_0 \in H^1(\Omega)$ satisfies $\int_\Omega h_0^{2-n}\,\mathrm{d}x < \infty, n > 2$ and has finite mass. Then, there exists a constant $C = C(\int_\Omega h_0^{2-n}\,\mathrm{d}x, p, a, n) > 0$ such that

$$\frac{\mathrm{d}H_p[h(t,x)]}{\mathrm{d}t} \leqslant -C(H_p[h(t,x)])^2, \qquad \forall t > 0. \tag{1.31}$$

where $h(t,x)$ is a weak solution of the equation (1.9) with initial and boundary conditions (1.10) and (1.11).

Clearly the proposition yields that

$$H_p[h(t,x)] \leqslant H_p[h_0]\left(1 + \tau_1 H_p[h_0]t\right)^{-1}, \qquad \tau_1 > 0. \tag{1.32}$$

This implies that whenever $H_p[h(t,x)]$ is small enough $h(t,x)$ becomes bounded below away from 0, and after this point on we have exponential decay by linearization. The remaining case for n is left for the upcoming paper.

We now provide the details of our analysis.

2. Nonnegativity and positivity of solutions

Let E_0 be defined by (1.12). Multiplying the equation (1.9) formally by Φ' and integrating we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}E_0[h(t,x)] = (p-1)(p-2)\iint_{Q_T}(h_x^2)^{\frac{p}{2}-1}h_{xx}^2\,\mathrm{d}x\,\mathrm{d}t + (p-1) \times \iint_{Q_T}(h_x^2)^{\frac{p}{2}-1}h_x h_{xxx}\,\mathrm{d}x\,\mathrm{d}t. \tag{2.1}$$

Here we used the fact that $\Phi'' = \frac{1}{h^n}$. From (2.1) one deduces that (1.13) is satisfied and this clearly implies that E_0 dissipates whenever $p \geqslant 1$. Thus, by using the Hölder continuity[2] of $h(t,x)$ in x, we deduce that for $n \geqslant 2 + \frac{p}{(p-1)}$ there cannot be singularity formation. Note that this was proved by Bernis and Friedman in [6] for the $p = 2$ case.

3. Approximation by positive h_ϵ

The above calculations suggest the following regularization of the problem. Let $P_\epsilon(h)$ be given by (1.14) and consider the regularized problem (1.15). The initial condition of the problem is also modified; indeed we define $h_{0\epsilon}(x)$ by (1.16).

Since $\lim_{s\to 0}\frac{P_\epsilon(s)}{s^{2+\frac{p}{(p-1)}}} = \frac{1}{\epsilon}$ if $1 \leqslant n < 2+\frac{p}{(p-1)}$ and if $n \geqslant 2+\frac{p}{(p-1)}$ then $\lim_{s\to 0}\frac{P_\epsilon(s)}{s^{2+\frac{p}{(p-1)}}} = 0$ and $h_{0\epsilon} > 0$, there exists a unique, positive smooth solution h_ϵ of the problem (1.15) combined with the initial condition (1.16) and with *no-flux* boundary conditions (1.11). We can modify the calculations done in [19] to deduce that there exists a limit function h such that $h_\epsilon \to h$ uniformly. Moreover, we will show that this limit function is a weak solution of the problem (1.9), (1.10) and (1.11).

Proof of theorem 1. Take ϕ as indicated. One can easily obtain that

$$||P_\epsilon(h)[(h_x^2)^{p/2-1}h_{xx}]_x||_{L^2(Q_T)} \leqslant A,$$

[2] One can show that $|h(t,x) - h(t,y)| < C|x-y|^{(p-1)/p}, \forall x, y \in \Omega, \forall t > 0$. We refer to [19] for details.

where A is independent of ϵ and T. Let $Z_\epsilon := P_\epsilon(h)[(h_x^2)^{p/2-1}h_{xx}]_x$. Then, for a subsequence such that $Z_\epsilon \to Z$ weakly in $L^2(Q_T)$. By regularity theory of uniformly parabolic equations and uniform Hölder continuity of h_ϵ we deduce that

$$h_{\epsilon t}, h_{\epsilon x}, h_{\epsilon xx}, h_{\epsilon xxx}, h_{\epsilon xxxx}$$

are uniformly convergent in any compact subset of $P = \bar{Q}_T - (\{h = 0\} \cup \{t = 0\})$. Hence, $h^n[(h_x^2)^{p/2-1}h_{xx}]_x = Z$, on P. Hence,

$$h_t, h_x, h_{xx}, h_{xxx}, h_{xxxx} \in C(P)$$

and

$$h^n[(h_x^2)^{p/2-1}h_{xx}]_x \in L^2(P).$$

Hence, we also see that h solves the problem (1.9) in the *weak* sense. For any $\delta > 0$ one has that

$$(p-1)\iint_{\{|h|>\delta\}} P_\epsilon(h_\epsilon)[(h_{\epsilon x}^2)^{p/2-1}h_{\epsilon xx}]_x\phi_x \,\mathrm{d}x\,\mathrm{d}t \to (p-1) \times \iint_{\{|h|>\delta\}} h^n[(h_x^2)^{p/2-1}h_{xx}]_x\phi_x \,\mathrm{d}x\,\mathrm{d}t.$$

On the other hand, if ϵ is small enough(depending on δ), then by the Cauchy–Schwartz inequality

$$\left|(p-1)\iint_{\{|h|\leqslant\delta\}} P_\epsilon(h_\epsilon)[(h_{\epsilon x}^2)^{p/2-1}h_{\epsilon xx}]_x\phi_x \,\mathrm{d}x\,\mathrm{d}t\right| \leqslant C|(p-1)|\delta^{n/2} \to 0 \qquad \text{as } \delta \to 0. \tag{3.1}$$

Recall also that we have

$$\int_\Omega |h_{\epsilon x}|^p(t,x)\,\mathrm{d}z \leqslant \int_\Omega |h_{0\epsilon x}|^p\,\mathrm{d}x,$$

and $h_{\epsilon 0} \to h_0$ in $H^p(\Omega)$. Combining these we deduce that

$$\limsup_{t\to 0}\int_\Omega |h_x|^p(t,x)\,\mathrm{d}x \leqslant \int_\Omega |h_{0x}(x)|^p\,\mathrm{d}x$$

and

$$h_x(t,.) \to h_{0x} \qquad \text{weakly in } L^p(\Omega) \text{ as } t \to 0.$$

We see a weakly convergent sequence which is bounded. In fact, in this case the sequence converges strongly to the same limit.

Taking $\epsilon \to 0$, since $\delta > 0$ is arbitrary, we conclude that (1.20) is satisfied. This completes the proof of the theorem. □

We now use this regularization scheme to improve the result of singularity formation. To this end, define the natural entropy by

$$H_\epsilon(h) := \int_\Omega G_\epsilon(h(t,x))\,\mathrm{d}x, \tag{3.2}$$

where G_ϵ satisfies

$$G_\epsilon(h(t,x))'' P_\epsilon(h(t,x)) = h^\beta(t,x), \qquad \beta < 0. \tag{3.3}$$

The more negative the β, the better the result we obtain for the singularity formation. We can easily determine $G_\epsilon(h(t,x))$ using (3.3), where we choose the constant of the integration so that $\int_\Omega G_\epsilon(h(t,x))\,\mathrm{d}x \geqslant 0$. Multiplying the equation (1.9) formally by $G'(h(x,t))$ and applying integration by parts we obtain that

$$\begin{aligned}\int_\Omega G'_\epsilon(h)h_t\,\mathrm{d}x &= -(p-1)^2\int_\Omega h^\beta (h_x^2)^{p/2-1}h_{xx}^2\,\mathrm{d}x\\ &\quad+\frac{\beta(\beta-1)(p-1)^2}{(p+1)}\int_\Omega h^{\beta-2}(h_x^2)^{p/2-1}h_x^4\,\mathrm{d}x.\\ &=: -c_1J_1(h(t,x))+c_2J_2(h(t,x)).\end{aligned}\tag{3.4}$$

To proceed further we need the following lemma.

Lemma 6.2 (Negative term beats in (3.4)). One has the following inequality for $0\leqslant h\in H^3(\Omega)$ and $h_x(\pm a)=0$

$$J_1(h(t,x))\geqslant CJ_2(h(t,x)),\tag{3.5}$$

where

$$C:=\frac{(1-\beta)^2}{(p+1)^2}.$$

Proof. Inequality (3.5) can be verified easily by considering that, for any constant $A>0$,

$$0\leqslant\int_\Omega[h^{\beta/2}((h_x^2)^{p/2-1})^{1/2}h_{xx}-Ah^{\beta/2-1}((h_x^2)^{p/2-1})^{1/2}h_x^2]^2\,\mathrm{d}x.\tag{3.6}$$

By employing integration by parts we see that (3.6) is equivalent to

$$J_1(h(t,x))+\left(A^2-\frac{2A(1-\beta)}{(p+1)}\right)J_2(h(t,x))\geqslant 0.\tag{3.7}$$

Optimizing over A, in (3.7) we obtain (3.5). □

Using (3.5) we finally have that

$$\frac{\mathrm{d}}{\mathrm{d}t}H_\epsilon(h(t,x))\leqslant C_\beta J_2(h(t,x)),\tag{3.8}$$

where

$$C_\beta:=-\frac{(p-1)^2(1-\beta)^2}{(p+1)^2}+\frac{\beta(\beta-1)(p-1)^2}{(p+1)}.$$

Note that C_β is nonpositive if and only if $\beta\geqslant-\frac{1}{p}$. Using the definition of G_ϵ and the Hölder continuity of $h(t,x)$ in x we deduce that there is no singularity formation of the form $h\to 0$ if

$$n\geqslant 2+p'-\frac{1}{p},$$

where $p'=\frac{p}{p-1}$. Note that this obeys the $p=2$ case where $n\geqslant 7/2$ implies no singularity formation in this case. Moreover, when $p=3$ this shows that $n\geqslant 3.1666...$ implies no singularity formation in this case.

4. An entropy dissipation–entropy estimate for (1.8)

The term 'entropy' is frequently used for a Lyapunov functional whose rate of decrease can be bounded in terms of itself. That is, if $H(f)$ is some functional of f, and along the flow of some evolution we have

$$\frac{\mathrm{d}}{\mathrm{d}t}H(f) \leqslant -\Phi(H(f)), \tag{4.1}$$

with Φ some continuous strictly monotone increasing function on $\mathbb{R}_+$, then the functional $H(f)$ is called an entropy, and the inequality (4.1) is called an entropy dissipation–entropy inequality. The point is that (4.1) can be used to quantitatively estimate the rate of decay of $H(f)$.

Consider again a smooth solution of (1.9) with $p = 3$ and $n = 2$ and define the functional $K_q(h(t, x))$ by

$$K_q(h(t,x)) := \int_\Omega \frac{h_x^2}{h^q}\,\mathrm{d}x. \tag{4.2}$$

We note that this functional has been discovered by Laugesen [12] for the thin-film equation. Laugesen showed that K_q is a Lyapunov functional for the thin-film equation provided that $q \in [0, 1/2]$. Moreover, K_q was used in [10] to prove an entropy dissipation–entropy estimate for the same equation. Recently, the author did the same thing for the following equation

$$h_t = -(h^n h_{xxx})_x + (h^m)_{xx}, \qquad n > 0, \quad 1 < m < 2 \tag{4.3}$$

with periodic boundary conditions [18].

Differentiating K_q along a smooth positive solution of (1.8) with $p = 3$ and $n = 2$ and applying integration by parts whenever necessary we obtain, after some algebra, that

$$\begin{aligned}\frac{\mathrm{d}K_q(h)}{\mathrm{d}t} &= -4\int \frac{(h_x^2)^{\frac{1}{2}} h_{xxx}^2}{h^{q-n}}\,\mathrm{d}x - 2q(q+1)\int \frac{(h_x^2)^{\frac{1}{2}} h_x^3 h_{xxx}}{h^{q-n+2}}\,\mathrm{d}x \\ &\quad - \frac{4}{5}(4q+1)\int \frac{(h_x^2)^{\frac{1}{2}} h_x^2 h_{xx}^2}{h^{q-n+2}}\,\mathrm{d}x + \frac{4}{3}(q-2)\int \frac{(h_x^2)^{\frac{1}{2}} h_x h_{xx} h_{xxx}}{h^{q-n+1}}\,\mathrm{d}x. \end{aligned} \tag{4.4}$$

We note that using the notation in (4.9) and (4.10) below, we can rewrite (4.4) as

$$\frac{\mathrm{d}K_q(h)}{\mathrm{d}t} = -4I_1 + \frac{4}{3}(q-2)J_{12} - 2q(q+1)J_{13} - \frac{4}{5}(4q+1)I_2. \tag{4.5}$$

Step 1. We will show that

$$\frac{\mathrm{d}K_q(h)}{\mathrm{d}t} \leqslant -C_q I_3, \tag{4.6}$$

where C_q is a positive constant which depends on q and I_3 is given in (4.9).

Proof of step 1. To show that the right-hand side of (4.4) is negative, we will try to write it as a sum of negative squares, which is the same strategy used in [10, 12, 18]. To do this, define the nonnegative quantity A by

$$A := \int \left[\alpha h_{xxx} + \beta \frac{h_x h_{xx}}{h} + \gamma \frac{h_x^3}{h^2}\right]^2 (h_x^2)^{\frac{1}{2}} h^{q-n}\,\mathrm{d}x, \tag{4.7}$$

where the numbers α, β and γ will be chosen below. Equation (4.7) can be written as

$$A = \alpha^2 I_1 + \beta^2 I_2 + \gamma^2 I_3 + 2\alpha\beta J_{12} + 2\alpha\gamma J_{13} + 2\beta\gamma J_{23}, \tag{4.8}$$

where

$$I_1 = \int (h_x^2)^{1/2} \frac{h_{xxx}^2}{h^{q-n}}\, \mathrm{d}x, \qquad I_2 = \int (h_x^2)^{1/2} \frac{h_x^2 h_{xx}^2}{h^{q-n+2}}\, \mathrm{d}x, \qquad I_3 = \int (h_x^2)^{1/2} \frac{h_x^6}{h^{q-n+4}}\, \mathrm{d}x, \tag{4.9}$$

$$J_{12} = \int (h_x^2)^{1/2} \frac{h_x h_{xx} h_{xxx}}{h^{q-n+1}}\, \mathrm{d}x, \qquad J_{13} = \int (h_x^2)^{1/2} \frac{h_x^3 h_{xxx}}{h^{q-n+2}}\, \mathrm{d}x,$$

$$J_{23} = \int (h_x^2)^{1/2} \frac{h_x^4 h_{xx}}{h^{q-n+3}}\, \mathrm{d}x. \tag{4.10}$$

Lemma 4.1. *Integration by parts yields the following relations:*

$$I_2 = -\frac{1}{4} J_{13} + \frac{q}{4} J_{23}, \tag{4.11}$$

$$J_{23} = \left(\frac{q+1}{6}\right) I_3. \tag{4.12}$$

Proof. This is a straightforward computation. □

There is no useful integration by parts identity relating J_{12} to the other integrals in the lists (4.9) and (4.10)—integrating by parts in J_{12}, no matter how it is done, would introduce other integrals into the game. Since there is no useful integration by parts identities for I_1 and J_{12}, we use the definition of A appropriately to eliminate these terms from the right-hand side of (4.5). We have that

$$-4I_1 + \frac{4}{3}(q-2)J_{12} = -A + \left(\frac{2-q}{3}\right)^2 I_2 + \gamma^2 I_3 + 4\gamma J_{13} + 2\left(\frac{2-q}{3}\right)\gamma J_{23}. \tag{4.13}$$

Using the relation (4.13) and the integration by parts relations (4.11) and (4.12) we obtain, after some algebra, that

$$\frac{\mathrm{d}K_q(h)}{\mathrm{d}t} \leqslant S(q,\gamma) J_{13} + \left[\gamma^2 + \left(\frac{q+1}{6}\right) R(q,\gamma)\right] I_3 \tag{4.14}$$

where

$$L(q) := \left(\frac{2-q}{3}\right)^2 + \frac{2}{5}(5q+2)(q-1) - 2q(q+1),$$

$$S(q,\gamma) := 4\gamma - 2q(q+1) - \tfrac{1}{4}L(q,n)$$

and

$$R(q,\gamma) := 2\frac{(2-q)}{3}\gamma + \frac{q}{4}L(q).$$

Since J_{13} can have either sign we choose γ so that the multiple of J_{13} vanishes. This leads to the following choice for γ:

$$\gamma = \frac{2q(q+1) + \frac{1}{4}L(q)}{4}. \tag{4.15}$$

Plugging (4.15) into (4.14) yields that

$$\frac{\mathrm{d}K_q(h)}{\mathrm{d}t} \leqslant C_q^* I_3, \tag{4.16}$$

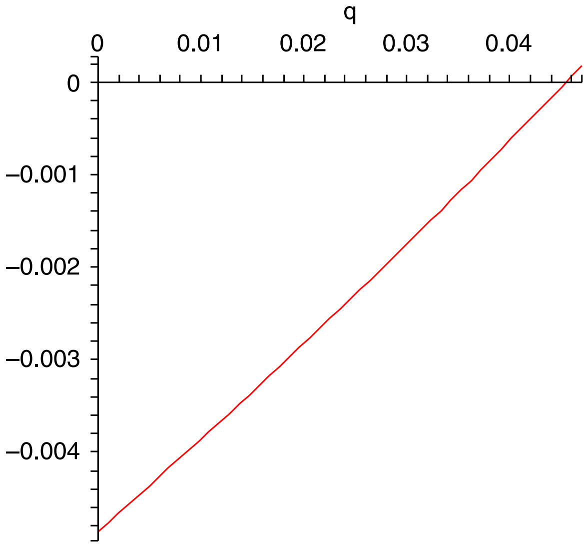

Figure 1. Graph of C_q^* as a function of q.

(This figure is in colour only in the electronic version)

where C_q^* is given by

$$C_q^* := \left(\left(\frac{2q(q+1)+\frac{1}{4}L(q)}{4}\right)^2 + 2\left(\frac{2-q}{3}\right)\left(\frac{2q(q+1)+\frac{1}{4}L(q)}{4}\right) + \frac{q}{4}\right)\left(\frac{q+1}{6}\right). \tag{4.17}$$

From figure 1 it is clear that $C_q^* < 0$ for $q \in (0, q^*)$, where q^* is the critical value close to 0.046.

Thus, for $q \in (0, q^*)$, we can define $C_q^* =: -C_q$, where $C_q > 0$. With this choice we have

$$\frac{\mathrm{d}K_q(h)}{\mathrm{d}t} \leqslant -C_q I_3. \tag{4.18}$$

Step 2. Now, we show that

$$I_3(h(t,x)) \geqslant N_q K_q^{7/2}(h(t,x)), \tag{4.19}$$

where N_q is a positive constant.

Proof of step 2. Notice that

$$I_3 \geqslant \int \frac{|h_x|^7}{h^{q-n+4}}\,\mathrm{d}x = \int \left(\frac{h_x^2}{h^q}\right)^{7/2} \frac{1}{h^r}\,\mathrm{d}x.$$

Letting $u = \frac{h_x^2}{h^q}$, and letting $v = h$, we have that

$$I_3 \geqslant \int u^{7/2} v^{-r}\,\mathrm{d}x. \tag{4.20}$$

The function $(v, s) \to v^{7/2}s^{-r}$ is jointly convex if $r \leqslant 5/2$, so that by Jensen's inequality

$$\frac{1}{2a}\int_{-a}^{a} u^{7/2}v^{-r}\,\mathrm{d}x \geqslant \left(\frac{1}{2a}\int_{-a}^{a} u\,\mathrm{d}x\right)^{7/2}\left(\frac{1}{2a}\int_{-a}^{a} v\,\mathrm{d}x\right)^{-r}$$
$$= \frac{1}{2a(\int_{-a}^{a} h_0(x)\,\mathrm{d}x)^r}(K_q(h))^{7/2}. \tag{4.21}$$

Combination of (4.20) and (4.21) proves (4.19).

Step 3. Consequence. Combining the inequalities (4.6) and (4.19) we deduce that there exists a constant $C > 0$ such that

$$K_q \leqslant \left[\frac{2}{5(Ct + \frac{2}{5}[K_q(0)]^{-5/2})}\right]^{2/5}. \tag{4.22}$$

This clearly gives an initial polynomial decay(like $t^{-2/5}$) of solutions to the equilibrium and once $K_q(h)$ is small enough we can then use linearization to obtain an exponential decay.

Remark 1. For the $p = 3$ case numerical calculations strongly suggest that one can obtain an entropy dissipation–entropy estimate for a wider range of n values.

Remark 2. Consider the general case. By a similar kind of calculations we may obtain, as in [11], an algebraic decision problem for the parameters p, q and n. In other words, the parameters p, n and q should satisfy certain inequalities. We leave such a study to an upcoming paper [20].

Remark 3. We note that for the thin film equation and for the equation (4.3) one obtains $t^{-1/2}$ initial decay towards the equilibrium [10, 18]. This is different from our decay rate(of the particular case $p = 3$ and $n = 2$).

Remark 4. For the modified thin film equation [3, 4, 5, 8, 18], i.e.

$$h_t = -h^n h_{xxxx}, \qquad n > 0, \tag{4.23}$$

one can try similar calculations for the energy $I_q := \int \frac{h_{xx}^2}{h^q}\,\mathrm{d}x$, but not for K_q. This is because I_0 is dissipated but K_0 is not. This work is in progress.

5. Integral estimates

Proposition 5.1 (Integral estimate for h_ϵ). Let h_0 satisfy (1.25), P_ϵ be defined by (1.14) and let h_ϵ be the solution of the regularized problem (1.15) with the initial condition

$$h(0, x) = h_{0\epsilon}(x), \qquad x \in \Omega,$$

and with boundary conditions (1.11).

Suppose that

$$h_{0\epsilon} \in C^\infty([-a, a]), \quad h_{0\epsilon} > 0, \qquad \text{for } x \in [-a, a],\ h_{0\epsilon} \to h_0 \text{ in } H^p((-a, a)) \text{ as } \epsilon \to 0 \tag{5.1}$$

and moreover suppose that $h_{0\epsilon}$ satisfies the corresponding boundary conditions (1.11).

Let $\alpha \neq 0$ be a real number such that

$$\frac{p-1}{p} \leqslant \alpha + n \leqslant 2, \tag{5.2}$$

let $T > 0$ and let $\zeta \in C^4(\Omega)$ be a nonnegative function with support in $(-a, a)$. Assume either

$$h_0 > 0 \qquad \text{in } \operatorname{supp}(\zeta) \tag{5.3}$$

or $h_{0\epsilon}$ satisfies

$$h_{0\epsilon}(x) \geqslant h_0(x) + \epsilon^\theta, \qquad 0 < \theta \leqslant \tfrac{2}{5}, \tag{5.4}$$

and h_0 satisfies

$$\int_\Omega \zeta^4 h_0^{\alpha+1}(x)\,\mathrm{d}x < \infty, \alpha \neq -1, \tag{5.5}$$

$$\int_\Omega \zeta^4 |\ln(h_0(x))|\,\mathrm{d}x < \infty, \alpha = -1. \tag{5.6}$$

Then, there exist constants C_1^* and C_2^* which are independent of ϵ such that

$$\int_\Omega \zeta^4 h_\epsilon^{\alpha+1}(t,x)\,\mathrm{d}x \leqslant C_1^*, \qquad 0 < t \leqslant T, \quad \alpha \neq -1, \tag{5.7}$$

$$\int_\Omega \zeta^4 |\ln(h_\epsilon(t,x)|\,\mathrm{d}x \leqslant C_2^*, \qquad 0 < t \leqslant T, \quad \alpha = -1. \tag{5.8}$$

If γ is a real number satisfying

$$\gamma_1 \leqslant \gamma \leqslant \gamma_2, \tag{5.9}$$

where

$$\gamma_1 := \frac{(\alpha + n + p - 1) - \sqrt{(\alpha + n - 2)(p - 1 - p(\alpha + n))}}{(p+1)} \tag{5.10}$$

and

$$\gamma_2 := \frac{(\alpha + n + p - 1) + \sqrt{(\alpha + n - 2)(p - 1 - p(\alpha + n))}}{(p+1)}, \tag{5.11}$$

then

$$\int_0^t \int_\Omega \zeta^4 h_\epsilon^{\alpha+n-2\gamma+1} (h_\epsilon^\gamma)_{xx}^2 (h_{\epsilon x}^2)^{p/2-1}\,\mathrm{d}x\,\mathrm{d}t \leqslant C_3^*, \tag{5.12}$$

and if

$$\frac{p-1}{p} < \alpha + n < 2,$$

then

$$\int_0^t \int_\Omega \zeta^4 h_\epsilon^{\alpha+n-3} h_{\epsilon x}^4 (h_{\epsilon x}^2)^{p/2-1}\,\mathrm{d}x\,\mathrm{d}t \leqslant C_4^*, \tag{5.13}$$

where C_3^* and C_4^* are positive constants independent of ϵ.

Remark 1. If the conditions of the proposition are satisfied and h is a solution of the corresponding limiting case where $\epsilon \to 0$, then, by *Fatou's lemma*, one deduces that

$$\int_\Omega \zeta^4 h^{\alpha+1}(t,x)\,\mathrm{d}x < \infty, \qquad t > 0, \quad \alpha \neq -1, \tag{5.14}$$

$$\int_\Omega \zeta^4 |\ln(h(t,x))|\,\mathrm{d}x < \infty, \qquad t > 0, \quad \alpha = -1. \tag{5.15}$$

Remark 2. The inequality (5.12) becomes

$$\int_0^t\int_\Omega \zeta^4 h_\epsilon^{1/p}(h_\epsilon^{(p-1)/p})_{xx}^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C_3^* \tag{5.16}$$

if $\alpha+n=\frac{p-1}{p}$, $\gamma=\frac{p-1}{p}$ and $n\neq\frac{p-1}{p}$(i.e. $\alpha\neq 0$), and it becomes

$$\int_0^t\int_\Omega \zeta^4 h_\epsilon h_{\epsilon xx}^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C_3^* \tag{5.17}$$

if $\alpha+n=2$, $\gamma=1$ and $n\neq 2$(i.e. $\alpha\neq 0$).

Proof. Define the following function

$$g_\epsilon(s) := -\int \frac{\alpha r^{\alpha+n-1}}{P_\epsilon(r)}\,\mathrm{d}r = c_1 s^{c_2} + s^\alpha - c_1 A^{c_2} - A^\alpha, \tag{5.18}$$

where the constants c_1 and c_2 are given by

$$c_1 := \frac{\alpha\epsilon}{n+\alpha-2-\frac{p}{(p-1)}}, \qquad c_2 := n+\alpha-2-\frac{p}{(p-1)}.$$

Now define

$$G_\epsilon(s) := -\int_s^A g_\epsilon(r)\,\mathrm{d}r, \tag{5.19}$$

where $A > \max h_\epsilon$ and $0 < s < A$. As $h_\epsilon > 0$ the functions g_ϵ and G_ϵ are well defined.

Let $\alpha\neq -1$. Multiplying the equation (1.15) by $\zeta^4 g_\epsilon(h)$ and integrating by parts, for any $t\in(0,T]$, one has that(note that we represent the solution of (1.15) by h to simplify notation; at the end of the proof we will return to the original notation)

$$\begin{aligned}
&\frac{1}{\alpha}\int_\Omega \zeta^4 G_\epsilon(h(t,x))\,\mathrm{d}x - \frac{1}{\alpha}\int_\Omega \zeta^4 G_\epsilon(h_{0\epsilon}(x))\,\mathrm{d}x\\
&\quad= (p-1)\int_0^t\int_\Omega \zeta^4 h^{\alpha+n-1} h_x[(h_x^2)^{p/2-1}h_{xx}]_x\,\mathrm{d}x + \frac{(p-1)}{\alpha}\\
&\quad\times\int_0^t\int_\Omega (\zeta^4)_x g_\epsilon(h)P_\epsilon(h)[(h_x^2)^{p/2-1}h_{xx}]_x\,\mathrm{d}x\\
&\quad=: L_1 + L_2.
\end{aligned} \tag{5.20}$$

We can integrate by parts and write L_1 as

$$\begin{aligned}
L_1 = &-(p-1)\int_0^t\int_\Omega (\zeta^4)_x h^{\alpha+n-1}(h_x^2)^{p/2-1}h_x h_{xx}\,\mathrm{d}x\,\mathrm{d}t\\
&-(p-1)(\alpha+n-1)\int_0^t\int_\Omega \zeta^4 h^{\alpha+n-2}(h_x^2)^{p/2-1}h_x^2 h_{xx}\,\mathrm{d}x\,\mathrm{d}t\\
&-(p-1)\int_0^t\int_\Omega \zeta^4 h^{\alpha+n-1}(h_x^2)^{p/2-1}h_{xx}^2\,\mathrm{d}x\,\mathrm{d}t\\
&=: -c_1 L_{1,1} - c_2 L_{1,2} - c_1 L_{1,c}.
\end{aligned} \tag{5.21}$$

To benefit fully from the sign of the term $L_{1,c}$ in (5.21) we use the following substitution, which was used in [2],

$$h_{xx}^2 = \frac{1}{\gamma^2}h^{2-2\gamma}(h^\gamma)_{xx}^2 - (\gamma-1)^2 h^{-2}h_x^4 - 2(\gamma-1)h^{-1}h_x^2 h_{xx}, \tag{5.22}$$

where γ is a positive constant.

Using (5.22) in (5.21) and collecting the likely terms together we obtain that

$$\begin{aligned}
L_1 = & -c_1 L_{1,1} - c_1(\alpha+n-2\gamma+1)L_{1,2} \\
& - \frac{(p-1)}{\gamma^2}\int_0^t\int_\Omega \zeta^4 h^{\alpha+n-2\gamma+1}(h_x^2)^{p/2-1}(h^\gamma)_{xx}^2\,\mathrm{d}x\,\mathrm{d}t \\
& + (p-1)(\gamma-1)^2\int_0^t\int_\Omega \zeta^4 h^{\alpha+n-3}(h_x^2)^{p/2-1}h_x^4\,\mathrm{d}x\,\mathrm{d}t \\
& =: -c_1L_{1,1} - c_3L_{1,2} - c_4L_{1,3} + c_5L_{1,4},
\end{aligned} \tag{5.23}$$

where in (5.23) we define the quantities on the right-hand side according to the occurrence of the quantities on the left-hand side.

Before proceeding further we prove the following result.

Lemma 5.2. One has the following integration by parts relations:

$$\begin{aligned}
L_{1,2} &= -\frac{(\alpha+n-2)}{(p+1)}L_{1,4} - \frac{1}{(p+1)}\int_0^t\int_\Omega (\zeta^4)_x h^{\alpha+n-2}(h_x^2)^{p/2-1}h_x^3\,\mathrm{d}x\,\mathrm{d}t \\
&=: -c_6L_{1,4} - c_7L_{1,5},
\end{aligned} \tag{5.24}$$

$$\begin{aligned}
L_{1,1} &= \frac{(\alpha+n-1)}{p}L_{1,5} - \frac{1}{p}\int_0^t\int_\Omega (\zeta^4)_{xx} h^{\alpha+n-1}(h_x^2)^{p/2-1}h_x^2\,\mathrm{d}x\,\mathrm{d}t \\
&=: -c_8L_{1,5} - \frac{1}{p}L_{1,6}.
\end{aligned} \tag{5.25}$$

Proof. This is a straightforward calculation. □

Using (5.24) and (5.25) and also collecting the likely terms together we finally obtain that

$$L_1 = -c_4L_{1,3} - c(\alpha+n,\gamma)L_{1,4} + R_1. \tag{5.26}$$

Here,

$$c(\alpha+n,\gamma) := -\left[(p-1)(\gamma-1)^2 + \frac{(p-1)}{(p+1)}(\alpha+n-2\gamma+1)(\alpha+n-2)\right] \tag{5.27}$$

and

$$R_1 = K_1L_{1,5} + \left(\frac{p-1}{p}\right)L_{1,6}, \tag{5.28}$$

where

$$K_1 := \frac{(p-1)}{p}(\alpha+n-1) - \frac{(p-1)}{(p+1)}(2\gamma-1-(\alpha+n)). \tag{5.29}$$

One can easily obtain that

$$c(\alpha+n,\gamma) = 0 \Leftrightarrow \gamma = \gamma_1 \text{ or } \gamma = \gamma_2, \tag{5.30}$$

where γ_1 and γ_2 are given by (5.10) and (5.11), respectively. Moreover,

$$c(\alpha+n,\gamma) \geqslant 0 \Longleftrightarrow \gamma_1 \leqslant \gamma \leqslant \gamma_2. \tag{5.31}$$

Now we start estimating L_2. For this purpose we write

$$\frac{1}{\alpha}P_\epsilon(h)g_\epsilon(h) = m(h) + c_\epsilon P_\epsilon(h), \tag{5.32}$$

where

$$m(h) := \frac{1}{\alpha} P_\epsilon(h) \left[\frac{\alpha\epsilon}{(\alpha+n-2-p/(p-1))} h^{(\alpha+n-2-p/(p-1))} + h^\alpha \right] \tag{5.33}$$

and

$$c_\epsilon := -\frac{1}{\alpha} A^\alpha - \frac{\epsilon}{(\alpha+n-2-p/(p-1))} A^{(\alpha+n-2-p/(p-1))}. \tag{5.34}$$

Using these we can rewrite L_2 as

$$\begin{aligned} L_2 &= \int_0^t \int_\Omega (\zeta^4)_x m(h) [(h_x^2)^{p/2-1} h_{xx}]_x \, \mathrm{d}x \, \mathrm{d}t + c_\epsilon \int_0^t \int_\Omega (\zeta^4)_x P_\epsilon(h) [(h_x^2)^{p/2-1} h_{xx}]_x \, \mathrm{d}x \, \mathrm{d}t \\ &=: L_{2,1} + c_\epsilon L_{2,2}. \end{aligned} \tag{5.35}$$

One keeps the second term and integrates by parts the first term to obtain

$$L_2 = c_\epsilon L_{2,2} - \int_0^t \int_\Omega [(\zeta^4)_x m(h)]_x [(h_x^2)^{p/2-1} h_{xx}] \, \mathrm{d}x \, \mathrm{d}t =: c_\epsilon L_{2,2} - L_{2,3}. \tag{5.36}$$

To proceed further we need to prove the following result.

Lemma 5.3. One has the following integration by parts relation:

$$\begin{aligned} -L_{2,3} &= \frac{1}{(p-1)} \int_0^t \int_\Omega (\zeta^4)_{xxx} m(h) (h_x^2)^{p/2-1} h_x \, \mathrm{d}x \, \mathrm{d}t \\ &\quad + \frac{2p-1}{p(p-1)} \int_0^t \int_\Omega (\zeta^4)_{xx} m'(h) (h_x^2)^{p/2-1} h_x^2 \, \mathrm{d}x \, \mathrm{d}t \\ &\quad + \frac{1}{p} \int_0^t \int_\Omega (\zeta^4)_x m''(h) (h_x^2)^{p/2-1} h_x^3 \, \mathrm{d}x \, \mathrm{d}t \\ &\quad =: c_4 L_{2,4} + c_5 L_{2,5} + c_6 L_{2,6}. \end{aligned} \tag{5.37}$$

Proof. This is a straightforward calculation. □

Using (5.37) and also collecting the likely terms together, we finally deduce that

$$\begin{aligned} L_2 &= c_\epsilon L_{2,2} + c_5 L_{2,5} + c_6 L_{2,6} - \frac{1}{(p-1)} \int_0^t \int_\Omega (\zeta^4)_{xxxx} (h_x^2)^{p/2-1} M_1(h) \, \mathrm{d}x \, \mathrm{d}t \\ &\quad - \frac{(p-2)}{(p-1)} \int_0^t \int_\Omega (\zeta^4)_{xxx} (h_x^2)^{p/2-2} h_x h_{xx} M_1(h) \, \mathrm{d}x \, \mathrm{d}t \\ &=: c_\epsilon L_{2,2} + c_5 L_{2,5} + c_6 L_{2,6} - c_7 L_{2,7} - c_8 L_{2,8}, \end{aligned} \tag{5.38}$$

where

$$M_1(h) := \int_0^h m(r) \, \mathrm{d}r.$$

Let $s \in (0, A)$; by considering the definitions of $m(h)$ and P_ϵ, we can deduce the following estimates

$$|m(s)| \leqslant K_2 s^{n+\alpha}, \qquad |m'(s)| \leqslant K_3 s^{n+\alpha-1}, \tag{5.39}$$

$$|m''(s)| \leqslant K_4 s^{n+\alpha-2}, \qquad |M_1(s)| \leqslant K_5 s^{n+\alpha+1}. \tag{5.40}$$

Using these estimates we will bound $R_1 + L_2$. But first we state the following result.

Lemma 5.4. One has the following integration by parts relation:

$$L_{2,8} = -\frac{1}{(p-2)} L_{2,7} - \frac{1}{(p-2)} \int_0^t \int_\Omega (\zeta^4)_{xxx} (h_x^2)^{p/2-2} m(h) h_x^3 \, \mathrm{d}x \, \mathrm{d}t$$
$$=: -\frac{1}{(p-2)} \left(L_{2,7} + L_{2,9}\right). \tag{5.41}$$

Proof. This is a straightforward calculation. □

Using these, together with the smoothness of ζ, we obtain that

$$|R_1 + L_2| \leqslant |c_\epsilon| \int_0^t \int_\Omega (\zeta^4)_x P_\epsilon(h) [(h_x^2)^{p/2-1} h_{xx}]_x \, \mathrm{d}x \, \mathrm{d}t$$
$$+ C_2 \int_0^t \int_\Omega \zeta^2 h^{\alpha+n-1} (h_x^2)^{p/2-1} h_x^2 \, \mathrm{d}x \, \mathrm{d}t + C_3 \int_0^t \int_\Omega \zeta^3 h^{\alpha+n-2} (h_x^2)^{p/2-1} |h_x^3| \, \mathrm{d}x \, \mathrm{d}t$$
$$+ C_4 \int_0^t \int_\Omega h^{\alpha+n+1} (h_x^2)^{p/2-1} \, \mathrm{d}x \, \mathrm{d}t + C_5 \int_0^t \int_\Omega \zeta h^{\alpha+n} (h_x^2)^{p/2-1} |h_x| \, \mathrm{d}x \, \mathrm{d}t$$
$$=: |c_\epsilon| L_{2,r,1} + C_2 L_{2,r,2} + C_3 L_{2,r,3} + C_4 L_{2,r,4} + C_5 L_{2,r,5}. \tag{5.42}$$

The first term in (5.42) is uniformly bounded by the dissipation result and uniform boundedness of the terms c_ϵ and $P_\epsilon(h)$. Indeed, by the Hölder's inequality we have that

$$|c_\epsilon| L_{2,r,1} \leqslant |c_\epsilon| \left(\int_0^t \int_\Omega P_\epsilon(h) ([(h_x^2)^{p/2-1} h_{xx}]_x)^2 \, \mathrm{d}x \, \mathrm{d}t \right)^{1/2} \left(\int_0^t \int_\Omega |(\zeta^4)_x|^2 P_\epsilon(h) \, \mathrm{d}x \, \mathrm{d}t \right)^{1/2}$$
$$\leqslant C' |c_\epsilon| \left(\int_0^t \int_\Omega |(\zeta^4)_x|^2 P_\epsilon(h) \, \mathrm{d}x \, \mathrm{d}t \right)^{1/2} \leqslant C'', \tag{5.43}$$

where we have used the dissipation result and the uniform boundedness of c_ϵ, $P_\epsilon(h)$ and $|(\zeta^4)_x|$.

Now, we will show that the last two terms in (5.42) are uniformly bounded. Indeed, by the Hölder's inequality we have that

$$C_4 L_{2,r,4} \leqslant C_4 \left(\int_0^t \int_\Omega (h_x^2)^{p/2} \, \mathrm{d}x \, \mathrm{d}t \right)^{(p-2)/p} \left(\int_0^t \int_\Omega h^{\frac{p}{2}(\alpha+n+1)} \, \mathrm{d}x \, \mathrm{d}t \right)^{2/p} \leqslant C, \tag{5.44}$$

where we have used the energy dissipation and the fact that $\frac{p}{2}(\alpha + n + 1) > 0$.

Similarly, we have that

$$C_5 L_{2,r,5} \leqslant C_5 \left(\int_0^t \int_\Omega (h_x^2)^{p/2} \, \mathrm{d}x \, \mathrm{d}t \right)^{(p-1)/p} \left(\int_0^t \int_\Omega \zeta h^{p(\alpha+n)} \, \mathrm{d}x \, \mathrm{d}t \right)^{1/p} \leqslant C, \tag{5.45}$$

where again we used the energy dissipation and the fact that $p(\alpha + n) > 0$. Collecting these results together and using the fact that ζ is a smooth bounded function and h_ϵ is a smooth positive function(so that it is bounded from below), we deduce from (5.42) that

$$|R_1 + L_2| \leqslant C_1' + c_6 \int_0^t \int_\Omega (h_x^2)^{p/2-1} h_x^2 \, \mathrm{d}x \, \mathrm{d}t + c_7 \int_0^t \int_\Omega (h_x^2)^{p/2-1} |h_x^3| \, \mathrm{d}x \, \mathrm{d}t$$
$$=: C_1' + c_6 L_{2,r,6} + c_7 L_{2,r,7}, \tag{5.46}$$

where C_1', c_6, c_7 are constants. To bound the last two terms in (5.46), we let $\beta = 0$ in (3.3) and we deduce that

$$\int_0^t \int_\Omega [(h_x^2)_x^{p/4}]^2 \, \mathrm{d}x \, \mathrm{d}t < \infty. \tag{5.47}$$

Using this we obtain that

$$\int_0^t ||h_x||^p_{L^\infty}\,\mathrm{d}t = \int_0^t ||(h_x^2)^{p/2}||^2_{L^\infty}\,\mathrm{d}t \leqslant c\int_0^t\int_\Omega [(h_x^2)_x^{p/4}]^2\,\mathrm{d}x\,\mathrm{d}t \leqslant C. \tag{5.48}$$

Now, rewriting the second term in (5.46) we have that

$$|c_6|L_{2,r,6} = |c_6|\int_0^t\int_\Omega |h_x|^p\,\mathrm{d}x\,\mathrm{d}t \leqslant |c_6|\int_0^t\int_\Omega ||h_x||^p_{L^\infty}\,\mathrm{d}x\,\mathrm{d}t \leqslant 2aC. \tag{5.49}$$

Similarly, the last term in (5.46) can be bounded by

$$|c_7|L_{2,r,7} = \int_0^t\int_\Omega |h_x|^{p+1}\,\mathrm{d}x\,\mathrm{d}t \leqslant \int_0^t ||h_x||_{L^\infty}(\int_\Omega |h_x|^p\,\mathrm{d}x)\,\mathrm{d}t \leqslant C\int_0^t ||h_x||_{L^\infty}\,\mathrm{d}t \leqslant C_1. \tag{5.50}$$

Collecting what we have obtained so far we finally deduce that

$$\frac{1}{\alpha}\int_\Omega \zeta^4 G_\epsilon(h(x,t))\,\mathrm{d}x + c_4L_{1,3} + c(\alpha+n,\gamma)L_{1,4} \leqslant \frac{1}{\alpha}\int_{-a}^{a}\zeta^4 G_\epsilon(h_{0\epsilon}(x))\,\mathrm{d}x + \bar{K}, \tag{5.51}$$

where $\bar{K}$ is a constant. Notice that by assumption

$$\frac{1}{\alpha}\int_\Omega \zeta^4 G_\epsilon(h_{0\epsilon}(x))\,\mathrm{d}x \leqslant K', \tag{5.52}$$

where K' is a constant independent of ϵ. Finally using (5.52) in (5.51) and the definition of G_ϵ we deduce that(now we start using the original notation, etc)

$$c_4L_{1,3} + c(\alpha+n,\gamma)L_{1,4} + \frac{\epsilon}{c_*(c_*-1)}\int_\Omega \zeta^4 h_\epsilon^{c_*}(t,x)\,\mathrm{d}x \leqslant -\frac{1}{\alpha(\alpha+1)}\int_\Omega \zeta^4 h_\epsilon^{\alpha+1}(t,x)\,\mathrm{d}x + \tilde{K}$$

$$\Longleftrightarrow$$

$$c_4L_{1,3} + c(\alpha+n,\gamma)L_{1,4} + \tilde{c_*}\tilde{L} \leqslant -c_\alpha L_\alpha + \tilde{K}, \tag{5.53}$$

where $c_* = \alpha + n - \frac{p}{(p-1)} - 1$ and $\tilde{K}$ is a constant independent of ϵ. Note that L_α is uniformly bounded if $\alpha+1>0$ and has a negative coefficient when $\alpha+1<0$; we then deduce that

$$c_4L_{1,3} + c(\alpha+n,\gamma)L_{1,4} + \tilde{c_*}\tilde{L} + |c_\alpha|L_\alpha \leqslant K^*, \tag{5.54}$$

where K^* is a constant independent of ϵ. Since the terms on the left-hand side of (5.54) are nonnegative, we obtain that for $t\in(0,T]$

$$L_\alpha \leqslant C_1^*, \tag{5.55}$$

where C_1^* is a constant independent of ϵ.

Moreover, we also obtain that

$$L_{1,3} \leqslant C_3^*, \tag{5.56}$$

where again C_3^* is a constant independent of ϵ. Choosing α, n and γ so that $c(\alpha+n,\gamma)>0$, we finally deduce that

$$L_{1,4} \leqslant C_4^*, \tag{5.57}$$

where C_4^* is a constant independent of ϵ.

One can modify the calculations for $\alpha \neq 1$ and obtain $\int_\Omega \zeta^4|\ln(h_\epsilon(t,x))| \leqslant C_2^*$ for $t\in(0,T]$. □

Corollary 5.5 (A useful integral estimate). Let $\alpha \neq 0$ and γ be real numbers satisfying

$$\frac{p-1}{p} < \alpha + n < 2,$$

$$\alpha + n + p - 1 < 3\gamma < \alpha + n + p - 1 + \sqrt{(\alpha + n - 2)(p - 1 - p(\alpha + n))}. \tag{5.58}$$

Let $T > 0$ and assume that $h_0, h_{0\epsilon}, P_\epsilon, \zeta$ satisfy the conditions of the proposition and let h_ϵ be the solution of the regularized problem (1.15) with the initial condition

$$h(0, x) = h_{0\epsilon}(x), \qquad x \in \Omega,$$

and with boundary conditions (1.11). Then, there exists a constant C, independent of ϵ, such that

$$\int_0^t \int_\Omega \zeta^4 |(|(h_\epsilon^\gamma)_x|^{(4-q)/q})_x|^q (h_{\epsilon x}^2)^{p/2-1} \,\mathrm{d}x\,\mathrm{d}t \leqslant C, \tag{5.59}$$

where

$$q = \frac{4\gamma - 1 - n - \alpha}{\gamma} \in (1, 2). \tag{5.60}$$

Proof. By proposition 5.1 h_ϵ satisfies the integral estimates. We can rewrite (5.12) as

$$\int_0^t \int_\Omega \zeta^4 (h_\epsilon^\gamma)_x^4 h_\epsilon^{\alpha+n+1-4\gamma} (h_{\epsilon x}^2)^{p/2-1} \,\mathrm{d}x\,\mathrm{d}t \leqslant C_3^*.$$

Note also that we write (5.59) in the given form as constants were worked out in [2]. This simplifies some of the calculations below.

We will choose $q \in (1, 2)$ and $\lambda > 0$ below and we set $p^* = \frac{4-q}{q}$. We apply Hölder's inequality with exponents $p' = 2/q$ and $q' = 2/(2-q)$ to obtain that

$$\begin{aligned}
&\int_0^t \int_\Omega \zeta^4 |(|(h_\epsilon^\gamma)_x|^{p^*})_x|^q (h_{\epsilon x}^2)^{p/2-1} \,\mathrm{d}x\,\mathrm{d}t \\
&\quad\leqslant C \left(\int_0^t \int_\Omega \zeta^4 |(h_\epsilon^\gamma)_{xx}|^2 h_\epsilon^{2\gamma/q} (h_{\epsilon x}^2)^{p/2-1} \,\mathrm{d}x\,\mathrm{d}t \right)^{q/2} \\
&\quad\times \left(\int_0^t \int_\Omega \zeta^4 |(h_\epsilon^\gamma)_x|^4 h_\epsilon^{-2\gamma/(2-q)} (h_{\epsilon x}^2)^{p/2-1} \,\mathrm{d}x\,\mathrm{d}t \right)^{(2-q)/q}.
\end{aligned} \tag{5.61}$$

Hence, to proceed further we need to show that we can choose $\lambda > 0$ and $1 < q < 2$ such that

$$-\lambda \frac{2}{2-q} \geqslant \alpha + n + 1 - 4\gamma, \qquad \frac{2\lambda}{q} \geqslant \alpha + n + 1 - 2\gamma. \tag{5.62}$$

Setting q as in (5.60) and

$$\lambda = \frac{2-q}{2}(4\gamma - 1 - n - \alpha),$$

we see that in order to show that (5.62) is satisfied we need to assume (5.58). This completes the proof. □

Corollary 5.6 (The case $T = +\infty$). Let $\alpha \neq 0$ and $n > 0$ satisfy

$$\frac{p-1}{p} \leqslant \alpha + n \leqslant 2$$

and let γ satisfy

$$\gamma_1 \leqslant \gamma \leqslant \gamma_2, \tag{5.63}$$

where γ_1 and γ_2 are given by (5.10) and (5.11) respectively.

Let h_0, $h_{0\epsilon}$ and P_ϵ satisfy the conditions given in proposition 5.1, and let $\zeta = 1$ in $[-a, a]$. If h_ϵ is the solution of (1.15) with no-flux boundary conditions then there exist constants C_1, C_2 which are independent of ϵ such that

$$\int_0^\infty \int_\Omega h_\epsilon^{\alpha+n-2\gamma+1}(h_\epsilon^\gamma)_{xx}^2 (h_{\epsilon x}^2)^{p/2-1}\,\mathrm{d}x\,\mathrm{d}t \leqslant C_1, \tag{5.64}$$

and if $\frac{p-1}{p} < \alpha + n < 2$, then

$$\int_0^\infty \int_\Omega h_\epsilon^{\alpha+n-3} h_{\epsilon x}^4 (h_{\epsilon x}^2)^{p/2-1}\,\mathrm{d}x\,\mathrm{d}t \leqslant C_2. \tag{5.65}$$

If α, n and γ satisfy (5.60), then there exists a constant C_3, which is independent of ϵ such that

$$\int_0^\infty \int_\Omega |(|(h_\epsilon^\gamma)_x|^{(4-q)/q})_x|^q (h_{\epsilon x}^2)^{p/2-1}\,\mathrm{d}x\,\mathrm{d}t \leqslant C_3, \tag{5.66}$$

where q is defined by (5.60).

Proof. The proof of proposition 5.1 and the proof of corollary 5.5 work if we take $Q = (0, \infty) \times \Omega$ instead of $Q_T = (0, T) \times \Omega$ and take $\zeta \equiv 1$ in $[-a, a]$. □

6. Support properties of solutions

First of all, we prove the following lemma which says that we can pass to the limit $\epsilon \to 0$ in some of the estimates of the previous section.

Lemma 6.1 (Integral estimates for h). Let $\alpha \neq 0$ and $n > 0$ satisfy

$$\frac{p-1}{p} < \alpha + n < 2.$$

Let h_0 satisfy the conditions of proposition 5.1 of the previous section with $\zeta \equiv 1$ in $[-a, a]$, and let h be a solution of the problem (1.9), (1.10) and (1.11). Then

$$\int_0^\infty \int_\Omega h^{\alpha+n-3} h_x^4 (h_x^2)^{p/2-1}\,\mathrm{d}x\,\mathrm{d}t < \infty, \tag{6.1}$$

and for almost every $t > 0$ there exists a constant $C(t) < \infty$ such that

if $h(t, y) = 0$ for some $y \in [-a, a]$, then

$$|h(t, x)| \leqslant C(t)|x - y|^m, \qquad \text{for } x \in [-a, a], \tag{6.2}$$

where

$$m := \frac{p+1}{\alpha + n + p - 1}.$$

Moreover, if α, n and γ satisfy (5.58), then

$$\int_0^\infty \int_\Omega |(|(h^\gamma)_x|^{(4-q)/q})_x|^q (h_x^2)^{p/2-1}\,\mathrm{d}x\,\mathrm{d}t < \infty, \tag{6.3}$$

where q is defined by (5.60).

Proof. Notice that we write (6.3) in the above form because the constants are already worked out in [2]. By (1.27) and (5.13), we deduce that

$$(h_{\epsilon_k}^{\tau_1})_x \to (h^{\tau_1})_x, \qquad \text{weakly in } L_{\mathrm{loc}}^{p+2}(\bar{Q}) \text{ as } \epsilon_k \to 0,$$

where

$$\tau_1 := \frac{\alpha+n+p-1}{p+2}. \tag{6.4}$$

This shows that (6.1) holds.

Claim 1. We have the following convergence result:

$$(h_{\epsilon_k}^{\tau})_x \to (h^{\tau})_x$$

strongly in $L_{\mathrm{loc}}^{p+2}(\bar{Q})$, as $\epsilon_k \to 0$ for any $\tau > \tau_1$, where τ_1 is given by (6.4).

Proof of claim 1. We have

$$|(h_{\epsilon_k}^{\tau})_x - (h^{\tau})_x| \leqslant |h_{\epsilon_k}^{\tau-\tau_1} - h^{\tau-\tau_1}||(h_{\epsilon_k}^{\tau_1})_x| + h^{\tau-\tau_1}|(h_{\epsilon_k}^{\tau_1})_x - (h^{\tau_1})_x|. \tag{6.5}$$

Notice that (1.27) and (5.13) imply that the first term on the right-hand side of (6.5) converges strongly to 0 in $L_{\mathrm{loc}}^{p+2}(\bar{Q})$.

For the second term we fix T and consider the sets

$$Q_r^1 := \{(t,x) \in Q_T : 0 \leqslant h \leqslant r\}$$

and

$$Q_r^2 := \{(t,x) \in Q_T : h > r\},$$

where r is an arbitrary positive number. Since $\tau > \tau_1$ we deduce from (6.1) and (5.13) that

$$\iint_{Q_r^1} h^{(p+2)(\tau-\tau_1)}|(h_{\epsilon_k}^{\tau_1})_x - (h^{\tau_1})_x|\,\mathrm{d}x\,\mathrm{d}t \to 0$$

uniformly in ϵ_k as $r \to 0$. On the other hand, since the derivatives of h_{ϵ_k} converge uniformly on compact subsets of the set Q_r^2 we deduce immediately that

$$\iint_{Q_r^2} h^{(p+2)(\tau-\tau_1)}|(h_{\epsilon_k}^{\tau_1})_x - (h^{\tau_1})_x|\,\mathrm{d}x\,\mathrm{d}t \to 0 \qquad \text{as } \epsilon_k \to 0.$$

Hence, the proof of the claim is complete.

Now note that we can rewrite (6.3) as follows:

$$\iint_Q |(((h^r)_x)^b)_x|^q\,\mathrm{d}x\,\mathrm{d}t \leqslant C < \infty, \tag{6.6}$$

where

$$r := 1 + \frac{(1-\frac{1}{\gamma})(\alpha+n+1)}{p+2-q}, \tag{6.7}$$

$$b := \frac{p+2-q}{q}, \tag{6.8}$$

and q is given by (5.60).

As $\gamma > \tau_1$ and $\frac{2+p-q}{q} < p+2$, by claim 1 we deduce that $(h_{\epsilon_k}^{\gamma})_x \to (h^{\gamma})_x$ strongly in $L_{\mathrm{loc}}^{\frac{p+2-q}{q}}(\bar{Q})$ as $\epsilon_k \to 0$. Hence, we may pass to the limit as $\epsilon_k \to 0$ and deduce that (6.1) holds.

Since $\frac{p-1}{p} < \alpha + n < 2$ there exists γ as assumed in the statement of lemma. Also , by (6.6) and the Sobolev embedding theorem, for almost every $t > 0$, there exists a constant $C_1(t) < \infty$ such that

$$||(h^r)_x|^b(t,x) - |(h^r)_x|^b(t,y)| \leqslant C_1(t)|x-y|^{\frac{q-1}{q}}, \qquad \forall x, \quad y \in [-a,a],$$

where b and r are given by (6.8) and (6.7), respectively. Hence, assuming $h(t,y) = 0$ and integrating this inequality yield (6.2). □

Proof of theorem 2.

(i) Let $h_0(x) > 0$ in some interval $(b,c) \subset (-a,a)$ and let ζ be a smooth nonnegative function with support in (b,c). First assume that $n > 1 + \frac{(p-1)}{p}$. Then, we can find an $\alpha < -1$ satisfying $\frac{p-1}{p} \leqslant \alpha + n < 2$ such that

$$\int_\Omega \zeta^4 h^{\alpha+1}(t,x)\,\mathrm{d}x < \infty, \qquad \text{for } t > 0. \tag{6.9}$$

Since $\alpha < -1$ we obtain the result in this case.
Now, if $n = 1 + \frac{(p-1)}{p}$, then we choose $\alpha = -1$ and we know that

$$\int_\Omega \zeta^4 |\ln(h(t,x))|\,\mathrm{d}x \qquad \text{for } t > 0. \tag{6.10}$$

But this implies the result.

(ii) Since $h_0(x_0) > 0$ and h_0 is continuous there exists a $\delta > 0$ such that $h_0(x) > 0$ for $x \in (x_0 - 2\delta, x_0 + 2\delta)$. Let $\zeta(x)$ be a smooth nonnegative function with support in $(x_0 - 2\delta, x_0 + 2\delta)$ satisfying

$$0 \leqslant \zeta \leqslant 1 \text{ in}[-a,a], \qquad \zeta \equiv 1 \text{ in } [x_0 - \delta, x_0 + \delta]. \tag{6.11}$$

Then $h_0 > 0$ in the support of ζ and hence (6.9) holds for $\alpha \neq -1$. Since $n > \frac{p}{p-1}$ we can choose $\frac{p-1}{p} - n < \alpha < -1$ such that $\frac{p+1}{\alpha+n+p-1}(\alpha+1) \leqslant -1$. By (6.2) we get a contradiction and this proves the result.

(iii) Suppose that the assertion is not true, i.e. $h(t,x_0) = 0$. Since $n \geqslant 1 + \frac{(p-1)}{p} + \frac{p}{(p-1)}$ we can choose $\alpha > -1$ such that $\frac{p-1}{p} < \alpha < -1$ such that $\frac{p-1}{p}(\alpha+1) \leqslant -1$ and this yields a contradiction to (6.9), by the uniform Hölder continuity of $h(t,x)$ in x. □

7. Asymptotic behaviour of nonnegative solutions

In this section first we consider nonnegative smooth solutions and prove asymptotic decay and then we do the same thing for nonnegative weak solutions. Some preliminary results proved in the smooth case will also be useful for the weak solution case. The main idea is to control the rate of decrease of the energy functional in terms of itself, which was the same motivation in [17]. For certain reasons we divide this section into two parts.

Asymptotic behaviour of nonnegative smooth solutions

In this section we will use the energy functional $H_p(f)$, defined in (1.7), to obtain asymptotic behaviour of the nonnegative smooth solutions of the equation (1.9) with initial and boundary conditions (1.10) and (1.11).

Note that

$$\frac{\mathrm{d}H_p}{\mathrm{d}t} = -(p-1)^2 \int_\Omega h^n[(h_x^2)^{p/2-1}h_{xx}]_x^2\,\mathrm{d}x \leqslant 0.$$

Thus, $H_p(h(t,x))$ is a Lyapunov functional for nonnegative smooth solutions of (1.9).

Lemma 7.1 (A zeroth order dissipated energy). Let h be a nonnegative smooth solution of the problem (1.9), (1.10) and (1.11) with $p \geqslant 1$. Then

$$t \to \int_\Omega h^{2-n}\mathrm{d}x, \qquad n \geqslant 2$$

is nonincreasing.

Proof. Indeed by differentiating and integrating by parts we have that, for simplicity we write $h = h(t,x)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega h^{2-n}\,\mathrm{d}x = (2-n)\int_\Omega h^{1-n}h_t\,\mathrm{d}x$$

$$= -(n-2)(n-1)(p-1)\int_\Omega (h_x^2)^{p/2-1}h_{xx}^2\,\mathrm{d}x \leqslant 0,$$

as $n \geqslant 2$ and $p \geqslant 1$. □

Proof of lemma 7.2. Note that an integration by parts yields that

$$pH_p(u) = -(p-1)\int_\Omega u[(u_x^2)^{p/2-1}u_{xx}]\,\mathrm{d}x.$$

Keeping this in mind we also note that for all $x_0, x \in \Omega$ one has by the Cauchy–Schwartz inequality that

$$-\int_{x_0}^x u[(u_x^2)^{p/2-1}u_{xx}]_x \leqslant \left(\int_\Omega \frac{u^2}{\psi(u)}\,\mathrm{d}x\right)^{1/2}\left(\int_\Omega \psi(u)[(u_x^2)^{p/2-1}u_{xx}]_x^2\,\mathrm{d}x\right)^{1/2}. \tag{7.1}$$

On the other hand, by integration by parts we have

$$-\int_{x_0}^x u[(u_x^2)^{p/2-1}u_{xx}]_x\,\mathrm{d}x = -u(x)[(u_x^2)^{p/2-1}u_{xx}](x) + u(x_0)[(u_x^2)^{p/2-1}u_{xx}](x_0)$$

$$+\int_{x_0}^x u_x(u_x^2)^{p/2-1}u_{xx}\,\mathrm{d}x.$$

Thus, if we denote the right-hand side of (7.1) by $A \in [0,\infty)$ we have, by assuming $u_x(x_0) = 0$,

$$A \geqslant -u(x)(u_x^2)^{p/2-1}u_{xx}(x) + \frac{1}{p}(u_x^2)^{p/2}(x).$$

Now integrating this in x over Ω and applying integration by parts to the first integral we finally deduce that

$$A \geqslant CH_p(u),$$

where

$$C := \frac{2p-1}{2ap(p-1)}.$$

This immediately gives the result. □

Proof of proposition 7.3.

(i) By the energy dissipation, we deduce that $t \to ||h_x||_{L^p(\Omega)}$ is nonincreasing. Moreover, since the mass of h_0 is finite there exists a constant $K = K(p)$ such that

$$||h(t,x)||_{L^\infty(\Omega)} \leqslant K||h_{0x}||_{L^p} =: R_0.$$

Using lemma 7.2 with $\psi(h) \equiv h^n$ and $u \equiv h$ we deduce that

$$E[h]D[h] \geqslant C H_p^2[h],$$

where

$$E[h] := \int_\Omega h^{2-n}\,\mathrm{d}x, \tag{7.2}$$

$$D[h] := \int_\Omega h^n[(h_x^2)^{p/2-1}h_{xx}]_x^2\,\mathrm{d}x \tag{7.3}$$

and

$$C_1 := \frac{2p-1}{2pa(p-1)}.$$

On the other hand, since $0 < n < 2$ we have

$$E[h] \leqslant R_0^{2-n}\int_\Omega \mathrm{d}x = 2aR_0^{2-n}.$$

Thus, we have that

$$D[h] \geqslant C[H_p(h)]^2,$$

where

$$C := \frac{\left[\frac{2p-1}{2pa(p-1)}\right]^2}{2aR_0^{2-n}}p^2.$$

(ii) Note that the above proof works and $C = [\frac{2p-1}{2pa(p-1)}]^2\frac{p^2}{2a}$.

(iii) By lemmas 7.1 and 7.2 we deduce that

$$D[h] \geqslant C_2 H_p^2[h],$$

where

$$C_2 := \frac{\left[\frac{2p-1}{2pa(p-1)}\right]^2}{\int_\Omega h_0^{2-n}\,\mathrm{d}x}.$$

This easily gives the result. Note that we can get the proof of (ii) from here as well.

□

By proposition 7.3 we deduce that

$$H_p[h(t,x)] \leqslant [H_p[h_0]^{-1} + Ct]^{-1}, t > 0. \tag{7.4}$$

Hence, from this, $H_p(h)$ becomes sufficiently small after some finite time and so $h(t,x)$ becomes uniformly bounded from below away from 0. From this point on we can then deduce from linearization that there is an exponential decay.

Asymptotic behaviour of nonnegative weak solutions

We now consider a weak solution of the problem (1.9), (1.10) and (1.11). We assume that

$$\int_\Omega h_0(x)^{2-n}\,\mathrm{d}x < \infty, \qquad n > 2, \tag{7.5}$$

$$\int_\Omega |\log(h_0(x))|\,\mathrm{d}x < \infty, \qquad n = 2 \tag{7.6}$$

and also $\int_\Omega h_0(x)\,\mathrm{d}x =: M < \infty$. These assumptions guarantee the existence of a weak solution.

Recall also the entropy H_ϵ defined by

$$H_\epsilon(h) = \int_\Omega G_\epsilon(h)\,\mathrm{d}x,$$

where

$$G''_\epsilon(h) = \frac{1}{P_\epsilon(h)},$$

and $P_\epsilon(h)$ is given by (1.14).

Case $0 < n < 1$ *or* $n > 2$. In this case we have that

$$H_\epsilon[h] = \int_\Omega \frac{\epsilon}{c(c-1)} h^c + c_n h^{2-n}\,\mathrm{d}x, \tag{7.7}$$

where

$$c := 2 - (p + p/(p-1)) \tag{7.8}$$

and

$$c_n := \frac{1}{(n-1)(n-2)}. \tag{7.9}$$

Note also that both the terms appearing in (7.7) are positive and we have, by the dissipation of the entropy H_ϵ, that

$$H_\epsilon[h_{0\epsilon}] \geqslant \begin{cases} \dfrac{1}{c(c-1)} \displaystyle\int_\Omega [\epsilon h^c + h^{2-n}]\,\mathrm{d}x & \text{if } \dfrac{1}{c(c-1)} > c_n, \\ c_n \int_\Omega [\epsilon h^c + h^{2-n}]\,\mathrm{d}x & \text{if } c_n > \dfrac{1}{c(c-1)}. \end{cases}$$

Note also that it is not difficult to show

$$H_\epsilon[h_{0\epsilon}] \leqslant \int_\Omega [C_p \epsilon^{(1-c)\theta} + c_n h_0^{2-n}]\,\mathrm{d}x,$$

where c and c_n are given by (7.8) and (7.9), respectively, and C_p is a finite constant depending on p. This clearly gives a uniform upper bound on $H_\epsilon[h_\epsilon]$ as $\epsilon \searrow 0$.

Proof of proposition 7.4. Given $t > 0$, we first note that it is not difficult to show that $H_\epsilon[h_{0\epsilon}]$ is bounded from above and from below uniformly in ϵ as $\epsilon \searrow 0$. Note also that

$$H_p[h_{0\epsilon}] = H_p[h_0],$$

$$H_\epsilon[h_{0\epsilon}] \to c_n E[h_0],$$

where c_n and E are given by (7.8) and (7.2), respectively. Applying lemma 7.2 with $u \equiv h_\epsilon(t,x)$ and $\psi \equiv P_\epsilon(h_\epsilon(t,x))$ and noticing that

$$\frac{h_\epsilon^2}{P_\epsilon(h_\epsilon)} = \frac{\epsilon}{h_\epsilon^c} + h_\epsilon^{2-n},$$

where c is given by (7.8), we deduce that

$$c_{pn} H_\epsilon[h_{0\epsilon}] D_\epsilon[h_\epsilon] \geqslant K_p H_p^2[h_\epsilon(t,x)], \tag{7.10}$$

where c_{pn}, K_p are positive constants which can be determined explicitly(we leave this for the reader) and

$$D_\epsilon[h_\epsilon] := \int_\Omega P_\epsilon(h_\epsilon(t,x))\{[(h_{\epsilon x}^2)^{p/2-1} h_{\epsilon xx}]_x\}^2 \,\mathrm{d}x. \tag{7.11}$$

On the other hand, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} H_p[h_\epsilon(t,x)] = -(p-1) D_\epsilon[h_\epsilon(t,x)], \tag{7.12}$$

where D_ϵ is given by (7.11). Combination of (7.10) and (7.12) and letting $\epsilon \to 0$ finishes the proof. □

Clearly proposition 7.4 yields that

$$H_p[h(t,x)] \leqslant H_p[h_0](1 + \tau_1 H_p[h_0]t)^{-1}, \qquad \tau_1 > 0. \tag{7.13}$$

This implies that whenever $H_p[h(t,x)]$ is small enough $h(t,x)$ becomes bounded below away from 0, and after this point on we have exponential decay by linearization.

The remaining case for n is left for an upcoming paper [20].

Remark 1. We note that we obtain the same initial decay rates as in the thin-film equation case [17], being t^{-1}. But unlike [17], we do not try to obtain the exponential decay directly.

Remark 2. We note that this approach is also used in [18] for the modified thin-film equation (4.23). But the energy considered there is $I_0 := \int h_{xx}^2 \,\mathrm{d}x$.

8. Final remarks and future research

We have obtained the existence of solutions for a nonlinear degenerate higher order parabolic equation. We use the ideas employed in the analysis of the thin-film case, but we note that most of the results we obtain here do not follow directly from that case. We also compare and comment on the corresponding results for these two cases.

Some further research needs to be done for equation (1.9). We mention a few of them here. First of all, we will extend the entropy dissipation–entropy estimate results to the general p and n values. As in [11], this problem will be analysed by considering a decision problem for polynomial systems. We note also that we have one more parameter in the problem, being p. This makes the analysis more subtle. Currently, we are unable to deduce a regularity theorem similar to theorem 3.1 of [2], which we think is true. The calculations are more tedious in the case at hand so we postpone it for the moment. It is also worth noting that the long time asymptotics for weak solutions in the case $1 \leqslant n \leqslant 2$ will be considered in [20]. The analysis here is harder due to the signs of the terms appearing in the definition of H_ϵ. Moreover, as in [1], one can try to prove results on the 'finite speed of propagation' and the 'waiting-time phenomena' for the case at hand. These problems are being worked on [20].

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