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J. Differential Equations 241 (2007) 279–292
Journal of Differential Equations
www.elsevier.com/locate/jde

Asymptotic equipartition and long time behavior of solutions of a thin-film equation ☆

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Received 16 January 2007; revised 13 June 2007

Available online 20 August 2007

Abstract

We investigate the large-time behavior of classical solutions to the thin-film type equation $u_t = -(uu_{xxx})_x$. It was shown in previous work of Carrillo and Toscani that for non-negative initial data u_0 that belongs to $H^1(\mathbb{R})$ and also has a finite mass and second moment, the strong solutions relax in the $L^1(\mathbb{R})$ norm at an explicit rate to the unique self-similar source type solution with the same mass. The equation itself is gradient flow for an energy functional that controls the $H^1(\mathbb{R})$ norm, and so it is natural to expect that one should also have convergence in this norm. Carrillo and Toscani raised this question, but their methods, using a different Lyapunov functions that arises in the theory of the porous medium equation, do not directly address this since their Lyapunov functional does not involve derivatives of u. Here we show that the solutions do indeed converge in the $H^1(\mathbb{R})$ norm at an explicit, but slow, rate. The key to establishing this convergence is an *asymptotic equipartition of the excess energy*. Roughly speaking, the energy functional whose dissipation drives the evolution through gradient flow consists of two parts: one involving derivatives of u, and one that does not. We show that these must decay at related rates—due to the asymptotic equipartition—and then use the results of Carrillo and Toscani to control the rate for the part that does not depend on derivatives. From this, one gets a rate on the dissipation for all of the excess energy.

MSC: 35B40; 35K25; 35K45; 35K55; 35K65

☆ Work partially supported by U.S. National Science Foundation grant DMS 06-00037.
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0022-0396/$ – see front matter
doi:10.1016/j.jde.2007.07.005

1. Introduction

In this paper we study the asymptotic behavior of classical solutions $u(x,t)$ to the thin-film equation

$$u_t = -(uu_{xxx})_x, \quad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

with

$$u(x,0) = u_0(x) \geqslant 0, \quad x \in \mathbb{R}. \tag{1.2}$$

Equation (1.1) is a special case of the so-called thin-film equation

$$u_t = -\big(u^n u_{xxx}\big)_x, \quad x \in \mathbb{R},\ t > 0, \tag{1.3}$$

for $n > 0$. (1.3) has been derived from a lubrication approximation to model the surface tension dominated motion of viscous liquid films and spreading droplets [1,3,7].

We show that for a fairly general class of initial data, the classical solutions of (1.1) converge toward certain self-similar solutions in the $H^1(\mathbb{R})$ norm. We also estimate the rate of convergence. Previous work [6] had established this convergence in the $L^1(\mathbb{R})$, and while these authors raised the question of $H^1(\mathbb{R})$ convergence, which is natural for the equation, their methods did not address the issue.

In what follows here, we make use of functionals involving higher-order derivatives, and to justify the calculations we make, we must assume that the solutions with which we work are classical. This is in contrast to the work in [6], where strong solutions were treated. The results in [4], where the issue of finite time blow up of solutions for the thin-film type equations has been discussed, show that in general it is possible for classical solutions to break down in finite time. However, the equipartition mechanism that we introduce here provides a new perspective on equilibration, and it may well be possible to establish it for a more general class of solutions.

Equation (1.1) is gradient flow for the energy $E_0(u)$ where

$$E_0(u) = \frac{1}{2}\int_{\mathbb{R}} u_x^2(x)\,\mathrm{d}x,$$

in that

$$u_t = \left(u\left(\frac{\delta E_0(u)}{\delta u}\right)_x\right)_x.$$

This has the consequence that for solutions $u(\cdot,t)$, $E_0(u(\cdot,t))$ is monotone decreasing in time. Also, since the equation is also a conservation law, the total mass

$$M = \int_{\mathbb{R}} u(x,t)\,\mathrm{d}x$$

is conserved.

Moreover, Eq. (1.1) has a scale invariance, and self-similar solutions. If one introduces

$$v(x,t) = \alpha(t)u\big(\alpha(t)x, \beta(t)\big), \tag{1.4}$$

where

$$\alpha(t) = e^t \quad \text{and} \quad \beta(t) = \frac{e^{5t}-1}{5}, \tag{1.5}$$

it becomes

$$v_t = (xv - vv_{xxx})_x, \quad x \in \mathbb{R},\ t > 0, \tag{1.6}$$

$$v(x,0) = v_0(x), \quad x \in \mathbb{R}. \tag{1.7}$$

Equation (1.6) has a unique steady state, found by Smyth and Hill [8]:

$$v^{(\infty)}(x) = \frac{1}{24}\big(C^2 - x^2\big)_+^2, \tag{1.8}$$

where g_+ indicates the positive part of g, and where the constant $C = C(M)$ is determined by the requirement that $\int_{\mathbb{R}} v^{(\infty)}(x)\,\mathrm{d}x = \int_{\mathbb{R}} u_0(x)\,\mathrm{d}x$. Source type solutions of the thin-film equation (1.3) have been studied in [2] and the uniqueness of the steady states of the rescaled equation in the general case is derived from the uniqueness of source type solutions $U(x,t)$ for (1.3), requiring $U_x(x,t) = 0$ at the edge of the support.

Clearly, if a solution $v(x,t)$ of (1.6) approaches $v^{(\infty)}$, the corresponding solution $u(x,t)$ of (1.1) approaches to the corresponding self-similar solution. For the investigation of the rates at which this takes place, it is important that (1.6) also describes a gradient flow: Introduce the energy functional $E(v)$ where

$$E(v) = \frac{1}{2}\int_{\mathbb{R}} \big(v_x^2(x) + x^2 v(x)\big)\,\mathrm{d}x.$$

Then, (1.6) can be rewritten as

$$v_t = \left(v\left(\frac{\delta E(v)}{\delta v}\right)_x\right)_x.$$

Clearly then, for any solution $v(x,t)$ of (1.6), $E(v(\cdot,t))$ is non-increasing in t. Define

$$E\big(v|v^{(\infty)}\big) = \frac{1}{2}\int_{\mathbb{R}} \big|v_x - v_x^{(\infty)}\big|^2\,\mathrm{d}x,$$

where $v^{(\infty)}$ is the stationary solution with the same mass as v. Our goal is to estimate the rate at which $E(v(\cdot,t)|v^{(\infty)})$ converges to zero. Indeed, our analysis will provide the first proof that for general initial conditions this convergence does indeed take place. Note that this convergence is exactly the convergence of $v(\cdot,t)$ to $v^{(\infty)}$ in the $H^1(\mathbb{R})$ norm.

By using the explicit formula for the function $v^{(\infty)}$ and proceeding as in the analysis of second-order degenerate diffusion in [5] we obtain

$$\begin{aligned} E\big(v|v^{(\infty)}\big) &= \frac{1}{2}\int_{\mathbb{R}} \big|v_x - v_x^{(\infty)}\big|^2\,\mathrm{d}x \\ &= E(v) - E\big(v^{(\infty)}\big) - \int_{\{v^{(\infty)}=0\}} \frac{x^2}{2} v\,\mathrm{d}x - \frac{C^2}{6}\int_{\{v^{(\infty)}=0\}} v\,\mathrm{d}x \\ &\leqslant E(v) - E\big(v^{(\infty)}\big), \end{aligned} \tag{1.9}$$

where C is the constant appearing in the definition of $v^{(\infty)}$.

To estimate the rate of convergence in $H^1(\mathbb{R})$, it therefore suffices to prove that the *excess energy*, $E(v) - E(v_\infty)$, decreases to zero. Toward this end we define the *energy dissipation*, $D_E(v)$, given by

$$D_E\big(v(\cdot,t)\big) = -\frac{\mathrm{d}}{\mathrm{d}t}\big(E(v) - E(v_\infty)\big) = -\frac{\mathrm{d}}{\mathrm{d}t}E(v).$$

It follows from (1.6) that $D_E(v)$ is given by

$$D_E(v) := \int_{\mathbb{R}} v(v_{xxx} - x)^2\,\mathrm{d}x. \tag{1.10}$$

Our object here is to prove a lower bound on $D_E(v(\cdot,t))$ in terms of $E(v(\cdot,t)|v^{(\infty)})$ which we shall use to prove that for a broad class of initial data, $\lim_{t\to\infty} E(v(\cdot,t)|v^{(\infty)}) = 0$, and to estimate the rate at which this convergence takes place.

In obtaining our energy dissipation bound, we shall make crucial use of an entropy dissipation bound. Indeed, as shown in [6], Eq. (1.6) can be written as

$$v_t = -\left(\Phi(v)\left[\frac{x^2}{2} + h(v)\right]_{xx}\right)_{xx} + \left(v\left[\frac{x^2}{2} + h(v)\right]_x\right)_x,$$

with $h(v) = \sqrt{6}v^{1/2}$ and $\Phi(v) = vh'(v)$. This leads to the exact form of the entropy associated to the unique steady state $v^{(\infty)}$, given in (1.8), which is

$$H(v) = \int_{\mathbb{R}} \left(\frac{x^2}{2}v(x) + 2\sqrt{\frac{2}{3}}v^{3/2}(x)\right)\mathrm{d}x.$$

One defines the relative entropy by

$$H\big(v|v^{(\infty)}\big) = H(v) - H\big(v^{(\infty)}\big).$$

As one can check, $v^{(\infty)}$ minimizes H for given total mass. This relative entropy had already been investigated earlier in the context of a second-order evolution equation, namely a special case of the porous medium equation for which $v^{(\infty)}$ is also a stationary solution. In fact, the porous medium equation in question is simply the gradient flow for $H(v)$ in the same way that (1.6) is gradient flow for the energy E. A truly remarkable discovery [6] of Carrillo and Toscani is that $H(v(\cdot,t))$ is also monotone decreasing for solutions of (1.6), despite the fact that this equation is gradient flow for the energy and not the entropy. Indeed, Carrillo and Toscani have proved that

$$\frac{\mathrm{d}}{\mathrm{d}t}H\big(v(\cdot,t)|v^{(\infty)}\big) \leqslant -D_H\big(v(\cdot,t)\big)$$

where the *partial entropy dissipation* $D_H(v)$ is given by

$$D_H(v) := \int_{\mathbb{R}} v\left(\frac{x^2}{2} + \sqrt{6}v^{1/2}\right)_x^2 \mathrm{d}x. \tag{1.11}$$

We use the term *partial* since full entropy dissipation is the sum of two positive terms, one of which is D_H. Interestingly enough, D_H is the exact entropy dissipation for $H(v|v^{(\infty)})$ for solutions of a porous medium equation. Moreover, as was already established in the investigation of the porous medium equation, one has the entropy–entropy dissipation bound

$$H\big(v|v^{(\infty)}\big) \leqslant \frac{1}{2}D_H(v). \tag{1.12}$$

This has the consequence that for solution of (1.6)

$$H\big(v(\cdot,t)|v^{(\infty)}\big) \leqslant e^{-2t}H\big(v_0|v^{(\infty)}\big).$$

Unfortunately, D_E is a much more complicated functional than D_H, and we do not possess a bound of this simple type relating $E(v|v^{(\infty)})$ and $D_E(v)$, and it is not even clear at this point that $E(v(\cdot,t)|v^{(\infty)})$ will generally tend to zero at all.

We shall show here that this convergence does occur, and estimate the rate, using an *equipartition theorem* for solutions of (1.6).

To explain, consider any classical solution of (1.6) with finite energy $E(v(\cdot,t))$. Define

$$\alpha(v) = \frac{1}{2}\int_{\mathbb{R}} x^2 v(x)\,\mathrm{d}x \quad \text{and} \quad \beta(v) = \frac{1}{2}\int_{\mathbb{R}} v_x^2(x)\,\mathrm{d}x$$

so that

$$E(v) = \alpha(v) + \beta(v).$$

By a simple computation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha\big(v(\cdot,t)\big) = -2\alpha\big(v(\cdot,t)\big) + 3\beta\big(v(\cdot,t)\big).$$

It follows that

$$2\alpha\big(v^{(\infty)}\big) = 3\beta\big(v^{(\infty)}\big). \tag{1.13}$$

Analogously to the way to we defined relative entropies and energies, we define $\alpha(v|v^{(\infty)})$ and $\beta(v|v^{(\infty)})$ respectively by

$$\alpha\big(v|v^{(\infty)}\big) = \alpha\big(v(\cdot,t)\big) - \alpha\big(v^{(\infty)}\big) \qquad \beta\big(v|v^{(\infty)}\big) = \beta\big(v(\cdot,t)\big) - \beta\big(v^{(\infty)}\big).$$

Then, by (1.13),

$$2\alpha\big(v|v^{(\infty)}\big) - 3\beta\big(v|v^{(\infty)}\big) = 2\alpha(v) - 3\beta(v). \tag{1.14}$$

We shall prove here that for a general class of classical solutions to (1.6),

$$\lim_{t\to\infty}\big(2\alpha\big(v(\cdot,t)\big) - 3\beta\big(v(\cdot,t)\big)\big) = 0. \tag{1.15}$$

We refer to this as *asymptotic equipartition* for solutions of (1.6).

To employ this, we use the entropic convergence result of Carrillo and Toscani to show that furthermore,

$$\lim_{t\to\infty}\big(\alpha\big(v(\cdot,t)\big) - \alpha\big(v^{(\infty)}\big)\big) = 0. \tag{1.16}$$

Combining (1.13), (1.15) and (1.16), we then have that

$$\lim_{t\to\infty}\big(\beta\big(v(\cdot,t)\big) - \beta\big(v^{(\infty)}\big)\big) = 0. \tag{1.17}$$

Combining (1.16) and (1.17), we then have that $\lim_{t\to\infty} E(v(\cdot,t)|v^{(\infty)}) = 0$. In proving all of this we shall keep track of the rate, so that our final result is quantitative.

The key to all of this is an identity expressing $2\alpha(v) - 3\beta(v)$ in terms of a iterated integrals. Suppose that v is a non-negative smooth function, then as we shall see,

$$\begin{aligned} &2\alpha(v) - 3\beta(v) \\ &\quad = 2\int_{-\infty}^{0}\left(\int_{-\infty}^{x} v(z)\big(v_{zzz}(z) - z\big)\,\mathrm{d}z\right)\mathrm{d}x - 2\int_{0}^{\infty}\left(\int_{x}^{\infty} v(z)\big(v_{zzz}(z) - z\big)\,\mathrm{d}z\right)\mathrm{d}x. \end{aligned} \tag{1.18}$$

Comparing this with (1.10), one sees the possibility of estimating $2\alpha(v) - 3\beta(v)$ in terms of something involving $D_E(v)$. In fact, we shall show, under the condition the fourth moment of the initial data is finite, that there is a finite constant K so that

$$\big|2\alpha(v) - 3\beta(v)\big| \leqslant K\big(D_E(v)\big)^{1/2}. \tag{1.19}$$

This shall be enough to deduce (1.16).

The paper is organized as follows. In Section 2, we prove some a priori bounds that we shall use later on. In Section 3, we prove the general version of the iterated integral identity (1.18), and then prove (1.19). In Section 4, we recall the results from [6] needed to prove (1.18), and to estimate the rate of convergence. Finally, in Section 5, we put everything together, and prove the main result, which is

1.1. Theorem. *For all classical solutions $v(x,t)$ of* (1.6) *with smooth, non-negative initial data v_0 such that the mass $M_0(v_0)$, the fourth moment $M_4(v_0)$ and $E(v_0)$ are all finite, there is a finite constant C, depending only on $M_0(v_0)$, $M_4(v_0)$ and $E(v_0)$, such that for all $t > 0$, $v(x,t)$ satisfies*

$$E\big(v(\cdot,t)|v^{(\infty)}\big) \leqslant \frac{C}{\sqrt{t}}. \tag{1.20}$$

As noted above, $E(v(\cdot,t)|v^{(\infty)}) = \frac{1}{2}\int_{\mathbb{R}} |v_x(\cdot,t) - v_x^{(\infty)}|^2\,\mathrm{d}x$, so this explicitly estimates the rate of convergence of $v(\cdot,t)$ to $v^{(\infty)}$ in the $H^1(\mathbb{R})$ norm. The fact that this is only power law decay reflects the fact that our proof only gives a power law decay on the rate of equipartition. If one could show the equipartition to take place exponentially fast, then one would get exponential convergence in Theorem 1.1. But we do not, at present, know whether the equipartition will in general take place exponentially fast. However, it is possible to get a slightly better rate with the present methods: As we shall explain following the proof of Theorem 1.1, one can improve the right-hand side to $C_\epsilon/T^{1-\epsilon}$ for any $\epsilon > 0$.

2. Some a priori bounds

The main result in the section is the moment bound in Lemma 2.3. Its proof requires some simpler bounds which we give in the first two lemmas.

2.1. Lemma. *Any integrable non-negative function v on $\mathbb{R}$ for which $E(v) < \infty$ is bounded. More precisely,*

$$\|v\|_\infty \leqslant M + \left(\int_{\mathbb{R}} v_x^2(x)\,\mathrm{d}x\right)^{1/2} \leqslant M + \big(E(v)\big)^{1/2}$$

where M is the total mass $\int_{\mathbb{R}} v(x)\,\mathrm{d}x$.

Proof. For any x_0, the average value of v over the interval $[x_0, x_0+1]$ is no greater than M since the length of the interval is 1 and

$$\int_{[x_0,x_0+1]} v(x)\,\mathrm{d}x \leqslant M.$$

Hence there is a point $y_0 \in [x_0, x_0+1]$ such that $v(y_0) \leqslant M$. But then

$$v(x_0) = v(y_0) - \int_{x_0}^{y_0} v_x(x)\,\mathrm{d}x \leqslant M + \left(\int_{\mathbb{R}} v_x^2(x)\,\mathrm{d}x\right)^{1/2}. \qquad \square$$

2.2. Lemma. *Any integrable non-negative function v on $\mathbb{R}$ for which $E(v) < \infty$ and $D_E(v) < \infty$ satisfies*

$$\int_{\mathbb{R}} v^{-3/2} v_x^4\,\mathrm{d}x \leqslant 2D_E(v) + 36H(v).$$

Proof. By the Minkowskii inequality in $L^2(\mathbb{R}, v(x)\,\mathrm{d}x)$,

$$\left(\int_{\mathbb{R}} v(v_{xxx})^2 \mathrm{d}x\right)^{1/2} = \left(\int_{\mathbb{R}} v\big((v_{xxx}-x)+x\big)^2 \mathrm{d}x\right)^{1/2} \leqslant \big(D_E(v)\big)^{1/2} + \left(\int_{\mathbb{R}} vx^2\,\mathrm{d}x\right)^{1/2}.$$

Now, for $A > 0$ consider the following inequality:

$$\int_{\mathbb{R}} \big(v^{1/2}v_{xxx} + Av_x\big)^2\,\mathrm{d}x \geqslant 0.$$

Integrating this by parts, we deduce that

$$\int_{\mathbb{R}} v(v_{xxx})^2\,\mathrm{d}x + A^2 \int_{\mathbb{R}} v_x^2\,\mathrm{d}x \geqslant \frac{A}{6}\int_{\mathbb{R}} v^{-3/2}v_x^4\,\mathrm{d}x + 2A\int_{\mathbb{R}} v^{1/2}v_{xx}^2\,\mathrm{d}x. \tag{2.1}$$

Choosing $A = 6$, we deduce the result. □

We shall need certain moment bounds. For future use, let us define for all positive integers k,

$$M_k(v) = \int_{\mathbb{R}} x^k v(x)\,\mathrm{d}x.$$

Since our goal is to show that a solution $v(x,t)$ of (1.6) is tending towards a functions of compact support, namely $v^{(\infty)}$, one would expect to be able to show that $M_k(v(\cdot,t))$ stays bounded uniformly in t for all k. For $k = 2$, this is obvious since $E(v) \geqslant M_2(v)$, and $E(v(\cdot,t))$ is non-increasing. Our analysis shall require a bound in $M_4(v(\cdot,t))$.

2.3. Lemma. *Let $v(x,t)$ be any classical solution of* (1.6) *for which the initial data v_0 is integrable and non-negative, and satisfies $M_4(v_0) < \infty$ and $E(v_0) < \infty$. Then*

$$M_4\big(v(\cdot,t)\big) \leqslant 2D_E(v) + 36E(v).$$

Proof. We first compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}M_4\big(v(\cdot,t)\big) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} x^4 v(x,t)\,\mathrm{d}x = \int_{\mathbb{R}} x^4(xv - vv_{xxx})_x\,\mathrm{d}x \\ &= -4\int_{\mathbb{R}} x^4 v\,\mathrm{d}x - 4\int_{\mathbb{R}} v^2\,\mathrm{d}x + 18\int_{\mathbb{R}} x^2 v_x^2\,\mathrm{d}x. \end{aligned}\tag{2.2}$$

The next to last term on the right can be discarded, but the last term requires further analysis. Using Lemmas 2.1 and 2.2, and the Cauchy–Schwarz inequality, we deduce that

$$\begin{aligned}
\int_{\mathbb{R}} x^2 v_x^2 \, \mathrm{d}x &= \int_{\mathbb{R}} x^2 v^{3/4} v^{-3/4} v_x^2 \, \mathrm{d}x \\
&\leqslant \Big(\int_{\mathbb{R}} x^4 v^{3/2} \, \mathrm{d}x \Big)^{1/2} \Big(\int_{\mathbb{R}} \frac{v_x^4}{v^{3/2}} \, \mathrm{d}x \Big)^{1/2} \\
&\leqslant C_1 \big(C_2 + D_E\big(v(\cdot, t)\big)\big)^{1/2} \big(M_4\big(v(\cdot, t)\big)\big)^{1/2},
\end{aligned} \tag{2.3}$$

where C_1 and C_2 are constants depending only on $E(v_0)$ and the total mass of v_0. Now, define

$$\phi(t) = \big(M_4\big(v(\cdot, t)\big)\big)^{1/2} \quad \text{and} \quad f(t) = 18C_1\big(C_2 + D_E\big(v(\cdot, t)\big)\big)^{1/2}.$$

Then we deduce from (2.2) and (2.3) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \leqslant -4\phi(t) + f(t).$$

Therefore,

$$\phi(t) \leqslant \phi(0) + e^{-4t} \int_0^t e^{4s} f(s) \, \mathrm{d}s.$$

Note that $f(t) \leqslant f_1(t) + f_2(t)$ where

$$f_1(t) = 18C_1(C_2)^{1/2} \quad \text{and} \quad f_2(t) = 18C_1\big(D\big(v(\cdot, t)\big)\big)^{1/2}.$$

Note that f_1 is bounded on $\mathbb{R}_+$, and f_2 is square integrable on $\mathbb{R}_+$:

$$\int_0^\infty f_2^2(t) \, \mathrm{d}t = (18C_1)^2 \int_0^\infty D\big(v(\cdot, t)\big) \, \mathrm{d}t \leqslant (18C_1)^2 H(v_0).$$

But clearly,

$$e^{-4t} \int_0^t e^{4s} f_1(s) \, \mathrm{d}s \leqslant \|f_1\|_\infty e^{-4t} \int_0^t e^{4s} \, \mathrm{d}s \leqslant \frac{\|f_1\|_\infty}{4},$$

and

$$e^{-4t} \int_0^t e^{4s} f_1(s) \, \mathrm{d}s \leqslant e^{-4t} \Big(\frac{e^{8t} - 1}{8} \Big)^{1/2} \|f\|_2 \leqslant \frac{\|f\|_2}{\sqrt{8}}.$$

Hence we have

$$\phi(t) \leqslant \phi(0) + \frac{\|f_1\|_\infty}{4} + \frac{\|f\|_2}{\sqrt{8}}$$

uniformly in t. The right-hand side is a constant depending only on $M_4(v_0)$, $E(v_0)$, and the total mass of v_0, $M_0(v_0)$. $\square$

3. The iterated integral identity

The key to result in this section is an identity for $2\alpha(v) - 3\beta(v)$ in terms of iterated integrals, where the integrand is related the integrand in $D_E(v)$.

3.1. Lemma. *For any smooth function v that vanishes at $\pm\infty$*

$$\begin{aligned}
&2\alpha(v) - 3\beta(v) \\
&\quad = 2\int_{-\infty}^{0}\left(\int_{-\infty}^{x} v(z)\big(v_{zzz}(z) - z\big)\,\mathrm{d}z\right)\mathrm{d}x - 2\int_{0}^{\infty}\left(\int_{x}^{\infty} v(z)\big(v_{zzz}(z) - z\big)\,\mathrm{d}z\right)\mathrm{d}x. \qquad (3.1)
\end{aligned}$$

Proof. We first compute

$$J_1 := \int_{-\infty}^{0}\left(\int_{-\infty}^{x} v(z)v_{zzz}(z)\,\mathrm{d}z\right)\mathrm{d}x - \int_{0}^{\infty}\left(\int_{x}^{\infty} v(z)v_{zzz}(z)\,\mathrm{d}z\right)\mathrm{d}x.$$

Integrating by parts in the inner integrals, we obtain respectively that

$$\begin{aligned}
\int_{-\infty}^{x} v(z)v_{zzz}(z)\,\mathrm{d}z &= v(x)v_{xx}(x) - \int_{-\infty}^{x} v_z(z)v_{zz}(z)\,\mathrm{d}z \\
&= v(x)v_{xx}(x) - \int_{-\infty}^{x} \big(v_z(z)/2\big)^2_z\,\mathrm{d}z \\
&= v(x)v_{xx}(x) - \big(v_x(x)/2\big)^2, \qquad (3.2) \\
\int_{x}^{\infty} v(z)v_{zzz}(z)\,\mathrm{d}z &= -v(x)v_{xx}(x) - \int_{x}^{\infty} v_z(z)v_{zz}(z)\,\mathrm{d}z \\
&= -v(x)v_{xx}(x) - \int_{x}^{\infty} \big(v_z(z)/2\big)^2_z\,\mathrm{d}z \\
&= -v(x)v_{xx}(x) + \big(v_x(x)/2\big)^2. \qquad (3.3)
\end{aligned}$$

Therefore, integrating by parts once more,

$$J_1 = -\frac{3}{2}\int_{-\infty}^{+\infty} v_x^2(x)\,\mathrm{d}x.$$

Next, we compute

$$J_2 := -\int_{-\infty}^{0}\left(\int_{-\infty}^{x} v(z)z\,\mathrm{d}z\right)\mathrm{d}x + \int_{0}^{\infty}\left(\int_{x}^{\infty} v(z)z\,\mathrm{d}z\right)\mathrm{d}x.$$

Changing the order of integration we easily find

$$J_2 = \int_{-\infty}^{+\infty} x^2 v(x)\,\mathrm{d}x.$$

Combining the pieces, the identity is proved. □

3.2. Lemma. *For any smooth, non-negative v such that $M_0(v)$, $M_4(v)$ and $E(v)$ are all finite, there is a finite constant K, depending only on $M_0(v)$, $M_4(v)$ and $E(v)$, such that*

$$\left|2\alpha(v) - 3\beta(v)\right| \leqslant K\big(D_E(v)\big)^{1/2}. \tag{3.4}$$

Proof. We first apply our uniform bound on the $M_4(v(\cdot,t))$ coming from Lemma 2.3 to conclude respectively that

$$\int_{-\infty}^{x} v\,\mathrm{d}t \leqslant \int_{-\infty}^{x}\left(\frac{t}{x}\right)^4 v(t)\,\mathrm{d}t \leqslant \frac{1}{x^4}\int_{-\infty}^{0} t^4 v(t)\,\mathrm{d}t \leqslant \min\left\{\frac{C_2}{x^4}, M\right\} \leqslant \frac{C_3}{1+x^4}, \tag{3.5}$$

$$\int_{x}^{\infty} v\,\mathrm{d}t \leqslant \int_{x}^{\infty}\left(\frac{t}{x}\right)^4 v(t)\,\mathrm{d}t \leqslant \frac{1}{x^4}\int_{0}^{\infty} t^4 v(t)\,\mathrm{d}t \leqslant \min\left\{\frac{C_2^*}{x^4}, M\right\} \leqslant \frac{C_3^*}{1+x^4}. \tag{3.6}$$

Hence, by Lemma 3.1 and the Cauchy–Schwarz inequality,

$$\begin{aligned}
\left|\int_{-\infty}^{0}\left(\int_{-\infty}^{x} v(v_{xxx}-t)\,\mathrm{d}t\right)\mathrm{d}x\right| &\leqslant \int_{-\infty}^{0}\left[\left(\int_{-\infty}^{x} v\,\mathrm{d}t\right)^{1/2}\left(\int_{-\infty}^{x} v(v_{xxx}-t)^2\,\mathrm{d}t\right)^{1/2}\right]\mathrm{d}x \\
&\leqslant \int_{-\infty}^{0}\left[\left(\int_{-\infty}^{x} \frac{C_3}{1+t^4}\,\mathrm{d}t\right)^{1/2}\left(\int_{-\infty}^{0} v(v_{xxx}-t)^2\,\mathrm{d}t\right)^{1/2}\right]\mathrm{d}x \\
&\leqslant \left[\int_{-\infty}^{0}\left(\int_{-\infty}^{x} \frac{C_3}{1+t^4}\,\mathrm{d}t\right)^{1/2}\mathrm{d}x\right]\big(D_E(v)\big)^{1/2},
\end{aligned} \tag{3.7}$$

$$\left|\int_0^\infty\left(\int_x^\infty v(v_{xxx}-t)\,\mathrm{d}t\right)\mathrm{d}x\right| \leqslant \int_0^\infty\left[\left(\int_x^\infty v\,\mathrm{d}t\right)^{1/2}\left(\int_x^\infty v(v_{xxx}-t)^2\,\mathrm{d}t\right)^{1/2}\right]\mathrm{d}x$$
$$\leqslant \int_0^\infty\left[\left(\int_x^\infty \frac{C_3^*}{1+t^4}\,\mathrm{d}t\right)^{1/2}\left(\int_{-\infty}^0 v(v_{xxx}-t)^2\,\mathrm{d}t\right)^{1/2}\right]\mathrm{d}x$$
$$\leqslant \left[\int_0^\infty\left(\int_x^\infty \frac{C_3^*}{1+t^4}\,\mathrm{d}t\right)^{1/2}\mathrm{d}x\right]\bigl(D_E(v)\bigr)^{1/2}. \tag{3.8}$$

The remaining iterated integrals in (3.7) and (3.8) are clearly finite. The result then follows by the triangle inequality. □

4. Asymptotic equipartition

4.1. Lemma. *Under the same conditions imposed in Lemma* 3.2*, with the same constant K, we have that for all $T > 0$,*

$$\inf_{T\leqslant t\leqslant 2T}\bigl\{\bigl|2\alpha\bigl(v(\cdot,t)\bigr)-3\beta\bigl(v(\cdot,t)\bigr)\bigr|\bigr\} \leqslant \frac{KE^{1/2}(v_0)}{\sqrt{T}}. \tag{4.1}$$

Proof. For any $T > 0$, we have from Lemma 3.2 that

$$\frac{1}{T}\int_T^{2T}\bigl|2\alpha\bigl(v(\cdot,t)\bigr)-3\beta\bigl(v(\cdot,t)\bigr)\bigr|\,\mathrm{d}t \leqslant \frac{1}{T}\int_T^{2T} KD_E^{1/2}\bigl(v(\cdot,t)\bigr)\,\mathrm{d}t.$$

By Cauchy–Schwarz inequality,

$$\int_T^{2T} D_E^{1/2}\bigl(v(\cdot,t)\bigr)\,\mathrm{d}t \leqslant \sqrt{T}\left(\int_T^{2T} D_E\bigl(v(\cdot,t)\bigr)\,\mathrm{d}t\right)^{1/2}$$
$$\leqslant \sqrt{T}\left(\int_0^{\infty} D_E\bigl(v(\cdot,t)\bigr)\,\mathrm{d}t\right)^{1/2}$$
$$\leqslant \sqrt{T}\bigl(E(v_0)\bigr)^{1/2}. \tag{4.2}$$

Finally, $\inf_{T\leqslant t\leqslant 2T}\{|2\alpha(v(\cdot,t))-3\beta(v(\cdot,t))|\}$ is no greater than the average $|2\alpha(v(\cdot,t)) - 3\beta(v(\cdot,t))|$ over the interval $[T,2T]$. □

4.2. Lemma. *Under the same conditions imposed in Lemma* 3.2*, for all $t > 0$,*

$$\bigl|\beta\bigl(v(\cdot,t)\bigr)-\beta\bigl(v^{(\infty)}\bigr)\bigr| \leqslant Ce^{-t/2}. \tag{4.3}$$

Proof. As Carrillo and Toscani [6] have shown, there is a constant C so that

$$\big\|v(\cdot,t)-v^{(\infty)}\big\|_{L^1(\mathbb{R})}\leqslant CH\big(v(\cdot,t)|v^{(\infty)}\big),$$

and thus there is another constant C so that

$$\big\|v(\cdot,t)-v^{(\infty)}\big\|_{L^1(\mathbb{R})}\leqslant Ce^{-t}.$$

Now, for any $R>0$,

$$\begin{aligned}
&\big|\beta\big(v(\cdot,t)\big)-\beta\big(v^{(\infty)}\big)\big| \\
&\quad\leqslant \int\limits_{|x|<R} x^2\big|v(x,t)-v^{(\infty)}(x)\big|\,\mathrm{d}x+\int\limits_{|x|>R} x^2\big|v(x,t)-v^{(\infty)}(x)\big|\,\mathrm{d}x \\
&\quad\leqslant R^2\int\limits_{|x|<R} \big|v(x,t)-v^{(\infty)}(x)\big|\,\mathrm{d}x+\frac{1}{R^2}\int\limits_{|x|>R} x^4\big(\big|v(x,t)\big|+\big|v^{(\infty)}(x)\big|\big)\,\mathrm{d}x \\
&\quad\leqslant c\left(R^2s^{-t}+\frac{1}{R^2}\right).
\end{aligned}\tag{4.4}$$

The optimal choice of R^2 is $R^2=e^{t/2}$, which yields the result. □

Proof of Theorem 1.1. Since

$$\begin{aligned}
2\big(\alpha\big(v|v^{(\infty)}\big)+\beta\big(v|v^{(\infty)}\big)\big)&=5\beta\big(v|v^{(\infty)}\big)+\big(2\alpha\big(v|v^{(\infty)}\big)-3\beta\big(v|v^{(\infty)}\big)\big) \\
&\leqslant 5\beta\big(v|v^{(\infty)}\big)+\big|2\alpha\big(v|v^{(\infty)}\big)-3\beta\big(v|v^{(\infty)}\big)\big|,
\end{aligned}$$

it follows from the last two lemmas and (1.14) that for some t in the interval $[T,2T]$,

$$\begin{aligned}
2E\big(v|v^{(\infty)}\big)&\leqslant\big\{\alpha\big(v|v^{(\infty)}\big)+\beta\big(v|v^{(\infty)}\big)\big\} \\
&\leqslant 5Ce^{-t/2}+\frac{KE^{1/2}(v_0)}{\sqrt{T}}.
\end{aligned}\tag{4.5}$$

Since $E(v|v^{(\infty)})$ is monotone decreasing, this implies that

$$E\big(v(\cdot,2T)|v^{(\infty)}\big)\leqslant Ce^{-T/2}+\frac{KE^{1/2}(v_0)}{\sqrt{T}}.$$

Now, possessing this bound, we can go back and use it to improve (4.2). Doing so will give a bound in terms of $T^{-3/4}$. Returning to (4.2) again and using this yields a bound in terms of $T^{-7/8}$. Continuing, we can obtain a bound in terms of $T^{\epsilon-1}$ for any $\epsilon>0$. □

Acknowledgment

The authors thank the referee for the valuable suggestions to improve the presentation of the original manuscript.

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