PROPAGATION OF L^1 AND L^∞ MAXWELLIAN WEIGHTED BOUNDS FOR DERIVATIVES OF SOLUTIONS TO THE HOMOGENEOUS ELASTIC BOLTZMANN EQUATION

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ABSTRACT. We consider the n-dimensional space homogeneous Boltzmann equation for elastic collisions for variable hard potentials with Grad (angular) cutoff. We prove sharp moment inequalities, the propagation of L^1-Maxwellian weighted estimates, and consequently, the propagation L^∞-Maxwellian weighted estimates to all derivatives of the initial value problem associated to the afore mentioned equation.

More specifically, we extend to all derivatives of the initial value problem associated to this class of Boltzmann equations corresponding sharp moment (Povzner) inequalities and time propagation of L^1-Maxwellian weighted estimates as originally developed Bobylev [2] in the case of hard spheres in 3 dimensions. To achieve this goal we implement the program presented in Bobylev-Gamba-Panferov [3], which includes a full analysis of the moments by means of sharp moment inequalities and the control of L^1-exponential bounds, in the case of stationary states for different inelastic Boltzmann related problems with 'heating' sources where high energy tail decay rates depend on the inelasticity coefficient and the type of 'heating' source. More recently, this work was extended to variable hard potentials with angular cutoff by Gamba-Panferov-Villani [5] in the elastic case collision case where the L^1-Maxwellian weighted norm was shown to propagate if initial states have such property. In addition, we also extend to all derivatives the propagation of L^∞-Maxwellian weighted estimates, proven in [5], to solutions of the initial value problem to the Boltzmann equations for elastic collisions for variable hard potentials with Grad (angular) cutoff.

RÉSUMÉ: Nous considerons l'équation de Boltzmann homogène en espace, en dimension n en vitesse, avec potentiels durs variables et troncature angulaire de Grad. Nous montrons des estimations fines sur les moments et sur la propagation en temps de normes L^1 à poids maxwelliens. Ceci nous permet de déduire la propagation en temps de normes L^∞ à poids maxwelliens pour toutes les dérivées de la solution du problème de Cauchy.
En particulier, Nous étendons à toutes les dérivées les estimations fines de moments (inégalités de Pozner) et les résultats de propagation en temps de normes L^1 à poids maxwelliens obtenus précédement par Bobylev [2] dans le cas des spères dures en dimension 3. Ce résultat utilise des techniques développées par Bobylev-Gamba-Panferov [3] pour étudier, dans une classe plus générale de sections angulaires et de contrôle L^1 à poids maxwelliens, les états stationnaires de l'équation de Boltzmann avec interactions inélastiques et chauffage. Ceci correspond à des situations où le taux de décroissance des queues d'énergie dépend du coefficient d'inélasticité et du type de chauffage. Nous généralisons aux cas des potentiels durs variables avec cutoff angulaire, le résultat de Gamba-Panferov-Villani [5] pour les collisions élastiques qui montre la propagation en temps de normes L^1 à poids maxwelliens. Enfin, toujours dans le cas élastique, nous généralisons les résultats de propagation de [5] à toutes les dérivees avec des normes L^∞ à poids maxwelliens.

1. INTRODUCTION

The study of propagation of L^1-Maxwellian weighted estimates to solutions of the initial value for the n-dimensional space homogeneous Boltzmann equation for elastic collisions for variable hard potentials with Grad (angular) cutoff entices the study of summability properties of a corresponding series of the solution moments to all orders. This problem was addressed for the first time by Bobylev in [2] in the case of 3 dimension for the hard sphere problem, i.e. for constant angular cross section in the collision kernel.

Previously, the behavior of time propagating properties for the moments of the solution to the initial value problem for the elastic Boltzmann transport equation, in the space homogeneous regime for hard spheres and variable hard potentials and integrable angular cross sections (Grad cutoff assumption) had been extensively studied, but not their summability properties. In fact, the study of Povzner estimates and propagation of moments of the solution to the of variable hard potentials with Grad cutoff assumption, was progressively understood in the work of Desvillettes [4] and Wennberg [14], where the Povzner estimates, a crucial tool for the moments control in the case of variable hard spheres with the Grad cutoff assumption, where based on pointwise estimates on the difference between pre and post-collisional velocities of convex, isotropic weights functions of the velocity in oreder to control their integral on the $n-1$ dimensional sphere, and consequently, and not sharp enough to obtain summability of moments.

A significant leap was developed by Bobylev [2] where the first proof of summability properties of moments was established, in the case of hard spheres in 3 dimensions, showing that L^1-Maxwellian weighted estimates propagates if the initial data is in within that class. Among several new crucial techniques that were developed in that fundamental paper, the is a significant improvement of the Pozvner estimates based on the averaging (integrals) on the $n-1$ dimensional sphere of convex, isotropic weights functions of the velocity, for the case of variable hard potentials with the Grad cutoff assumption. As a consequence it is possible to established that, in the case of three dimensions velocity, for hard spheres, the moments of the gain operator will decay proportional to the order of the moment with respect to the loss term uniformly in time, by means of infinity evolution inequalities in terms of moments. That key estimate yields summability of a moments series, uniformly in time. Later, Bobylev, Gamba and Panferov [3], establish the sharpest version of the Povzner inequality for elastic or inelastic collisions, using the approach of [2], by a somehow reduced argument, under the conditions that both the convex, isotropic weights functions of the velocity and the angular part of the angular cross section are non-decreasing. The two main ideas are to pass to the center of mass relative velocity variables and to use the angular integration in to obtain more precise constants in the corresponding inequalities. The summability property, which in the work of Bobylev [3], was done only for hard spheres in three dimensions whose the angular cross section is constant, was extended, in [2], to the case of bounded angular section. In addition, the problem of stationary states to Boltzmann equations for inelastic interaction problems with 'heating' sources, such as random heat bath, shear flow or self-similar transformed problems, was addressed in [3], where L^1-exponential bounds with decay rates depending of the inelasticity coefficient and the the type of 'heating' source were shown as well. In these cases the authors showed L^1-exponential weighted decay bounds with slower decay than Maxwellian (i.e. $s < 2$).

In an interesting application of these moments summability formulas, estimates and techniques, Mouhot [9], was able to establish (for the elastic case) a result that proves the instantaneous 'generation', of L^1-exponential bounds uniformly in time, with only $L^1_2 \cap \mathrm{L}^2$ initial data, where the exponential rate is half of the variable hard sphere exponent, under the same assumptions on the angular function as in [3].

However, still for the elastic case, of variable hard potentials and Grad cutoff assumption, neither [2] nor [3], addressed the propagation of L^1-Maxwellian weighted bounds, uniform in time to solutions of the corresponding initial value problem in n-dimensions with more realistic intramolecular potentials. In a recent manuscript by Gamba, Panferov and Villani [5], showed the L^1-Maxwellian weighted propagation estimates and the provided a proof to the open problem of propagation of L^∞-Maxwellian weighted bounds, uniformly in time. The Grad cutoff assumption was still assumed (integrability of the angular part of the collision kernel) without the boundedness condition, but a growth rate assumption on the angular singularity, depending only on the velocity space dimension, that still keeps integrability. The propagation of L^∞-Maxwellian weighted bounds combines the classical Carleman representation of the gain operator with the L^1-Maxwellian bounds.

More specifically, the behavior for large velocities is commonly called "high energy tails". Under precise conditions, described in [2] and [5]), it is known that for a solution this asymptotic behavior is comparable in some way to $\exp(-r|\xi|^s)$ with r, s positive numbers. In the case of elastic interactions, it is known that $s = 2$, provided the initial state also has that behavior, i.e. it decays as a Maxwellian. This is a revealing fact that says that a solution of the elastic initial value problem for the n-dimensional Boltzmann equation, with variable hard potential kernels and singular integrable angular cross section, decays like a Maxwellian for all times as long as the initial state does it as well.

In this present manuscript, we extend the results of [5] to show both propagation of of L^1-Maxwellian and L^∞-Maxwellian weighted estimates to all derivatives of the solution to the initial value problem to the space homogeneous Boltzmann equations for elastic collisions for variable hard potentials with an integrable angular singularity condition as in [5].

We first note that sharp Povzner inequalities ([2]-[3]-[5]) are, indeed, the main tool for the study of the solution's moments for the variable hard potential models. They control the decay of the moments of the gain collision operator with respect to the moments of the loss collision operator. This technique yields a control of the time derivative of any higher order moment using the lower order ones. In particular, one uses the Boltzmann equation, in order to build an infinite system of sharp Povzner inequalities for each moments which can be used, by arguing inductively, to control each moment uniformly in time. As a result one obtains L^1-Maxwellian weighted estimates and the corresponding L^∞-Maxwellian weighted estimates in the elastic interaction models in n-dimensions and for variable hard potential collision kernels with an integrable angular singularity condition depending on the dimension n.

Here we show that these results extend to the study of propagation of L^1-Maxwellian and L^∞-Maxwellian weighted estimates to any high order derivatives of the solution to the n-dimensional elastic Boltzmann equation for variable hard potentials. In particular, these bounds imply that if the initial derivatives of the solution are controlled pointwise by the derivatives of a Maxwellian then this control propagates for all times.

The paper is organized as follows. After this introduction, section 2 presents the problem and the main Theorem 1. Section 3 focus in finding sharp Povzner inequalities for the solution's derivatives. All the computations regarding the derivatives of the collision operator and the action of the differential collision operator on test functions are presented in Lemmas 1, 2 and 3. Lemmas 4, 5, 6 and 7 are devoted to provide a suitable expression ready to use for the construction of the mentioned system of inequalities on the derivative's moments. In Lemma 8 such a system of inequalities is presented.

Then in section 4, all previous results used to obtain information for the solution's derivatives in the elastic case. Theorem 2 proves the control of moment's growth, and Theorem 3 uses Lemma 8 to obtain a global in time bound for the derivative's moments in the elastic

case yielding the L^1-Maxwellian bounds to derivatives of any order. Finally in section 5, we show that uniform bounds on the moments of these derivatives lead to a pointwise estimate. This is possible after using an $L^\infty - L^1$ Maxwellian weighted control on the gain collision operator as shown in [5] (see Theorems 5 and 6 in the Appendix A). The Boltzmann equation and this control are sufficient to find a time uniform pointwise control by Maxwellians for the solution's derivatives of any order. Calculations of this fact are performed in Theorem 1, where the L^∞-Maxwellian bound is shown.

2. PRELIMINARIES AND MAIN RESULT

This section presents the assumptions and notation used along the paper. Assume that the function $0 \le f(\xi, t)$ with $(\xi, t) \in \mathbb{R}^n \times \mathbb{R}^+$ solves the homogeneous Boltzmann problem

$$(1) \qquad \frac{\partial f}{\partial t} = Q(f, f) \text{ on } (0, T) \times \mathbb{R}^n, \quad f(\xi, 0) = f_0,$$

where $Q(f, f) := Q^+(f, f) - Q^-(f, f)$ is the standard Boltzmann collision operator for variable hard potentials. It is defined for any two measurable functions f and g by the formula

$$(2) \qquad Q(f, g) = \int_{\mathbb{R}^n} \int_{S^{n-1}} (f'g'_* - fg_*) B(\xi - \xi_*, \sigma) d\sigma d\xi_*.$$

In particular,

$$(3) \qquad Q^+(f, g) = \int_{\mathbb{R}^n} \int_{S^{n-1}} f'g'_* B(\xi - \xi_*, \sigma) d\sigma d\xi_*$$

and,

$$(4) \qquad Q^-(f, g) = \int_{\mathbb{R}^n} \int_{S^{n-1}} fg_* B(\xi - \xi_*, \sigma) d\sigma d\xi_*.$$

The classical notation $'f$, $'f_*$, f' and f'_* is adopted to imply that the distributional function f has the pre-collision velocity arguments $'\xi$, $'\xi_*$ or the post collision velocity arguments ξ', ξ'_*. Recall that the dependence of post and pre-collision velocities is given by the formulas

$$\xi' = \xi + \frac{1}{2}(|u|\sigma - u)), \quad \xi'_* = \xi_* - \frac{1}{2}(|u|\sigma - u))$$

where $\sigma \in S^{n-1}$ is a vector describing the geometry of the collisions, see for example [13], for a complete description.

Intramolecular potentials are modeled by the collision kernel as a non negative function given by

$$B(\xi - \xi_*, \sigma) = |\xi - \xi_*|^\alpha h(\hat{u} \cdot \sigma) \qquad \text{and} \quad \hat{u} = \frac{\xi - \xi_*}{|\xi - \xi_*|}$$

with $\alpha \in (0, 1]$ and $\hat{u}$ is the renormalized relative velocity. It is assumed that the angular cross section $h(\cdot)$ has the following properties

    (i) $h(z) \ge 0$ is nonnegative on $(-1, 1)$ such that

$$h(z) + h(-z) \text{ is nondecreasing on } (0, 1)$$

    (ii)

$$0 \le h(z)(1 - z^2)^{\mu/2} \le C \text{ for } z \in (-1, 1)$$

where $\mu < n - 1$ and $C > 0$ constant.

Note that assumption *(i)* implies that $h(\hat{u} \cdot \sigma) \in L^1(S^{n-1})$. For convenience we normalized its mass as follows

$$\int_{S^{n-1}} h(\hat{u} \cdot \sigma) d\sigma = \omega_{n-2} \int_{-1}^{1} h(z)(1-z^2)^{\frac{n-3}{2}} dz = 1$$

where ω_{n-2} is measure of the $n-2$ dimensional sphere.
In the case of three dimensional collisional models for variable hard potential, condition *(ii)* simplifies to

$$(5) \qquad \int_{-1}^{1} h(z) dz = 1/2\pi.$$

usually referred as the *Grad cutoff assumption*. With these assumptions the collision model kernel used still falls in the category of variable hard potential with angular cut-off. The reader may go to [5] for a recent, complete discussion on the behavior of the moments of the solution for variable hard potential with cut-off in any dimension.

The standard integrability conditions on the initial datum f_0 are assumed to be

$$\int_{\mathbb{R}^n} f_0 d\xi = 1, \quad \int_{\mathbb{R}^n} f_0 \xi d\xi = 0, \quad \int_{\mathbb{R}^n} f_0 |\xi|^2 d\xi < \infty.$$

In other words, f_0 has finite mass, which is normalized to one for convenience, and finite energy. These conditions can be addressed in a compact manner using the weighted Lebesgue space L_k^p with $p \geq 1$ and $k \in \mathbb{R}$, defined by the norm

$$\|f\|_{L_k^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f|^p (1 + |\xi|^2)^{pk/2} d\xi \right)^{1/p}.$$

In particular the initial datum can be referred as $f_0 \in L_2^1$.
Following these ideas, the weighted Sobolev spaces $W_k^{s,p}$, with $s \in N$, are used to work with the weak derivatives of f. These spaces are defined by the norm

$$\|f\|_{W_k^{s,p}(\mathbb{R}^n)} = \left(\sum_{|\nu| \leq s} \|\partial^\nu f\|_{L_k^p}^p \right)^{1/p}$$

where the symbol ∂^η is understood as $\partial^\eta = \partial_{\xi_1}^{\eta_1} \partial_{\xi_2}^{\eta_2} \cdots \partial_{\xi_n}^{\eta_n}$ for a multi-index η of n dimensions. The usual notation is used for the Hilbert space $H_k^s \equiv W_k^{s,2}$.

Throughout the paper, the order of the multi-index is defined as $|\eta| = \sum_{1 \leq i \leq n} \eta_i$, in addition, the comparison between multi-indexes is denoted as $\nu < \eta$ or $\nu \leq \eta$. This is understood as $\nu_i \leq \eta_i$ for all $1 \leq i \leq n$ and $|\nu| < |\eta|$ or $|\nu| \leq |\eta|$ respectively.
Regarding the regularity of the initial datum, it is required that $f_0 \in W_2^{s,1}$ for some $s \geq 1$ to be chosen afterwards. The additional assumption $f_0 \in H^s$ is required for the final result.

Definition 1. *Define for any sufficiently regular function f, multi-index η and $p > 0$ the moment of order p for the η derivative of f as the time dependent function*

$$(6) \qquad \delta^\eta m_p(t) \equiv \int_{\mathbb{R}^n} |\partial^\eta f| |\xi|^{2p} d\xi.$$

This definition tries to generalize the classical definition for the moments, $m_p(t) \equiv \int_{\mathbb{R}^n} f |\xi|^{2p} d\xi$, however an absolute value is imposed in $\partial^\eta f$ since this function in general does not have sign. Observe that the condition $f_0 \in W_2^{s,1}$ is equivalent to say that $\delta^\nu m_0(0)$ and $\delta^\nu m_1(0)$ are bounded for $|\nu| \leq s$.

The next definition is related to the exponential tail concept introduced in [2] in the study of the solution's moments of the elastic homogenous Boltzmann equation, and later in [3] in the study of large velocity tails for solutions of the inelastic homogeneous Boltzmann equation with source terms.

Definition 2. *The function f has an L^1 exponential (weighted) tail of order $s > 0$ in $[0, T]$ if*

(7) $$\bar{r}_s = \sup_{r>0} \left\{ r : \sup_{0 \leq t \leq T} \| f \, \exp(r|\xi|^s) \|_{L^1(\mathbb{R}^n)} < +\infty \right\}$$

is positive.
In particular, for $s = 2$ we simply say that f has an L^1-Maxwellian (weighted) bound.

This definition is equivalent to that one used in [2] and [3] for the solution of the homogeneous Boltzmann equation. Observe that the definition does not requires non negativity in the function f. This is important since the main purpose of this paper is to study the derivatives of the solution of problem (1), functions that do not have sign.

We point out that Bobylev proved in [2] the propagation L^1-Maxwellian tails for the hard sphere problem in three dimentsions. That is, he showed the existence of L^1 exponential tail of order 2 for the solution of the homogeneous Boltzmann equation provided the initial data has L^1 exponential tail of order 2 the case is special in the sense that the angular cross section function $h(z)$ in (2) is constant. Recently, this result was extended was in [5] under the conditions (i-ii) for the angular cross section, to further show that the tail behavior is in fact 'pointwise' for all times if initially so. This fact motivates the following natural definition.

Definition 3. *The function f has an L^∞ exponential (weighted) tail of order $s > 0$ in $[0, T]$ if*

(8) $$\bar{r}_s = \sup_{r>0} \left\{ r : \sup_{0 \leq t \leq T} \| f \, \exp(r|\xi|^s) \|_{L^\infty(\mathbb{R}^n)} \right\} < \infty$$

is positive.
In particular, for $s = 2$ we simply say that f has an L^∞-Maxwellian (weighted) bound.

As it was just mentioned above, the elastic, space homogeneous Boltzmann equation for variable hard spheres (i.e. $\alpha \in (0, 1]$ in equation (2)) it has been shown that the solution has an L^∞ exponential tail of order 2 in $[0, \infty)$. This is a consequence of the rather strong fact that L^1 exponential tail implies the L^∞ exponential tail in the solution by means of a result like Theorem 5 in the Appendix. Another example of the strong relation of $L^1 - L^\infty$ exponential tails is given in [8]. In this work the authors proved the existence of self-similar solutions for the inelastic homogeneous Boltzmann equation with constant restitution coefficient. Using the results from [5], where it was shown proved that an steady state of a self- similar solution must has all moments bounded and an L^1 exponential tail of order 1, the authors went further to show the existence of such steady states and that also it has an L^∞ exponential tail of order 1.

We are ready to formulate the main result of this work after the introduction of the short

notation for the Maxwellian (i.e. exponential of order 2) weight,

$$M_r \equiv M_r(\xi) = \exp(-r|\xi|^2) \text{ with } r \in \mathbb{R}.$$

Theorem 1. *Let η any multi-index and assume that $f_0 \in H^{|\eta|}_{(|\eta|-1)(1+\alpha/2)}$. In addition, assume that for all $\nu \leq \eta$ we have that $|\partial^\nu f_0|/M_{r_0} \in L^1$ and $|\partial^\nu f_0|/\left\{(1+|\xi|^2)^{|\nu|/2}M_{r_0}\right\} \in L^\infty$ for some $r_0 > 0$. Then, there exist $r \leq r_0$ such that*

$$\sup_{t \geq 0} \frac{|\partial^\nu f|}{(1+|\xi|^2)^{|\nu|/2}M_r} \leq K_{\eta,r_0}$$

for all $\nu \leq \eta$, where K_{η,r_0} is a positive constant depending on η, r_0 and the kernel $h(\cdot)$. In particular, for $\nu \leq \eta$ and $t > 0$

$$\lim_{|\xi|\to\infty} |\partial^\nu f(\xi,t)| \leq K_{\eta,r_0} \lim_{|\xi|\to\infty} M_{\bar{r}^\nu_2}(\xi)$$

where the constant $\bar{r}^\nu_2$ is given by (8) for the function $\partial^\nu f$.

Remarks :

- In other words, if $\partial^\nu f_0$ has a L^∞ exponential tail of order 2 for all $\nu \leq \eta$, then the ν derivative of the solution will propagate such behavior, that is the $\partial^\nu f(t,v)$ still has an L^1 exponential tail of order 2. In addition, using related arguments to the ones in [5], yields the propagation of L^∞ exponential tails of order 2 for the ν derivative of the solution, for all $\nu \leq \eta$.
- It is clear that the property of having L^1 or L^∞ exponential tail is transparent to the polynomial weight that we include. Indeed, a function has any of the previous properties if an only if the product of the function with a polynomial also has the property. We include the weight in the statement of Theorem 1 since it appears naturally for variable hard sphere kernels with an angular cross section function $h(z)$ satisfying *(i)-(ii)*, as the proof of the Theorem shows. In addition, emphasis has been done about the fact that the ν derivative of the solution is being compared with the ν derivative of the Maxwellian.

As it was noticed in [2] and [3], the existence of L^1 exponential tails for a solution f of the space homogeneous Boltzmann equation, is closely related to the existence of all its moments and its summability properties. Following that line of work in the such of properties of such nature for $\partial^\eta f$, we also observe that

$$\int_{\mathbb{R}^n} |\partial^\eta f| \, \exp\left(r|\xi|^s\right)d\xi = \sum_{k=0}^{\infty} \frac{\delta^\eta m_{sk/2}}{k!}r^k.$$

Thus, in order to show that there is a the choice of $s > 0$ for which the summability of moments, or equivalently, a bound for the right hand side of (2) uniformly in time t, one would need to show that there exist positive constants K and Q, independent of t, such that $\delta^\eta m_{sk/2}/k! < KQ^k$, $k = 1, 2, 3 \cdots$. Hence, if that is the case, the sum in the right hand side of (2) converges choosing, uniformly in time, for any r such that $0 < r < 1/Q$. This fact will imply that the integral is finite and therefore $\bar{r}^\eta_s > 0$.

Bobylev et al, in [2], and in [3], proved that under precise conditions the moments of f satisfy estimates

(9) $\qquad\qquad m_{sk/2}/k! < KQ^k \quad$ uniformly in time for $s = 2$.

This paper intends to do the same for $\delta^\eta m_{sk/2}$ as defined in (6).

Conversely, if the integral in the left hand side is bounded on $[0,T]$ for some positive r, s,

the terms in the sum must be controlled in the form $\delta^\eta m_{sk/2}/k! < K\,Q^k$, $\quad k = 1,2,3\cdots$ for some constants $K, Q > 0$. Thus, the moments $\delta^\eta m_{sk/2}$ with $k = 1,2,3,\cdots$ are uniformly bounded on $[0, T]$ if and only if $\partial^\eta f$ has an L^1 exponential tail of some order $s > 0$ in $[0, T]$. Before continuing with the technical work the reader may go to Appendix A. and see some of the classical results known for a distributional solution f of (1) used throughout this work.

3. Sharp Povzner-type inequalities for the solution's derivatives

The purpose of this section is to give technical Lemmas regarding the derivative of the collision operator $\partial^\eta Q(f, f)$. The idea of the following Lemmas is to obtain expressions for this operator as close as possible to those already given in [3] for $Q(f, f)$.

Lemma 1. *Let f a sufficiently smooth function. Then, the following expressions hold for the positive and negative parts of the collision operator*

$$\partial^\eta Q^\pm(f, f) = \sum_{\nu \leq \eta} \binom{\eta}{\nu} Q^\pm(\partial^\nu f, \partial^{\eta-\nu} f).$$

In particular,

$$\partial^\eta Q(f, f) = \sum_{\nu \leq \eta} \binom{\eta}{\nu} Q(\partial^\nu f, \partial^{\eta-\nu} f).$$

Proof. This is a direct consequence of the invariance property $\tau_\Delta Q(f, f) = Q(\tau_\Delta f, \tau_\Delta f)$, where τ_Δ is the translation operator defined by $\tau_\Delta g(\xi) = g(\xi - \Delta)$, for ξ and Δ in $\mathbb{R}^n$. For details see [12]. $\qquad\square$

Next, we need a suitable form for the action of the derivative of the collision operator $\partial^\nu Q(f, f)$ on test functions.

Lemma 2. *Let f a sufficiently smooth function. Then, the action of the η derivative of the collision operator on any test function ϕ is given by*

$$(10) \quad \int_{\mathbb{R}^n} \partial^\eta Q(f, f)\phi d\xi = \int\int_{\mathbb{R}^n \times \mathbb{R}^n} f_* \, \partial^\eta f \, A[\phi]|u|^\alpha d\xi_* d\xi$$

$$+ 1/2 \sum_{0 < \nu < \eta} \binom{\eta}{\nu} \int\int_{\mathbb{R}^n \times \mathbb{R}^n} \partial^\nu f \, \partial^{\eta-\nu} f_* \, A[\phi]|u|^\alpha d\xi_* d\xi$$

where

$$A[\phi] \;=\; A^+[\phi] \;-\; A^-[\phi]$$

with

$$(11) \qquad A^+[\phi] \;=\; \int_{S^{n-1}} (\phi' + \phi'_*) h(\hat{u} \cdot \sigma) d\sigma \quad \text{and} \quad A^-[\phi] \;=\; \phi + \phi_*$$

Proof. For f and g any smooth functions we have after the regular change of variables $\xi \to \xi'$

$$(12) \qquad \int_{\mathbb{R}^n} Q^+(f, g)\phi d\xi = \frac{1}{2} \int\int\int_{\mathbb{R}^n \times \mathbb{R}^n \times S^{n-1}} (fg_*\phi' + f_*g\phi'_*) h(\hat{u} \cdot \sigma) d\sigma |u|^\alpha d\xi_* d\xi.$$

Also, using the change of variables $\xi \to \xi_*$ the action of the negative collision part is given by

$$(13) \qquad \int_{\mathbb{R}^n} Q^-(f, g)\phi d\xi = \frac{1}{2} \int\int_{\mathbb{R}^n \times \mathbb{R}^n} (fg_*\phi + f_*g\phi_*)|u|^\alpha d\xi_* d\xi.$$

Now, using Lemma 1

$$\int_{\mathbb{R}^n} \partial^\eta Q^+(f,f)\phi d\xi = \sum_{\nu \leq \eta} \binom{\eta}{\nu} \int_{\mathbb{R}^n} Q^+(\partial^\nu f, \partial^{\eta-\nu} f)\phi d\xi.$$

Let $f \equiv \partial^\nu f$ and $g \equiv \partial^{\eta-\nu} f$ in (12) to get

(14) $$\int_{\mathbb{R}^n} \partial^\nu Q^+(f,f)\phi d\xi = 1/2 \sum_{\nu \leq \eta} \binom{\eta}{\nu} \int\int_{\mathbb{R}^n \times \mathbb{R}^n} \partial^\nu f \, \partial^{\eta-\nu} f_* \, A^+[\phi] |u|^\alpha d\xi_* d\xi.$$

Following the same idea, and using (13) and the renormalization of angular cross section

(15) $$\int_{\mathbb{R}^n} \partial^\eta Q^-(f,f)\phi d\xi = 1/2 \sum_{\nu \leq \eta} \binom{\eta}{\nu} \int\int_{\mathbb{R}^n \times \mathbb{R}^n} \partial^\nu f \, \partial^{\eta-\nu} f_* \, A^-[\phi] |u|^\alpha d\xi_* d\xi.$$

Subtract (15) from (14) and split the total sum to conclude. □

The moments of the derivative of the collision operator need to be controlled in order to find a bound for the moments of the solution's derivatives. The following Lemma is a first step in this direction.

Lemma 3. *Assume $\phi \geq 0$, then for any multi-index η*

(16) $$\int_{\mathbb{R}^n} \partial^\eta Q(f,f)\mathrm{sgn}(\partial^\eta f)\phi d\xi \leq$$

$$\int\int_{\mathbb{R}^n \times \mathbb{R}^n} f_* |\partial^\eta f| A[\phi] |u|^\alpha d\xi_* d\xi + 2\int\int_{\mathbb{R}^n \times \mathbb{R}^n} f_* |\partial^\eta f| \phi_* |u|^\alpha d\xi_* d\xi$$

$$+ 1/2 \sum_{0 < \nu < \eta} \binom{\eta}{\nu} \int\int_{\mathbb{R}^n \times \mathbb{R}^n} |\partial^\nu f \partial^{\eta-\nu} f_*| A[\phi] |u|^\alpha d\xi_* d\xi$$

$$+ \sum_{0 < \nu < \eta} \binom{\eta}{\nu} \int\int_{\mathbb{R}^n \times \mathbb{R}^n} |\partial^\nu f \partial^{\eta-\nu} f_*| \phi_* |u|^\alpha d\xi_* d\xi$$

Proof. Let $\Psi \geq 0$ and $\phi = \mathrm{sgn}(\partial^\eta f)\Psi$ in Lemma 2. In one hand, observe that for the first term in (10) $|A^+[\mathrm{sgn}(\partial^\eta f)\Psi]| \leq A^+[\Psi]$, hence

$$\partial^\eta f A^+[\mathrm{sgn}(\partial^\eta f)\Psi] \leq |\partial^\eta f| A^+[\Psi].$$

On the other hand

$$\partial^\eta f A^-[\mathrm{sgn}(\partial^\eta f)\Psi] = |\partial^\eta f| A^-[\Psi] - \Psi_* \left\{ |\partial^\eta f| - \mathrm{sgn}(\partial^\eta f_*)\partial^\eta f \right\}.$$

Gathering these two inequations we have

(17) $$\partial^\eta f A[\mathrm{sgn}(\partial^\eta f)\Psi] \leq |\partial^\eta f| A[\Psi] + 2|\partial^\eta f| \Psi_*.$$

Note that (17) yields a control for the first term in (10) of Lemma 2. Moreover, a similar argument also works for the second term in (10). □

Although the expression (16) may look cumbersome, we point out that the main idea here is to separate the terms that depend on the actual derivative of order η from the lower order derivatives. In this way, it is possible to take advantage of the expression (16) when an induction argument is used.

We are now ready to study the moments of the solution's derivatives for $p > 1$. The idea is to follow the work [3] adapting the results to this extended case. Several Lemmas are needed before attempting to prove a time uniform control on these moments. Let us first consider, in the following Lemma, test functions of the form $\phi_p = |\xi|^{2p}$, with $p > 1$.

For a detailed proof see [3] for bounded angular cross section function $h(z)$ and from [5] for $h(z)$ satisfying conditions *(i)-(ii)*. Nevertheless, we present a slightly modified argument from the one in [5] to handle condition *(ii)*:

Lemma 4. *Under the previous assumptions on $h(\cdot)$, for every $p \geq 1$,*

$$(18) \qquad A[\phi_p] = A[|\xi|^{2p}] \leq -(1 - \gamma_p)(|\xi|^{2p} + |\xi_*|^{2p}) + \gamma_p((|\xi|^2 + |\xi_*|^2)^p - |\xi|^{2p} - |\xi_*|^{2p})$$

where the constant γ_p is given by the formula

$$(19) \qquad \gamma_p = \omega_{n-2} \int_{-1}^{1} \left(\frac{1+z}{2}\right)^p \bar{h}(z)(1 - z^2)^{\frac{n-3}{2}} dz$$

with $\bar{h}(z) = \frac{1}{2}(h(z) + h(-z))$. In particular, for $\epsilon = n - 1 - \mu > 0$

$$(20) \qquad \lim_{p \to \infty} \gamma_p \sim p^{-\epsilon/2} \searrow 0,$$

where μ is the growth exponent of condition (ii) on $h(z)$.
Furthermore if $h(z)$ is bounded, the following estimate holds for for $p > 1$

$$\gamma_p < \min\left\{1, \frac{16\pi \|h\|_\infty}{p+1}\right\}.$$

Proof. It is easy to see that $\lim_{p \to \infty} \gamma_p \searrow 0$, since by conditions *(i-ii)* in $h(z)$ it follows that γ_1 is bounded. In particular

$$\left(\frac{1+z}{2}\right)^p \searrow 0 \quad \text{a.e. in } (-1, 1) \text{ as } p \to \infty$$

so γ_p is decreasing on p. Using Lebesgue's Dominated Convergence the decay of $\gamma_p \searrow 0$ follows.
However, using *(ii)* on $h(z)$ we can say more about the decreasing rate of γ_p to zero. Since

$$\bar{h}(z) \leq C(1 - z^2)^{-\mu/2}$$

then

$$\gamma_p = \omega_{n-2} \int_{-1}^{1} \left(\frac{1+z}{2}\right)^p \bar{h}(z)(1 - z^2)^{\frac{n-3}{2}} dz$$

$$\leq 2^{\epsilon-1} C \omega_{n-2} \int_{0}^{1} s^{p+\epsilon/2-1}(1 - s)^{\epsilon/2-1} ds$$

$$(21) \qquad = 2^{\epsilon-1} C \omega_{n-2} \beta(p + \epsilon/2, \epsilon/2)$$

where $\epsilon = n - 1 - \mu > 0$. Then we can estimate

$$\beta(p + \epsilon/2, \epsilon/2) = \frac{\Gamma(p + \epsilon/2)\Gamma(\epsilon/2)}{\Gamma(p + \epsilon)} \sim p^{-\epsilon/2} \text{ for large } p.$$

It is concluded that $\gamma_p \sim p^{-\epsilon/2}$ when $p \to \infty$ and (20) holds. □

Remarks :

- Lemma 4 is a sharp version of the so called Povzner inequalities. It is proved after a clever manipulation of the post collision variables in the positive part of the collision operator. The convexity of $h(\cdot)$ plays an essential role to conclude Lemma (4) because it provides a non decreasing property to the action of the gain operator on convex functions.
- For hard spheres $h(\hat{u} \cdot \sigma) = 1/4\pi$, hence $\gamma_p < \min\left\{1, \frac{4}{p+1}\right\}$.

Lemma 5. *Assume that $p > 1$, and let $k_p = [\frac{p+1}{2}]$. Then for all $x, y > 0$ the following inequalities hold*

$$(22) \quad \sum_{k=1}^{k_p-1} \binom{p}{k}(x^k y^{p-k} + x^{p-k}y^k) \leq (x+y)^p - x^p - y^p \leq \sum_{k=1}^{k_p} \binom{p}{k}(x^k y^{p-k} + x^{p-k}y^k).$$

Remark : The binomial coefficient for a non integer p is defined for $k \geq 1$ as

$$(23) \quad \binom{p}{k} = \frac{p(p-1)\cdots(p-k+1)}{k!}, \text{ and } \binom{p}{0} = 1.$$

The following Lemma is a consequence of the previous two results. It provides a control on the collision operator's moments of order p using moments of the solution's derivatives of order strictly less that p.

Lemma 6. *Assume $h(z)$ fulfill all the conditions discussed, then for every $p > 1$ and multi-index η*

$$(24) \quad \int_{\mathbb{R}^n} \partial^\eta Q(f,f) sgn(\partial^\eta f)|\xi|^{2p}d\xi \leq -(1-\gamma_p)k_\alpha \delta^\eta m_{p+\alpha/2} + \gamma_p \delta^\eta S_p$$

$$+ \delta^{\eta^-}(m_{\alpha/2}m_p) + \delta^{\eta^-}(m_0 m_{p+\alpha/2})$$

where k_α is a positive constant depending on α but not on p.
In addition,

$$\delta^\eta S_p \equiv \sum_{k=1}^{k_p} \binom{p}{k}\left\{\delta^\eta(m_k m_{p-k+\alpha/2}) + \delta^\eta(m_{k+\alpha/2}m_{p-k})\right\} \text{ with } k_p = \left[\frac{p+1}{2}\right]$$

and

$$\delta^{\eta^-}(m_{\alpha/2}m_p) + \delta^{\eta^-}(m_0 m_{p+\alpha/2}) \equiv 2\sum_{\nu<\eta}\binom{\eta}{\nu}\left\{\delta^{\eta-\nu}m_{\alpha/2}\delta^\nu m_p + \delta^{\eta-\nu}m_0\delta^\nu m_{p+\alpha/2}\right\}.$$

Remarks :

- The notation

$$\delta^\eta(m_p m_q) \equiv \sum_{\nu\leq\eta}\binom{\eta}{\nu}\delta^\nu m_p \delta^{\eta-\nu}m_q \text{ and }$$

$$(25) \quad \delta^{\eta^-}(m_p m_q) \equiv \sum_{\nu<\eta}\binom{\eta}{\nu}\delta^\nu m_p \delta^{\eta-\nu}m_q$$

  has been chosen so that the "product rule of differentiation" holds. Expression (24) makes it clear that this notation is very convenient to maintain the length of expressions short and at the same time easy to remember. The minus sign next to the upper script η in the last term of (24) was introduced to stress the fact that the sum is done on the multi-index $\nu < \eta$.
- Observe that none of the two last terms in (24) depends on $\delta^\eta m_{p+\alpha/2}$ for $p > 1$. This is important for the induction arguments used later on.

Proof. Let $\phi = |\xi|^{2p}$ in Lemma 3. The sum of terms two and four in the right-hand side of (16) is bounded by $\delta^{\eta^-}(m_{\alpha/2}m_p) + \delta^{\eta^-}(m_0 m_{p+\alpha/2})$. Indeed, use the inequality $|u|^\alpha \leq |\xi|^\alpha + |\xi_*|^\alpha$, which is valid for $\alpha \in (0,1]$, expand the integrals, and use the definition of moments of the derivatives (6) to conclude directly.

Recall the inequality, found in [1] or [5, Lemma 7], valid for solutions of the Boltzmann equation with finite mass, energy and entropy,

$$(26) \qquad \int_{\mathbb{R}^n} f_*|u|^\alpha d\xi_* \geq k_\alpha |\xi|^\alpha$$

where the constant k_α depends, in addition to α, on the mass, energy and entropy of the initial datum f_0. Thus, it follows that

$$\int\int_{\mathbb{R}^n \times \mathbb{R}^n} f_*|\partial^\eta f|(|\xi|^{2p} + |\xi_*^{2p}|)|u|^\alpha d\xi_* d\xi \geq \int\int_{\mathbb{R}^n \times \mathbb{R}^n} f_*|\partial^\eta f||\xi|^{2p}|u|^\alpha d\xi_* d\xi$$

$$(27) \qquad\qquad\qquad\qquad \geq k_\alpha \delta^\eta m_{p+\alpha/2}.$$

Use (27), Lemma 4 and Lemma 5, to control the first term in the right hand side of (16)

$$\int\int_{\mathbb{R}^n \times \mathbb{R}^n} f_*|\partial^\eta f||A[\phi]||u|^\alpha d\xi_* d\xi \leq -(1-\gamma_p)k_\alpha \delta^\eta m_{p+\alpha/2}+$$

$$\gamma_p \sum_{k=1}^{k_p} \binom{p}{k} \delta^\eta m_{k+\alpha/2} m_{p-k} + \delta^\eta m_{p-k} m_{k+\alpha/2} + \delta^\eta m_k m_{p-k+\alpha/2} + \delta^\eta m_{p-k+\alpha/2} m_k.$$

Similarly, the following control is obtained for the third term of (16)

$$\sum_{0<\nu<\eta} \binom{\eta}{\nu} \int\int_{\mathbb{R}^n \times \mathbb{R}^n} |\partial^\nu f \partial^{\eta-\nu} f_*||A[\phi]||u|^\alpha d\xi_* d\xi \leq$$

$$\gamma_p \sum_{k=1}^{k_p} \binom{p}{k} \sum_{0<\nu<\eta} \binom{\eta}{\nu} \delta^\nu m_{k+\alpha/2} \delta^{\eta-\nu} m_{p-k} + \delta^\nu m_{p-k} \delta^{\eta-\nu} m_{k+\alpha/2}$$

$$+ \delta^\nu m_k \delta^{\eta-\nu} m_{p-k+\alpha/2} + \delta^\nu m_{p-k+\alpha/2} \delta^{\eta-\nu} m_k .$$

Combining these two inequalities, the sum of first and third term of (16) is bounded by

$$-(1-\gamma_p)k_\alpha \delta^\eta m_{p+\alpha/2} + \gamma_p \delta^\eta S_p.$$

This concludes the proof. □

Let us introduce the normalized moments, which are defined as

$$(28) \qquad\qquad \delta^\eta z_p \equiv \delta^\eta m_p / \Gamma(p+b)$$

where $\Gamma(\cdot)$ is the gamma function and $b > 0$ is a free parameter to be chosen in the sequel. These moments can be used to simplify the estimate obtained in Lemma (6).

Lemma 7. *For every $p > 1$ and multi-index η,*

$$\delta^\eta S_p \leq A\, \Gamma(p+\alpha/2+2b)\delta^\eta Z_p$$

where

$$(29) \qquad\qquad \delta^\eta Z_p = \max_{1 \leq k \leq k_p} \left\{ \delta^\eta(z_k z_{p-k+\alpha/2}), \delta^\eta(z_{k+\alpha/2} z_{p-k}) \right\}$$

and $A > 0$ is a constant depending only on b.

Proof. The proof is identical to that of Lemma 4 in [3]. First observe that

$$\delta^\eta S_p = \sum_{k=1}^{k_p} \binom{p}{k} \Gamma(k+b)\Gamma(p-k+\alpha/2+b)\delta^\eta(z_k z_{p-k+\alpha/2})$$

$$+ \Gamma(k+\alpha/2+b)\Gamma(p-k+b)\delta^\eta(z_{k+\alpha/2} z_{p-k}).$$

But the Beta and Gamma functions are related by

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

This allow us to reduce the right-hand side in the previous equality to

$$\Gamma(p+\alpha/2+2b)\sum_{k=1}^{k_p}\binom{p}{k}\beta(k+b,p-k+\alpha/2+b)\delta^\eta(z_k z_{p-k+\alpha/2})$$

$$+\,\beta(k+\alpha/2+b,p-k+b)\delta^\eta(z_{k+\alpha/2}z_{p-k}).$$

Therefore,

$$(30)\quad \delta^\eta S_p \leq \Gamma(p+\alpha/2+2b)\,\delta^\eta Z_p \sum_{k=1}^{k_p}\binom{p}{k}\beta(k+b,p-k+\alpha/2+b)$$

$$+\,\beta(k+\alpha/2+b,p-k+b).$$

Using the definition of the beta function it is possible to control the sum in (30) by constant A depending only on b, for details of this last step see [3, Lemma 4]. □

We are ready to construct the system of differential inequalities used in the proof of Theorem 2. The following Lemma follows by a direct application of Lemma 6 and Lemma 7 on the Boltzmann equation (1).

Lemma 8. *Let η any multi-index and assume that $\delta^\eta m_0 > 0$ and $\delta^\nu m_0$, $\delta^\nu m_{\alpha/2}$ uniformly bounded on time for all $\nu \leq \eta$, then*

$$(31)\quad \frac{d(\delta^\eta z_p)}{dt} + (1-\gamma_p)k_\alpha\Gamma(p+b)^{\alpha/2p}\delta^\eta m_0^{-\alpha/2p}(\delta^\eta z_p)^{1+\alpha/2p} \leq \gamma_p k_0 p^{\alpha/2+b}\,\delta^\eta Z_p$$

$$+\,k_1 p^{\alpha/2}\delta^{\eta^-}(m_0 z_{p+\alpha/2}) + \delta^{\eta^-}(m_{\alpha/2}z_p)$$

for all $p > 1$, with $k_1 > 0$ universal constant, $k_0 > 0$ depending only on b and k_α given in Lemma (6)

Proof. First, note that using Jensen's inequality

$$\delta^\eta m_{p+\alpha/2} \geq \delta^\eta m_0^{-\alpha/2p}\delta^\eta m_p^{1+\alpha/2p}.$$

Next, take the η derivative in velocity in both sides of the Boltzmann equation (1), then multiply it by $\operatorname{sgn}(\partial^\eta f)|\xi|^{2p}$ and integrate in velocity, then, use Lemma 6 to obtain

$$\frac{d(\delta^\eta m_p)}{dt} + (1-\gamma_p)k_\alpha\delta^\eta m_0^{-\alpha/2p}(\delta^\eta m_p)^{1+\alpha/2p} \leq$$

$$\gamma_p\delta^\eta S_p + \left\{\delta^{\eta^-}(m_{p+\alpha/2}m_0) + \delta^{\eta^-}(m_p m_{\alpha/2})\right\}.$$

Use the definition of $\delta^\eta z_p$ in the previous inequality, and combine it with Lemma 7 to get

$$\frac{d(\delta^\eta z_p)}{dt} + (1-\gamma_p)k_\alpha\Gamma(p+b)^{\alpha/2p}\delta^\eta m_0^{-\alpha/2p}(\delta^\eta z_p)^{1+\alpha/2p} \leq$$

$$\gamma_p\,K\frac{\Gamma(p+\alpha/2+2b)}{\Gamma(p+b)}\delta^\eta Z_p + \frac{\Gamma(p+\alpha/2+b)}{\Gamma(p+b)}\delta^{\eta^-}(m_0 z_{p+\alpha/2}) + \delta^{\eta^-}(m_{\alpha/2}z_p).$$

For the terms involving Gamma functions use the asymptotic formulas for large p

$$\frac{\Gamma(p+\alpha/2+2b)}{\Gamma(p+b)} \sim p^{\alpha/2+b},\quad \frac{\Gamma(p+\alpha/2+b)}{\Gamma(p+b)} \sim p^{\alpha/2}$$

to find the right polynomial grow. □

Remark : Lemma 8 is the equivalent result to Lemma 6.2 in [2]. Differential inequalities (31) have the additional complication that they are not of constant coefficients since $\delta^\eta m_0$, unlike m_0, is in general a function of t.

The following classical result on ODE's helps to implement an induction argument on inequalities of the form (33-34).

Lemma 9. *Let $\mathbf{a}$ and $\mathbf{b}$ positive continuous functions in t such that*

$$\mathbf{a}_* = \inf_{t>0} \mathbf{a} > 0, \quad \mathbf{b}_* = \sup_{t>0} \mathbf{b} < +\infty$$

and let $\mathbf{c}$ a positive constant. In addition assume that $y \geq 0 \in C^1([0,\infty))$ solves the differential inequality

$$(32) \qquad\qquad y' + \mathbf{a}y^{1+\mathbf{c}} \leq \mathbf{b}, \quad y(0) = y_0$$

then $y \leq \max\left\{y_0, (\mathbf{b}_*/\mathbf{a}_*)^{1/(1+c)}\right\}$

Proof. Since $y' + \mathbf{a}_* y^{1+\mathbf{c}} \leq \mathbf{b}_*$, it suffices to prove the Lemma in the case that $\mathbf{a}$ and $\mathbf{b}$ are positive constants. As a first step assume equality in (32). Thus, from classical theory of differential equations this ODE has a unique C^1 solution y_* with the property stated by the Lemma, i.e.

$$y_* \leq \max\left\{y_0, (\mathbf{b}/\mathbf{a})^{1/(1+c)}\right\}.$$

If $y \in C([0,\infty))$ solves (32) we claim that $y \leq y_*$. Assume that there exist $T' > 0$ where the inequality $y(T') > y_*(T')$ holds. Let $T < T'$ such that $y(T) = y_*(T)$ and $y > y_*$ in (T, T'). The existence of such a point is assured by the continuity of the functions in addition to the fact that $y(0) = y_*(0) = y_0$. Therefore,

$$\int_T^{T'} y' = y(T') - y(T) > y_*(T') - y_*(T) = \int_T^{T'} y'_*.$$

Hence, there exist $T_0 \in (T, T')$ such that $y'(T_0) > y'_*(T_0)$. Thus, the inequalities

$$\mathbf{b} - y(T_0)^{1+\mathbf{c}} \geq y'(T_0) > y'_*(T_0) = \mathbf{b} - y_*(T_0)^{1+\mathbf{c}}$$

hold. Consequently, it is concluded that $y(T_0) < y_*(T_0)$ which is a contradiction. As a result, $y \leq y_*$. This proves the Lemma. $\qquad\square$

Remark : Same argument proves a similar result for $y \geq 0 \in C^1([0,\infty))$ solving the differential inequality $y' + \mathbf{a}y^{1+\mathbf{c}} \leq \mathbf{d}y + \mathbf{b}$ with $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{d}$ positive function on t (satisfying similar hypothesis) and $\mathbf{c}$ a positive constant. Of course the bound slightly changes to $y \leq \max\{y_0, \bar{y}\}$ where point $\bar{y}$ is given by the equation $\mathbf{a}_*\bar{y}^{1+\mathbf{c}} = \mathbf{d}_*\bar{y} + \mathbf{b}_*$. Here $\mathbf{d}_* = \sup_{t>0} \mathbf{d}$.

Next result relate the last two Lemmas and gives some orientation of the future application of Lemma 9.

Corollary 1. *Inequalities (31) can be written for $p = 3/2$ as*

$$(33) \qquad\qquad \frac{d(\delta^\eta z_{3/2})}{dt} + \mathbf{a}_{3/2}(\delta^\eta z_{3/2})^{1+c_{3/2}} \leq \mathbf{b}_{3/2} + \mathbf{d}_{3/2}(\delta^\eta z_{3/2})$$

and for $p \in \{2, 5/2, 3, ...\}$ as,

$$(34) \qquad\qquad \frac{d(\delta^\eta z_p)}{dt} + \mathbf{a}_p(\delta^\eta z_p)^{1+c_p} \leq \mathbf{b}_p$$

where $\mathbf{a}_p$, $\mathbf{b}_p$, $\mathbf{c}_p$ and $\mathbf{d}_{3/2} \geq 0$ are positive functions on t and b for $p \in \{3/2, 2, 5/2, ...\}$, and more importantly, they are independent of the normalized moment $\delta^\eta z_p$.

Proof. The expressions for $\mathbf{a}_p$ and $\mathbf{c}_p$ can be found by comparison between (33-34) and (31)

$$(35) \qquad \mathbf{a}_p(t) = (1 - \gamma_p)k_\alpha \Gamma(p+b)^{\alpha/2p}\delta^\eta m_0^{-\alpha/2p} \text{ and } \mathbf{c}_p = \alpha/2p.$$

Clearly they are positive functions of t and independent of $\delta^\eta z_p$. For $p = 3/2$ the short expression for $\delta^\eta Z_p$ is obtained

$$\begin{aligned}
\delta^\eta Z_{3/2} &= \max\left\{\delta^\eta(z_1 z_{(1+\alpha)/2}), \delta^\eta(z_{1+\alpha/2}z_{1/2})\right\} \\
&\leq \delta^\eta(z_1 z_{(1+\alpha)/2}) + \delta^\eta(z_{1+\alpha/2}z_{1/2}) \\
&= \delta^\eta(z_1 z_{(1+\alpha)/2}) + \delta^{\eta^-}(z_{1+\alpha/2}z_{1/2}) + z_{1/2}\delta^\eta z_{1+\alpha/2}.
\end{aligned}$$

But note that for $\alpha \in (0,1]$,

$$(36) \qquad \delta^\eta z_{1+\alpha/2} \leq 1 + \delta^\eta z_{3/2}.$$

Therefore, this together with Lemma 8 leads to (33) with

$$(37) \quad \mathbf{d}_{3/2}(t) = \gamma_{3/2}k_0(3/2)^{\alpha/2+b}z_{1/2}(t) \text{ and}$$

$$\begin{aligned}
\mathbf{b}_{3/2}(t) = \gamma_{3/2}k_0(3/2)^{\alpha/2+b}&\left\{\delta^\eta(z_1 z_{(1+\alpha)/2}) + \delta^{\eta^-}(z_{1+\alpha/2}z_{1/2}) + z_{1/2}\right\} \\
&+ k_1(3/2)^{\alpha/2}\delta^{\eta^-}(m_0 z_{(3+\alpha)/2}) + \delta^{\eta^-}(m_{\alpha/2}z_{3/2}).
\end{aligned}$$

For $p \in \{2, 5/2, 3, ...\}$ it is clear that for $1 \leq k \leq k_p$ the subindexes k, $p - k + \alpha/2$, $k + \alpha/2$ and $p - k$ used in the definition of $\delta^\eta Z_p$ are all strictly less that p. Hence the term $\delta^\eta Z_p$ do not depend on $\delta^\eta z_p$. Therefore the expression (34) follows if we select

$$(38) \qquad \mathbf{b}_p(t) = \gamma_p k_0 p^{\alpha/2+b}\delta^\eta Z_p + \left\{k_1 p^{\alpha/2}\delta^{\eta^-}(m_0 z_{p+\alpha/2}) + \delta^{\eta^-}(m_{\alpha/2}z_p)\right\}.$$

Recall in equation (38) that the functions $\delta^{\eta^-}(m_0 m_{p+\alpha/2})$ and $\delta^{\eta^-}(m_{\alpha/2}m_p)$ depend on lower derivatives moments. $\qquad\qquad\square$

4. Main Results

Theorem 2 states that if the initial moments of the derivatives of any order are finite, they will continue finite through time. Moreover, these moments are controlled by the initial datum in a specific way. The result is an extension of Bobylev work, Theorem 4 (item 1), which assures this behavior for the regular p-moments.

Theorem 2. *Let η any multi-index and assume that $\delta^\eta m_0 > 0$, and $\delta^\nu m_0, \delta^\nu m_1$ uniformly bounded in $[0, T]$ for $T > 0$ and all $\nu \leq \eta$. Also assume that for some constants $k > 0$ and $q \geq 1$ the initial renormalized moments of the solution's derivatives satisfy the grow condition on p*

$$\delta^\nu z_p(0) \leq kq^p, \quad p = 3/2, 2, 5/2, ...$$

Then we have the following uniform bound for the renormalized moments on $t \in [0, T]$

$$(39) \qquad \delta^\nu z_p(t) \leq KQ^p, \quad p = 3/2, 2, 5/2...$$

for all $\nu \leq \eta$, some $Q \geq q$ and a positive constant $K = K(\eta, \|f\|_{L^\infty([0,T];W_2^{|\eta|,1})}, k)$ depending on the multi-index η, the constant k, and on the $L^\infty([0,T];W_2^{|\eta|,1})$ norm of f.

Proof. Argue by induction on the multi-index order $|\eta|$. The case $|\eta| = 0$ follows from a direct application of Bobylev work [2] for the hard spheres case ($\alpha = 1$) or Gamba-Panferov-Villani work [5] for the general hard potential case $\alpha \in (0,1)$, see Theorem 4 (item 1) on Appendix A.

Thus, the induction hypothesis **(IH)** reads: Assume that Theorem 2 is true for any multi-index ν with $|\nu| < |\eta|$, therefore there exists $K_1 > 0$ and $Q \geq q$ depending on different parameters as stated above, such that for $t \in [0, T]$ and $|\nu| < |\eta|$

$$\delta^\nu z_p(t) \leq K_1 Q^p \quad \text{for} \quad p \geq 1$$

The purpose of the rest of the proof is to prove that

$$\delta^\eta z_p(t) \leq K Q^p \quad \text{for} \quad p = 1, 3/2, 2, 5/2, \ldots$$

for some $K > 0$. Recall the parameters $\mathbf{a}_p$ and $\mathbf{b}_p$ with $p \in 3/2, 2, 5/2, \ldots$ defined explicitly in the proof Corollary 1. The idea is to use induction on p to show that the quotient $\mathbf{a}_p / \mathbf{b}_p$ is bounded by $K Q^p$ and conclude by using Lemma 9.
Define

(40)
$$\mathbf{a}_*^p \equiv \inf_{t \in [0,T]} \mathbf{a}_p(t) = \|\delta^\eta m_0\|_{L^\infty[0,T]}^{-\alpha/2p} (1 - \gamma_p) \Gamma(p+b)^{\alpha/2p}.$$

Observe that due to the induction hypothesis **(IH)**

$$\delta^{\eta^-}(m_0 z_{p+\alpha/2}) \leq 2^{|\eta|} K_1 \|f\|_{L^\infty([0,T];W_1^{|\eta|,1})} Q^{p+\alpha/2}, \quad \text{and}$$

$$\delta^{\eta^-}(m_{\alpha/2} z_p) \leq 2^{|\eta|} K_1 \|f\|_{L^\infty([0,T];W_1^{|\eta|,1})} Q^p.$$

where we have used the control of the moments 0 and $\alpha/2$ of the η derivative of f provided by the $L^\infty([0,T]; W_1^{|\eta|,1})$ norm of f

$$\sum_{i=1,\alpha/2} \max_{\nu \leq \eta} \left\{ \|\delta^\nu m_i\|_{L^\infty[0,T]} \right\} \leq \|f\|_{L^\infty([0,T];W_1^{|\eta|,1})}.$$

Hence, for $p \in 2, 5/2, 3, \ldots$

(41)
$$\mathbf{b}_p(t) \leq \gamma_p k_0 p^{\alpha/2+b} \delta^\eta Z_p + 2^{|\eta|} K_1 \|f\|_{L^\infty([0,T];W_1^{|\eta|,1})} Q^p \left\{ k_1 Q^{\alpha/2} p^{\alpha/2} + 1 \right\}.$$

Substitute (40) and (41) in (34) to conclude that

(42)
$$\frac{d(\delta^\eta z_p)}{dt} + \mathbf{a}_*^p (\delta^\eta z_p)^{1+\alpha/2p} \leq \gamma_p k_0 p^{\alpha/2+b} \delta^\eta Z_p$$
$$+ 2^{|\eta|} K_1 \|f\|_{L^\infty([0,T];W_1^{|\eta|,1})} Q^p \left\{ k_1 Q^{\alpha/2} p^{\alpha/2} + 1 \right\}.$$

Next, define the following sequences of p

$$A_p^1 = \frac{k_0 \gamma_p p^{\alpha/2+b}}{\mathbf{a}_*^p} \quad \text{and} \quad A_p^2 = 2^{|\eta|} K_1 \|f\|_{L^\infty([0,T];W_1^{|\eta|,1})} \frac{k_1 Q^{\alpha/2} p^{\alpha/2} + 1}{\mathbf{a}_*^p},$$

in this way equation (42) can be written as

(43)
$$\frac{d(\delta^\eta z_p)}{dt} + \mathbf{a}_*^p (\delta^\eta z_p)^{1+\alpha/2p} \leq \mathbf{a}_*^p A_p^1 \delta^\eta Z_p + \mathbf{a}_*^p A_p^2 Q^p.$$

Let us, for the moment, divert our attention from equation (43) and make an observation regarding the sequences $\{A_p^1\}$ and $\{A_p^2\}$. Recall the asymptotic formula for $\Gamma(p+b)$ with $b > 0$

$$\Gamma(p+b)^{\alpha/2p} \sim p^{\alpha/2} \quad \text{for large } p,$$

also recall that in the prove of Lemma (4)

$$\gamma_p \sim p^{-\epsilon/2} \quad \text{for large } p.$$

Therefore, by letting $b - \epsilon/2 < 0$ we have,

$$A_p^1 \sim p^{b-\epsilon/2} \to 0 \quad \text{as} \quad p \to \infty.$$

Meanwhile, the sequence $\{A_p^2\}$ is bounded. Define,

$$\left|A_p^2\right|_\infty \equiv \sup_{p \geq 2} A_p^2 < \infty.$$

Thus, there exists p_0 such that

$$2^{|\eta|} K_1 Q^{\alpha/2} A_p^1 \leq 1/2 \quad \text{if} \ \ p \geq p_0.$$

Now, it is claimed that it is possible to take a number $K \geq \max\left\{1, k, K_1, 2\left|A_p^2\right|_\infty\right\}$ such that

(44) $$\delta^\eta z_p(t) \leq KQ^p \quad \text{for} \ \ p = 3/2, 2, 5/2... \ \ \text{and} \ \ p \leq p_0.$$

Let us prove that this K actually exists by arguing as follows: When $p = 3/2$

$$\delta^\eta Z_p = \max\left\{\delta^\eta(z_1 z_{(1+\alpha)/2}), \delta^\eta(z_{1+\alpha/2} z_{1/2})\right\}.$$

In the one hand, by hypothesis

$$\delta^\eta(z_1 z_{(1+\alpha)/2}) \leq \frac{2^{|\eta|}}{\Gamma((1+\alpha)/2 + b)} \|f\|^2_{L^\infty([0,T];W_2^{|\eta|,1})} < +\infty.$$

and, in the other hand, by **(IH)**

$$\delta^\eta(z_{1+\alpha/2} z_{1/2}) \leq \frac{2^{|\eta|} K_1 Q^{1+\alpha/2}}{\Gamma(1/2 + b)}(1 + \delta^\eta z_{1+\alpha/2}) \|f\|_{L^\infty([0,T];W_1^{|\eta|,1})}$$

$$\leq \frac{2^{|\eta|} K_1 Q^{1+\alpha/2}}{\Gamma(1/2 + b)}(2 + \delta^\eta z_{3/2}) \|f\|_{L^\infty([0,T];W_1^{|\eta|,1})}.$$

Combine these bounds with the definitions for $\mathbf{b}_{3/2}$ and $\mathbf{d}_{3/2}$ in Corollary 1 and apply Lemma 9 to find that $\delta^\eta z_{3/2}$ is bounded. But once $\delta^\eta z_{3/2}$ is bounded, $\delta^\eta Z_2$ is immediately bounded and hence, by a new application of Lemma 9 on (43) for $p = 2$, $\delta^\eta z_2$ is bounded. Repeat this process up to p_0 to find that $\delta^\eta z_p(t)$ is bounded for $p = 3/2, 2, 5/2, ..., p_0$. So it is just a matter of choosing $K > 0$ sufficiently large so that (44) is fulfill.

Let us argue by induction on the integrability index p to show that the same constants K and Q also hold for $p > p_0$ with $p \in \{3/2, 2, ...\}$. Assume that (44) holds. Hence, observing that the term $\delta^\eta Z_p$ does not depend on $\delta^\eta z_p$ for $p > 3/2$ and using **(IH)** one concludes that

$$\delta^\eta Z_p < 2^{|\eta|} K_1 K Q^{p+\alpha/2}.$$

Therefore, inequality (43) reads for $p > p_0$

(45) $$\frac{d(\delta^\eta z_p)}{dt} + \mathbf{a}_*^p(\delta^\eta z_p)^{1+\alpha/2p} \leq 2^{|\eta|}\mathbf{a}_*^p A_p^1 K_1 K Q^{p+\alpha/2} + \mathbf{a}_*^p A_p^2 Q^p = \mathbf{b}_*^p$$

thus by Lemma 9

$$\delta^\eta z_p(t) \leq \max\left\{(\mathbf{b}_*^p/\mathbf{a}_*^p)^{2p/(2p+\alpha)}, \delta^\eta z_p(0)\right\}.$$

But the condition $p > p_0$ and the choice of K implies that

$$\mathbf{b}_p(t)/\mathbf{a}_p(t) \leq \mathbf{b}_*^p/\mathbf{a}_*^p = \left\{2^{|\eta|} A_p^1 K_1 Q^{\alpha/2} + \frac{A_p^2}{K}\right\} K Q^p \leq KQ^p \quad \text{for} \ \ p > p_0.$$

Since same inequality holds for $p \leq p_0$ one concludes that,

$$\delta^\eta z_p \leq \max\{KQ^p, kq^p\} = KQ^p \quad \text{for} \ \ p = 1, 3/2, 2, ...$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remarks :

- For any $p > 1$ a simple Lebesgue interpolation argument together with Theorem 2 shows that $\delta^\eta z_p \leq K_\eta Q^p$.
- The growing constant q for the initial datum is in general smaller that the one obtained for the differential moments. Thus, the control on the differential moments may worsen depending on the initial conditions.
- The following is a different way to state Theorem 2: Let η any multi-index and assume that $f_0 \in L_2^1$ and $f \in L^\infty([0,T]; W_2^{|\eta|,1})$, if for some $r_0 > 0$ we have that $\int_{\mathbb{R}^n} |\partial^\nu f_0| \exp(r_0 |\xi|^2) d\xi < \infty$ for all $\nu \leq \eta$ then

$$\sup_{[0,T]} \left\{ \int_{\mathbb{R}^n} |\partial^\nu f| \exp(r|\xi|^2) d\xi \right\} < \infty$$

for some $r \leq r_0$ and all $\nu \leq \eta$.

Lemma 10 and Lemma 11 prove that, for a solution of Boltzmann equation f, the differential moments $\delta^\nu m_0$ and $\delta^\nu m_1$ are uniformly bounded on time for $\nu \leq \eta$, provided we have sufficient regularity in the initial datum f_0. In other words, given sufficient regularity on f_0 we should have that $f \in L^\infty([0,T]; W_2^{|\eta|,1})$.

Lemma 10. *Let η any multi-index and suppose that $f_0 \in W_{2+\alpha}^{|\eta|,1}$, then for any $T \in (0,\infty)$ we have that $f \in L^\infty([0,T]; W_{2+\alpha}^{|\eta|,1})$. Moreover, $\delta^\eta m_0(t) > 0$ always holds.*

Proof. First note that if $\delta^\eta m_0(t') = 0$ for some fixed $t' > 0$, we have that $\partial^\eta f(\xi, t') = 0$. Therefore $f(\xi, t')$ would be a polynomial in the variables ξ_i with $i = 1, 2, \cdots, n$. Hence $f(\xi, t')$ would not be integrable unless $f(\xi, t') = 0$. But $0 = \|f(\cdot, t')\|_{L^1} = \|f_0\|_{L^1}$ due to mass conservation. This is impossible for a non zero initial datum.

Next, since $A[1] = 0$ and $A[|\xi|^2] \leq 0$, one uses Lemma 3 to obtain the following inequalities

$$1/2 \frac{d(\delta^\eta m_0)}{dt} \leq \delta^\eta m_0 m_{\alpha/2} + \delta^\eta m_{\alpha/2} m_0$$

$$+ \sum_{0 < \nu < \eta} \binom{\eta}{\nu} \left(\delta^\nu m_{\alpha/2} \delta^{\eta-\nu} m_0 + \delta^\nu m_0 \delta^{\eta-\nu} m_{\alpha/2} \right)$$

and

$$(46) \quad 1/2 \frac{d(\delta^\eta m_1)}{dt} \leq \delta^\eta m_0 m_{1+\alpha/2} + \delta^\eta m_{\alpha/2} m_1$$

$$+ \sum_{0 < \nu < \eta} \binom{\eta}{\nu} \left(\delta^\nu m_{\alpha/2} \delta^{\eta-\nu} m_1 + \delta^\nu m_0 \delta^{\eta-\nu} m_{1+\alpha/2} \right).$$

We can now conclude the proof by using inequality (46) in order to implement an induction argument on the index order $|\eta|$. Note that for the case $|\eta| = 0$, the conservation of mass and dissipation of energy implies that $f \in L^\infty([0,T]; L_2^1)$. In addition, since $f_0 \in L_{2+\alpha}^1$, the moment $1 + \alpha/2$ is finite in the initial datum, then we must have that this moment is uniformly bounded in time, for this is precisely the work of Gamba-Panferov-Villani [5]. Hence, $f \in L^\infty([0,T]; L_{2+\alpha}^1)$.

For $|\eta| > 0$, take $f_0 \in W_{2+\alpha}^{|\eta|,1}$ and assume that the result is valid for all $|\nu| < |\eta|$. Since $W_{2+\alpha}^{|\eta|,1} \subset W_{2+\alpha}^{|\nu|,1}$ then $f_0 \in W_{2+\alpha}^{|\nu|,1}$, thus by induction hypothesis we have that $f \in L^\infty([0,T]; W_{2+\alpha}^{|\nu|,1})$ for all $|\nu| < |\eta|$. Therefore, $\delta^\nu m_0$, $\delta^\nu m_1$ and $\delta^\nu m_{1+\alpha/2}$ are uniformly

bounded on $[0,T]$ as long as $|\nu| < |\eta|$. Note, that

$$\delta^\eta m_{\alpha/2} \leq \delta^\eta m_0 + \delta^\eta m_1.$$

As a result, inequalities (46) imply that $\delta^\eta m_0$ and $\delta^\eta m_1$ are uniformly bounded on $[0,T]$, i.e. $f \in L^\infty([0,T]; W_2^{|\eta|,1})$. But $\delta^\eta m_{1+\alpha/2}(0)$ is finite by hypothesis, thus we can apply Theorem 2 again to get that $\delta^\eta m_{1+\alpha/2}(t)$ is finite in $[0,T]$. We conclude that $f \in L^\infty([0,T]; W_{2+\alpha}^{|\eta|,1})$. $\qquad\square$

Lemma 11 shows that it is possible to go further and obtain a global in time result for the elastic case, provided that more regularity on f_0 is imposed.

Lemma 11. *Let η any multi-index and assume that $f_0 \in W_{2+\alpha}^{|\eta|,1} \cap H_{(|\eta|-1)(1+\alpha/2)}^{|\eta|}$ then $f \in L^\infty(\mathbb{R}^+; W_{2+\alpha}^{|\eta|,1})$.*

Proof. In the one hand, for all multi-index ν satisfying $\nu \leq \eta$ we have by Cauchy-Schwartz inequality that

$$\delta^\nu m_p \leq C_{s,n} \|f\|_{H_{2p+s/2}^{|\eta|}}$$

for any $s > n$ and some constant $C_{s,n}$ depending on s and the dimension n. Therefore, by letting $p = 1 + \alpha/2$ we obtain,

$$\max_{\nu \leq \eta}\left\{\delta^\nu m_0(t), \delta^\nu m_1(t), \delta^\nu m_{1+\alpha/2}(t)\right\} \leq C_{s,n}\|f(t,\cdot)\|_{H_{2+\alpha+s/2}^{|\eta|}}.$$

Then, using Theorem (6) in Appendix A

$$\sup_{t \geq t_0}\left\{\max_{\nu \leq \eta}\left\{\delta^\nu m_0(t), \delta^\nu m_1(t), \delta^\nu m_{1+\alpha/2}(t)\right\}\right\} < +\infty.$$

On the other hand, the differential moments are bounded for $t \leq t_0$ by Lemma 10 under these assumptions on f_0. Hence, they are bounded uniformly for all $t > 0$. As a result, $f \in L^\infty(\mathbb{R}^+; W_{2+\alpha}^{|\eta|,1})$. $\qquad\square$

The results of Theorem 2, Lemma 10 and Lema 11 can be readily used to obtain the L^1-Maxwellian bound for derivatives of any order.

Theorem 3. *Let η any multi-index and assume that $f_0 \in W_{2+\alpha}^{|\eta|,1}$. In addition, assume the grow condition on the initial moments*

$$\delta^\nu m_p(0)/p! \leq k\, q^p$$

for $p \geq 3/2$, all $\nu \leq \eta$ and some positive constants k and q. Then, $\partial^\nu f$ has exponential tail of order 2 in $[0,T]$ for $\nu \leq \eta$ and $T \in (0,T)$. Moreover, if we additionally assume $f_0 \in H_{(|\eta|-1)(1+\alpha/2)}^{|\eta|}$ then the conclusion can be extended to $T = +\infty$.

Proof. Fix $T \in (0,\infty)$ and observe that using Lemma 10 it is possible to conclude that $f \in L^\infty([0,T]; W_{2+\alpha}^{|\eta|,1})$. From this follows that the moments $\delta^\nu m_0$ and $\delta^\nu m_1$ are bounded in $[0,T]$ for all $\nu \leq \eta$. Therefore, the conditions of Theorem 2 are fulfilled and we can use it to conclude that for all $\nu \leq \eta$ the following inequality holds in $[0,T]$

$$(47) \qquad \int_{\mathbb{R}^n} |\partial^\nu f| e^{r|v|^2} dv = \sum_i \frac{\delta^\nu m_i}{i!}\, r^i \leq K \sum_i \frac{\Gamma(i+b)}{\Gamma(i+1)}\, (Qr)^i,$$

where $Q \geq q$ and $K > 0$ are constants that depend on different parameters as discussed in Theorem 2. But,

$$\frac{\Gamma(i+b)}{\Gamma(i+1)} \sim i^{b-1} \text{ for large } i.$$

Consequently, the sum behave like

$$\sum_i i^{b-1}(Qr)^i$$

Thus, it suffices to choose $r > 0$ such that $Qr < 1$ so that the sum in (47) converges.

Use the assumption that $f_0 \in H^{|\eta|}_{(|\eta|-1)(1+\alpha/2)}$ and apply Lemma 11 to extend the result to the limit case $T = +\infty$. $\qquad\square$

Remark :

- As a final remark on Theorem 2, Lemma 10 and Lemma 11, observe that for any multi-index η and $k \geq 2 + \alpha$, Theorem 2 implies that if $f_0 \in W^{|\eta|,1}_k$, then $f \in C([0,T]; W^{|\eta|,1}_k)$ for any $T < \infty$. For the elastic case $T = \infty$ is also allowed, provided we have that $f_0 \in H^{|\eta|}_{(|\eta|-1)(1+\alpha/2)}$.

5. Proof of Theorem 1

In order to simplify the notation set $Q^-(f,g) = f \cdot L(g)$ where

$$L(g) = \int_{\mathbb{R}^n} g_* |\xi - \xi_*|^\alpha d\xi_*.$$

Proof. Differentiate the equation (1) η times in velocity and multiply the result by $\mathrm{sgn}(\partial^\eta f)$ to obtain

$$(48) \quad \partial_t(|\partial^\eta f|) + |\partial^\eta f| \, L(f) \leq Q^+(|\partial^\eta f|, f) + Q^+(f, |\partial^\eta f|) + f \cdot L(|\partial^\eta f|)$$

$$+ \sum_{0 < \nu < \eta} \binom{\eta}{\nu} \left\{ Q^+(|\partial^\nu f|, |\partial^{\eta-\nu} f|) + Q^-(|\partial^\nu f|, |\partial^{\eta-\nu} f|) \right\}.$$

We use equation (48) to argue by induction on the index order $|\eta|$. The case $|\eta| = 0$ follows directly from Theorem 4, item (2).

Next, let f_0 fulfilling all the conditions of the Theorem and assume the result for $|\nu| < |\eta|$. Then, there exists $r' \leq r_0$ such that for any $|\nu| < |\eta|$

$$|\partial^\nu f| \leq K^1_{\eta,r_0}(1 + |\xi|^2)^{|\nu|/2} M_{r'}$$

where K^1_{η,r_0} is a positive constant depending on η and r_0.

By hypothesis, $|\partial^\nu f_0|/M_{r_0} \in L^1$ for all $\nu \leq \eta$. Thus, the grow condition required in Theorem 3 on the derivative moments of the initial datum f_0 is satisfied, namely, that for some positive constants k and q,

$$\delta^\nu m_p(0)/p! \leq kq^p \quad \text{for} \quad p \geq 0.$$

Furthermore, $f_0 \in H^{|\eta|}_{(|\eta|-1)(1+\alpha/2)}$, as a result, Theorem 3 applies to obtain that for some $r'' \leq r_0$

$$(49) \qquad \sup_{t \geq 0} \int_{\mathbb{R}^n} |\partial^\nu f| \, \exp(r''|\xi|^2) d\xi = \sup_{t \geq 0} \|\partial^\nu f / M_{r''}\|_{L^1} < \infty$$

for all $\nu \leq \eta$. Indeed, recall that in Theorem 2 a bigger grow constant $Q \geq q$ was obtained for controlling the derivative's moments through time. Hence, previous integral must converge in general for $r'' \leq r_0$.

Let $r = \min\{r', r''\}$ and divide inequality (48) by M_r. Using the induction hypothesis we can bound the derivatives of lower order in (48) to get the inequality,

$$\partial_t(|\partial^\eta f|/M_r) + |\partial^\eta f/M_r|\, L(f) \leq$$

$$\frac{K^1_{\eta,r_0}}{M_r}\left\{Q^+(|\partial^\eta f|, M_r) + Q^+(M_r, |\partial^\eta f|)\right\} + K^1_{\eta,r_0}\, L(|\partial^\eta f|) +$$

$$\frac{K^1_{\eta,r_0}}{M_r} \sum_{0<\nu<\eta} \binom{\eta}{\nu} Q^+((1+|\xi|^2)^{|\nu|/2}M_r, |\partial^{\eta-\nu}f|) + Q^-((1+|\xi|^2)^{|\nu|/2}M_r, |\partial^{\eta-\nu}f|).$$

Use Theorem 5 and Theorem 6 in the Appendix A, to obtain the following L^1 control from the previous inequality

(50) $\quad \partial_t(|\partial^\eta f|/M_r) + |\partial^\eta f/M_r|\, L(f) \leq$

$$K^2_{\eta,r_0}(1+|\xi|^2)^{(|\eta|-1)/2} \sum_{0<\nu\leq\eta} \binom{\eta}{\nu} \|\partial^\nu f/M_r\|_{L^1} + L(|\partial^\nu f|)$$

where $K^2_{\eta,r_0} > 0$ is a constant depending on η, r_0 and on the kernel $b(\cdot)$, as Theorem 5 states.

However, observe that for all ν

$$L(|\partial^\nu f|) \leq |\xi|^\alpha \delta^\nu m_0 + \delta^\nu m_{\alpha/2} \leq \text{Const. } \|\partial^\nu f/M_r\|_{L^1} (1+|\xi|^2)^{\alpha/2}$$

$$\leq \text{Const. } \|\partial^\nu f/M_r\|_{L^1} (1+|\xi|^2)^{1/2}.$$

Therefore, combining this inequality with by (49) we conclude that the right-hand side of (50) is bounded by $K^3_{\eta,r_0}(1+|\xi|^2)^{|\eta|/2}$. Specifically,

(51) $$\partial_t(|\partial^\eta f|/M_r) + |\partial^\eta f/M_r|\, L(f) \leq K^3_{\eta,r_0}(1+|\xi|^2)^{|\eta|/2}.$$

Fix $t_0 > 0$ and integrate (51) over $[0, t_0]$. It follows that for any $t \in [0, t_0]$

$$|\partial^\eta f|/M_r \leq K^3_{\eta,r_0}\, t_0(1+|\xi|^2)^{|\eta|/2} + |\partial^\eta f_0|/M_r \leq K^4_{\eta,r_0}\, t_0(1+|\xi|^2)^{|\eta|/2}$$

where K^4_{η,r_0} is a positive constant that depends on η, r_0 and the kernel $h(\cdot)$.
For $t > t_0$ use the lower bound that provides Theorem 4 (item 3) in the Appendix to conclude that $C \equiv \inf_{\xi, t\geq t_0} L(f) > 0$, thus using the full differential inequality (51)

$$|\partial^\eta f|/M_r \leq \max\left\{C^{-1}K^3_{\eta,r_0}(1+|\xi|^2)^{|\eta|/2}, |\partial^\eta f_0|/M_r\right\} \leq K^5_{\eta,r_0}(1+|\xi|^2)^{|\eta|/2}.$$

Therefore, $K_{\eta,r_0} = \max\{K^4_{\eta,r_0} \cdot t_0, K^5_{\eta,r_0}\}$ provides a sufficiently large constant for any $t \geq 0$. Since it is possible to fix any time t_0 to perform these calculations, this constant just depends on η, r_0 and the kernel $h(\cdot)$. $\qquad\square$

Remarks :

- If assumption $f_0 \in H^{|\eta|}_{(|\eta|-1)(1+\alpha/2)}$ is not imposed, Theorem 1 is still valid changing in the conclusion " $\sup_{t\geq 0}$ " for " $\sup_{0\leq t\leq T}$ " with T finite. This is a direct consequence of the fact that Theorem 3 is valid under these conditions for any finite time T.

- Take as hypothesis of Theorem 1 only that $f_0 \in H^{|\eta|}_{(|\eta|-1)(1+\alpha/2)}$ and

$$|\partial^\nu f_0|/\left\{(1+|\xi|^2)^{|\nu|/2}M_{r_0}\right\} \in L^\infty$$

for $\nu \leq \eta$ and some positive r_0. Since for any $r' \in (0, r_0)$ the last hypothesis implies that $|\partial^\nu f_0|/M_{r'} \in L^1$ and

$$|\partial^\nu f_0|/\left\{(1+|\xi|^2)^{|\nu|/2}M_{r'}\right\} \in L^\infty$$

for all $\nu \le \eta$. Thus, using Theorem 1, there exist $r \le r' < r_0$ such that

$$\sup_{t \ge 0} \frac{|\partial^\nu f|}{(1 + |\xi|^2)^{|\nu|/2} M_r} \le K_{\eta,r'}.$$

- Mischler et al. [7] proved that for inelastic collisions the solution of the problem (1) converges to the Dirac delta distribution as the time goes to infinity (see [7]). This is a consequence of the energy loss and therefore the cooling process that is taking place in the gas. Thus, for this case, it is not possible to obtain results like Theorem 1 which involve bounds that are uniformly in time for the solution f. In the elastic case, the gas does not have this cool down phenomena hence uniform bounds on the derivatives can be proved in $[0, \infty)$.

APPENDIX A. FACTS FOR A SOLUTION f OF THE HOMOGENEOUS BOLTZMANN PROBLEM

The homogeneous Boltzmann problem for hard and Maxwellian potentials is nowadays pretty well understood, in addition to existence and uniqueness of solutions [6] many other results are available like positive estimates [11] and propagation of regularity [10]. The most useful results used in this work are stated by the following theorems.

Theorem 4. *Assume that f_0 and $h(\cdot)$ have the properties discussed in the introduction and that $\alpha \in (0, 1]$. Then the following properties holds for a solution f of the elastic homogeneous Boltzmann problem:*

(1): If f_0 satisfies $\int_{\mathbb{R}^n} f_0 \exp(r_0|\xi|^2)d\xi < \infty$ for some $r_0 > 0$, then there exist $r \le r_0$ such that $\sup_{t \ge 0} \int_{\mathbb{R}^n} f \exp(r|\xi|^2)d\xi < \infty$.

(2): If $f_0 \le K_0 \exp(-r_0|\xi|^2)$ for some $K_0, r_0 > 0$ then there exist $r \le r_0$ such that $f \le K \exp(-r|\xi|^2)$ for all $t \ge 0$ and some positive constants K.

(3): For every $t_0 > 0$ there are positive constants K, r_0 such that $f(t, \xi) \ge K \exp(-r_0|\xi|^2)$ for all $t \ge t_0$.

These are precisely the results that we want to extend for the derivative of f and their proof can be found in [3] for item 1, also [5] for item 2 and [11] for item 3. Of course item 3 is not true in general for $|\partial^\eta f|$, for example as shown by a Maxwellian solution, the gradient can be in general zero in some points of the velocity space at a given time. However, this result will prove to be helpful in showing pointwise bounds for the derivatives of a solution. Observe also that in items 1 and 2 in Theorem 4 the rate of decay r_0 that controls f_0 is worsen in general to $r \le r_0$ for controlling f.

Next, we state a remarkable result essential to prove item 2 in the previous Theorem, in particular, essential to control the gain collision operator.

Theorem 5. *Assume $B(u, \sigma) = |u|^\alpha h(\hat{u} \cdot \sigma)$ with $h(\cdot)$ satisfying the conditions stated in the introduction. Then for any measurable function $g \ge 0$,*

$$\left\| \frac{Q^+(g, M_r)}{M_r} \right\|_{L^\infty} \le K \left\| \frac{g}{M_r} \right\|_{L^1}$$

for some positive constant K depending on α and r.

As usual in the L^∞ bounds for $Q^+(f, f)$, this result is a direct application of the Carleman representation formula and clever manipulations of it. This Theorem is very helpful when we try to prove an L^∞ bound for the derivatives of f. The proof of Theorem 5 can be found on [5, Lemma 12].

It is clear that same result holds for $Q^+(M_r, g)$, moreover, and slightly modification of the proof can be used to obtain the following Theorem.

Theorem 6. *Assume $B(u, \sigma) = |u|^\alpha h(\hat{u} \cdot \sigma)$ with $h(\cdot)$ satisfying the conditions stated in the introduction, then for any measurable function $g \geq 0$*

$$\left\| \frac{Q^+(g, (1 + |\xi|^2)^s M_r)}{(1 + |\xi|^2)^s M_r} \right\|_{L^\infty} \leq K \left\| \frac{g}{M_r} \right\|_{L^1},$$

for any $s > 0$ and some positive constant K depending on s, α and β.

Finally, a powerful result proved by Mouhot and Villani [10, Theorem 4.2] is also used. This result helps to obtain uniform bounds for infinite time for the derivative's moments. A small piece of this theorem, which is the one of use for us, is stated below.

Theorem 7. *Let $\alpha \in (0, 2)$, $s \in \mathbb{N}$ and assume that $f_0 \in L^1_2 \cap H^s_{(s-1)(1+\alpha/2)}$. Then for any $t_0 > 0$ and $k > 0$,*

$$\sup_{t \geq t_0} \|f(t, \cdot)\|_{H^s_k} < +\infty.$$

This quantity depends on an upper bound on L^1_2 and $H^s_{(s-1)(1+\gamma/2)}$ norms of f_0 and a lower bound on t_0.

The proof of this Theorem is rather technical and requires several previous results on the control of the positive collision operator including the gain of regularity of the positive operator, however its spirit is, as in this work, to find a stable differential equation for the H^s norm of f and proceed by induction.

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