LOCAL WELL-POSEDNESS OF A DISPERSIVE NAVIER-STOKES SYSTEM

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Abstract. We establish local well-posedness and smoothing results for the Cauchy problem of a degenerate dispersive Navier-Stokes system that arises from kinetic theory. Under assumptions that the initial data satisfy asymptotic flatness and nontrapping conditions, we show there exists a unique classical solution for a finite time. Due to degeneracies in both dissipation and dispersion for the system, different components of the solution gain different regularity. Couplings of these components are analyzed using pseudodifferential operators.

1. Introduction

In this paper we establish the local well-posedness of the Cauchy problem for a dispersive Navier-Stokes (DNS) system that has the form

$$
\begin{aligned}
\partial_t\rho + \nabla_x \cdot (\rho u) &= 0\,, \\
\partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x(\rho\theta) &= \nabla_x \cdot \Sigma + \nabla_x \cdot \tilde{\Sigma}\,, \\
\partial_t(\rho e) + \nabla_x \cdot (\rho e u + \rho\theta u) &= \nabla_x \cdot (\Sigma u + q) + \nabla_x \cdot (\tilde{\Sigma} u + \tilde{q})\,, \\
(\rho, u, \theta)(x, 0) &= (\rho^{in}, u^{in}, \theta^{in})(x)\,,
\end{aligned}
\tag{1.1}
$$

where $\rho(x,t)$ is the mass density, $u(x,t)$ is the bulk velocity, and $\theta(x,t)$ is the temperature at a position $x \in \mathbb{R}^d$ and time $t \geq 0$. We assume that $d \geq 2$. Here the total energy density ρe is given by

$$
\rho e = \tfrac{1}{2}\rho|u|^2 + \tfrac{d}{2}\rho\theta\,,
$$

while the classical Navier-Stokes stress tensor $-\Sigma$ and heat flux $-q$ are given by

$$
\Sigma = \mu(\theta)\, D_x u\,, \qquad q = \kappa(\theta)\, \nabla_x\theta\,,
\tag{1.2}
$$

where $D_x u = \nabla_x u + (\nabla_x u)^T - \frac{2}{d}\nabla_x \cdot u I$ is the strain-rate tensor while $\mu(\theta) \geq 0$ and $\kappa(\theta) \geq 0$ are the coefficients of shear viscosity and heat conductivity. Dispersive corrections to the stress tensor $\tilde{\Sigma}$ and the heat flux $\tilde{q}$ are given by

$$
\begin{aligned}
\tilde{\Sigma} &= \tau_1(\rho,\theta)\Big(\nabla_x^2\theta - \tfrac{1}{d}\Delta_x\theta I\Big) + \tau_2(\rho,\theta)\,\Big(\nabla_x\theta \otimes \nabla_x\theta - \tfrac{1}{d}|\nabla_x\theta|^2 I\Big) \\
&\quad + \tau_3(\rho,\theta)\Big(\nabla_x\rho \otimes \nabla_x\theta + \nabla_x\theta \otimes \nabla_x\rho - \tfrac{2}{d}\nabla_x\rho \cdot \nabla_x\theta I\Big)\,, \\
\tilde{q} &= \tau_4(\rho,\theta)\Big(\Delta_x u + \tfrac{d-2}{d}\nabla_x\nabla_x \cdot u\Big) + \tau_5(\rho,\theta)\, D_x u \cdot \nabla_x\theta + \tau_6(\rho,\theta)\, D_x u \cdot \nabla_x\rho \\
&\quad + \tau_7(\rho,\theta)\Big(\nabla_x u - (\nabla_x u)^T\Big) \cdot \nabla_x\theta\,,
\end{aligned}
\tag{1.3}
$$

where $\tau_i(\rho,\theta)$ for $i = 1, \cdots, 7$ are additional transport coefficients.

Gas dynamical systems with terms of the form (1.1) can be systematically derived from classical kinetic equations such as the Boltzmann equation in small mean-free-path regimes [12]. They arise as the first correction to the classical compressible Navier-Stokes system. When so derived, the transport coefficients $\mu(\theta)$, $\kappa(\theta)$, and $\tau_i(\rho,\theta)$ for $i = 1, \cdots, 7$ have forms that depend on details of the underlying

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kinetic equation. In particular, the transport coefficients $\tau_i(\rho,\theta)$ for $i = 1,\cdots,6$ will satisfy the relations

$$\tau_4 = \frac{\theta}{2}\tau_1\,, \qquad \frac{\tau_2}{\theta} + \frac{2\tau_5}{\theta^2} = \partial_\theta\left(\frac{\tau_4}{\theta^2}\right), \qquad \theta\tau_3 + \tau_6 = 2\partial_\rho\tau_4\,. \tag{1.4}$$

The resulting DNS system (1.1) inherits an entropy structure from the kinetic equation in which the mathematical entropy density η is given by

$$\eta = \rho\log\left(\frac{\rho}{\theta^{d/2}}\right).$$

Direct calculation from system (1.1) shows that η satisfies

$$\partial_t\eta + \nabla_x\cdot\left(\eta\, u + \frac{q}{\theta} + \frac{\tilde{q}}{\theta}\right) = -\left(\frac{\Sigma}{\theta} : \nabla_x u + \frac{q}{\theta^2}\cdot\nabla_x\theta\right) - \left(\frac{\tilde{\Sigma}}{\theta} : \nabla_x u + \frac{\tilde{q}}{\theta^2}\cdot\nabla_x\theta\right). \tag{1.5}$$

It follows from the constitutive relations (1.2) that

$$\frac{\Sigma}{\theta} : \nabla_x u + \frac{q}{\theta^2}\cdot\nabla_x\theta = \frac{\mu}{2\theta}|D_x u|^2 + \frac{\kappa}{\theta^2}|\nabla_x\theta|^2 \geq 0\,,$$

while it follows from constitutive relations (1.3) and (1.4) that

$$\tilde{\Sigma} : \frac{\nabla_x u}{\theta} + \tilde{q}\cdot\frac{\nabla_x\theta}{\theta^2} = \nabla_x\cdot\left(\frac{\tau_1}{2\theta}D_x u\cdot\nabla_x\theta\right).$$

One thereby sees that the dispersion terms containing $\tilde{\Sigma}$ and $\tilde{q}$ contribute only to the entropy flux in the entropy equation (1.5). DNS systems (1.1) derived from kinetic equations therefore formally dissipate the entropy in the same way as the compressible Navier-Stokes system, but transport it differently.

The above calculation indicates that the DNS system is formally well-posed over domains without boundary. The main goal of our paper is to establish the local well-posedness of the DNS system. Because our theory is local in time, we will not need the entropy structure of the system, and so will not assume that (1.4) holds. We will however assume that $\mu(\theta)$, $\kappa(\theta)$, and $\tau_i(\rho,\theta)$ for $i = 1,\cdots,7$ are smooth functions of ρ and θ with $\mu(\theta)$, $\kappa(\theta)$, and $\tau_1(\rho,\theta)\tau_4(\rho,\theta)$ being strictly positive whenever ρ and θ are bounded away from zero.

Remark 1.1. We believe that the entropy structure would play an important role in any global well-posedness result for the DNS system with large initial data.

In our proof of local well-posedness, dispersive regularization plays a crucial role. We use the fact that solutions of dispersive equations gain spatial differentiability provided the initial data satisfy certain asymptotic flatness conditions at infinity. This type of smoothing was noticed by Kato when he showed in [7] that solutions of the 1D KdV equation gain half a spatial derivative compared to its initial data. This kind of smoothing has since been generalized by various authors to more general dispersive equations and systems [4, 8]. In general, solutions of dispersive equations with order m gain $\frac{m-1}{2}$ derivatives locally for positive times [4].

Based on Kato's smoothing effect, various well-posedness results have been established for semi-linear or quasi-linear dispersive equations and systems with strict or uniform dispersive effects [10, 5]. However, these existing results do not apply directly to the DNS system because its dispersion is degenerate. To see this, one first observes that the mass equation has no terms that are dissipative or dispersive. Another degeneracy occurs in the energy equation where the dispersive term $\nabla_x\cdot\tilde{q}$ is

$$\nabla_x\cdot\tilde{q} = \tfrac{2(d-1)}{d}\tau_4\,\Delta_x\nabla_x\cdot u + \text{lower order terms}\,.$$

The leading order term in $\nabla_x\cdot\tilde{q}$ gives the dispersive effect for the velocity field. It is clear that the incompressible part, *i.e.*, the divergence free part of the velocity field u vanishes in this term. Therefore, if u is decomposed into the divergence free part and the gradient part as done in the Hodge decomposition, then only the gradient component of u has a dispersive effect. These degeneracies suggest to decompose the DNS system into a strictly dispersive subsystem and a nondispersive subsystem. We apply the technique by Kenig, Ponce, and Vega in [10] to treat the principle part of the strictly dispersive subsystem. The coupling of the strictly dispersive and nondispersive parts will be treated using both dissipative and dispersive regularization. Our main theorem will imply the following.

Theorem 1.1. Well-Posedness Theorem. *In dimension $d \geq 2$, let $s_1,\ s \in \mathbb{R}_+$ such that $s_1 \geq d/2 + 6$ and $s = \max\{s_1 + 6, N + d/2 + 4\}$ where $N = N(d, 3, 0) \in \mathbb{N}$ is given in Theorem 2.1. Let $\bar{\rho} > 0$ and $\bar{\theta} > 0$ be constants. Let the functions ρ^{in}, u^{in}, and θ^{in} satisfy:*

- *the boundedness and asymptotic flatness condition*

$$\|\rho^{in} - \bar{\rho}\|_{H^{s+1}} + \|(u^{in}, \theta^{in} - \bar{\theta})\|_{H^s} + \sum_{1\leq|\alpha|\leq s_1} \left(\left\|\langle x\rangle^2 \partial_x^\alpha \rho^{in}\right\|_{H^1} + \left\|\langle x\rangle^2 \partial_x^\alpha (u^{in}, \theta^{in})\right\|_{L^2} \right) \leq C^{in} < \infty\,, \tag{1.6}$$

where $\alpha \in \mathbb{N}^d$ denote multi-indices with $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$ and we define $\langle x\rangle^2 \triangleq 1 + |x|^2$;
- *there exists a constant $\alpha^{in} > 0$ such that for every $x \in \mathbb{R}^d$*

$$\begin{gathered} \alpha^{in} \leq \rho^{in}(x)\,, \qquad \alpha^{in} \leq \theta^{in}(x)\,, \qquad \alpha^{in} \leq \mu(\theta^{in})\,, \qquad \alpha^{in} \leq \kappa(\theta^{in})\,, \\ \alpha^{in} \leq \tfrac{4(d-1)^2}{d^3} \cdot \frac{\tau_1(\rho^{in}(x), \theta^{in}(x))\tau_4(\rho^{in}(x), \theta^{in}(x))}{\rho^{in}(x)^2}\,; \end{gathered} \tag{1.7}$$

- *the Hamiltonian defined by*

$$h^{in}(\xi, x) = \tfrac{2(d-1)}{d^{3/2}} \left(\frac{\tau_1(\rho^{in}(x), \theta^{in}(x))\tau_4(\rho^{in}(x), \theta^{in}(x))}{\rho^{in}(x)^2} \right)^{\frac{1}{2}} |\xi|^3 \tag{1.8}$$

generates a flow that is nontrapping.

Then for some $T_0 > 0$ depending only on C^{in}, α^{in}, and d there exist unique functions ρ, u, and θ with

$$\begin{gathered} \rho - \bar{\rho} \in C([0, T_0]; H^{s'+1}) \cap L^\infty([0, T_0]; H^{s'+1})\,, \\ (u,\ \theta - \bar{\theta}) \in C([0, T_0]; H^{s'}) \cap L^\infty([0, T_0]; H^{s'})\,, \end{gathered} \tag{1.9}$$

for any $0 \leq s' < s$ such that (ρ, u, θ) solves the DNS initial-value problem (1.1).

Here L^2 denotes the Lebesgue space $L^2(\mathbb{R}^d; \mathbb{R}^m)$ where $\mathbb{R}^m$ is the Euclidian space implied by the context, and $\|\cdot\|_{L^2}$ denotes its norm. Similarly, H^s denotes the Sobolev space $H^s(\mathbb{R}^d; \mathbb{R}^m)$ where $\mathbb{R}^m$ is the Euclidian space implied by the context, and $\|\cdot\|_{H^s}$ denotes its norm.

To prove the above theorem, we construct an approximating sequence of solutions by adding an artificial hyperviscosity term to the DNS system (1.1). An *a priori* estimate is established that is independent of the artificial hyperviscosity. Then using this a priori estimate and letting the artificial hyperviscosity term vanish, we show that the approximating sequence converges to a solution of the original system. Uniqueness is also shown by the a priori estimate.

This paper is laid out as follows. In Section 2 we establish an estimate for a linear system that we will later use to construct our approximating sequence of solutions to the DNS system (1.1) plus an artificial hyperviscosity. In Section 3 we establish the a priori estimate for this regularized DNS system. In section 4 we show the existence of the approximating sequence and the convergence of this sequence to the unique solution to the original DNS system.

2. Estimate for an Associated Linear System

In this section we establish the key estimate for a linear system associated with a regularization of the DNS system (1.1). One can see from the proof that the same estimate holds for the analogous linear system associated with the original DNS system (1.1).

2.1. **Regularized DNS System.** Our regularized system is obtained by first expressing the DNS system (1.1) as a system for the evolution of the fluid variables (ρ, u, θ) and then adding a fourth-order artificial hyperviscosity term to each dynamical equation. The result is the regularized DNS system

$$
\begin{aligned}
&\partial_t \rho = -\epsilon\, \Delta_x^2 \rho - \rho \nabla_x \cdot u - u \cdot \nabla_x \rho\,,\\
&\partial_t u = -\epsilon\, \Delta_x^2 u + \frac{1}{\rho}\nabla_x \cdot \Sigma + \frac{1}{\rho}\nabla_x \cdot \tilde{\Sigma} - \frac{1}{\rho}\nabla_x(\rho\theta) - u \cdot \nabla_x u\,,\\
&\partial_t \theta = -\epsilon\, \Delta_x^2 \theta + \tfrac{2}{d}\frac{1}{\rho}\nabla_x \cdot q + \tfrac{2}{d}\frac{1}{\rho}\nabla_x \cdot \tilde{q} + \tfrac{2}{d}\frac{\tilde{\Sigma} : \nabla_x u}{\rho} + \tfrac{2}{d}\frac{\Sigma : \nabla_x u}{\rho} - \tfrac{2}{d}\theta \nabla_x \cdot u - u \cdot \nabla_x \theta\,,\\
&(\rho, u, \theta)(x, 0) = (\rho^{in}, u^{in}, \theta^{in})(x)\,,
\end{aligned}
\tag{2.1}
$$

where Σ and q are given by (1.2) while $\tilde{\Sigma}$ and $\tilde{q}$ are given by (1.3). The structure of this system becomes explicit if we express Σ, $\tilde{\Sigma}$, q, and $\tilde{q}$ in terms of the fluid variables (ρ, u, θ). It follows from (1.3) that $\nabla_x \cdot \tilde{\Sigma}$ and $\nabla_x \cdot \tilde{q}$ have the forms

$$
\begin{aligned}
\nabla_x \cdot \tilde{\Sigma} &= \tfrac{d-1}{d}\tau_1(\rho,\theta)\Delta_x \nabla_x \theta + A^\rho(\rho,\theta,\nabla_x\rho,\nabla_x\theta) : \nabla_x^2 \rho + A^\theta(\rho,\theta,\nabla_x\rho,\nabla_x\theta) : \nabla_x^2\theta\\
&\quad + B^\rho(\rho,\theta,\nabla_x\rho,\nabla_x\theta)\cdot\nabla_x\rho + B^\theta(\rho,\theta,\nabla_x\rho,\nabla_x\theta)\cdot\nabla_x\theta\,,\\
\nabla_x \cdot \tilde{q} &= \tfrac{2(d-1)}{d}\tau_4(\rho,\theta)\Delta_x \nabla_x \cdot u + A^u(\rho,\theta,\nabla_x\rho,\nabla_x\theta) \,\vdots\, \nabla_x^2 u + \tau_5(\rho,\theta) D_x u : \nabla_x^2\theta\\
&\quad + \tau_6(\rho,\theta) D_x u : \nabla_x^2 \rho + B^u(\rho,\theta,\nabla_x\rho,\nabla_x\theta) : \nabla_x u\,,
\end{aligned}
\tag{2.2}
$$

where

$$
\begin{aligned}
A^\rho(\rho,\theta,\nabla_x\rho,\nabla_x\theta) : \nabla_x^2\rho &= \tau_3(\rho,\theta)\nabla_x\theta\Delta_x\rho + \tfrac{d-2}{d}\tau_3(\rho,\theta)\nabla_x\theta : \nabla_x^2\rho\,,\\
A^\theta(\rho,\theta,\nabla_x\rho,\nabla_x\theta) : \nabla_x^2\theta &= \Big[\Big(\partial_\rho\tau_1 + \tfrac{d-2}{d}\tau_3\Big)\nabla_x\rho + \Big(\partial_\theta\tau_1 + \tfrac{d-2}{d}\tau_2\Big)\nabla_x\theta\Big]\cdot\nabla_x^2\theta\\
&\quad + \Big[\Big(\tfrac{1}{d}\partial_\rho\tau_1 + \tau_3\Big)\nabla_x\rho + \Big(\tfrac{1}{d}\partial_\theta\tau_1 + \tau_2\Big)\nabla_x\theta\Big]\Delta_x\theta\,,\\
A^u(\rho,\theta,\nabla_x\rho,\nabla_x\theta)\,\vdots\,\nabla_x^2 u &= \Big[\Big(\partial_\rho\tau_4 + \tau_6\Big)\nabla_x\rho + \Big(\partial_\rho\tau_4 + \tau_5 + \tau_7\Big)\nabla_x\theta\Big]\cdot\Delta_x u\\
&\quad + \Big[\Big(\tfrac{d-2}{d}\partial_\rho\tau_4 + \tfrac{d-2}{d}\tau_6\Big)\nabla_x\rho + \Big(\tfrac{d-2}{d}\partial_\theta\tau_4 + \tfrac{d-2}{d}\tau_5 - \tau_7\Big)\nabla_x\theta\Big]\cdot(\nabla_x\nabla_x\cdot u)\,,
\end{aligned}
$$

while B^ρ, B^θ have the forms

$$
\begin{aligned}
&a_1(\rho,\theta)(\nabla_x\rho\cdot\nabla_x\theta)I + a_2(\rho,\theta)|\nabla_x\theta|^2 I + a_3(\rho,\theta)\nabla_x\rho\otimes\nabla_x\theta\\
&\quad + a_4(\rho,\theta)\nabla_x\theta\otimes\nabla_x\rho + a_5(\rho,\theta)\nabla_x\theta\otimes\nabla_x\theta\,,
\end{aligned}
$$

with $a_1, \cdots, a_5$ being given by the functional forms of τ_1, τ_2, τ_3 and B^u is of the form

$$
b_1(\rho,\theta)\nabla_x\rho\otimes\nabla_x\theta + b_2(\rho,\theta)\nabla_x\theta\otimes\nabla_x\rho + b_3(\rho,\theta)\nabla_x\theta\otimes\nabla_x\theta\,,
$$

where b_1, b_2, b_3 are determined by the functional forms of τ_5, τ_6, and τ_7. Notice that the forms of B^ρ and B^θ are not uniquely specified above, but the specific choice of B^ρ and B^θ does not affect our subsequent arguments. The main structure of B^θ, B^ρ, and B^u is that they are $d \times d$ tensors of linear combinations of quadratic forms of $\nabla_x\rho$, $\nabla_x\theta$.

The regularized DNS system (2.1) thereby has the form

$$
\begin{aligned}
\partial_t\rho &= -\epsilon\,\Delta_x^2\rho - \rho\nabla_x\cdot u - u\cdot\nabla_x\rho\,,\\
\partial_t u &= -\epsilon\,\Delta_x^2 u + \frac{1}{\rho}\nabla_x\cdot[\mu D_x u] + \tfrac{d-1}{d}\frac{\tau_1}{\rho}\Delta_x\nabla_x\theta + \frac{A^\rho}{\rho} : \nabla_x^2\rho + \frac{A^\theta}{\rho} : \nabla_x^2\theta\\
&\quad + \frac{B^\rho}{\rho}\cdot\nabla_x\rho + \frac{B^\theta}{\rho}\cdot\nabla_x\theta - \frac{1}{\rho}\nabla_x(\rho\theta) - u\cdot\nabla_x u\,,\\
\partial_t\theta &= -\epsilon\,\Delta_x^2\theta + \tfrac{2}{d}\frac{1}{\rho}\nabla_x\cdot[\kappa\nabla_x\theta] + \tfrac{4(d-1)}{d^2}\frac{\tau_4}{\rho}\Delta_x\nabla_x\cdot u\\
&\quad + \tfrac{2}{d}\frac{A^u}{\rho}\,\vdots\,\nabla_x^2 u + \tfrac{1}{d}\frac{\tau_1+2\tau_5}{\rho}D_x u : \nabla_x^2\theta + \tfrac{2}{d}\frac{\tau_6}{\rho}D_x u : \nabla_x^2\rho\\
&\quad + \tfrac{2}{d}\frac{B^u}{\rho} : \nabla_x u + \tfrac{1}{d}\frac{\tau_2}{\rho}\nabla_x\theta\cdot D_x u\cdot\nabla_x\theta + \tfrac{1}{d}\frac{\tau_3}{\rho}\nabla_x\rho\cdot D_x u\cdot\nabla_x\theta\\
&\quad + \tfrac{1}{d}\frac{\mu}{\rho}|D_x u|^2 - \tfrac{2}{d}\theta\nabla_x\cdot u - u\cdot\nabla_x\theta\,.\\
&(\rho,u,\theta)(x,0) = (\rho^{in},u^{in},\theta^{in})(x)\,,
\end{aligned}
\tag{2.3}
$$

2.2. **Associated Linear System.** The linear system associated with the regularized DNS system (2.3) is obtained by replacing certain (ρ,u,θ) by a given state $(\hat\rho,\hat u,\hat\theta)$. Specifically, it is the linear system for $(\tilde\rho,\tilde u,\tilde\theta)$ given by

$$
\begin{aligned}
\partial_t\tilde\rho &= -\epsilon\,\Delta_x^2\tilde\rho - \hat\rho\nabla_x\cdot\tilde u - \hat u\cdot\nabla_x\tilde\rho\,,\\
\partial_t\tilde u &= -\epsilon\,\Delta_x^2\tilde u + \frac{1}{\hat\rho}\nabla_x\cdot\Big[\mu(\hat\theta)D_x\tilde u\Big] + \hat\tau_1\Delta_x\nabla_x\tilde\theta + \hat A^\rho : \nabla_x^2\tilde\rho + \hat A^\theta : \nabla_x^2\tilde\theta\\
&\quad + \hat B^\rho\cdot\nabla_x\tilde\rho + \hat B^\theta\cdot\nabla_x\tilde\theta - \nabla_x\tilde\theta - \frac{\hat\theta}{\hat\rho}\nabla_x\tilde\rho - \hat u\cdot\nabla_x\tilde u\,,\\
\partial_t\tilde\theta &= -\epsilon\,\Delta_x^2\tilde\theta + \tfrac{2}{d}\frac{1}{\hat\rho}\nabla_x\cdot\Big[\kappa(\hat\theta)\nabla_x\tilde\theta\Big] + \hat\tau_4\Delta_x\nabla_x\cdot\tilde u + \hat A^u\,\vdots\,\nabla_x^2\tilde u + \hat\tau_5 D_x\hat u : \nabla_x^2\tilde\theta + \hat\tau_6 D_x\hat u : \nabla_x^2\tilde\rho\\
&\quad + \hat B^u : \nabla_x\tilde u + \hat\tau_2\nabla_x\hat\theta\cdot D_x\tilde u\cdot\nabla_x\hat\theta + \hat\tau_3\nabla_x\hat\rho\cdot D_x\tilde u\cdot\nabla_x\hat\theta\\
&\quad + \tfrac{1}{d}\frac{\mu(\hat\theta)}{\hat\rho}D_x\hat u : D_x\tilde u - \tfrac{2}{d}\hat\theta\nabla_x\cdot\tilde u - \hat u\cdot\nabla_x\tilde\theta\,.\\
&(\tilde\rho,\tilde u,\tilde\theta)(x,0) = (\rho^{in}-\bar\rho,u^{in},\theta^{in}-\bar\theta)(x)\,,
\end{aligned}
\tag{2.4}
$$

where

$$
\begin{aligned}
\hat\tau_1 &= \tfrac{d-1}{d}\frac{\tau_1(\hat\rho,\hat\theta)}{\hat\rho}\,, & \hat\tau_2 &= \tfrac{1}{d}\frac{\tau_2(\hat\rho,\hat\theta)}{\hat\rho}\,, & \hat\tau_3 &= \tfrac{1}{d}\frac{\tau_3(\hat\rho,\hat\theta)}{\hat\rho}\,,\\
\hat\tau_4 &= \tfrac{4(d-1)}{d^2}\frac{\tau_4(\hat\rho,\hat\theta)}{\hat\rho}\,, & \hat\tau_5 &= \tfrac{1}{d}\frac{\tau_1(\hat\rho,\hat\theta)+2\tau_5(\hat\rho,\hat\theta)}{\hat\rho}\,, & \hat\tau_6 &= \tfrac{2}{d}\frac{\tau_6(\hat\rho,\hat\theta)}{\hat\rho}\,,\\
\hat A^\rho &= \frac{A^\rho(\hat\rho,\hat\theta,\nabla_x\hat\rho,\nabla_x\hat\theta)}{\hat\rho}\,, & \hat A^\theta &= \frac{A^\theta(\hat\rho,\hat\theta,\nabla_x\hat\rho,\nabla_x\hat\theta)}{\hat\rho}\,, & \hat A^u &= \tfrac{2}{d}\frac{A^u(\hat\rho,\hat\theta,\nabla_x\hat\rho,\nabla_x\hat\theta)}{\hat\rho}\,,\\
\hat B^\rho &= \frac{B^\rho(\hat\rho,\hat\theta,\nabla_x\hat\rho,\nabla_x\hat\theta)}{\hat\rho}\,, & \hat B^\theta &= \frac{B^\theta(\hat\rho,\hat\theta,\nabla_x\hat\rho,\nabla_x\hat\theta)}{\hat\rho}\,, & \hat B^u &= \tfrac{2}{d}\frac{B^u(\hat\rho,\hat\theta,\nabla_x\hat\rho,\nabla_x\hat\theta)}{\hat\rho}\,.
\end{aligned}
\tag{2.5}
$$

The linear system (2.4) is satisfied by $(\tilde\rho,\tilde u,\tilde\theta) = (\rho-\bar\rho,u,\theta-\bar\theta)$ when $(\hat\rho,\hat u,\hat\theta) = (\rho,u,\theta)$ solves (2.3).

Notation. Henceforth we will use Ψ_m (with various sup-indices to specify the equations in which they appear) to denote any pseudo-differential operator (ΨDO) of order m. The reason to unify the lower order terms in this way is because the specific forms of those terms change when we apply various operators to either system (2.1) or its linearized form (2.4). However, it will be clear from the calculation that the estimates of various norms depend only on the orders of those lower order terms while their specific forms are not essential. We will denote the space of all m^{th} order symbols as S^m. For every $p(\xi, x) \in S^m$, we define the norm $|p|^{(j)}_{S^m}$ as

$$|p|^{(j)}_{S^m} = \sup\left\{ \|\langle\xi\rangle^{-m+\alpha}\partial_\xi^\alpha \partial_x^\beta p(\cdot,\cdot)\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)} : \alpha, \beta \in \mathbb{N}^d, |\alpha+\beta| \le j \right\}, \tag{2.6}$$

where $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$. The following theorem is a classical result for ΨDO's (c.f. [2, 6] for a proof).

Theorem 2.1. *Let $m, s \in \mathbb{R}$. Let $p(\xi, x) \in S^m$ be the symbol of a pseudo-differential operator Ψ_p. Then Ψ_p is a bounded linear operator from $H^{m+s}(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$. Moreover, there exist $N = N(d, m, s) \in \mathbb{N}$ and $c = c(d, m, s)$ such that*

$$\|\Psi_p f\|_{H^s} \le c\,|p|^{(N)}_{S^m}\,\|f\|_{H^{m+s}} \quad \textit{for every } f \in H^{m+s}(\mathbb{R}^d)\,. \tag{2.7}$$

With this notation the linear system (2.4) has the form

$$\begin{aligned}
\partial_t\tilde\rho &= -\epsilon\,\Delta_x^2\tilde\rho + \Psi_1^\rho(\tilde\rho,\tilde u)\,,\\
\partial_t\tilde u &= -\epsilon\,\Delta_x^2\tilde u + \frac{1}{\hat\rho}\nabla_x\cdot\left[\mu(\hat\theta)D_x\tilde u\right] + \hat\tau_1\Delta_x\nabla_x\tilde\theta + \Psi_2^u(\tilde\rho,\tilde\theta) + \Psi_1^u(\tilde\rho,\tilde\theta) - \hat u\cdot\nabla_x\tilde u\,,\\
\partial_t\tilde\theta &= -\epsilon\,\Delta_x^2\tilde\theta + \tfrac{2}{d}\frac{1}{\hat\rho}\nabla_x\cdot\left[\kappa(\hat\theta)\nabla_x\tilde\theta\right] + \hat\tau_4\Delta_x\nabla_x\cdot\tilde u + \Psi_2^\theta(\tilde\rho,\tilde u,\tilde\theta) + \Psi_1^\theta\tilde u - \hat u\cdot\nabla_x\tilde\theta\,,
\end{aligned}\tag{2.8}$$

where

$$\begin{aligned}
\Psi_1^\rho(\tilde\rho,\tilde u) &= -\hat\rho\nabla_x\cdot\tilde u - \hat u\cdot\nabla_x\tilde\rho\,,\\
\Psi_2^u(\tilde\rho,\tilde\theta) &= \hat A^\rho : \nabla_x^2\tilde\rho + \hat A^\theta : \nabla_x^2\tilde\theta\,,\\
\Psi_1^u(\tilde\rho,\tilde\theta) &= \hat B^\rho\cdot\nabla_x\tilde\rho + \hat B^\theta\cdot\nabla_x\tilde\theta - \nabla_x\tilde\theta - \frac{\hat\theta}{\hat\rho}\nabla_x\tilde\rho\,,\\
\Psi_2^\theta(\tilde\rho,\tilde u,\tilde\theta) &= \hat A^u \vdots \nabla_x^2\tilde u + \hat\tau_5 D_x\hat u : \nabla_x^2\tilde\theta + \hat\tau_6 D_x\hat u : \nabla_x^2\tilde\rho\,,\\
\Psi_1^\theta\tilde u &= \hat B^u : \nabla_x\tilde u + \hat\tau_2\nabla_x\hat\theta\cdot D_x\tilde u\cdot\nabla_x\hat\theta + \hat\tau_3\nabla_x\hat\rho\cdot D_x\tilde u\cdot\nabla_x\hat\theta\\
&\quad + \tfrac{1}{d}\frac{\mu(\hat\theta)}{\hat\rho}D_x\hat u : D_x\tilde u - \tfrac{2}{d}\hat\theta\nabla_x\cdot\tilde u - \hat u\cdot\nabla_x\tilde\theta\,.
\end{aligned}\tag{2.9}$$

One can see that the symbols of these first and second order operators are first and second order polynomials in ξ respectively with their coefficients depending algebraically upon $(\hat\rho, \hat\theta, \nabla_x\hat\rho, D_x\hat u, \nabla_x\hat\theta)$. We drop the tildes on $(\tilde\rho, \tilde u, \tilde\theta)$ and write the regularized system (2.8) as

$$\partial_t(\rho, u, \theta) = -\epsilon\,\Delta_x^2(\rho, u, \theta) + \mathcal{L}(\hat\rho, \hat u, \hat\theta)(\rho, u, \theta)\,, \tag{2.10}$$

where the linear operator $\mathcal{L}$ is defined through (2.8) and has the form

$$\mathcal{L}(\hat\rho,\hat u,\hat\theta)(\rho,u,\theta) = \begin{pmatrix}\mathcal{L}_1(\hat\rho,\hat u,\hat\theta)(\rho,u,\theta)\\ \mathcal{L}_2(\hat\rho,\hat u,\hat\theta)(\rho,u,\theta)\\ \mathcal{L}_3(\hat\rho,\hat u,\hat\theta)(\rho,u,\theta)\end{pmatrix} = \begin{pmatrix}\Psi_1^{\rho,0}(\rho,u)\\ \Psi_D^u u + \Psi_3^u\theta + \Psi_2^{u,0}(\rho,\theta) + \Psi_1^{u,0}(\rho,u,\theta)\\ \Psi_D^\theta\theta + \Psi_3^\theta u + \Psi_2^{\theta,0}(\rho,u,\theta) + \Psi_1^{\theta,0}(u,\theta)\end{pmatrix}, \tag{2.11}$$

where

$$\Psi_1^{\rho,0}(\rho,u) = \Psi_1^\rho(\rho,u)\,,\quad \Psi_2^{u,0}(\rho,\theta) = \Psi_2^u(\rho,\theta)\,,\quad \Psi_2^{\theta,0}(\rho,u,\theta) = \Psi_2^\theta(\rho,u,\theta)\,,$$

$$\Psi_1^{u,0}(\rho, u, \theta) = \Psi_1^u(\rho, u, \theta) - \hat{u} \cdot \nabla_x \tilde{u}\,, \quad \Psi_1^{\theta,0}(u, \theta) = \Psi_1^\theta(u, \theta) - \hat{u} \cdot \nabla_x \tilde{\theta}\,,$$
$$\Psi_D^u u = \frac{1}{\hat{\rho}} \nabla_x \cdot \left[\mu(\hat{\theta}) D_x u\right]\,, \quad \Psi_3^u \theta = \hat{\tau}_1 \Delta_x \nabla_x \theta\,,$$
$$\Psi_D^\theta \theta = \tfrac{2}{d} \frac{1}{\hat{\rho}} \nabla_x \cdot \left[\kappa(\hat{\theta}) \nabla_x \theta\right]\,, \quad \Psi_3^\theta u = \hat{\tau}_4 \Delta_x \nabla_x \cdot u\,.$$

Here Ψ_1^ρ, Ψ_2^u, Ψ_2^θ are defined in (2.9).

The discussion of degeneracies in the introduction suggests that we decompose the velocity field u into its divergence free part Pu and its gradient part Qu. Because $\nabla_x \cdot (D_x u) = \Delta_x u + \frac{d-2}{d} \nabla_x \nabla_x \cdot u$, we have

$$P \nabla_x \cdot (D_x u) = \Delta_x P u\,, \qquad Q \nabla_x \cdot (D_x u) = 2 \tfrac{d-1}{d} \Delta_x Q u\,. \tag{2.12}$$

We can then decompose system (2.8) by using the facts that

$$\begin{aligned} \frac{1}{\hat{\rho}} \nabla_x \cdot \left[\mu(\hat{\theta}) D_x u\right] &= \hat{\mu}\, \nabla_x \cdot (D_x u) + \frac{1}{\hat{\rho}} \nabla_x \mu(\hat{\theta}) \cdot D_x u\,, \\ \tfrac{2}{d} \frac{1}{\hat{\rho}} \nabla_x \cdot \left[\kappa(\hat{\theta}) \nabla_x \theta\right] &= \hat{\kappa}\, \Delta_x \theta + \tfrac{2}{d} \frac{1}{\hat{\rho}} \nabla_x \kappa(\hat{\theta}) \cdot \nabla_x \theta\,, \end{aligned} \tag{2.13}$$

where

$$\hat{\mu} = \frac{\mu(\hat{\theta})}{\hat{\rho}}\,, \qquad \hat{\kappa} = \tfrac{2}{d} \frac{\kappa(\hat{\theta})}{\hat{\rho}}\,,$$

and that

$$\begin{aligned} P(\hat{\mu}\, \nabla_x \cdot (D_x u)) &= \hat{\mu}\, P \nabla_x \cdot (D_x u) + [P\,, \hat{\mu}]\, \nabla_x \cdot (D_x u)\,, \\ Q(\hat{\mu}\, \nabla_x \cdot (D_x u)) &= \hat{\mu}\, Q \nabla_x \cdot (D_x u) + [Q\,, \hat{\mu}]\, \nabla_x \cdot (D_x u)\,. \end{aligned} \tag{2.14}$$

Although the symbols of the projection operators P and Q have singularities at $\xi = 0$, we can still treat them as pseudo-differential operators of order zero because they are homogeneous operators of order zero [11]. Therefore, by noticing that $|\xi| = \xi \cdot \frac{\xi}{|\xi|}$ where $\frac{\xi}{|\xi|}$ is homogeneous of order zero, one can view $|\xi|$ as the symbol of a first order pseudo-differential operator. This further implies that $|\xi|^k$ for any integer k can be viewed as the symbol of an operator of order k. Therefore by the symbolic calculus, the above commutator operators $[P\,, \hat{\mu}]$ and $[Q\,, \hat{\mu}]$ are of order -1. It follows from (2.12), (2.13), and (2.14), that system (2.8) decomposes into its nondispersive part

$$\begin{aligned} &\partial_t \rho = -\epsilon\, \Delta_x^2 \rho - \hat{u} \cdot \nabla_x \rho - \hat{\rho} \nabla_x \cdot Qu\,, \\ &\partial_t Pu = -\epsilon\, \Delta_x^2 Pu + \hat{\mu} \Delta_x Pu + \Psi_2^{Pu,1}(\rho, \theta) + \Psi_1^{Pu,1}(\rho, Pu, Qu, \theta)\,, \\ &(\rho, Pu)(x, 0) = (\rho^{in}, Pu^{in})(x)\,, \end{aligned} \tag{2.15}$$

and its strictly dispersive part

$$\begin{aligned} &\partial_t Qu = -\epsilon\, \Delta_x^2 Qu + 2 \tfrac{d-1}{d} \hat{\mu} \Delta_x Qu + \nabla_x (\hat{\tau}_1 \Delta_x \theta) + \Psi_2^{Qu,1}(\rho, \theta) + \Psi_1^{Qu,1}(\rho, Pu, Qu, \theta)\,, \\ &\partial_t \theta = -\epsilon\, \Delta_x^2 \theta + \hat{\kappa} \Delta_x \theta + \hat{\tau}_4 \Delta_x \nabla_x \cdot Qu + \Psi_2^{\theta,1}(\rho, Pu, Qu, \theta) + \Psi_1^{\theta,1}(Pu, Qu, \theta)\,, \\ &(Qu, \theta)(x, 0) = (Qu^{in}, \theta^{in})(x)\,, \end{aligned} \tag{2.16}$$

where

$$\Psi_2^{Pu,1}(\rho, \theta) = P\, \Psi_2^{u,0}(\rho, \theta) - P(\nabla_x \hat{\tau}_1 \Delta_x \theta)\,,$$
$$\Psi_2^{Qu,1}(\rho, \theta) = Q\, \Psi_2^{u,0}(\rho, \theta) - Q(\nabla_x \hat{\tau}_1 \Delta_x \theta)\,,$$
$$\Psi_2^{\theta,1}(\rho, Pu, Qu, \theta) = \Psi_2^{\theta,0}(\rho, Pu, \theta) + \Psi_2^{\theta,0}(\rho, Qu, \theta)\ ,$$

$$\Psi_1^{Pu,1}(\rho, Pu, Qu, \theta) = P\,\Psi_1^{u,0}(\rho, Pu, \theta) + P\left(\frac{1}{\hat\rho}\,\nabla_x\mu(\hat\theta)\cdot D_x(Pu)\right) + [P\,,\,\hat\mu]\,\nabla_x\cdot D_x(Pu)\,,$$
$$+ P\,\Psi_1^{u,0}(\rho, Qu, \theta) + P\left(\frac{1}{\hat\rho}\,\nabla_x\mu(\hat\theta)\cdot D_x(Qu)\right) + [P\,,\,\hat\mu]\,\nabla_x\cdot D_x(Qu)\,,$$

$$\Psi_1^{Qu,1}(\rho, Pu, Qu, \theta) = Q\,\Psi_1^{u,0}(\rho, Pu, \theta) + Q\left(\frac{1}{\hat\rho}\,\nabla_x\mu(\hat\theta)\cdot D_x(Pu)\right) + [Q\,,\,\hat\mu]\,\nabla_x\cdot D_xPu$$
$$+ Q\,\Psi_1^{u,0}(\rho, Qu, \theta) + Q\left(\frac{1}{\hat\rho}\,\nabla_x\mu(\hat\theta)\cdot D_x(Qu)\right) + [Q\,,\,\hat\mu]\,\nabla_x\cdot D_xQu\,,$$

$$\Psi_1^{\theta,1}(Pu, Qu, \theta) = \Psi_1^{\theta,0}(Pu, \theta) + \Psi_1^{\theta,0}(Qu, \theta) + \tfrac{2}{d}\frac{1}{\hat\rho}\,\nabla_x\kappa(\hat\theta)\cdot\nabla_x\theta\,.$$

Notice that the nondispersive part and strictly dispersive part couple through the lower order terms $\Psi_2^{\theta,1}$, $\Psi_1^{Pu,1}$, $\Psi_1^{Qu,1}$, and $\Psi_1^{\theta,1}$.

2.3. **Assumptions for the Estimate.** In order to obtain bounds on the solutions of linear system (2.8) we make the following assumptions on $(\hat\rho, \hat u, \hat\theta)$. These assumptions are the key to choosing the proper space for our well-posedness result.

$\mathcal{A}_1$. *Asymptotic flatness*. There exists constants $c_A, T_1 > 0$ such that $\forall (x,t) \in \mathbb{R}^d \times [0, T_1]$,

$$\left|\partial_t(\hat\rho, \hat u, \hat\theta)(x,t)\right| + \left|\nabla_x(\hat\rho, \hat u, \hat\theta)(x,t)\right| + \left|\partial_t\nabla_x(\hat\rho, \hat u, \hat\theta)(x,t)\right| \le \frac{c_A}{\langle x\rangle^2} \tag{2.17}$$

with $\langle x\rangle^2 \triangleq 1 + |x|^2$.

$\mathcal{A}_2$. *Regularity*. There exists $T_2 > 0$ such that $(\hat\rho, \hat u, \hat\theta)(x,t) \in C_b^{N+1}(\mathbb{R}^d \times [0, T_2])$ where $N = N(d, 3, 0)$ is again given by Theorem 2.1. Here $C_b^{N+1}(\mathbb{R}^d \times [0, T_2])$ is the set of functions that have continuous bounded derivatives up to order $N+1$. Again use c_A to denote the uniform upper bound of the coefficients of Ψ_2 in $C_b^N(\mathbb{R}^d \times [0, T_2])$.

$\mathcal{A}_3$. *Lower bounds*. There exists a constant $\alpha_0 > 0$ such that $\hat\rho, \hat\theta \ge \alpha_0 > 0$. This together with the uniform bounds on $\hat\rho, \hat\theta$ guarantees the existence of a constant $\tau_0 > 0$ such that $\frac{1}{\tau_0} \ge \hat\tau_1/\hat\tau_4 \ge \tau_0 > 0$.

$\mathcal{A}_4$. *Nontrapping condition*. Let $h^{in}(\xi, x) = \sqrt{\hat\tau_1(x,0)\hat\tau_4(x,0)}\,|\xi|^3$ as defined in (1.8) and $H_{h^{in}}$ be the corresponding Hamiltonian flow. Then $H_{h^{in}}$ is nontrapping, that is, if $(\Xi, X)(t; \xi, x)$ is a solution to

$$\frac{\mathrm{d}\Xi}{\mathrm{d}t} = -\nabla_x h^{in}(\Xi, X)\,, \qquad \Xi(0) = \xi\,,$$
$$\frac{\mathrm{d}X}{\mathrm{d}t} = \nabla_\xi h^{in}(\Xi, X)\,, \qquad X(0) = x\,,$$

then for any $\xi \neq 0$,

$$|X(t)| \to \infty \quad \text{as} \quad t \to \pm\infty\,.$$

Remark 2.1. It is sufficient to verify the nontrapping condition for $|\xi| = 1$ because the Hamiltonian flow satisfies the dilation scaling $\Xi(t; \lambda\xi, x) = \lambda\Xi(\lambda^2 t; \xi, x)$, $X(t; \lambda\xi, x) = X(\lambda^2 t; \xi, x)$.

Remark 2.2. Let $q(\xi, x, t)$ be the symbol of any of the second order ΨDO's in the system except the dissipation. Then by (2.17), q is a homogeneous second order polynomial in ξ and

$$|\partial_t q(\xi, x, t)| + |q(\xi, x, t)| \le \frac{c_A}{\langle x\rangle^2}|\xi|^2\,, \qquad \forall (x,t) \in \mathbb{R}^d \times [0, T_1]\,.$$

Notation. Henceforth constants that depend only on the initial data will have a 0 subscript. Constants that depend on the constants that appear in the above assumptions will have an A subscript.

The asymptotic flatness assumption $\mathcal{A}_1$ and the nontrapping condition $\mathcal{A}_4$ will be required by the following lemma, which is due to Chihara [3]. We refer to [3, 5, 10] and references therein for discussions about the necessity of the nontrapping condition for the L^2-well-posedness of dispersive equations.

Lemma 2.1. *Let $\theta_R(\xi) \in C^\infty(\mathbb{R}^d)$ be a cutoff function such that $\theta_R(\xi) = 1$ for $|\xi| \geq 2R$, $\theta_R(\xi) = 0$ for $|\xi| < R$, and $0 < \theta_R(\xi) < 1$ otherwise. Let C^{in} and α^{in} be the constants in (1.6) and (1.7) such that*

$$\sqrt{\hat{\tau}_1(x,0)\hat{\tau}_4(x,0)} \geq \sqrt{\alpha^{in}}\,, \qquad |\nabla_x(\hat{\rho},\hat{u},\hat{\theta})(x,0)| \leq \frac{2C^{in}}{\langle x\rangle^2}\,,$$

and $H_{h^{in}}$ nontrapping. Then there exists $p(\xi, x) \in S^0$ real and constants c_1, c_2 which depend on C^{in} and α^{in}, such that

$$H_{\theta_R h^{in}} p \geq c_1 \frac{|\xi|^2}{\langle x\rangle^2} - c_2\,, \qquad \forall(\xi, x) \in \mathbb{R}^d \times \mathbb{R}^d\,. \tag{2.18}$$

Moreover, finitely many seminorms of p given by (2.6) have bounds that depend only on the constants C^{in} and α^{in}.

Remark 2.3. We will show in Lemma 2.2 that given the assumptions $\mathcal{A}_1$ and $\mathcal{A}_2$, the bound (2.18) propagates for a finite time.

2.4. **Linear Estimate.** The main result of this section is the following bound on solutions of the linear system (2.10). Because this estimate is a priori, we assume that (ρ, u, θ) is a smooth solution to (2.10). Note that the regularity guaranteed by $\mathcal{A}_2$ is enough for the proof.

Theorem 2.2. *Let $(\hat{\rho}, \hat{u}, \hat{\theta})(t, x) \in C([0, T]; H^\infty)$ be functions that satisfy assumptions $\mathcal{A}_1 - \mathcal{A}_4$. Then for every solution $(\rho, u, \theta) \in C([0, T]; H^\infty)$ of the linear system (2.10) there exists $T > 0$ depending on the constants c_0, c_A, and α^{in} in the assumptions and $c > 0$ depending on C^{in} and α^{in} such that*

$$\sup_{[0,T]} \left(\|\rho\|_{H^1}^2 + \|(u,\theta)\|_{L^2}^2\right)(t) + \int_0^T \|\nabla_x(u,\theta)\|_{L^2}^2(s)\,\mathrm{d}s \leq c\left(\|\rho^{in} - \bar{\rho}\|_{H^1}^2 + \|(u^{in}, \theta^{in} - \bar{\theta})\|_{L^2}^2\right)\,. \tag{2.19}$$

Both c and T are independent of ϵ.

Proof. The proof of Theorem 2.2 has six steps. We begin with estimates for (Qu, θ) derived from subsystem (2.16). This subsystem has nondegenerate dispersive terms.

<u>*Step 1. Projection of the equation for Qu.*</u> Let $\xi = (\xi_1, \xi_2, \ldots, \xi_d) \in \mathbb{R}^d$ be the Fourier variable. For $1 \leq k \leq d$, let $p_k(\xi)$ be the symbol of the Riesz transform

$$p_k(\xi) = -i\frac{\xi_k}{|\xi|}\,.$$

Then p_k's are homogenous of order zero. Let Ψ_{p_k} be the ΨDO with the symbol $p_k(\xi)$. For $u = (u_1, u_2, \ldots, u_d)$, abbreviate the scalar function $\sum_{k=1}^d \Psi_{p_k} u_k$ as $\Psi_{p_0} u$. Apply Ψ_{p_0} to the equation for Qu. The resulting equations for $\Psi_{p_0} u$ is

$$\begin{aligned} \partial_t(\Psi_{p_0}u) = &-\epsilon\,\Delta_x^2(\Psi_{p_0}u) + 2\tfrac{d-1}{d}\hat{\mu}\Delta_x(\Psi_{p_0}u) + (-\Delta_x)^{\frac{1}{2}}\left(\hat{\tau}_1\Delta_x\theta\right) \\ &+ \Psi_2^{p_0}(\rho,\theta) + \Psi_1^{p_0}\left(\rho,\ Pu,\ \Psi_{p_0}u,\ \theta\right)\,, \end{aligned} \tag{2.20}$$

where

$$\Psi_2^{p_0}(\rho,\theta) = \Psi_{p_0}\,\Psi_2^{u,0}(\rho,\theta) - \Psi_{p_0}\,(\nabla_x \hat{\tau}_1 \Delta_x \theta) \triangleq \Psi_2^{p_0,1}(\rho) + \Psi_2^{p_0,2}(\theta)\,,$$

$$\begin{aligned}\Psi_1^{p_0}(\rho,\, Pu,\, \Psi_{p_0}u,\, \theta) &= \Psi_{p_0}\,\Psi_1^{Qu,1}\left(\rho,\, Pu,\, \left(\Psi_{p_1}\Psi_{p_0}u,\, \cdots,\, \Psi_{p_d}\Psi_{p_0}u\right),\, \theta\right)\\ &\quad + 2\tfrac{d-2}{d}[\Psi_{p_0},\hat{\mu}]\Delta_x Pu + 2\tfrac{d-2}{d}\sum_{k=1}^{d}[\Psi_{p_k},\hat{\mu}]\Delta_x \Psi_{p_k}\Psi_{p_0}u\,.\end{aligned}$$

Here we applied that $Qu = \left(\Psi_{p_1}\Psi_{p_0}u,\, \cdots,\, \Psi_{p_d}\Psi_{p_0}u\right)$ and the operators Ψ_{p_0} and Δ_x^2 commute over $\mathbb{R}^d$. Next, we rewrite the equation for θ as

$$\partial_t\theta = -\epsilon\,\Delta_x^2\theta + \hat{\kappa}\Delta_x\theta + \hat{\tau}_4(-\Delta_x)^{\frac{3}{2}}(\Psi_{p_0}u) + \Psi_2^{\theta,2}\left(\rho,\, Pu,\, \Psi_{p_0}u,\, \theta\right) + \Psi_1^{\theta,2}\left(\rho,\, Pu,\, \Psi_{p_0}u,\, \theta\right),$$

where

$$\begin{aligned}\Psi_2^{\theta,2}\left(\rho,\, Pu,\, \Psi_{p_0}u,\, \theta\right) &= \Psi_2^{\theta,1}\left(\rho,\, Pu,\, \left(\Psi_{p_1}\Psi_{p_0}u,\, \cdots,\, \Psi_{p_d}\Psi_{p_0}u\right),\, \theta\right)\\ &\triangleq \Psi_2^{\theta,2,1}\left(\rho,\, Pu\right) + \Psi_2^{\theta,2,2}\left(\Psi_{p_0}u,\, \theta\right),\\ \Psi_1^{\theta,2}\left(\rho,\, Pu,\, \Psi_{p_0}u,\, \theta\right) &= \Psi_1^{\theta,1}\left(\rho,\, Pu,\, \left(\Psi_{p_1}\Psi_{p_0}u,\, \cdots,\, \Psi_{p_d}\Psi_{p_0}u\right),\, \theta\right).\end{aligned}$$

Let $\vec{\omega} = (\Psi_{p_0}u,\, \theta)^T = (\omega_1,\omega_2)^T$. Then the system for $\vec{\omega}$ has the form

$$\partial_t\vec{\omega} = -\epsilon\,\Delta_x^2\vec{\omega} + \Psi_D\vec{\omega} + \Psi_{L_0}\vec{\omega} + \Psi_{B_0}\vec{\omega} + \Psi_2^{\omega}(\rho, Pu) + \Psi_1^{\omega}(\rho, Pu, \vec{\omega})\,,$$

with

$$\Psi_D\vec{\omega} = \begin{pmatrix} 2\frac{d-1}{d}\hat{\mu}\Delta_x\omega_1 & 0\\ 0 & \hat{\kappa}\Delta_x\omega_2\end{pmatrix},\quad \Psi_{L_0}\vec{\omega} = \begin{pmatrix}(-\Delta_x)^{\frac{1}{2}}\left(\hat{\tau}_1\,\Delta_x\omega_2\right)\\ \hat{\tau}_4\,(-\Delta_x)^{\frac{3}{2}}\omega_1\end{pmatrix},$$

$$\Psi_{B_0}\vec{\omega} = \begin{pmatrix}\Psi_2^{p_0,2}\omega_2\\ \Psi_2\vec{\omega}\end{pmatrix},\quad \Psi_2^{\omega}(\rho, Pu) = \begin{pmatrix}\Psi_2^{p_0,1}(\rho)\\ \Psi_2^{\theta,2,1}(\rho, Pu)\end{pmatrix},\quad \Psi_1^{\omega}(\rho,\vec{\omega}, Pu) = \begin{pmatrix}\Psi_1^{p_0}(\rho,\, Pu,\, \vec{\omega})\\ \Psi_1^{\theta,2}\,(\rho,\, Pu,\, \vec{\omega})\end{pmatrix}.$$

The leading order of the symbol matrix L_0 is given by

$$L_1 = \begin{pmatrix}0 & -\hat{\tau}_1\,|\xi|^3\\ \hat{\tau}_4\,|\xi|^3 & 0\end{pmatrix}.$$

Here the off-diagonal terms of L_0 can be viewed as third order operators by the comment we made after (2.14). By the symbol calculus, $L_1 - L_0$ is a second order ΨDO. Let B_1 be the leading order term of the second order symbol $B_0 + (L_1 - L_0)$. Then B_1 is homogeneous of order 2 on $\vec{\omega}$. Therefore the system for $\vec{\omega}$ has the form

$$\partial_t\vec{\omega} = -\epsilon\,\Delta_x^2\vec{\omega} + \Psi_D\vec{\omega} + \Psi_{L_1}\vec{\omega} + \Psi_{B_1}\vec{\omega} + \Psi_2^{\omega}(\rho,\, Pu) + \Psi_1^{\omega}(\rho,\vec{\omega}, Pu)\,. \tag{2.21}$$

Henceforth, we will use Ψ_m without any sup-index to denote m-th order operators for the reason we stated before Theorem 2.1.

Step 2. Diagonalization of Ψ_{L_1}. The matrix L_1 has eigenvalues $\lambda_\pm = \pm i\,\sqrt{\hat{\tau}_1\hat{\tau}_4}\,|\xi|^3\,\theta_R(\xi)$. Let

$$A = \begin{pmatrix}-i & \sqrt{\hat{\tau}_1/\hat{\tau}_4}\\ i & \sqrt{\hat{\tau}_1/\hat{\tau}_4}\end{pmatrix},\qquad L = \begin{pmatrix}\lambda_+ & 0\\ 0 & \lambda_-\end{pmatrix}.$$

Then

$$A^{-1} = \tfrac{1}{2}\begin{pmatrix}i & -i\\ \sqrt{\hat{\tau}_4/\hat{\tau}_1} & \sqrt{\hat{\tau}_4/\hat{\tau}_1}\end{pmatrix},\qquad AL_1 = LA\,.$$

Notation. For any $w(x)$ nonnegative, let H^s_w denote the weighted Sobolev space defined as

$$H^s_w \triangleq \{u(x)\;:\; wu \in H^s\}\,,$$

with the norm

$$\|u\|_{H^s_w} = \|wu\|_{H^s}\,.$$

Note that for the weight function $w = \langle x\rangle^2 = 1 + |x|^2$, the norm $\|u\|_{H^s_{\langle x\rangle^2}}$ is equivalent to the norm $\sum_{|\alpha|\le s} \|\langle x\rangle^2 D^\alpha_x u\|_{L^2}$ for every $u \in H^s_{\langle x\rangle^2}$.

Both A and A^{-1} are zeroth order symbols. The operator Ψ_A is therefore a multiplication — i.e. $\Psi_A\vec{\omega} = A\vec{\omega}$. Then as an operator, A is invertible on H^s, $H^s_{\langle x\rangle^{-2}}$, and $H^s_{\langle x\rangle^2}$ for every s by $\mathcal{A}_2$ and $\mathcal{A}_3$.

Multiply (2.21) by A to obtain

$$\begin{aligned}\partial_t\vec{\beta} = \partial_t(A\vec{\omega}) = &-\epsilon A\Delta_x^2\vec{\omega} + A\Psi_D\vec{\omega} + A\Psi_{L_1}\vec{\omega} + A\Psi_{B_1}\vec{\omega} + (\partial_t A)\vec{\omega}\\ &+ A\Psi_2\rho + A\Psi_2 Pu + \Psi_1(\rho, Pu, \vec{\omega}) + \Psi_0\left(\Psi_q Qu\right)\,.\end{aligned} \tag{2.22}$$

Rewrite each term on the right-hand side of (2.22) respectively to obtain the system for $\vec{\beta}$. First,

$$\epsilon A\Delta_x^2\vec{\omega} = \epsilon\,\Delta_x^2 A\vec{\omega} + \epsilon\left(\Psi_{R_1}A^{-1}\right)A\vec{\omega} = \epsilon\,\Delta_x^2\vec{\beta} + \epsilon\,\Psi_{R_2}\vec{\beta}\,,$$

where $\Psi_{R_1} = A\Delta_x^2 - \Delta_x^2 A$ and $\Psi_{R_2} = \Psi_{R_1}A^{-1}$ are third order ΨDO's with seminorms bounded by the constants c_A and τ_0.

Second,

$$A\Psi_{B_1}\vec{\omega} = A\Psi_{B_1}A^{-1}\vec{\beta}\,.$$

Because both A and A^{-1} are of zeroth order, $A\Psi_{B_1}A^{-1}$ is still a second order (matrix) operator and we use Ψ_{B_2} to denote this operator. Notice that there exists a $T_1 > 0$ depending on c_A such that B_2 satisfies

$$|B_2(\xi, x, t)| \le \frac{c_{0,1}|\xi|^2}{\langle x\rangle^2}\,,\qquad \forall (x,t)\in\mathbb{R}^d\times[0,T_1]\,,\quad |\xi|\ge 2R\,, \tag{2.23}$$

with $c_{0,1}$ depending only on the initial data.

Next,

$$\begin{aligned}A\Psi_D A^{-1} &= \tfrac{1}{2}\begin{pmatrix}-i & \sqrt{\hat{\tau}_1/\hat{\tau}_4}\\ i & \sqrt{\hat{\tau}_1/\hat{\tau}_4}\end{pmatrix}\begin{pmatrix}2\frac{d-1}{d}\hat{\mu}\Delta_x & 0\\ 0 & \hat{\kappa}\Delta_x\end{pmatrix}\begin{pmatrix} i & -i\\ \sqrt{\hat{\tau}_4/\hat{\tau}_1} & \sqrt{\hat{\tau}_4/\hat{\tau}_1}\end{pmatrix}\\ &= \begin{pmatrix}\left(\frac{d-1}{d}\hat{\mu} + \frac{1}{2}\hat{\kappa}\right)\Delta_x & 0\\ 0 & \left(\frac{d-1}{d}\hat{\mu} + \frac{1}{2}\hat{\kappa}\right)\Delta_x\end{pmatrix} + \begin{pmatrix}0 & \left(-\frac{d-1}{d}\hat{\mu} + \frac{1}{2}\hat{\kappa}\right)\Delta_x\\ \left(-\frac{d-1}{d}\hat{\mu} + \frac{1}{2}\hat{\kappa}\right)\Delta_x & 0\end{pmatrix} + \Psi_{r_1}\,,\end{aligned}$$

where Ψ_{r_1} is a first order ΨDO. Therefore,

$$A\Psi_D\vec{\omega} = \left(\tfrac{d-1}{d}\hat{\mu} + \tfrac{1}{2}\hat{\kappa}\right)\Delta_x\vec{\beta} + \begin{pmatrix}0 & \left(-\frac{d-1}{d}\hat{\mu} + \frac{1}{2}\hat{\kappa}\right)\Delta_x\\ \left(-\frac{d-1}{d}\hat{\mu} + \frac{1}{2}\hat{\kappa}\right)\Delta_x & 0\end{pmatrix}\vec{\beta} + \Psi_{r_1}A^{-1}\vec{\beta}\,.$$

Notice that although the second term on the right side of the above equation is of second order, there is no contribution from the diagonal. Combine this term with Ψ_{B_2} and use Ψ_{B_3} to denote this new second order operator and B_3 satisfies the same property as B_2 in (2.23).

Next, we study the structure of $A\Psi_{L_1}$. Using the fact that $AL_1 = LA$, we have

$$A\Psi_{L_1} = \Psi_L A + (\Psi_{AL_1} - \Psi_{LA}) + (\Psi_{LA} - \Psi_L A) = \Psi_L A + (\Psi_{LA} - \Psi_L A)\,.$$

Let B_4 be the symbol matrix corresponding to the leading order of $\Psi_{LA} - \Psi_L A$. By the symbol calculus,

$$B_4 = \sum_{k=1}^{d} (\partial_{\xi_k}L)\,(\partial_{x_k}A)\,.$$

By assumptions $\mathscr{A}_1 - \mathscr{A}_3$, there exist $T_2, c_{0,2} > 0$ such that

$$|B_4(\xi, x, t)| \leq \frac{c_{0,2}|\xi|^2}{\langle x\rangle^2}, \qquad \forall (x,t) \in \mathbb{R}^d \times [0, T_2], \quad |\xi| \geq 2R.$$

Combine Ψ_{B_3} with Ψ_{B_4} and use Ψ_B to denote this second order operator. Let

$$T_3 = \min\{T_1, T_2\}, \qquad c_{0,3} = \min\{c_{0,1}, c_{0,2}\}.$$

Then the diagonal of B satisfies that

$$|B_{diag}(\xi, x, t)| \leq \frac{c_{0,3}|\xi|^2}{\langle x\rangle^2}, \qquad \forall (x,t) \in \mathbb{R}^d \times [0, T_3], \quad |\xi| \geq 2R.$$

For the rest of the terms on the right-hand side of (2.22),

$$(\partial_t A)\,\vec{\omega} = (\partial_t A)A^{-1}\vec{\beta} = \Psi_0\vec{\beta}, \qquad A\Psi_2 A^{-1} = \Psi_2, \qquad A\Psi_1 A^{-1} = \Psi_1.$$

Overall the system for $\vec{\beta}$ has the form

$$\begin{aligned} \partial_t\vec{\beta} = {} & -\epsilon\,\Delta_x^2\vec{\beta} + \epsilon\,\Psi_{R_2}\vec{\beta} + \left(\tfrac{d-1}{d}\hat{\mu} + \tfrac{1}{2}\hat{\kappa}\right)\Delta_x\vec{\beta} + \Psi_L\vec{\beta} + \Psi_B\vec{\beta} \\ & + \Psi_2(\rho, Pu) + \Psi_1(\rho, Pu, \vec{\beta}) + \Psi_0\left(\Psi_q Qu\right). \end{aligned} \tag{2.24}$$

<u>*Step 3. Diagonalization of* Ψ_B</u>. Write

$$\Psi_B = \Psi_{B_{diag}} + \Psi_{B_{anti}} = \begin{pmatrix} \Psi_{B_{11}} & 0 \\ 0 & \Psi_{B_{22}} \end{pmatrix} + \begin{pmatrix} 0 & \Psi_{B_{12}} \\ \Psi_{B_{21}} & 0 \end{pmatrix}.$$

We show in the following that $\Psi_{B_{anti}}$ can be eliminated using Ψ_L. To this end, let

$$h(\xi, x, t) = \sqrt{\hat{\tau}_1\hat{\tau}_4}|\xi|^3, \qquad \tilde{h}(\xi, x, t) = h^{-1}(\xi, x, t)\,\theta_R(\xi).$$

Then $\Psi_{\tilde{h}}$ is of order -3 uniformly in t and $\Psi_{\tilde{h}}\Psi_h = I + \Psi_{r_2}$ with Ψ_{r_2} of order -1 uniformly in t.

Define the operators

$$T_{12} = \tfrac{i}{2}\,\Psi_{B_{12}}\Psi_{\tilde{h}}, \qquad T_{21} = -\tfrac{i}{2}\,\Psi_{B_{21}}\Psi_{\tilde{h}}, \qquad T = \begin{pmatrix} 0 & T_{12} \\ T_{21} & 0 \end{pmatrix},$$

and the diagonalizing transformation Λ of order 0

$$\Lambda = I - T.$$

Because T is of order -1, its S^0 seminorm is of order $O(R^{-1})$. Therefore by taking R large enough one can assume that Λ is invertible on H^s, $H^s_{\langle x\rangle^2}$, and $H^s_{\langle x\rangle^{-2}}$ with the operator norm between $1/2$ and 2. The inverse of Λ is also of order 0 with operator norm between $1/2$ and 2.

To diagonalize Ψ_B, apply the transformation Λ to system (2.24). First,

$$\epsilon\Lambda\Delta_x^2 + \epsilon\Lambda\Psi_{R_2} = \epsilon\Delta_x^2\Lambda + \epsilon(\Lambda\Delta_x^2 - \Delta_x^2\Lambda)\Lambda^{-1}\Lambda + \epsilon(\Lambda\Psi_{R_2}\Lambda^{-1})\Lambda = \epsilon\Delta_x^2\Lambda + \epsilon\Psi_{R_3}\Lambda,$$

with $\Psi_{R^3} = (\Lambda\Delta_x^2 - \Delta_x^2\Lambda)\Lambda^{-1} + \Lambda\Psi_{R_2}\Lambda^{-1}$ being a third order ΨDO. The seminorms of Ψ_{R^3} depend on the constants c_A and τ_0.

Second,

$$\Lambda\partial_t\vec{\beta} = \partial_t(\Lambda\vec{\beta}) - (\partial_t\Lambda)\Lambda^{-1}\Lambda\vec{\beta},$$

where $(\partial_t\Lambda)\Lambda^{-1}$ is a zeroth order operator.

Next, by the facts that the symbol of T is in S^{-1} and that

$$\Lambda\Psi_{B_{diag}} - \Psi_{B_{diag}}\Lambda = -T\Psi_{B_{diag}} + \Psi_{B_{diag}}T,$$

it is clear that $\Lambda\Psi_{B_{diag}} = \Psi_{B_{diag}}\Lambda + \Psi_1\Lambda$.

Similarly,

$$\Lambda\Psi_{B_{anti}} = \Psi_{B_{anti}}\Lambda + \Psi_1\Lambda\,, \qquad \Lambda\left(\tfrac{d-1}{d^2}\hat{\mu} + \hat{\kappa}\right)\Delta_x = \left(\tfrac{d-1}{d^2}\hat{\mu} + \hat{\kappa}\right)\Delta_x\Lambda + \Psi_1\Lambda\,,$$
$$\Lambda\Psi_1 = \Psi_1\Lambda + \Psi_0\Lambda\,, \qquad \Lambda\Psi_2 = \Psi_2 + \Psi_1\,,$$

For the term $\Lambda\Psi_L - \Psi_L\Lambda$, we have

$$\begin{aligned}\Lambda\Psi_L - \Psi_L\Lambda = \Psi_L T - T\Psi_L &= i\begin{pmatrix}\Psi_h & 0\\ 0 & -\Psi_h\end{pmatrix}\begin{pmatrix}0 & T_{12}\\ T_{21} & 0\end{pmatrix} - i\begin{pmatrix}0 & T_{12}\\ T_{21} & 0\end{pmatrix}\begin{pmatrix}\Psi_h & 0\\ 0 & -\Psi_h\end{pmatrix}\\ &= i\begin{pmatrix}0 & \Psi_h T_{12} + T_{12}\Psi_h\\ -[\Psi_h T_{21} + T_{21}\Psi_h] & 0\end{pmatrix}.\end{aligned}$$

Because $\Psi_h T_{12} = T_{12}\Psi_h + \Psi_1$, we have

$$\begin{aligned} i(\Psi_h T_{12} + T_{12}\Psi_h) &= 2iT_{12}\Psi_h + \Psi_1 = -\Psi_{B_{12}} + \Psi_1\,,\\ -i(\Psi_h T_{21} + T_{21}\Psi_h) &= -2iT_{21}\Psi_h + \Psi_1 = -\Psi_{B_{21}} + \Psi_1\,.\end{aligned}$$

Therefore,

$$\Lambda\Psi_L + \Lambda\Psi_{B_{anti}} = \Psi_L\Lambda + \Psi_1\,.$$

Let $\vec{z} = \Lambda\vec{\beta}$. Then the system for $\vec{z}$ has the form

$$\begin{aligned}\partial_t\vec{z} &= -\epsilon\,\Delta_x^2\vec{z} + \epsilon\Psi_{R_3}\vec{z} + \left(\tfrac{d-1}{d^2}\hat{\mu} + \hat{\kappa}\right)\Delta_x\vec{z} + \Psi_L\vec{z}\\ &\quad + \Psi_{B_{diag}}\vec{z} + \Psi_2\rho + \Psi_2 Pu + \Psi_1(\rho, Pu, \vec{z}) + \Psi_0\left(\rho,\ Pu,\ \vec{z},\ \Psi_q Qu\right)\,,\end{aligned}\tag{2.25}$$

where R_3 is a third order operator.

<u>*Step 4. Regularized Pu and* ρ.</u> By utilizing the dispersive operator Ψ_L as in *Step 3*, we can eliminate the second order term including θ in the equation for Pu and the first order term $\rho\nabla_x\cdot u$ in the mass equation.

To this end, write θ in terms of $\vec{z} = (z_1, z_2)^T$. Recall that $\vec{z} = \Lambda A\vec{\omega}$ where Λ and A are both invertible. Solve $\vec{\omega}$ in terms of $\vec{z}$ to obtain

$$\vec{\omega} = A^{-1}\Lambda^{-1}\vec{z} = A^{-1}\begin{pmatrix}\Lambda_{11} & \Lambda_{12}\\ \Lambda_{21} & \Lambda_{22}\end{pmatrix}\vec{z} = \frac{1}{2}\begin{pmatrix} i & -i\\ \sqrt{\hat{\tau}_4/\hat{\tau}_1} & \sqrt{\hat{\tau}_4/\hat{\tau}_1}\end{pmatrix}\begin{pmatrix}\Lambda_{11}z_1 + \Lambda_{12}z_2\\ \Lambda_{21}z_1 + \Lambda_{22}z_2\end{pmatrix},$$

which shows that

$$\theta = \sqrt{\hat{\tau}_4/\hat{\tau}_1}\left((\Lambda_{11} + \Lambda_{21})z_1 + (\Lambda_{12} + \Lambda_{22})z_2\right).$$

Hence,

$$\begin{aligned}\Psi_2\theta &= \Psi_2\left(\tfrac{\sqrt{\tau_4/\tau_1}}{d}(\Lambda_{11} + \Lambda_{21})\right)z_1 + \Psi_2\left(\tfrac{\sqrt{\tau_4/\tau_1}}{d}(\Lambda_{12} + \Lambda_{22})\right)z_2\\ &\triangleq \Psi_{\Gamma_1}z_1 + \Psi_{\Gamma_2}z_2\,.\end{aligned}$$

Because Λ_{kl} is of zeroth order for all $1 \le k, l \le 2$, Ψ_{Γ_1} and Ψ_{Γ_2} are both second order operators with seminorms bounded by the constants c_A and τ_0.

Now define

$$T_1 = i\Psi_{\Gamma_1}\Psi_{\tilde{h}}\,, \qquad T_2 = i\Psi_{\Gamma_2}\Psi_{\tilde{h}}\,.$$

The operators T_1 and T_2 are of order -1 and

$$T_1\Psi_{ih} + \Psi_{\Gamma_1} = -\Psi_{\Gamma_1}\Psi_{\tilde{h}}\Psi_h + \Psi_{\Gamma_1}\,, \qquad -T_2\Psi_{ih} + \Psi_{\Gamma_2} = -\Psi_{\Gamma_2}\Psi_{\tilde{h}}\Psi_h + \Psi_{\Gamma_2}\,,$$

are both first order operators. Upon applying T_1 and T_2 to the respective equations of system (2.25), we find that T_1z_1 and T_2z_2 obey

$$\begin{aligned}\partial_t(T_1z_1) &= -\epsilon\,\Delta_x^2(T_1z_1) + \epsilon\,\Psi_{R_4}z_1 + T_1\Psi_{ih}z_1 + \Psi_1(\rho, Pu, \vec{z}) + \Psi_0(\rho,\ Pu,\ \vec{z},\ \Psi_q Qu)\,,\\ \partial_t(T_2z_2) &= -\epsilon\,\Delta_x^2(T_2z_2) + \epsilon\,\Psi_{R_5}z_2 - T_2\Psi_{ih}z_2 + \Psi_1(\rho, Pu, \vec{z}) + \Psi_0(\rho,\ Pu,\ \vec{z},\ \Psi_q Qu)\,,\end{aligned}\tag{2.26}$$

where $R_4, R_5 \in S^2$. Upon adding the equations in (2.26) to the equation for Pu in system (2.15), we see that $\vec{y} = Pu + T_1 z_1 + T_2 z_2$ satisfies

$$\partial_t \vec{y} = -\epsilon\, \Delta_x^2 \vec{y} + \epsilon\, \Psi_{R_4} z_1 + \epsilon\, \Psi_{R_5} z_2 + \hat{\mu}\Delta_x \vec{y} + \Psi_2 \rho + \Psi_1(\rho, \vec{y}, \vec{z}) + \Psi_0(\rho,\, \vec{y},\, \vec{z},\, \Psi_q Qu)\,, \tag{2.27}$$

where $R_4, R_5 \in S^3$.

To eliminate the term $\hat{\rho}\nabla_x \cdot u$ in the mass equation, write

$$\hat{\rho}\nabla_x \cdot u = \hat{\rho}\nabla_x \cdot Qu \triangleq \Psi_{\Gamma_3} z_1 + \Psi_{\Gamma_4} z_2\,.$$

Define the operators

$$T_3 = i\Psi_{\Gamma_3}\Psi_{\tilde{h}}\,, \qquad T_4 = i\Psi_{\Gamma_4}\Psi_{\tilde{h}}\,.$$

Then T_3 and T_4 are of order -2, while the operators $T_3\Psi_{ih} + \Psi_{\Gamma_3}$ and $T_4\Psi_{ih} + \Psi_{\Gamma_4}$ are of order zero. The equation for $\varrho = \rho + T_3 z_1 + T_4 z_2$ is

$$\partial_t \varrho = -\epsilon\, \Delta_x^2 \varrho + \epsilon\, \Psi_{R_6} z_1 + \epsilon\, \Psi_{R_7} z_2 - \nabla_x \varrho \cdot u + \Psi_0(\varrho,\, \vec{y},\, \vec{z},\, \Psi_q Qu)\,, \tag{2.28}$$

where $R_6, R_7 \in S^2$.

Step 5. A further transformation. Before defining a further transformation, we prove the following lemma which is an extension of Lemma 2.1.

Lemma 2.2. *There exits $T^* > 0$, depending only on the constants in $\mathcal{A}_1$, $\mathcal{A}_3$, and Lemma 1, such that for every $t \in [0, T^*)$ one has*

$$H_{\theta_R h} p = \{\theta_R h, p\}\,(\xi, x, t) \geq \frac{c_1}{2}\frac{|\xi|^2}{\langle x\rangle^2} - \tilde{c}_2\,, \qquad \forall(\xi, x) \in \mathbb{R}^d \times \mathbb{R}^d\,, \quad |\xi| \geq 2R\,,$$

where $\tilde{c}_2$ depends only on α_0 and c_0.

Proof. By definition,

$$H_{\theta_R h} p = \sum_{j=1}^{d} \left(\partial_{\xi_j} h\, \partial_{x_j} p - \partial_{x_j} h\, \partial_{\xi_j} p\right) \theta_R + r\,,$$
$$H_{\theta_R h^{in}} p = \sum_{j=1}^{d} \left(\partial_{\xi_j} h^{in}\, \partial_{x_j} p - \partial_{x_j} h^{in}\, \partial_{\xi_j} p\right) \theta_R + r^{in}\,,$$

where

$$r = \sum_{j=1}^{d} h\, \partial_{\xi_j}\theta_R\, \partial_{x_j} p\,, \qquad r^{in} = \sum_{j=1}^{d} h^{in}\, \partial_{\xi_j}\theta_R\, \partial_{x_j} p\,.$$

By the definition of $\theta_R(\xi)$, we have $\partial_{\xi_j}\theta_R$ compactly supported. Therefore, $r, r^{in} \in S^0$ and the bound of their S^0 seminorms depends only on the constants c_0 and α_0. For $|\xi| > 2R$ we have

$$\partial_{\xi_j} h = 3\sqrt{\hat{\tau}_1\hat{\tau}_4}|\xi|\xi_j\,, \qquad \partial_{x_j} h = \partial_{x_j}\left(\sqrt{\hat{\tau}_1\hat{\tau}_4}\right)|\xi|^3\,.$$

Thus, it follows from assumption $\mathcal{A}_1$ that

$$|\partial_{\xi_j} h(\xi, x, t) - \partial_{\xi_j} h(\xi, x, 0)| \leq \frac{c_{0,4}T^*}{\langle x\rangle^2}|\xi|^2\,,$$
$$|\partial_{x_j} h(\xi, x, t) - \partial_{x_j} h(\xi, x, 0)| \leq \frac{c_{0,5}T^*}{\langle x\rangle^2}|\xi|^3\,.$$

Therefore, there exist $c_{0,6}, c_{0,7} > 0$ such that

$$\begin{aligned}\left|H_{\theta_R h}p - H_{\theta_R h^{in}}p\right| &\le \sum_{j=1}^{d}\left|\partial_{\xi_j}h - \partial_{\xi_j}h^{in}\right|\left|\partial_{x_j}p\right| + \sum_{j=1}^{d}\left|\partial_{x_j}h - \partial_{x_j}h^{in}\right|\left|\partial_{\xi_j}p\right| + |r| + |r^{in}| \\ &\le \frac{c_{0,6}T^*}{\langle x\rangle^2}|\xi|^2 + c_{0,7}\,.\end{aligned}$$

Choosing T^* small enough and applying Lemma 2.1, we have

$$H_{\theta_R h}p \ge H_{\theta_R h^{in}}p - \left|H_{\theta_R h}p - H_{\theta_R h^{in}}p\right| \ge (c_1 - c_{0,6}T^*)\frac{|\xi|^2}{\langle x\rangle^2} - c_2 - c_{0,7} \ge \frac{c_1}{2}\frac{|\xi|^2}{\langle x\rangle^2} - \tilde{c}_2$$

where $\tilde{c}_2 = c_2 + c_{0,7}$ depends only on α_0 and c_0. □

To construct a further transformation, let

$$q_1(\xi, x) = \exp(Mp(\xi, x)\theta_R(\xi))\,, \qquad q_2(\xi, x) = \exp(-Mp(\xi, x)\theta_R(\xi))\,,$$

where $\theta_R(\xi)$ is again the cutoff function and $M > 0$ to be chosen. Then

$$\Psi_{q_1}\Psi_{q_2} = I + \Psi_{r_2}\,, \qquad \Psi_{q_2}\Psi_{q_1} = I + \Psi_{r_3}\,,$$

with $r_2, r_3 \in S^{-1}$. Thus Ψ_{q_1} and Ψ_{q_2} are invertible and their inverses are of order 0 for large R.

By the calculus of symbols, we have

$$\Psi_h\Psi_{q_1} - \Psi_{q_1}\Psi_h = \Psi_{-i\{h,q_1\}} + \Psi_1\,,$$

with

$$\begin{aligned}\{h, q_1\} &= \sum_{j=1}^{d}\left(\partial_{\xi_j}h\,\partial_{x_j}q_1 - \partial_{x_j}h\,\partial_{\xi_j}q_1\right) \\ &= \sum_{j=1}^{d}\left(\partial_{\xi_j}h\,M\,\partial_{x_j}p\theta_R - \partial_{x_j}h(M\partial_{\xi_j}p)\theta_R\right)q_1 - \sum_{j=1}^{d}\left(\partial_{x_j}h\,Mp\partial_{\xi_j}\theta_R\right)q_1\,.\end{aligned}$$

Hence,

$$\{h, q_1\} = M\theta_R(\xi)\,(H_h p)\;q_1 + \Psi_0\,.$$

Therefore

$$\Psi_{ih}\Psi_{q_1} - \Psi_{q_1}\Psi_{ih} = \Psi_{M\theta_R H_h p}\Psi_{q_1} + \Psi_1\,.$$

A similar computation shows that

$$\Psi_{ih}\Psi_{q_2} - \Psi_{q_2}\Psi_{ih} = -\Psi_{M\theta_R H_h p}\Psi_{q_2} + \Psi_1\,.$$

Step 6. Energy estimate. Consider the system in the following variable

$$\vec{\alpha} = \begin{pmatrix}\Psi_{q_1} & 0\\ 0 & \Psi_{q_2}\end{pmatrix}\vec{z} \triangleq \Psi\vec{z}.$$

Notice that Ψ is invertible and $\Psi^{-1} = \begin{pmatrix}\Psi_{q_1}^{-1} & 0\\ 0 & \Psi_{q_2}^{-1}\end{pmatrix}$ is also a matrix of order 0.

To compute the system for $\vec{\alpha}$, multiply system (2.25) by Ψ.

$$\begin{aligned}\partial_t\vec{\alpha} = \Psi\partial_t\vec{z} = &-\epsilon\,\Psi\Delta_x^2\vec{z} + \epsilon\,\Psi\Psi_{R_3}\vec{z} + \Psi\left(\tfrac{d-1}{d}\hat{\mu} + \tfrac{1}{2}\hat{\kappa}\right)\Delta_x\vec{z} + \Psi\Psi_L\vec{z} \\ &+ \Psi\Psi_{B_{diag}}\vec{z} + \Psi\Psi_2(\varrho, \vec{y}) + \Psi\Psi_1(\varrho, \vec{y}, \vec{z}) + \Psi\Psi_0(\varrho, \vec{y}, \vec{z}, \Psi_q Qu)\,.\end{aligned}$$

Evaluate each term on the right as follows. First, there exists a $R_6 \in S^3$ such that

$$-\epsilon\Psi\Delta_x^2\vec{z} + \epsilon\Psi\Psi_{R_3}\vec{z} = -\epsilon\Delta_x^2\vec{\alpha} + \epsilon\Psi_{R_6}\vec{\alpha}\,.$$

Second,

$$\begin{aligned}\Psi\left(\tfrac{d-1}{d}\hat{\mu} + \hat{\kappa}\right)\Delta_x\vec{z} &= \left(\tfrac{d-1}{d}\hat{\mu} + \hat{\kappa}\right)\Psi\Delta_x\Psi^{-1}\vec{\alpha} + \Psi_1\vec{\alpha}\\ &= \left(\tfrac{d-1}{d}\hat{\mu} + \hat{\kappa}\right)\begin{pmatrix}\Psi_{q_1}\Delta_x\Psi_{q_1}^{-1} & 0\\ 0 & \Psi_{q_2}\Delta_x\Psi_{q_2}^{-1}\end{pmatrix}\vec{\alpha} + \Psi_1\vec{\alpha}\\ &= \left(\tfrac{d-1}{d}\hat{\mu} + \hat{\kappa}\right)\Delta_x\vec{\alpha} + \Psi_1\vec{\alpha}\,.\end{aligned}$$

Similarly,

$$\Psi\Psi_{B_{diag}}\vec{z} = \Psi\Psi_{B_{diag}}\Psi^{-1}\vec{z} = \Psi_{B_{diag}}\vec{\alpha} + \Psi_1\vec{\alpha}\,.$$

Next,

$$\begin{aligned}\Psi\Psi_2(\varrho,\vec{y}) &= \Psi_2(\varrho,\vec{y})\,, & \Psi\Psi_1(\varrho,\vec{y}) &= \Psi_1(\varrho,\vec{y})\,,\\ \Psi\Psi_1\vec{z} = \Psi\Psi_1\Psi^{-1}\vec{\alpha} &= \Psi_1\vec{\alpha}\,, & \Psi\Psi_0\vec{z} = \Psi\Psi_0\Psi^{-1}\vec{\alpha} &= \Psi_0\vec{\alpha}\,,\end{aligned}$$

For the dispersive part,

$$\begin{aligned}\Psi\Psi_L\vec{z} &= \Psi_L\vec{\alpha} + \begin{pmatrix}\Psi_{q_1}\Psi_{ih} - \Psi_{ih}\Psi_{q_1} & 0\\ 0 & -(\Psi_{q_2}\Psi_{ih} - \Psi_{ih}\Psi_{q_2})\end{pmatrix}\vec{z}\\ &= \Psi_L\vec{\alpha} + \begin{pmatrix}-\Psi_{M\theta_R H_h p} & 0\\ 0 & -\Psi_{M\theta_R H_h p}\end{pmatrix}\Psi\vec{z} + \Psi_1\vec{\alpha}\,.\end{aligned}$$

Therefore, the system for $\vec{\alpha}$ has the form

$$\begin{aligned}\partial_t\vec{\alpha} = &-\epsilon\,\Delta_x^2\vec{\alpha} + \epsilon\,\Psi_{R_6}\vec{\alpha} + \left(\tfrac{d-1}{d}\hat{\mu} + \tfrac{1}{2}\hat{\kappa}\right)\Delta_x\vec{\alpha} + \Psi_L\vec{\alpha} + \begin{pmatrix}-\Psi_{M\theta_R H_h p} & 0\\ 0 & -\Psi_{M\theta_R H_h p}\end{pmatrix}\vec{\alpha}\\ &+ \Psi_{B_{diag}}\vec{\alpha} + \Psi_2(\varrho,\vec{y}) + \Psi_1(\varrho,\vec{\alpha},\vec{y}) + \Psi_0(\varrho,\vec{\alpha},\vec{y},\,\Psi_q Qu)\,.\end{aligned}\tag{2.29}$$

Now we derive the energy estimate for $(\varrho,\vec{\alpha},\vec{y},\Psi_q Qu)$. First, we multiply (2.29) by $\vec{\alpha}$ and integrate over $\mathbb{R}^d$ to obtain

$$\begin{aligned}\frac{d}{dt}\langle\vec{\alpha},\vec{\alpha}\rangle &= \langle\partial_t\vec{\alpha}\,,\,\vec{\alpha}\rangle + \langle\vec{\alpha}\,,\,\partial_t\vec{\alpha}\rangle\\ &= -\epsilon\,\langle\Delta_x^2\vec{\alpha}\,,\,\vec{\alpha}\rangle - \epsilon\,\langle\vec{\alpha}\,,\,\Delta_x^2\vec{\alpha}\rangle + \epsilon\,\langle\Psi_{R_6}\vec{\alpha}\,,\,\vec{\alpha}\rangle + \epsilon\,\langle\vec{\alpha}\,,\,\Psi_{R_6}\vec{\alpha}\rangle\\ &\quad + \left\langle\left(\tfrac{d-1}{d}\hat{\mu} + \tfrac{1}{2}\hat{\kappa}\right)\Delta_x\vec{\alpha}\,,\,\vec{\alpha}\right\rangle + \left\langle\vec{\alpha}\,,\,\left(\tfrac{d-1}{d}\hat{\mu} + \tfrac{1}{2}\hat{\kappa}\right)\Delta_x\vec{\alpha}\right\rangle\\ &\quad + \langle\Psi_L\vec{\alpha}\,,\,\vec{\alpha}\rangle + \langle\vec{\alpha}\,,\,\Psi_L\vec{\alpha}\rangle\\ &\quad + \left\langle\begin{pmatrix}-\Psi_{M\theta_R H_h p} & 0\\ 0 & -\Psi_{M\theta_R H_h p}\end{pmatrix}\vec{\alpha}\,,\,\vec{\alpha}\right\rangle + \left\langle\vec{\alpha}\,,\,\begin{pmatrix}-\Psi_{M\theta_R H_h p} & 0\\ 0 & -\Psi_{M\theta_R H_h p}\end{pmatrix}\vec{\alpha}\right\rangle\\ &\quad + \langle\Psi_{B_{diag}}\vec{\alpha}\,,\,\vec{\alpha}\rangle + \langle\vec{\alpha}\,,\,\Psi_{B_{diag}}\vec{\alpha}\rangle + \langle\Psi_2\vec{y}\,,\,\vec{\alpha}\rangle + \langle\vec{\alpha}\,,\,\Psi_2\vec{y}\rangle\\ &\quad + \langle\Psi_2\varrho\,,\,\vec{\alpha}\rangle + \langle\vec{\alpha}\,,\,\Psi_2\varrho\rangle + \langle\Psi_1\vec{\alpha}\,,\,\vec{\alpha}\rangle + \langle\vec{\alpha}\,,\,\Psi_1\vec{\alpha}\rangle\\ &\quad + \langle\Psi_1\vec{y}\,,\,\vec{\alpha}\rangle + \langle\vec{\alpha}\,,\,\Psi_1\vec{y}\rangle + \langle\Psi_1\varrho\,,\,\vec{\alpha}\rangle + \langle\vec{\alpha}\,,\,\Psi_1\varrho\rangle\\ &\quad + \langle\Psi_0\vec{\alpha}\,,\,\vec{\alpha}\rangle + \langle\vec{\alpha}\,,\,\Psi_0\vec{\alpha}\rangle + \langle\Psi_0\vec{y}\,,\,\vec{\alpha}\rangle + \langle\vec{\alpha}\,,\,\Psi_0\vec{y}\rangle\\ &\quad + \langle\Psi_0\varrho\,,\,\vec{\alpha}\rangle + \langle\vec{\alpha}\,,\,\Psi_0\varrho\rangle + \langle\Psi_0\Psi_q Qu\,,\,\vec{\alpha}\rangle + \langle\Psi_q Qu\,,\,\Psi_0\vec{\alpha}\rangle\,.\end{aligned}$$

Now estimate each term above. For the terms containing ϵ,

$$-\epsilon\,\langle\Delta_x^2\vec{\alpha}\,,\,\vec{\alpha}\rangle - \epsilon\,\langle\vec{\alpha}\,,\,\Delta_x^2\vec{\alpha}\rangle = -2\epsilon\,\|\Delta_x\vec{\alpha}\|_{L^2}^2\,.$$

Because $R_4 \in S^3$, it is clear that

$$\epsilon \left|\langle \Psi_{R_4}\vec{\alpha}\,,\,\vec{\alpha}\rangle\right| + \epsilon \left|\langle \vec{\alpha}\,,\,\Psi_{R_4}\vec{\alpha}\rangle\right| \le \epsilon\, c_{A,1}\|\vec{\alpha}\|^2_{H^{\frac{3}{2}}} \le 2\epsilon\,\|\Delta_x\vec{\alpha}\|^2_{L^2} + \epsilon\, c_{A,2}\|\vec{\alpha}\|^2_{L^2}\,.$$

For the dissipative term,

$$\left\langle \left(\tfrac{d-1}{d}\hat{\mu} + \tfrac{1}{2}\hat{\kappa}\right)\Delta_x\vec{\alpha}\,,\,\vec{\alpha}\right\rangle + \left\langle \vec{\alpha}\,,\,\left(\tfrac{d-1}{d}\hat{\mu} + \tfrac{1}{2}\hat{\kappa}\right)\Delta_x\vec{\alpha}\right\rangle \le -c_{0,7}\|\nabla_x\vec{\alpha}\|^2_{L^2(\mathbb{R}^2)} + c_{A,3}\|\vec{\alpha}\|^2_{L^2}\,,$$

where $c_{0,7} > 0$ depends only on α^{in}.

Next, by the calculus it can be shown that

$$\langle \Psi_L\vec{\alpha}\,,\,\vec{\alpha}\rangle + \langle \vec{\alpha}\,,\,\Psi_L\vec{\alpha}\rangle = \langle \vec{\alpha}\,,\,\Psi_{\hat{B}_{diag}}\vec{\alpha}\rangle\,,$$

where

$$\hat{B}_{diag} = \begin{pmatrix}\hat{B}_{11} & 0\\ 0 & \hat{B}_{22}\end{pmatrix}, \quad |\hat{B}_{kk}| \le \frac{c_{0,8}}{\langle x\rangle^2}|\xi|^2\,, \quad \forall (x,t)\in\mathbb{R}^d\times[0,T_4]\,, \quad |\xi|\ge 2R\,, \quad k=1,2\,.$$

By the fact that $\hat{B}_{diag}$ is real, we can combine $\frac{1}{2}\hat{B}_{diag}$ with B_{diag} and still denote it $B_{diag} = \begin{pmatrix}B_{11} & 0\\ 0 & B_{22}\end{pmatrix}$. Notice that $|B_{kk}| \le \frac{c_{0,10}|\xi|^2}{\langle x\rangle^2}$ for $k=1,2$. By taking M large enough we have

$$-M\theta_R H_h p + |B_{kk}| \le c_{0,11} - \tfrac{1}{2}c_{0,\eta}\frac{|\xi|^2}{\langle x\rangle^2}.$$

The choice of M depends only on the initial data. Let $c' = c_{0,\eta}$. Then c' depends only on the data.

By the *sharp Gårding* inequality,

$$\begin{aligned}
&\left\langle\begin{pmatrix}-\Psi_{M\theta_R H_h p} + B_{11} & 0\\ 0 & -\Psi_{M\theta_R H_h p} + B_{22}\end{pmatrix}\vec{\alpha}\,,\,\vec{\alpha}\right\rangle + \left\langle\vec{\alpha}\,,\,\begin{pmatrix}-\Psi_{M\theta_R H_h p} + B_{11} & 0\\ 0 & -\Psi_{M\theta_R H_h p} + B_{22}\end{pmatrix}\vec{\alpha}\right\rangle\\
&\le c\|\vec{\alpha}\|^2_{H^{\frac{1}{2}}} - \mathrm{Re}\left\langle\begin{pmatrix}\Psi_{c'|\xi|^2/\langle x\rangle^2} & 0\\ 0 & \Psi_{c'|\xi|^2/\langle x\rangle^2}\end{pmatrix}\vec{\alpha}\,,\,\vec{\alpha}\right\rangle\\
&\le \eta\|\vec{\alpha}\|^2_{H^1} + c_\eta\|\vec{\alpha}\|^2_{L^2} - \mathrm{Re}\left\langle\begin{pmatrix}\Psi_{c'|\xi|^2/\langle x\rangle^2} & 0\\ 0 & \Psi_{c'|\xi|^2/\langle x\rangle^2}\end{pmatrix}\vec{\alpha}\,,\,\vec{\alpha}\right\rangle.
\end{aligned}$$

For the operator $\Psi_{c'|\xi|^2/\langle x\rangle^2}$, because

$$\Psi_{c'|\xi|^2/\langle x\rangle^2} = \frac{1}{\langle x\rangle^2}\Psi_{c'|\xi|^2} + \Psi_1\,, \qquad \Psi_{c'|\xi|^2} = -c'\Delta_x\,,$$

and

$$-\frac{1}{\langle x\rangle^2}\Delta_x = -\nabla_x\cdot\left(\frac{1}{\langle x\rangle^2}\nabla_x\right) + \Psi_1\,,$$

we have

$$\mathrm{Re}\left\langle\begin{pmatrix}\Psi_{c'|\xi|^2/\langle x\rangle^2} & 0\\ 0 & \Psi_{c'|\xi|^2/\langle x\rangle^2}\end{pmatrix}\vec{\alpha}\,,\,\vec{\alpha}\right\rangle \ge c'\int_{\mathbb{R}^d}\frac{1}{\langle x\rangle^2}|\nabla_x\vec{\alpha}|^2\,\mathrm{d}x - \left(\eta\,\|\nabla_x\vec{\alpha}\|^2_{L^2} + c_\eta\,\|\vec{\alpha}\|^2_{L^2}\right).$$

For the first order terms $\Psi_1(\vec{\alpha},\vec{y},\varrho)$, let $\eta > 0$ be small enough. Then

$$\begin{aligned}
|\langle\Psi_1\vec{\alpha}\,,\,\vec{\alpha}\rangle| + |\langle\vec{\alpha}\,,\,\Psi_1\vec{\alpha}\rangle| &\le \eta\,\|\nabla_x\vec{\alpha}\|^2_{L^2} + c_{A,\eta}\,\|\vec{\alpha}\|^2_{L^2}\,,\\
|\langle\Psi_1\vec{y}\,,\,\vec{\alpha}\rangle| + |\langle\vec{\alpha}\,,\,\Psi_1\vec{y}\rangle| &\le \eta\,\|\nabla_x\vec{y}\|^2_{L^2} + c_{A,\eta}\,\|\vec{\alpha}\|^2_{L^2}\,,\\
|\langle\Psi_1\varrho\,,\,\vec{\alpha}\rangle| + |\langle\vec{\alpha}\,,\,\Psi_1\varrho\rangle| &\le \eta\,\|\nabla_x\vec{\alpha}\|^2_{L^2} + c_{A,\eta}\,\|\varrho\|^2_{L^2}\,,
\end{aligned}$$

with $c_{A,\eta}$ depending on η and the bounds c_A.

For the term $\Psi_2\vec{y}$, by assumptions $\mathcal{A}_1$ and $\mathcal{A}_3$, we can choose $T_5 > 0$ small enough such that the symbol of this second order ΨDO, denoted as F_1 satisfies $|F_1| \le \frac{c_{0,9}}{\langle x\rangle^2}|\xi|^2$ for every $(x,t) \in \mathbb{R}^d \times [0, T_5)$. Therefore we obtain the bound

$$\left|\langle \Psi_{F_1}\vec{y}\,,\,\vec{\alpha}\rangle\right| + \left|\langle \vec{\alpha}\,,\,\Psi_{F_1}\vec{y}\rangle\right| \le \eta\,\|\nabla_x\vec{y}\|_{L^2}^2 + c_{0,\eta}\int_{\mathbb{R}^d}\frac{1}{\langle x\rangle^2}|\nabla_x\vec{\alpha}|^2\,\mathrm{d}x\,,$$

where $c_{0,\eta}$ depends on η and the data.

For the $\Psi_2\varrho$ term, we need to estimate the H^1 norm of ϱ. First, the L^2 estimate of ϱ shows that

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varrho\|_{L^2}^2 \le c_A\|\varrho\|_{L^2}^2 + \|\vec{\alpha}\|_{L^2}^2 + \|\vec{y}\|_{L^2}^2 + \|\Psi_q Qu\|_{L^2}^2\,. \tag{2.30}$$

Differentiate the equation (2.28) for ϱ with respect to x_l, $1 \le l \le d$ and multiply by $\partial_l\varrho$ we obtain the following equality

$$\begin{aligned}&\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_l\varrho\|_{L^2}^2 - \tfrac{1}{2}\int_{\mathbb{R}^d}|\partial_l\varrho|^2\nabla_x\cdot u\;\mathrm{d}x + \int_{\mathbb{R}^d}\partial_l\varrho\nabla_x\varrho\cdot\partial_l u\;\mathrm{d}x\\ &\quad= -\epsilon\,\langle\Delta_x^2(\partial_l\varrho)\,,\,\partial_l\varrho\rangle + \langle\epsilon\,\partial_l\Psi_{R_6}z_1\,,\,\partial_l\varrho\rangle + \langle\epsilon\,\partial_l\Psi_{R_7}z_2\,,\,\partial_l\varrho\rangle + \langle\Psi_1\vec{y}\,,\,\partial_l\varrho\rangle + \langle\Psi_1\vec{z}\,,\,\partial_l\varrho\rangle\,.\end{aligned}$$

Therefore, the energy inequality shows that

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_l\varrho\|_{L^2}^2 \le c_{A,\eta}\,\|\partial_l\varrho\|_{L^2}^2 + \epsilon\,\|\vec{\alpha}\|_{L^2}^2 + \eta\,\|\nabla_x\vec{y}\|_{L^2}^2 + \eta\,\|\nabla_x\vec{\alpha}\|_{L^2}^2\,. \tag{2.31}$$

By combining (2.30) and (2.31) we obtain the energy estimate for $\|\varrho\|_{H^1}^2$ as

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varrho\|_{H^1}^2 \le c_{A,\eta}\,\|\varrho\|_{H^1}^2 + \eta\,\|\nabla_x\vec{y}\|_{L^2}^2 + \eta\,\|\nabla_x\vec{\alpha}\|_{L^2}^2 + \|\vec{\alpha}\|_{L^2}^2 + \|\vec{y}\|_{L^2}^2 + \|\Psi_q Qu\|_{L^2}^2\,. \tag{2.32}$$

Upon adding the above estimates together we conclude that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\vec{\alpha}\|_{L^2}^2 + \|\varrho\|_{H^1}^2\right) + \hat{c}\int_{\mathbb{R}^d}\|\nabla_x\vec{\alpha}\|^2\,\mathrm{d}x &\le -\tfrac{1}{2}\epsilon\,\|\Delta_x\vec{\alpha}\|_{L^2}^2 + \tilde{c}\left(\|\vec{\alpha}\|_{L^2}^2 + \|\varrho\|_{H^1}^2 + \|\Psi_q Qu\|_{L^2}^2\right)\\ &\quad + \eta\,\|\nabla_x(\Psi_{q_1}\vec{y})\|_{L^2}^2 + \eta\,\|\nabla_x(\Psi_{q_2}\vec{y})\|_{L^2}^2 + \eta\,\|\nabla_x\vec{\alpha}\|_{L^2}^2\,.\end{aligned}$$

That is

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\vec{\alpha}\|_{L^2}^2 + \|\varrho\|_{H^1}^2\right) + \tfrac{1}{2}\hat{c}\int_{\mathbb{R}^d}\|\nabla_x\vec{\alpha}\|^2\,\mathrm{d}x\\ &\qquad\le -\tfrac{1}{2}\epsilon\,\|\Delta_x\vec{\alpha}\|_{L^2}^2 + \tilde{c}\left(\|\vec{\alpha}\|_{L^2}^2 + \|\varrho\|_{H^1}^2 + \|\Psi_q Qu\|_{L^2}^2\right) + c_{0,\eta}\,\|\nabla_x\vec{y}\|_{L^2}^2\,,\end{aligned} \tag{2.33}$$

by the fact that $q_1, q_2 \in S^0$ and their norms are given by the initial data. Here $c_{0,\eta}$ depending on η and the initial data, $\tilde{c}$ depending on c_A, α_0, and $\hat{c}$ depending on μ, κ, α_0, and d.

Next we check the energy estimates of $\vec{y}$. From equation (2.27), it is easy to see that the energy estimate for $\vec{y}$ is as follows.

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\vec{y}\|_{L^2}^2 + c_{0,12}\|\nabla_x\vec{y}\|_{L^2}^2 \le \eta\epsilon\,\|\Delta_x\vec{\alpha}\|_{L^2}^2 + c_{A,\eta}\,\|\vec{\alpha}\|_{L^2}^2 + c_{A,\eta}\,\|\varrho\|_{L^2}^2 + c_A\left(\|\vec{y}\|_{L^2}^2 + \|\Psi_q Qu\|_{L^2}^2\right)\,. \tag{2.34}$$

Take η sufficiently small and add up (2.33) and (2.34).

Multiply the equation for $\Psi_q Qu$ in (2.20) and integrate over $\mathbb{R}^d$. The L^2 estimate for $\Psi_q Qu$ is

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Psi_q Qu\|_{L^2}^2 \le c\,\left(\|\vec{\alpha}\|_{L^2}^2 + \|\vec{y}\|_{L^2}^2 + \|\varrho\|_{L^2}^2 + \|\Psi_q Qu\|_{L^2}^2\right)\,. \tag{2.35}$$

Then the energy estimate for the entire system is written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\varrho\|_{H^1}^2 + \|(\vec{\alpha},\,\vec{y},\,\Psi_q Qu)\|_{L^2}^2\right) + c_0\|\nabla_x(\vec{\alpha},\,\vec{y},\,\Psi_q Qu)\|_{L^2}^2 \le c_A\left(\|\varrho\|_{H^1}^2 + \|(\vec{\alpha},\,\vec{y},\,\Psi_q Qu)\|_{L^2}^2\right)\,. \tag{2.36}$$

By the Gronwall inequality we have

$$\begin{aligned}
&\sup_{0\le t\le T}\left(\|\varrho\|_{H^1}^2+\|(\vec{\alpha},\,\vec{y},\,\Psi_q Qu)\|_{L^2}^2\right)+\int_0^T\|\nabla_x\vec{\alpha},\,\vec{y})\|_{L^2}^2(s)\,\mathrm{d}s\\
&\le c'e^{TK_0}\left(\|\varrho(0)\|_{H^1}^2+\|(\vec{\alpha}(0),\,\vec{y}(0),\,\Psi_q Qu(0))\|_{L^2}^2\right)\\
&\le c'\left(\|\varrho(0)\|_{H^1}^2+\|(\vec{\alpha}(0),\,\vec{y}(0),\,\Psi_q Qu(0))\|_{L^2}^2\right),
\end{aligned}$$

where c' depends only on the initial data and θ_0 which is the lower bound of ρ, θ, K_0 depends on c_A, α_0 and $T>0$ is chosen to be small enough such that the second inequality is true.

Hence, using the fact that the equivalence of $\|\varrho\|_{H^1}^2+\|\vec{\alpha}\|_{L^2}^2+\|\vec{y}\|_{L^2}^2+\|\Psi_q Qu\|_{L^2}^2$ and $\|\rho\|_{H^1}^2+\|u\|_{L^2}^2+\|\theta\|_{L^2}^2$ depends only on the data, we conclude that there exist $T>0$ depending on c_A, α_0, and $c>0$ depending only on the data and α_0 such that

$$\sup_{[0,T]}\left(\|\rho\|_{H^1}^2+\|(u,\theta)\|_{L^2}^2\right)(t)+\int_0^T\|\nabla_x(u,\theta)\|_{L^2}^2(s)\,\mathrm{d}s\le c\left(\|\rho^{in}\|_{H^1}^2+\|(u^{in},\theta^{in})\|_{L^2}^2)\right).$$

This completes the proof of Theorem 2.2. □

3. A Priori Estimate

Based on the linear estimate (2.19), we can now establish the a priori estimate for the nonlinear regularized system (2.3), which has the abstract form

$$\begin{aligned}
\partial_t U &= -\epsilon\,\Delta_x^2 U+\mathcal{L}(U)U\,,\\
U(x,0) &= (\rho^{in},u^{in},\theta^{in})\,,
\end{aligned}\tag{3.1}$$

where $U=(\rho,u,\theta)$ and $\mathcal{L}$ is given by (2.11).

We begin by defining the following norms. Let $s>s_1$ be two integers such that

$$s_1>d/2+6,\qquad s=\max\{s_1+6,N+d/2+4\}\,,\tag{3.2}$$

where $N=N(d)$ is given in $\mathscr{A}_2$. Let $\bar{\rho},\bar{\theta}>0$ be two constants such that the initial data $(\rho^{in},u^{in},\theta^{in})$ satisfy (1.6). For $(\rho,u,\theta):\mathbb{R}^d\times[0,T]\to\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}$ satisfying

$$\begin{aligned}
&\rho-\bar{\rho}\in C([0,T];H^{s+1})\,,\qquad (u,\,\theta-\bar{\theta})\in C([0,T];H^s)\,,\\
&\langle x\rangle^2\partial_x^\alpha(\rho,u,\theta)\in C([0,T];L^2)\,,\qquad \forall\alpha\in\mathbb{N}^d\,,\ 1\le|\alpha|\le s_1\,,
\end{aligned}\tag{3.3}$$

define

$$\begin{aligned}
&|||(\rho-\bar{\rho},\,u,\,\theta-\bar{\theta})|||_T\\
&=\sup_{[0,T]}\left(\|\rho(t)-\bar{\rho}\|_{H^{s+1}}+\|(u,\theta-\bar{\theta})(t)\|_{H^s}+\sum_{1\le|\alpha|\le s_1+1}\left(\|\langle x\rangle^2\partial_x^\alpha\rho(t)\|_{H^1}+\|\langle x\rangle^2\partial_x^\alpha(u,\theta)(t)\|_{L^2}\right)\right).
\end{aligned}$$

Define

$$\lambda=\|\rho^{in}-\bar{\rho}\|_{H^{s+1}}+\|(u^{in},\theta^{in}-\bar{\theta})\|_{H^s}+\sum_{1\le|\alpha|\le s_1}\left(\|\langle x\rangle^2\partial_x^\alpha\rho^{in}\|_{H^1}+\|\langle x\rangle^2\partial_x^\alpha(u^{in},\theta^{in})\|_{L^2}\right)<\infty\,,$$

Suppose there exists a constant $\alpha_0>0$ such that $\rho^{in},\theta^{in},\mu(\theta^{in}),\kappa(\theta^{in})\ge 2\alpha_0>0$. Given $T,M>0$, define the space $X_{T,M}$ by

$$X_{T,M}=\left\{(\rho,u,\theta)(t,x)\ :\ |||(\rho-\bar{\rho},u,\theta-\bar{\theta})|||_T\le M\,,\quad \rho,\theta\ge\alpha_0>0,\quad (\rho,u,\theta)(0)=(\rho^{in},u^{in},\theta^{in})\right\}.$$

Let

$$M_0 = 4c\lambda\,, \tag{3.4}$$

with c being the constant in (2.19). Suppose $2c > 1$. Then the a priori estimate for system (2.3) states

Lemma 3.1. *Let $\bar{\rho}, \bar{\theta} > 0$ be two constants and $\bar{U} = (\bar{\rho}, 0, \bar{\theta})$. Let $U = (\rho, u, \theta)$ be a solution to system (2.3) satisfying (3.3) with $\rho, \theta \geq \alpha_0 > 0$. Then there exists $T_{\alpha_0} > 0$ independent of ϵ such that $|||U - \bar{U}|||_{T_{\alpha_0}} \leq M_0$, where M_0 is defined in (3.4).*

Proof. First, by the linear estimate for (3.1), there exists $T_1 > 0$ independent of ϵ such that

$$\sup_{[0,T_1]} \left(\|\rho(t) - \bar{\rho}\|_{H^1} + \|(u,\, \theta - \bar{\theta})(t)\|_{L^2} \right) \leq M_0\,.$$

Next we check the bounds of $(\rho - \bar{\rho},\, u,\, \theta - \bar{\theta})$ in higher order norms and norms with the weight $\langle x \rangle^2$. For any multi-index α with $0 \leq |\alpha| \leq s$, apply ∂_x^α to the nonlinear system (2.3). The resulting system for $\partial_x^\alpha U = (\partial_x^\alpha \rho, \partial_x^\alpha u, \partial_x^\alpha \theta)$ is

$$\begin{aligned}
&\partial_t(\partial_x^\alpha \rho) = -\epsilon\, \Delta_x^2(\partial_x^\alpha \rho) + \mathcal{L}_1(U)(\partial_x^\alpha U) + \Psi_0^\rho(\partial_x^\gamma \rho,\, \partial_x^\gamma u) + f_{\alpha,0}\,,\\
&\partial_t(\partial_x^\alpha u) = -\epsilon\, \Delta_x^2(\partial_x^\alpha u) + \mathcal{L}_2(U)(\partial_x^\alpha U) + \Psi_2^u(\partial_x^\gamma \rho,\, \partial_x^\gamma \theta) + \Psi_1(\partial_x^\gamma U) + \Psi_0(\partial_x^\gamma U) + f_{\alpha,1}\,,\\
&\partial_t(\partial_x^\alpha \theta) = -\epsilon\, \Delta_x^2(\partial_x^\alpha \theta) + \mathcal{L}_3(U)(\partial_x^\alpha U) + \Psi_2(\partial_x^\gamma \rho,\, \partial_x^\gamma u,\, \partial_x^\gamma \theta) + \Psi_1(\partial_x^\gamma U) + \Psi_0(\partial_x^\gamma U) + f_{\alpha,2}\,,
\end{aligned} \tag{3.5}$$

where γ denotes any multi-index satisfying $|\gamma| = |\alpha|$, and $(f_{\alpha,0}, f_{\alpha,1}, f_{\alpha,2})$ are functions depending on $(\partial_x^\sigma U)_{|\sigma| \leq |\alpha|-1}$. Assuming that the lower derivatives of orders no more than $|\alpha| - 1$ of (ρ, u, θ) are bounded, we treat them as forcing terms. The coefficients of the additional second order terms Ψ_2 depend on $\nabla_x^{\gamma_1} U$ where γ_1 is any multi-index such that $|\gamma_1| \leq 3$. Those coefficients satisfy the assumptions $\mathcal{A}_1$, $\mathcal{A}_2$. The coefficient of Ψ_1, Ψ_0 in the above system also depend on $\nabla_x^{\gamma_2} U$ for any γ_2 such that $|\gamma_2| \leq 3$. The assumptions $\mathcal{A}_1$, $\mathcal{A}_2$ are also satisfied for $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ when the solution U satisfies (3.3) with s, s_1 given in (3.2). Therefore, we conclude that the same linear estimate (2.19) applies for $\partial_x^\alpha(\rho, u, \theta)$ for every $0 \leq |\alpha| \leq s$ — that is, there exists $T_2 > 0$ such that

$$\begin{aligned}
&\sup_{[0,T_2]} \left(\|\rho\|_{H^{|\alpha|+1}}^2 + \|(u, \theta)\|_{H^{|\alpha|}}^2 \right) + \int_0^{T_2} \|\nabla_x (u, \theta)\|_{H^{|\alpha|}}^2(s)\, \mathrm{d}s\\
&\leq c \left(\|\rho^{in}\|_{H^{|\alpha|+1}}^2 + \|(u^{in}, \theta^{in})\|_{H^{|\alpha|}}^2 + \int_0^{T_2} \|(f_{\alpha,0}, f_{\alpha,1}, f_{\alpha,2})\|_{L^2}^2(s)\, \mathrm{d}s \right),
\end{aligned}$$

where c depends only on the data and α_0. Because $(f_{\alpha,0}, f_{\alpha,1}, f_{\alpha,2}) \in L^\infty(0, T_1; L^2(\mathbb{R}^d))$, the last term including the forcing can be made arbitrarily small by taking T_2 small. Therefore, there exists a time $T_3 > 0$ independent of ϵ such that

$$\sup_{[0,T_3]} \left(\|\rho - \bar{\rho}\|_{H^{s+1}} + \|(u, \theta - \bar{\theta})\|_{H^s} \right) \leq M_0/2\,. \tag{3.6}$$

Next, we estimate the bounds on $\langle x \rangle^2 \partial_x^\alpha(\rho, u, \theta)$ for $1 \leq |\alpha| \leq s_1$. We will show that the system for $\langle x \rangle^2 \partial_x^\alpha(\rho, u, \theta)$ has a similar structure as those for (ρ, u, θ) and $\partial_x^\alpha(\rho, u, \theta)$ so that the linear estimate (2.19) again applies.

For each $1 \leq l \leq d$ and multi-index β such that $1 \leq |\beta| \leq s_1 + 3$, the system satisfied by $x_l\, \partial_x^\beta U$ has the form

$$\begin{aligned}
&\partial_t(x_l \partial_x^\beta \rho) = -\epsilon\, \Delta_x^2(x_l \partial_x^\beta \rho) + \mathcal{L}_1(U)(x_l \partial_x^\beta U) + \Psi_0(x_l \partial_x^\gamma \rho,\, x_l \partial_x^\gamma u) + f_{l,0}\,,\\
&\partial_t(x_l \partial_x^\beta u) = -\epsilon\, \Delta_x^2(x_l \partial_x^\beta u) + \mathcal{L}_2(U)(x_l \partial_x^\beta U) + \Psi_2(x_l \partial_x^\gamma \rho,\, x_l \partial_x^\gamma \theta) + \Psi_1(x_l \partial_x^\gamma U) + \Psi_0(x_l \partial_x^\gamma U) + f_{l,1}\,,\\
&\partial_t(x_l \partial_x^\beta \theta) = -\epsilon\, \Delta_x^2(x_l \partial_x^\beta \theta) + \mathcal{L}_3(U)(x_l \partial_x^\beta U) + \Psi_2(x_l \partial_x^\gamma U) + \Psi_1(x_l \partial_x^\gamma U) + \Psi_0(x_l \partial_x^\gamma U) + f_{l,2}\,,
\end{aligned}$$

with $\gamma \in \mathbb{N}^d$ such that $|\gamma| = |\beta|$. Similar as in (3.5), we have the coefficients of Ψ_2, Ψ_1, Ψ_0 satisfying $\mathcal{A}_1, \mathcal{A}_2$, and $(f_{l,0}, f_{l,1}, f_{l,2})$ depend only on the H^{s_1+6} norm of (ρ, u, θ) and thereby is well-controlled when $s \geq s_1 + 6$. Consequently, the linear estimate holds for $(x_l \partial_x^\beta \rho, x_l \partial_x^\beta u, x_l \partial_x^\beta \theta)$ for each $l = 1, 2, \cdots, d$, that is, there exists $T_4 > 0$ such that for every $1 \leq l \leq d$ and multi-index γ such that $0 \leq |\gamma| \leq s_1 + 2$,

$$\sup_{[0,T_4]} \left(\|x_l \nabla_x \rho\|_{H^{|\gamma|}}^2 + \|x_l \nabla_x (u, \theta)\|_{H^{|\gamma|-1}}^2 \right) + \int_0^{T_4} \|\nabla_x (x_l u, x_l \theta)\|_{H^{|\gamma|}}^2 (s) \, ds$$
$$\leq c \left(\|x_l \nabla_x \rho^{in}\|_{H^{|\gamma|}}^2 + \|x_l \nabla_x (u^{in}, \theta^{in})\|_{H^{|\gamma|-1}}^2 + \int_0^{T_4} \|(f_{l,0}, f_{l,1}, f_{l,2})\|_{L^2}^2 (s) \, ds \right).$$

Similarly, for each $1 \leq l \leq d$ and multi-index α satisfying $1 \leq |\alpha| \leq s_1$, the system for $x_l^2 \partial_x^\alpha U$ has the form

$$\partial_t (x_l^2 \partial_x^\alpha \rho) = -\epsilon \Delta_x^2 (x_l^2 \partial_x^\alpha \rho) + \mathcal{L}_1(U)(x_l^2 \partial_x^\alpha U) + \Psi_0(x_l^2 \partial_x^\gamma \rho, \ x_l^2 \partial_x^\gamma u) + g_{l,0} \, ,$$
$$\partial_t (x_l^2 \partial_x^\alpha u) = -\epsilon \Delta_x^2 (x_l^2 \partial_x^\alpha u) + \mathcal{L}_2(U)(x_l^2 \partial_x^\alpha U) + \Psi_2(x_l^2 \partial_x^\gamma \rho, \ x_l^2 \partial_x^\gamma \theta) + \Psi_1(x_l^2 \partial_x^\gamma U) + \Psi_0(x_l^2 \partial_x^\gamma U) + g_{l,1} \, ,$$
$$\partial_t (x_l^2 \partial_x^\alpha \theta) = -\epsilon \Delta_x^2 (x_l^2 \partial_x^\alpha \theta) + \mathcal{L}_3(U)(x_l^2 \partial_x^\alpha U) + \Psi_2(x_l^2 \partial_x^\gamma U) + \Psi_1(x_l^2 \partial_x^\gamma U) + \Psi_0(x_l^2 \partial_x^\gamma U) + g_{l,2} \, ,$$

where $g_{l,k} = g_{l,k} \left((x_l \partial_x^{\sigma_1} (\rho, u, \theta), \ \partial_x^{\sigma_2} (\rho, u, \theta))_{|\sigma_1| \leq s_1+3, \, |\sigma_2| \leq s_1+2} \right)$ for $k = 0, 1, 2$. Again the linear estimate applies. Therefore, there exists $T_5 > 0$ sufficiently small such that for every $1 \leq |\alpha| \leq s_1 - 1$, we have

$$\sup_{[0,T_5]} \left(\|x_l^2 \nabla_x \rho\|_{H^{|\alpha|}}^2 + \|x_l^2 \nabla_x (u, \theta)\|_{H^{|\alpha|-1}}^2 \right) + \int_0^{T_5} \left\| \nabla_x \left(x_l^2 u, x_l^2 \theta \right) \right\|_{H^{|\alpha|}}^2 (s) \, ds$$
$$\leq c \left(\|x_l^2 \nabla_x \rho^{in}\|_{H^{|\alpha|}}^2 + \|x_l^2 \nabla_x (u^{in}, \theta^{in})\|_{H^{|\alpha|-1}}^2 + \int_0^{T_5} \| (g_{l,0}, g_{l,1}, g_{l,2}) \|_{L^2}^2 (s) \, ds \right).$$

Thus, by taking $T_6 = \min\{T_3, T_4, T_5\}$ sufficiently small, we have

$$\sup_{[0,T_6]} \left(\sum_{1 \leq |\alpha_1| \leq s_1+1} \|\langle x \rangle^2 \partial_x^{\alpha_1} \rho\|_{L^2} + \sum_{1 \leq |\alpha_2| \leq s_1} \|\langle x \rangle^2 \partial_x^{\alpha_2} (u, \theta)\|_{L^2} \right) \leq M_0 / 2 \, . \tag{3.7}$$

Upon adding (3.6) and (3.7), we conclude that there exists $T_{\alpha_0} > 0$ independent of ϵ such that

$$|||(\rho - \bar{\rho}, u, \theta - \bar{\theta})|||_{T_{\alpha_0}} \leq M_0 \, . \tag{3.8}$$

We thereby finish the proof of Lemma 3.1. □

4. Local Existence Proof

Based on Lemma 3.1, we can now prove the local existence of classical solutions to the nonlinear system (1.1). To show this, we first establish the existence of solutions to the regularized DNS system in Lemma 4.1. Then using Lemma 3.1, we conclude that the sequence of solutions to the regularized system exists on a time interval which is independent of ϵ. Finally, in the Main theorem, the convergence of this sequence of solutions is proved. Uniqueness is also proved in the Main Theorem.

Let $U = (\rho, u, \theta)$, $\bar{U} = (\bar{\rho}, 0, , \bar{\theta})$, $U^{in} = (\rho^{in}, u^{in}, \theta^{in})$ where U^{in} satisfies the condition (1.6). Define the operator $\Gamma = \Gamma_\epsilon$ on $X_{T,M}$ by

$$\Gamma(\rho, u, \theta) = \Gamma(U) = e^{-\epsilon t \Delta_x^2} U^{in} + \int_0^t e^{-\epsilon (t-t') \Delta_x^2} (\mathcal{L}(U) U)(t') \, dt' \, . \tag{4.1}$$

Then

$$\Gamma(\rho, u, \theta) - \bar{U} = e^{-\epsilon t \Delta_x^2}(U^{in} - \bar{U}) + \int_0^t e^{-\epsilon(t-t')\Delta_x^2}(\mathcal{L}(U)(U - \bar{U}))(t')\,\mathrm{d}t' . \tag{4.2}$$

Use the contraction mapping theorem to show the local existence of the solution to the regularized DNS system for each $0 < \epsilon < 1$.

Lemma 4.1. *For each $\epsilon \in (0, 1)$ there exists $T_\epsilon = O(\epsilon^3)$ such that the operator Γ defined in (4.1) defines a contraction mapping on X_{T_ϵ, M_0} where M_0 is defined in (3.4). Therefore, the regularized system (3.1) has a unique solution in X_{T_ϵ, M_0}.*

Proof. Study the semigroup generated by $-\epsilon\Delta_x^2$. Let β be a multi-index such that $|\beta| = 3$. Then for any $g \in L^2$,

$$\|\partial_x^\beta e^{-\epsilon t \Delta_x^2} g\|_{L^2} \le \frac{C}{\epsilon^{3/4} t^{3/4}} \|g\|_{L^2} ,$$

where $C > 0$ is a generic constant. Because $\mathcal{L}$ is of order three, for any multi-indices α_1, α_2 such that $|\alpha_1| \le s + 1$, $|\alpha_2| \le s$, we have

$$\begin{aligned}
\sup_{[0, T_{\epsilon,1}]} \|\partial_x^{\alpha_1} (\Gamma(\rho) - \bar{\rho})\|_{L^2} &\le \|\partial_x^{\alpha_1} (\rho^{in} - \bar{\rho})\|_{L^2} + \int_0^{T_{\epsilon,1}} \left\| \partial_x^{\alpha_1} e^{-\epsilon(T_{\epsilon,1} - t')\Delta_x^2} (\mathcal{L}_1(U)(U - \bar{U})) \right\|_{L^2} (t')\,\mathrm{d}t' \\
&\le \|\partial_x^{\alpha_1} (\rho^{in} - \bar{\rho})\|_{L^2} + \frac{c_{0,s} T_{\epsilon,1}^{1/4}}{\epsilon^{3/4}} M_0^2 , \\
\sup_{[0, T_{\epsilon,1}]} \|\partial_x^{\alpha_2} \left(\Gamma(u, \theta) - (0, \bar{\theta}) \right)\|_{L^2} &\le \|\partial_x^{\alpha_2} (u^{in}, \theta^{in} - \bar{\theta})\|_{L^2} \\
&\quad + \int_0^{T_{\epsilon,1}} \left\| \partial_x^{\alpha_2} e^{-\epsilon(T_{\epsilon,1} - t')\Delta_x^2} (\mathcal{L}_2, \mathcal{L}_3)(U)(U - \bar{U}) \right\|_{L^2} (t')\,\mathrm{d}t' \\
&\le \|\partial_x^{\alpha_2} (u^{in}, \theta^{in} - \bar{\theta})\|_{L^2} + \frac{c_{0,s} T_{\epsilon,1}^{1/4}}{\epsilon^{3/4}} M_0^2 \left(1 + M_0^{k_s}\right) ,
\end{aligned}$$

where $c_{0,s} > 0$ depends on α_0, s, and $k_s > 0$ depends only on s. Therefore,

$$\sup_{[0, T_{\epsilon,1}]} \|\Gamma(U) - \bar{U}\|_{H^s} \le \|U^{in} - \bar{U}\|_{H^s} + \frac{c_{0,s} T_{\epsilon,1}^{1/4}}{\epsilon^{3/4}} M_0^2 \left(1 + M_0^{k_s}\right) \le \lambda + M_0/4 \le M_0/2 , \tag{4.3}$$

by choosing $T_{\epsilon,1} = O(\epsilon^3)$ small enough.

Next, we show that the weighted norm $\sum_{1 \le |\alpha| \le s_1} \|\langle x \rangle^2 \partial_x^\alpha \Gamma(U)\|_{L^2}$ is also bounded by $M_0/2$ for a sufficiently short time. The argument is similar to the one in Section 3. Notice that for any $1 \le l \le d$, $|\alpha| \le s_1 + 3$, a direct calculation shows that $x_l\, \partial_x^\alpha \Gamma(U)$ satisfies

$$\partial_t \left(x_l\, \partial_x^\alpha \Gamma(U)\right) = -\epsilon\, \Delta_x^2 \left(x_l \partial_x^\alpha \Gamma(U)\right) + \mathcal{L}(U) \left(x_l\, \partial_x^\alpha U\right) + F ,$$

where $F = F\left(\partial_x^\beta \Gamma(U),\, \partial_x^\gamma U\right)_{|\gamma|, |\beta| \le s_1 + 6}$ is a C^∞ function in its variables. Because $\Gamma(U)$ and U are bounded in $L^\infty([0, T_{\epsilon,1}]; H^s)$ for $s \ge s_1 + 6$, the function F is bounded in $L^\infty([0, T_{\epsilon,1}]; L^2)$. Moreover, there exists a polynomial $Q_1(x) > 0$ such that

$$\sup_{[0, T_{\epsilon,1}]} \|F\|_{L^2} \le Q_1(M_0) .$$

Using the fact that $x_l\,\partial_x^\alpha U$ is bounded in L^2 for any $\alpha \le s_1$, similar calculation as above shows that for $T_{\epsilon,2} \le T_{\epsilon,1}$,

$$\sup_{[0,T_{\epsilon,2}]} \sum_{1\le|\alpha|\le s_1} \|x_l\,\partial_x^\alpha \Gamma(U)\|_{L^2} \le \sum_{1\le|\alpha|\le s_1} \|x_l\,\partial_x^\alpha U^{in}\|_{L^2} + \frac{c_{0,s_1} T_{\epsilon,2}^{1/4}}{\epsilon^{3/4}} M_0^2 \left(1 + M_0^{k_{s_1}}\right) + T_{\epsilon,2} Q_1(M_0)$$
$$\le \lambda + M_0/4 \le M_0/2\,,$$

by choosing $T_{\epsilon,2} = O(\epsilon^3)$ small enough. Here $c_{0,s_1} > 0$ depends on α_0, s_1, and $k_{s_1} > 0$ depends only on s_1.

Similarly, for each $1 \le l \le d$, $x_l^2\,\partial_x^\alpha \Gamma(U)$ satisfies

$$\partial_t \left(x_l^2\,\partial_x^\alpha \Gamma(U)\right) = -\epsilon\,\Delta_x^2 \left(x_l^2\,\partial_x^\alpha \Gamma(U)\right) + \mathcal{L}(U)\left(x_l^2\,\partial_x^\alpha U\right) + G\,,$$

where $G = G\left(x_l\,\partial_x^{\sigma_1}\Gamma(U),\ x_l\,\partial_x^{\sigma_2} U,\ \partial_x^\beta \Gamma(U),\ \partial_x^\gamma U\right)_{|\sigma_1|,|\sigma_2|,|\gamma|,|\beta|\le s_1+3}$ is a C^∞ function in its variables. Because $x_l\,\partial_x^\sigma \Gamma(U)$ is shown above to be bounded in $L^\infty([0,T_{\epsilon,2}];L^2)$ for any $|\sigma| \le s_1+3$, and $\Gamma(U), U$ are bounded in $L^\infty([0,T_{\epsilon,2}];H^s)$ for $s \ge s_1+6$, the function G is bounded in $L^\infty([0,T_{\epsilon,2}];L^2)$. Moreover, there exists a polynomial $Q_2(x) > 0$ such that

$$\sup_{[0,T_{\epsilon,2}]} \|G\|_{L^2} \le Q_2(M_0)\,.$$

Using the fact that $x_l^2\,\partial_x^\alpha U$ is bounded in L^2 for any $\alpha \le s_1$, a calculation similar to the one above shows that for $0 < T_{\epsilon,3} \le T_{\epsilon,2}$,

$$\sup_{[0,T_{\epsilon,3}]} \sum_{1\le|\alpha|\le s_1} \|x_l\,\partial_x^\alpha \Gamma(U)\|_{L^2} \le \sum_{1\le|\alpha|\le s_1} \|x_l\,\partial_x^\alpha U^{in}\|_{L^2} + \frac{c_{0,s_1} T_{\epsilon,3}^{1/4}}{\epsilon^{3/4}} M_0^2 \left(1 + M_0^{k_{s_1}}\right) + T_{\epsilon,3} Q_2(M_0) \tag{4.4}$$
$$\le \lambda + M_0/4 \le M_0/2\,,$$

by choosing $T_{\epsilon,3} = O(\epsilon^3)$ small enough. Here $c_{0,s_1} > 0$ depends on α_0, s_1, and $k_{s_1} > 0$ depends only on s_1.

For the positivity of $\Gamma(\rho)$ and $\Gamma(\theta)$, notice that $\Gamma(U)$ satisfies the linear equation

$$\partial_t \Gamma(U) = -\epsilon\,\Delta_x^2 \Gamma(U) + \mathcal{L}(U)U\,,$$

with $\mathcal{L}(U)U$ sufficiently smooth. Therefore, for the initial data $\rho^{in}, \theta^{in} \ge 2\alpha_0 > 0$, if we choose $T_{\epsilon,4}$ small, we have $\Gamma(\rho), \Gamma(\theta) \ge \alpha_0 > 0$.

Upon combining the positivity with (4.3) and (4.4), we conclude that Γ maps X_{T_ϵ,M_0} into itself for $T_\epsilon = \min\{T_{\epsilon,k}\}_{k=1}^4$ sufficiently small.

To show $\Gamma(U)$ is a contraction mapping on X_{T_ϵ,M_0}, for any $U_1, U_2 \in X_{T_\epsilon,M_0}$ consider the difference equation for $\Gamma(U_1) - \Gamma(U_2)$:

$$\partial_t\Big(\Gamma(U_1) - \Gamma(U_2)\Big) = -\epsilon\,\Delta_x^2\Big(\Gamma(U_1) - \Gamma(U_2)\Big) + \Big(\mathcal{L}(U_1)U_1 - \mathcal{L}(U_2)U_2\Big)\,.$$

Similar calculation shows that there exists a polynomial $Q_3(M_0)$ such that

$$|||\Gamma(U_1) - \Gamma(U_2)|||_{X_{T_\epsilon,M_0}} \le c_{\epsilon,s,s_1} T_\epsilon^{1/4} Q_3(M_0) |||U_1 - U_2|||_{X_{T_\epsilon,M_0}}\,.$$

Therefore, by choosing T_ϵ sufficiently small, $\Gamma : X_{T_\epsilon,M_0} \to X_{T_\epsilon,M_0}$ is a contraction mapping. Therefore, there exists a solution $(\rho^\epsilon, u^\epsilon, \theta^\epsilon) \in X_{T_\epsilon,M_0}$ to the regularized nonlinear system (3.1). □

We show in the following lemma that the lifespan of U can be extended from T_ϵ to $T_0 > 0$ which is independent of ϵ.

Lemma 4.2. *There exists $T_0 > 0$ independent of ϵ such that the solution $U = (\rho, u, \theta)$ to the regularized system (2.3) exists on $[0, T_0]$ and satisfies that $|||U|||_{T_0} \leq M_0$.*

Proof. The proof is done by a bootstrapping argument together with Lemma 3.1. We need to show that there exists $T_0 > 0$ independent of ϵ such that the bounds $\rho, \theta \geq \alpha_0 > 0$ hold over $[0, T_0]$. These bounds together with Lemma 3.1 imply that $|||U|||_{T_0} \leq M_0$. Because the right-hand side of system (2.3) is bounded by a function of M_0 independent of ϵ, we see that $\|(\partial_t \rho, \partial_t \theta)\|_{L^\infty}$ is bounded with an upper bound independent of ϵ. Therefore, the time interval over which ρ, θ are strictly bounded below by α_0 depends only on M_0. In particular, it does not depend on ϵ. That is, given $\rho^{in}, \theta^{in} \geq 2\alpha_0 > 0$, and $|||U|||_{T_0} \leq M_0$, there exists $\hat{T}$ independent of ϵ such that $\rho, \theta \geq \alpha_0 > 0$ over $[0, \hat{T}]$. Therefore, there exists $T_0 \geq \min\{\hat{T}, T_{\alpha_0}\}$, where T_{α_0} is given by Lemma 3.1, such that $\rho, \theta > \alpha_0$ over $[0, T_0]$ and $|||U|||_{T_0} \leq M_0$. □

Finally, we state and prove the Main theorem.

Main Theorem. *Under the hypotheses that the Hamiltonian flow generated by the the symbol*

$$h^{in}(\xi, x) = \sqrt{\hat{\tau}_1(x, 0)\hat{\tau}_4(x, 0)}|\xi|^3$$

is nontrapping, there exists $N = N(d) \in \mathbb{Z}^+$ and two constants $\bar{\rho}, \bar{\theta} > 0$ such that given the initial data $(\rho^{in}, u^{in}, \theta^{in})$ satisfying the condition

$$\|\rho^{in} - \bar{\rho}\|_{H^{s+1}} + \|(u^{in}, \theta^{in} - \bar{\theta})\|_{H^s} + \sum_{1 \leq |\alpha| \leq s_1} \Big(\|\langle x \rangle^2 \partial_x^\alpha \rho^{in}\|_{H^1} + \|\langle x \rangle^2 \partial_x^\alpha (u^{in}, \theta^{in})\|_{L^2} \Big) < \infty ,$$

where $s_1 \geq d/2 + 6$, $s = \max\{s_1 + 6, N + d/2 + 4\}$, there exists $T_0 > 0$ independent of ϵ such that system (2.2) has a unique solution $(\rho^\epsilon, u^\epsilon, \theta^\epsilon)$ in X_{T_0, M_0}. Moreover, there exists (ρ, u, θ) such that $\rho - \bar{\rho} \in C([0, T_0]; H^s) \cap L^\infty([0, T_0]; H^{s+1}), (u, \theta - \bar{\theta}) \in C([0, T_0]; H^{s-1}) \cap L^\infty([0, T_0]; H^s)$ satisfying

$$\left.\begin{aligned} \rho^\epsilon - \bar{\rho} &\longrightarrow \rho - \bar{\rho} && \text{in } C([0, T_0]; H^s) , \\ (u^\epsilon, \theta^\epsilon - \bar{\theta}) &\longrightarrow (u, \theta - \bar{\theta}) && \text{in } C([0, T_0]; H^{s-1}) , \\ \langle x \rangle^2 \partial_x^\alpha \rho^\epsilon &\longrightarrow \langle x \rangle^2 \partial_x^\alpha \rho && \text{in } C([0, T_0]; H^1) , \\ \langle x \rangle^2 \partial_x^\alpha (u^\epsilon, \theta^\epsilon) &\longrightarrow \langle x \rangle^2 \partial_x^\alpha (u, \theta) && \text{in } C([0, T_0]; L^2) , \end{aligned}\right\} \quad \text{as } \epsilon \to 0 ,$$

for any $1 \leq |\alpha| \leq s_1$ and (ρ, u, θ) is the unique solution to the original DNS system (1.1).

Proof. The first part has been shown in Lemma 4.1 and Lemma 3.1. The convergence of approximate solutions $(\rho^\epsilon, u^\epsilon, \theta^\epsilon)$ is shown by the standard high-low technique [13]. Basically, we will show that $(\rho^\epsilon, u^\epsilon, \theta^\epsilon)$ converges in $C(0, T_0; L^2)$. Then by using the interpolation and uniform bounds on $(\rho^\epsilon, u^\epsilon, \theta^\epsilon)$, we prove the convergence of $(\rho^\epsilon, u^\epsilon, \theta^\epsilon)$ in $C(0, T_0; H^{s-1})$.

For $\epsilon, \epsilon' > 0$, let $\varrho = \rho^\epsilon - \rho^{\epsilon'}, v = u^\epsilon - u^{\epsilon'}, \eta = \theta^\epsilon - \theta^{\epsilon'}$ and study the system for (ϱ, v, η).

$$\begin{aligned} \partial_t \varrho &= -\epsilon \, \Delta_x^2 \varrho - (\epsilon - \epsilon') \, \Delta_x^2 \rho^{\epsilon'} + \mathcal{L}_1(\rho^\epsilon, u^\epsilon, \theta^\epsilon)(\varrho, v) + \Psi_{0,1}(\varrho, v) , \\ \partial_t v &= -\epsilon \, \Delta_x^2 v - (\epsilon - \epsilon') \, \Delta_x^2 u^{\epsilon'} + \mathcal{L}_2(\rho^\epsilon, u^\epsilon, \theta^\epsilon)(\varrho, v, \eta) + \Psi_{1,2}(\varrho, v, \eta) , \\ \partial_t \eta &= -\epsilon \, \Delta_x^2 \eta - (\epsilon - \epsilon') \, \Delta_x^2 \theta^{\epsilon'} + \mathcal{L}_3(\rho^\epsilon, u^\epsilon, \theta^\epsilon)(\varrho, v, \eta) + \Psi_{1,3}(\varrho, v, \eta) . \end{aligned} \tag{4.5}$$

where $\Psi_{k,j}$ are k^{th}-order ΨDOs with their coefficients depending on $(\rho^\epsilon, u^\epsilon, \theta^\epsilon)$ and $(\rho^{\epsilon'}, u^{\epsilon'}, \theta^{\epsilon'})$ for $k = 0, 1$ and $j = 1, 2, 3$. The terms in $\Psi_{1,2}(\varrho, v, \eta)$ involving μ and τ_1 have the forms

$$\frac{1}{\rho^\epsilon} \nabla_x \cdot \left[\eta \frac{\mu(\theta^\epsilon) - \mu(\theta^{\epsilon'})}{\theta^\epsilon - \theta^{\epsilon'}} D_x u^{\epsilon'} \right] - \varrho \frac{1}{\rho^\epsilon \rho^{\epsilon'}} \nabla_x \cdot [\mu(\theta^{\epsilon'}) D_x u^{\epsilon'}] .$$

The terms in $\Psi_{1,2}(\varrho, v, \eta)$ involving τ_1 have the forms

$$\tfrac{d-1}{d}\frac{1}{\rho^\epsilon}\frac{\tau_1(\theta^{\epsilon'}) - \tau_1(\theta^\epsilon)}{\theta^\epsilon - \theta^{\epsilon'}}\eta\Delta_x\nabla_x\theta^{\epsilon'} + \varrho\tfrac{d-1}{d}\frac{1}{\rho^\epsilon\rho^{\epsilon'}}\tau_1(\theta^{\epsilon'})\Delta_x\nabla_x\theta^{\epsilon'}\,.$$

The terms in $\Psi_{1,2}(\varrho, v, \eta)$ involving A^ρ defined in (2.2) have the forms

$$\begin{aligned}
&\sum_{m=1}^{d}\frac{A^\rho(\rho^\epsilon,\theta^\epsilon,\nabla_x\rho^\epsilon,\nabla_x\theta^\epsilon) - A^\rho(\rho^\epsilon,\theta^\epsilon,\nabla_x\rho^\epsilon,\nabla_x\theta^{\epsilon'})}{\partial_{x_m}\theta^\epsilon - \partial_{x_m}\theta^{\epsilon'}}\partial_{x_m}\eta : \nabla_x^2\rho^{\epsilon'}\\
&+\sum_{m=1}^{d}\frac{A^\rho(\rho^\epsilon,\theta^\epsilon,\nabla_x\rho^\epsilon,\nabla_x\theta^{\epsilon'}) - A^\rho(\rho^\epsilon,\theta^\epsilon,\nabla_x\rho^{\epsilon'},\nabla_x\theta^{\epsilon'})}{\partial_{x_m}\rho^\epsilon - \partial_{x_m}\rho^{\epsilon'}}\partial_{x_m}\varrho : \nabla_x^2\rho^{\epsilon'}\\
&+\frac{A^\rho(\rho^\epsilon,\theta^\epsilon,\nabla_x\rho^{\epsilon'},\nabla_x\theta^{\epsilon'}) - A^\rho(\rho^\epsilon,\theta^{\epsilon'},\nabla_x\rho^{\epsilon'},\nabla_x\theta^{\epsilon'})}{\theta^\epsilon - \theta^{\epsilon'}}\eta : \nabla_x^2\rho^{\epsilon'}\\
&+\frac{A^\rho(\rho^\epsilon,\theta^{\epsilon'},\nabla_x\rho^{\epsilon'},\nabla_x\theta^{\epsilon'}) - A^\rho(\rho^{\epsilon'},\theta^{\epsilon'},\nabla_x\rho^{\epsilon'},\nabla_x\theta^{\epsilon'})}{\rho^\epsilon - \rho^{\epsilon'}}\varrho : \nabla_x^2\rho^{\epsilon'}\,.
\end{aligned}$$

The rest of the terms in $\Psi_{1,2}(\varrho, v, \eta)$ involving A^θ, B^ρ, B^θ, as well as terms in $\Psi_{1,3}(\varrho, v, \eta)$ have similar forms as above. The zeroth-order operator in the ϱ-equation in (4.5) has the form

$$\Psi_{0,1}(\varrho, v) = -\varrho\nabla_x\cdot u^{\epsilon'} - v\cdot\nabla_x\rho^{\epsilon'}\,,$$

It is clear that given $(\rho^\epsilon - \bar\rho,\ u^\epsilon,\ \theta^\epsilon - \bar\theta), (\rho^{\epsilon'} - \bar\rho,\ u^{\epsilon'},\ \theta^{\epsilon'} - \bar\theta) \in X_{T_0,M_0}$, the linear estimate applies to the above system. Therefore we have

$$\begin{aligned}
\text{(4.6)}\qquad \sup_{[0,T_0]}\left(\|\varrho\|_{H^1}^2 + \|(v,\eta)\|_{L^2}^2\right) &\le c(\epsilon-\epsilon')\int_0^{T_0}\left(\|\Delta_x^2\rho^\epsilon(\cdot,s)\|_{H^1}^2 + \|\Delta_x^2 u^\epsilon(\cdot,s)\|_{L^2}^2 + \|\Delta_x^2\theta^\epsilon(\cdot,s)\|_{L^2}^2\right)\mathrm{d}s\\
&\le c(\epsilon-\epsilon')T_0M_0\,.
\end{aligned}$$

This shows that $(\rho^\epsilon - \bar\rho,\ u^\epsilon,\ \theta^\epsilon - \bar\theta)$ is a Cauchy sequence in $C([0,T_0]; L^2)$. Because it is also a bounded sequence in $L^\infty([0,T_0]; H^s)$ we conclude that it is a Cauchy sequence in $C([0,T_0]; H^{s-1})$. Thus, there exists (ρ, u, θ) such that $(\rho^\epsilon - \bar\rho,\ u^\epsilon,\ \theta^\epsilon - \bar\theta) \to (\rho - \bar\rho,\ u,\ \theta - \bar\theta)$ in $C([0,T_0]; H^{s-1})$. By the weak compactness of $(\rho^\epsilon - \bar\rho,\ u^\epsilon,\ \theta^\epsilon - \bar\theta)$ in $L^\infty([0,T_0]; H^s)$ we see that $(\rho, u, \theta) \in L^\infty([0,T_0]; H^s)$.

By Fatou's Lemma, for all $\alpha \in \mathbb{N}^d$, $1 \le |\alpha| \le s_1$

$$\begin{aligned}
\langle x\rangle^2\,\partial_x^\alpha(\rho - \bar\rho) &\in L^\infty([0,T_0]; H^1)\,,\\
\langle x\rangle^2\,\partial_x^\alpha(u, \theta - \bar\theta) &\in L^\infty([0,T_0]; L^2)\,,
\end{aligned}$$

By interpolation it is clear that for each $1 \le l \le d$ and each $\alpha \in \mathbb{N}^d$ with $1 \le |\alpha| \le s_1$ one has

$$\left.\begin{aligned}
x_l\,\partial_x^\alpha\rho^\epsilon &\longrightarrow x_l\,\partial_x^\alpha\rho &&\text{in } C([0,T_0]; H^1)\,,\\
x_l\,\partial_x^\alpha u^\epsilon &\longrightarrow x_l\,\partial_x^\alpha u &&\text{in } C([0,T_0]; L^2)\,,\\
x_l\,\partial_x^\alpha\theta^\epsilon &\longrightarrow x_l\,\partial_x^\alpha\theta &&\text{in } C([0,T_0]; L^2)\,,
\end{aligned}\right\}\quad\text{as } \epsilon \to 0\,.$$

Apply ∂_x^α to the system (4.5) for (ϱ, v, η) and multiply the result by $\langle x\rangle^2$. Using a similar argument as the L^2 convergence (4.6), we can show that for each $\alpha \in \mathbb{N}^d$ with $1 \le |\alpha| \le s_1$ one has

$$\left.\begin{aligned}
\langle x\rangle^2\partial_x^\alpha\rho^\epsilon &\longrightarrow \langle x\rangle^2\partial_x^\alpha\rho &&\text{in } C([0,T_0]; H^1)\,,\\
\langle x\rangle^2\partial_x^\alpha u^\epsilon &\longrightarrow \langle x\rangle^2\partial_x^\alpha u &&\text{in } C([0,T_0]; L^2)\,,\\
\langle x\rangle^2\partial_x^\alpha\theta^\epsilon &\longrightarrow \langle x\rangle^2\partial_x^\alpha\theta &&\text{in } C([0,T_0]; L^2)\,,
\end{aligned}\right\}\quad\text{as } \epsilon \to 0\,.$$

Based on the above results, we see that if we let $\epsilon \to 0$ then (ρ, u, θ) will be a classical solution to the nonlinear system with $(\rho, u, \theta) \in C([0, T_0]; H^{s-1}) \cap C^1((0, T_0]; H^{s-4})$.

To show the uniqueness of the classical solution, notice that if $U_1 = (\rho_1, u_1, \theta_1)$, $U_2 = (\rho_2, u_2, \theta_2)$ are two solutions with the same initial, then the difference $U_1 - U_2$ satisfies

$$\partial_t(U_1 - U_2) = \mathcal{L}(U_1)U_1 - \mathcal{L}(U_2)U_2 \,, \qquad (U_1 - U_2)(x, 0) = 0 \,, \tag{4.7}$$

Similar as we have done for (4.5), the difference equation (4.7) has the form

$$\partial_t(U_1 - U_2) = \mathcal{L}(U_1)(U_1 - U_2) + \Psi_1(U_1 - U_2) \,.$$

By the linear estimate in Section 2 with $\epsilon = 0$, we have

$$\sup_{[0,T_0]} \|U_1 - U_2\|_{L^2} \le c \, \|(U_1 - U_2)(x, 0)\|_{H^1} = 0 \,,$$

which proves the uniqueness.

The proof for the stability follows similarly by comparing the equations for $\partial_x^{\alpha_1} U_1$, $\partial_x^{\alpha} U_2$ and the equations for $\langle x\rangle^2 \partial_x^{\alpha} U_1$, $\langle x\rangle^2 \partial_x^{\alpha_2} U_2$ where $\alpha_1, \alpha_2 \in \mathbb{N}^d$ with $1 \le |\alpha_1| \le s$, $1 \le |\alpha_2| \le s_1$ and the linear estimates for the differences.

Hence, there exists a unique solution (ρ, u, θ) to the original DNS system (1.1) such that

$$\rho \in C\left([0, T_0]; H^s\right) \cap C^1\left((0, T_0]; H^{s-2}\right) \cap C\left([0, T_0]; H^{s_1}_{\langle x\rangle^2}\right) \,,$$
$$(u, \theta) \in C\left([0, T_0]; H^{s-1}\right) \cap C^1\left((0, T_0]; H^{s-4}\right) \cap C\left([0, T_0]; H^{s_1-1}_{\langle x\rangle^2}\right) \,,$$

with $s_1 \ge d/2 + 6$, $s = \max\{s_1 + 6; N + d/2 + 4\}$. □

References

[1] Beals, M., Reed, M.: *Microlocal regularity theorems for nonsmooth pseudodifferential operators and applications to nonlinear problems*, Trans. Amer. Math. Soc. **285** (1984), no. 1, 159-184.

[2] Calderón, A. P., Vaillancourt, R.: *A class of bounded pseudo-differential operators*, Proc. Natl. Acad. Sci. USA **69** (1972), 1185-1187.

[3] Chihara, H.: *Smoothing effects of dispersive pseudodifferential equations*, Comm. Partial Diff. Eq., **27** (2002), no. 9-10, 1953–2005.

[4] Constantin, P., Saut, J.-C.: *Local smoothing properties of dispersive equations*. J. Am. Math. Soc. **1** (1989), 413–446.

[5] Craig, W., Kappeler, T., Strauss, W.: *Microlocal dispersive smoothing for the Schrödinger equation*, Commun. Pure & Appl. Math. **48** (1995), 769–860.

[6] Hörmander, L.: *The analysis of linear partial differential operators*, Vol. III, Springer-Verlag, New York, 1985

[7] Kato, T.: *On the Cauchy problem for the (generalized) Korteweg-de Vries equation*. Advances in Math. Supp. Studies. Stud. Appl. Math. **8** (1983), 93–128.

[8] Kenig, C.E., Ponce, G., Vega, L.: *Oscillatory integrals and regulartiy of dispersive equations*, Indiana Univ. Math. J. **40** (1991), 33–69.

[9] Kenig, C.E., Ponce, G., Vega, L.: *Smoothing effects and local existence theory for the generalized nonlinear Schrödinger equations*, Invent. Math. **134** (1998), 489-545

[10] Kenig, C.E., Ponce, G., Vega, L.: *The Cauchy problem for quasi-linear Schrödinger equations*, Invent. Math. **158** (2004), 343–388.

[11] Kumano-go, H.: *Pseudo-differential Operators*, MIT Press, Cambridge, 1981.

[12] Levermore, C.D.: *Gas Dynamics Beyond Navier-Stokes*, preprint 2004.

[13] Majda, A.: *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables*, Applied Math. Sci., **53**, Springer-Verlag, New York, 1984.

[14] Strichartz, R.: *Restriction of Fourier transform to quadratic surfaces and decay of solutions of wave equations*, Duke Math. J., **44** (1977), 705–774.

[15] Taylor, M.: *Tools for PDE: pseudodifferential operators, paradifferential operators, and layer potentials*, Mathematical Surveys and Monographs, **81**, AMS, Providence, 2000.

[16] Terence, T.: *Nonlinear Dispersive Equations, Local and Global Analysis*, CBMS regional conference series in mathematics, **106**, AMS, Providence, 2006.

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