COMPACTNESS OF THE GAIN PARTS OF THE LINEARIZED BOLTZMANN OPERATOR WITH WEAKLY CUTOFF KERNELS

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A . We prove an L^p compactness result for the gain parts of the linearized Boltzmann collision operator associated with weakly cutoff collision kernels that derive from a power-law intermolecular potential. We replace the Grad cutoff assumption previously made by Caflisch [1], Golse and Poupaud [7], and Guo [11] with a weaker local integrability assumption. This class includes all classical kernels to which the DiPerna-Lions theory applies that derive from a repulsive inverse-power intermolecular potential. In particular, our approach allows the treatment of both hard and soft potential cases.

1. I

The linearized Boltzmann collision operator arises in the study of fluid dynamical approximations to solutions of the Boltzmann equation. That equation governs the kinetic density $F(v, x, t)$ of a gas composed of identical particles with velocities $v \in \mathbb{R}^D$ and positions $x \in \mathbb{R}^D$ at time $t \geq 0$ as [3, 4]

$$\partial_t F + v \cdot \nabla_x F = \mathcal{B}(F, F)\,, \tag{1.1}$$

where the collision operator $\mathcal{B}(F, F)$ is given by

$$\mathcal{B}(F, F) = \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} (F_1' F' - F_1 F)\, b(\omega, v_1 - v)\, \mathrm{d}\omega\, \mathrm{d}v_1\,. \tag{1.2}$$

Here $\omega \in \mathbb{S}^{D-1}$ is a unit vector, $\mathrm{d}\omega$ is the usual rotationally invariant Lebesgue measure on $\mathbb{S}^{D-1}$, while F_1', F', F_1, and F are the density $F(\cdot, x, t)$ evaluated at the velocities v_1', v', v_1, and v respectively. Here (v_1', v') are the velocities after an elastic binary collision between two molecules that had the velocities (v_1, v) before the collision, or vice versa. Because both momentum and energy are conserved in an elastic collision, one can express v_1' and v' in terms of v_1 and v as

$$v_1' = v_1 - \omega\, \omega \cdot (v_1 - v)\,, \qquad v' = v + \omega\, \omega \cdot (v_1 - v)\,, \tag{1.3}$$

where the unit vector $\omega \in \mathbb{S}^{D-1}$ is parallel to the deflections $v_1' - v_1$ and $v' - v$, and is thereby normal to the plane of reflection. The part of $\mathcal{B}(F, F)$ involving F_1' and F' is called the gain term, while the part involving F_1 and F is called the loss term.

By Galilean invariance and rotational invariance, every classical collision kernel $b(\omega, v_1 - v)$ has the general form [3]

$$b(\omega, v_1 - v) = |v_1 - v|\, \Sigma(|\omega \cdot n|, |v_1 - v|)\,, \qquad n = \frac{v_1 - v}{|v_1 - v|}\,,$$

where $\Sigma \geq 0$ is the differential scattering cross-section. When the molecules are hard spheres of mass m and radius r_o then b has the form

$$b(\omega, v_1 - v) = |\omega \cdot (v_1 - v)|\, \frac{(2r_o)^{D-1}}{2m}\,. \tag{1.4}$$

Date: July 23, 2009.

When the molecules are point particles with a repulsive intermolecular potential proportional to r^{-k} for some $k > 0$, where r is the intermolecular distance, then the kernel b has the factored form

$$b(\omega, v_1 - v) = \hat{b}(\omega \cdot n)\,|v_1 - v|^\beta\,, \tag{1.5}$$

where $\hat{b}$ is an even function of $\omega \cdot n$ and $\beta = 1 - \frac{2(D-1)}{k} < 1$. Notice that the hard sphere kernel (1.4) can be put into this form with $\beta = 1$. The cases $\beta < 0$, $\beta = 0$, and $\beta > 0$ are respectively referred to as the "soft", "Maxwell", and "hard" potential cases.

In this paper we assume that the collision kernel b has the factored form (1.5) where $\hat{b}$ and β satisfy

$$\hat{b}(\omega \cdot n) \in L^1(\mathrm{d}\omega)\,, \qquad -D < \beta\,. \tag{1.6}$$

These assumptions are necessary for b to be locally integrable in all of its variables. This is needed to seperately make sense of the gain and loss parts of $\mathcal{B}(F, F)$ for all continuous, rapidly decaying F. They include the so-called "super hard" kernels coresponding to $\beta > 1$, which do not arise classically. The first assumption in (1.6) is a so-called small deflection cutoff condition because the $\hat{b}$ that is derived from microscopic physics has a nonintegrable singularity at $\omega \cdot n = 0$ due to the large number of grazing collisions. Grad [10] observed that these grazing collisions should not appreciably affect the macroscopic dynamics, and therefore proposed that these singularities can be cutoff in order to make analysis of the Boltzmann equation more tractable. The small deflection cutoff condition in (1.6) is much weaker than the one introduced by Grad, which requires that $\hat{b}$ vanish like $|\omega \cdot n|$ as $|\omega \cdot n| \to 0$. It is therefore sometimes called the weak cutoff condition. In order to apply the DiPerna-Lions theroy of global renormalized solutions [5] to the Boltzmann equation with kernels in the factored form (1.5), it is necessary to impose our assumptions (1.6) and also to require that $\beta < 2$.

Now let M denote the Maxwellian given by

$$M(v) = \frac{1}{(2\pi)^{D/2}}\, e^{-\frac{1}{2}|v|^2}\,. \tag{1.7}$$

We consider the linearized Boltzmann operator $\mathcal{L}$ defined by

$$\mathcal{L} f = -\frac{2}{M}\,\mathcal{B}(M, Mf) = \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} (f + f_1 - f' - f_1')\, b(\omega, v_1 - v)\,\mathrm{d}\omega\, M_1 \mathrm{d}v_1\,. \tag{1.8}$$

This case is general because any Maxwellian can be put into the form (1.7) by applying a Galilean transformation to make its bulk velocity zero and a rescaling of units to make its mass density and temperature equal to unity, and because the collision operator $\mathcal{B}(F, F)$ commutes with Galilean boosts and is homogeneous under rescalings of density and temperature units. This last fact is a consequence of the factored form (1.5) of b. Finally, also without loss of generality, we normalize $\hat{b}$ so that

$$\int_{\mathbb{S}^{D-1}} \hat{b}(\omega \cdot n)\,\mathrm{d}\omega = 1\,. \tag{1.9}$$

We decompose the linearized Boltzmann operator $\mathcal{L}$ as

$$\mathcal{L} f = a(v)\,(I + \mathcal{K}_1 - \mathcal{K}_2 - \mathcal{K}_3)\, f\,, \tag{1.10}$$

where the attenuation coefficient $a(v)$ is given by

$$a(v) = \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} b(\omega, v_1 - v)\,\mathrm{d}\omega\, M_1 \mathrm{d}v_1\,, \tag{1.11}$$

while the loss operator $\mathcal{K}_1$ and the gain operators $\mathcal{K}_2$ and $\mathcal{K}_3$ are given by

$$
\begin{aligned}
\mathcal{K}_1 f &= \frac{1}{a(v)} \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} f_1 \, b(\omega, v_1 - v) \, \mathrm{d}\omega \, M_1 \, \mathrm{d}v_1 \,, \\
\mathcal{K}_2 f &= \frac{1}{a(v)} \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} f' \, b(\omega, v_1 - v) \, \mathrm{d}\omega \, M_1 \, \mathrm{d}v_1 \,, \\
\mathcal{K}_3 f &= \frac{1}{a(v)} \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} f_1' \, b(\omega, v_1 - v) \, \mathrm{d}\omega \, M_1 \, \mathrm{d}v_1 \,.
\end{aligned}
\tag{1.12}
$$

The main result of this paper is the following.

Main Theorem. *Let the collision kernel b have the factored form (1.5) and satisfy conditions (1.6). Then for every* $p \in (1, \infty)$ *and every* $j = 1, 2, 3$ *the operator*

$$
\mathcal{K}_j : L^p(aM\mathrm{d}v) \to L^p(aM\mathrm{d}v) \quad \textit{is compact}\,. \tag{1.13}
$$

The first result of this kind was given by Hilbert in the same paper in which he introduced what we now call the Hilbert expansion [13]. For the hard sphere case in $D = 3$ he applied his new theory of integral operators to essentially show that for the operator $\mathcal{K} = \mathcal{K}_1 - \mathcal{K}_2 - \mathcal{K}_3$ one has

$$
a\mathcal{K} : L^2(M\mathrm{d}v) \to L^2(M\mathrm{d}v) \quad \text{is compact}\,. \tag{1.14}
$$

More precisely, what he showed was isometrically equivalent to this. Also for the hard sphere case in $D = 3$, Hecke [12] gave a variant of Hilbert's result by showing that the kernel of $\mathcal{K}^5$ is Hilbert-Schmidt in $L^2(aM\mathrm{d}v)$, and Carleman [2] improved upon this by showing that the kernel of $\mathcal{K}^2$ is Hilbert-Schmidt in $L^2(aM\mathrm{d}v)$. Grad [10] extended Hilbert's result to the Maxwell and hard potential cases, $\beta \in [0, 1]$, by introducing his small deflection cutoff condition. This allowed him to adapt Hecke's argument and show that the kernel of $(a\mathcal{K})^3$ is Hilbert-Schmidt in $L^2(M\mathrm{d}v)$. By using similar methods, Caflisch [1] extended Grad's result (1.14) to the soft potential case by treating Grad cutoff kernels with $\beta \in (-1, 1]$ in $D = 3$, and Golse and Poupaud [7] established that $\mathcal{K}$ is compact over $L^2(aM\mathrm{d}v)$ for Grad cutoff kernels with $\beta \in (-2, 1]$ in $D = 3$. By using an approach that is closer to our own, Guo [11] extended Caflisch's result for Grad cutoff kernels to the full range $\beta \in (-3, 1]$ in $D = 3$. Our approach allows us to replace the Grad cutoff condition with the weaker cutoff condition given in (1.6). We do so for general dimension D.

The most important application of such compactness results has been to establish Fredholm alternative results for the linearized collision operator $\mathcal{L}$. Indeed, Grad showed that for Grad cutoff kernels with $\beta \in [0, 1]$ the operator $\mathcal{L}$ satisfied a Fredholm alternative in $L^2(M\mathrm{d}v)$. For soft potential case this does not hold, but it was pointed out by Golse and Poupaud in [7] that $\frac{1}{a}\mathcal{L}$ still satisfies a Fredholm alternative in $L^2(aM\mathrm{d}v)$. Fredholm alternatives and their related coercivity bounds play an important role in establishing hydrodynamic limits of the Boltzmann equation [4]. We mention that the Fredholm alternative has been established in $L^2(aM\mathrm{d}v)$ for an even broader class of collision kernels without any small deflection cutoff assumption by Mouhot and Strain [16] using a different approach, thereby improving upon an earlier result of Pao [17]. Our result yields a Fredholm alternative in $L^p(aM\mathrm{d}v)$ for every $p \in (1, \infty)$, but we require the weak cutoff condition. This L^p Fredholm alternative is used in [15] to prove an incompressible Navier-Stokes limit for the Boltzmann equation for the class of collision kernels considered here, thereby extending the results of Golse and Saint-Raymond [8, 9] for Grad cutoff kernels with $\beta \in [0, 1]$.

2. C P

2.1. **Preliminaries.** We begin with a basic fact that illustrates why the operators $\mathcal{K}_j$ defined by (1.12) and the spaces $L^p(aM\mathrm{d}v)$ are natural for this study.

Lemma 2.1. *Let the collision kernel b have the factored form (1.5) and satisfy conditions (1.6). Then for every $p \in [1,\infty]$ and every $j = 1,2,3$ the operator*

$$\mathcal{K}_j : L^p(aM\mathrm{d}v) \to L^p(aM\mathrm{d}v) \quad \textit{is bounded with } \|\mathcal{K}_j\|_{B(L^p)} \leq 1 \,. \tag{2.1}$$

Proof. First let $p \in (1,\infty)$ and set $p^* \in (1,\infty)$ such that $\frac{1}{p} + \frac{1}{p^*} = 1$. For every $f, g \in C_c(\mathbb{R}^D)$ we have

$$\int_{\mathbb{R}^D} g\,\mathcal{K}_1 f\; aM\mathrm{d}v = \iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} g\; f_1\, b\,\mathrm{d}\omega\, M_1\mathrm{d}v_1\, M\mathrm{d}v \,. \tag{2.2}$$

An application of the Hölder inequality then yields

$$\begin{aligned}\int_{\mathbb{R}^D} |g\,\mathcal{K}_1 f|\, aM\mathrm{d}v &\leq \left(\iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} |g|^{p^*}\, b\,\mathrm{d}\omega\, M_1\mathrm{d}v_1\, M\mathrm{d}v\right)^{\frac{1}{p^*}} \left(\iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} |f_1|^p\, b\,\mathrm{d}\omega\, M_1\mathrm{d}v_1\, M\mathrm{d}v\right)^{\frac{1}{p}} \\ &= \|g\|_{L^{p^*}(aM\mathrm{d}v)}\, \|f\|_{L^p(aM\mathrm{d}v)} \,.\end{aligned}$$

This inequality extends to every $f \in L^p(aM\mathrm{d}v)$ and $g \in L^{p^*}(aM\mathrm{d}v)$ by a density argument. By the Reisz Representation Theorem we see that assertion (2.1) for $\mathcal{K}_1$ holds with $\|\mathcal{K}_1\|_{B(L^p)} \leq 1$ for every $p \in (1,\infty)$. The extension to every $p \in [1,\infty]$ follows because (2.2) holds for every $f \in L^1(aM\mathrm{d}v)$ and $g \in L^\infty(aM\mathrm{d}v)$ or for every $f \in L^\infty(aM\mathrm{d}v)$ and $g \in L^1(aM\mathrm{d}v)$. The proofs of assertion (2.1) for $\mathcal{K}_2$ and $\mathcal{K}_3$ go similarly. □

Remark. This proof does not require the kernel $b(\omega, v_1 - v)$ to have the factored form (1.5). Rather, it only requires that the attenuation coefficient $a(v)$ given by (1.11) exists.

Given Lemma 2.1, our Main Theorem follows by a straightforward interpolation argument once we show that the compactness assertion (1.13) holds for $p = 2$. We will use the following compactness criterion [14], which is a generalization of the classical Hilbert-Schmidt criterion.

Lemma 2.2. *Let $\mathcal{K}$ be an integral operator given by*

$$\mathcal{K} f(v) = \int_{\mathbb{R}^D} K(v,v')\, f(v')\,\mathrm{d}\mu(v') \,,$$

where $\mathrm{d}\mu(v)$ is a σ-finite, positive measure over $\mathbb{R}^D$. Let the kernel $K(v,v')$ be symmetric in v and v' and for some $r \in [1,2]$ satisfy the bound

$$\|K\|_{L^{r^*}(L^r)} = \left(\int_{\mathbb{R}^D} \left(\int_{\mathbb{R}^D} |K(v,v')|^r\; \mathrm{d}\mu(v')\right)^{\frac{r^*}{r}} \mathrm{d}\mu(v)\right)^{\frac{1}{r^*}} < \infty \,, \tag{2.3}$$

where $r^* \in [2,\infty]$ satisfies $\frac{1}{r} + \frac{1}{r^*} = 1$. Let $p, p^* \in [r, r^*]$ satisfy $\frac{1}{p} + \frac{1}{p^*} = 1$. Then for every $f \in L^p(\mathrm{d}\mu)$ and $g \in L^{p^*}(\mathrm{d}\mu)$ one has

$$\int_{\mathbb{R}^D} |g(v)\,\mathcal{K} f(v)|\,\mathrm{d}\mu(v) \leq \iint_{\mathbb{R}^D\times\mathbb{R}^D} \left|K(v,v')\, f(v')\, g(v)\right| \mathrm{d}\mu(v')\,\mathrm{d}\mu(v) \leq \|K\|_{L^{r^*}(L^r)}\, \|f\|_{L^p}\, \|g\|_{L^{p^*}} \,, \tag{2.4}$$

whereby $\mathcal{K} : L^p(\mathrm{d}\mu) \to L^p(\mathrm{d}\mu)$ is bounded with $\|\mathcal{K}\|_{B(L^p)} \leq \|K\|_{L^{r^*}(L^r)}$. Moreover, if $r \in (1,2]$ then $\mathcal{K} : L^p(\mathrm{d}\mu) \to L^p(\mathrm{d}\mu)$ is compact.

The bounded (2.4) for either $p = r$ or $p = r^*$ is simply obtained by two applications of the Hölder inequality. Its extension to every $p \in [r, r^*]$ then follows by interpolation. The assertion that $\mathcal{K}$ maps $L^p(\mathrm{d}\mu)$ into itself and the bound on $\|\mathcal{K}\|_{B(L^p)}$ are consequences of the Riesz Representation Theorem. The compactness assertion holds because when $r \in (1,2]$ the finite-rank kernels are dense in the space $L^{r^*}(\mathrm{d}\mu; L^r(\mathrm{d}\mu))$ whose norm is defined by (2.3). The classical Hilbert-Schmidt compactness criterion is the special case $r = 2$. The above criterion often becomes easier to meet as r gets closer to 1.

2.2. **Compactness of the Loss Operator.** In this section we establish that the loss operator $\mathcal{K}_1$ is compact from $L^2(aM\mathrm{d}v)$ to $L^2(aM\mathrm{d}v)$. We begin with the following lemma, which plays a central role in our compactness proofs for the operators $\mathcal{K}_1$, $\mathcal{K}_2$, and $\mathcal{K}_3$.

Lemma 2.3. *Let $H \in L^\infty(\mathrm{d}v)$ such that $H \geq 0$, $H \neq 0$, and $(1+|v|)^r H \in L^\infty(\mathrm{d}v)$ for every $r > 0$. Let $\gamma > -D$ and define*

$$h(v) = \int_{\mathbb{R}^D} |v_1 - v|^\gamma \, H(v_1) \, \mathrm{d}v_1 \, . \tag{2.5}$$

Then $h \in C(\mathbb{R}^D)$ and there exist positive constants $\underline{C}$ and $\overline{C}$ such that

$$\underline{C}\,(1+|v|)^\gamma \leq h(v) \leq \overline{C}\,(1+|v|)^\gamma \,, \quad \textit{for every } v \in \mathbb{R}^D \, . \tag{2.6}$$

Proof. First consider the case when $\gamma \geq 0$. From the elementary bounds

$$|v_1 - v|^2 \leq (1+|v_1|^2)\,(1+|v|^2) \leq (1+|v_1|)^2\,(1+|v|)^2 \,, \quad \text{for every } v \in \mathbb{R}^D \,, \tag{2.7}$$

and the fact $H > 0$, we directly obtain the upper bound

$$h(v) = \int_{\mathbb{R}^D} |v_1 - v|^\gamma \, H(v_1) \, \mathrm{d}v_1 \leq \int_{\mathbb{R}^D} (1+|v_1|)^\gamma \, H(v_1) \, \mathrm{d}v_1 \, (1+|v|)^\gamma \,,$$

which yields the upper bound of (2.6).

Next, the bounds (2.7) and the fact $H > 0$ imply that $(1+|v|)^{-\gamma}\,|v_1 - v|^\gamma\,H(v_1) \leq (1+|v_1|)^\gamma\,H(v_1)$. The Lebesgue Dominated Convergence Theorem therefore implies that the positive function

$$v \mapsto \frac{h(v)}{(1+|v|)^\gamma} = \frac{1}{(1+|v|)^\gamma} \int_{\mathbb{R}^D} |v_1 - v|^\gamma \, H(v_1) \, \mathrm{d}v_1 \,, \tag{2.8}$$

is continuous over $\mathbb{R}^D$ and satisfies

$$\lim_{|v|\to\infty} \frac{h(v)}{(1+|v|)^\gamma} = \lim_{|v|\to\infty} \frac{1}{(1+|v|)^\gamma} \int_{\mathbb{R}^D} |v_1 - v|^\gamma \, H(v_1) \, \mathrm{d}v_1 = \int_{\mathbb{R}^D} H(v_1) \, \mathrm{d}v_1 > 0 \, .$$

The function (2.8) is thereby bounded away from zero, whereby the lower bound of (2.6) follows.

Now consider the case when $\gamma \in (-D, 0)$. From the elementary bounds (2.7) and the fact $H > 0$, we directly obtain the lower bound

$$\int_{\mathbb{R}^D} (1+|v_1|)^\gamma \, H(v_1) \, \mathrm{d}v_1 \, (1+|v|)^\gamma \leq \int_{\mathbb{R}^D} |v_1 - v|^\gamma \, H(v_1) \, \mathrm{d}v_1 = h(v) \,,$$

which yields the lower bound of (2.6).

Next, because $|v|^\gamma \in (L^p + L^q)$ while $H \in (L^{p^*} \cap L^{q^*})$ for every $p \in (1, \frac{D}{|\gamma|})$ and $q \in (\frac{D}{|\gamma|}, \infty)$, classical results on convolutions [6] show that $h \in C_0(\mathbb{R}^D)$. The function (2.8) is therefore continuous over $\mathbb{R}^D$. Let $s > D$ and $C_s < \infty$ such that $\|(1+|v|)^s H(v)\|_{L^\infty} \leq C_s$. Then for every nonzero $v \in \mathbb{R}^D$ we have

$$\frac{h(v)}{(1+|v|)^\gamma} \leq \frac{1}{(1+|v|)^\gamma} \int_{\mathbb{R}^D} |v_1 - v|^\gamma \, \frac{C_s}{(1+|v_1|)^s} \, \mathrm{d}v_1 = \frac{|v|^\gamma}{(1+|v|)^\gamma} \int_{\mathbb{R}^D} |w_1 - \hat{v}|^\gamma \, \frac{C_s\,|v|^D}{(1+|v||w_1|)^s} \, \mathrm{d}w_1 \,,$$

where $\hat{v} = v/|v|$ and $v_1 = |v|w_1$. By rotational invariance, the integral on the right-hand side above is independent of $\hat{v}$. Moreover, by classical "δ-function" approximation estimates we can show

$$\limsup_{|v|\to\infty} \frac{h(v)}{(1+|v|)^\gamma} \leq \lim_{|v|\to\infty} \int_{\mathbb{R}^D} |w_1 - \hat{v}|^\gamma \, \frac{C_s |v|^D}{(1+|v||w_1|)^s} \, \mathrm{d}w_1 = \int_{\mathbb{R}^D} \frac{C_s}{(1+|v_1|)^s} \, \mathrm{d}v_1 < \infty \, .$$

The function (2.8) is thereby uniformly bounded, whereby the upper bound of (2.6) follows. □

Compactness of $\mathcal{K}_1$ is a direct result of Lemmas 2.2 and 2.3.

Theorem 2.1. *Let the collision kernel b have the factored form (1.5) and satisfy conditions (1.6). Then*

$$\mathcal{K}_1 : L^2(aM\mathrm{d}v) \to L^2(aM\mathrm{d}v) \quad \textit{is compact}\,. \tag{2.9}$$

Proof. When b has the factored form (1.5) and satisfies (1.6) then with the normalization (1.9) the attenuation coefficient defined by (1.11) takes the form

$$a(v) = \int_{\mathbb{R}^D} |v_1 - v|^\beta \, M_1 \mathrm{d}v_1 \,, \tag{2.10}$$

while the loss operator $\mathcal{K}_1$ defined by (1.12) takes the form

$$\mathcal{K}_1 f(v) = \int_{\mathbb{R}^D} K_1(v, v_1) \, f(v_1) \, a_1 M_1 \mathrm{d}v_1 \,, \tag{2.11}$$

where the kernel $K_1(v, v_1)$ is given by

$$K_1(v, v_1) = \frac{1}{a(v)\, a(v_1)} \int_{\mathbb{S}^{D-1}} b(\omega, v_1 - v) \, \mathrm{d}\omega = \frac{|v_1 - v|^\beta}{a(v)\, a(v_1)} \,.$$

By Lemma 2.3 with $H = M$ and $\gamma = \beta$ there exist positive constants $\underline{C}_a$ and $\overline{C}_a$ such that

$$\underline{C}_a (1 + |v|)^\beta \leq a(v) \leq \overline{C}_a (1 + |v|)^\beta \,, \quad \text{for every } v \in \mathbb{R}^D \,. \tag{2.12}$$

Because $\beta \in (-D, \infty)$, there exists $r \in (1, 2]$ such that $\beta r \in (-D, \infty)$. Direct calculation shows that

$$\left(\int_{\mathbb{R}^D} |K_1(v, v_1)|^r \, a_1 M_1 \, \mathrm{d}v_1 \right)^{\frac{1}{r}} = \frac{1}{a(v)} \left(\int_{\mathbb{R}^D} |v_1 - v|^{\beta r} \, a_1^{1-r} M_1 \, \mathrm{d}v_1 \right)^{\frac{1}{r}} \,. \tag{2.13}$$

By Lemma 2.3 with $H = a^{1-r} M$ and $\gamma = \beta r$ there exist positive constants $\underline{C}_r$ and $\overline{C}_r$ such that

$$\underline{C}_r \, (1 + |v|)^{\beta r} \leq \int_{\mathbb{R}^D} |v_1 - v|^{\beta r} \, a_1^{1-r} M_1 \, \mathrm{d}v_1 \leq \overline{C}_r \, (1 + |v|)^{\beta r} \,, \quad \text{for every } v \in \mathbb{R}^D \,.$$

When this estimate is combined with estimate (2.12), we see from (2.13) that

$$0 < \frac{\underline{C}_r^{\frac{1}{r}}}{\overline{C}_a} \leq \left(\int_{\mathbb{R}^D} |K_1(v, v_1)|^r \, a_1 M_1 \, \mathrm{d}v_1 \right)^{\frac{1}{r}} \leq \frac{\overline{C}_r^{\frac{1}{r}}}{\underline{C}_a} < \infty \,, \quad \text{for every } v \in \mathbb{R}^D \,.$$

In particular, we see that $K_1 \in L^\infty(aM\mathrm{d}v; L^r(aM\mathrm{d}v)) \subset L^{r^*}(aM\mathrm{d}v; L^r(aM\mathrm{d}v))$, where $\frac{1}{r} + \frac{1}{r^*} = 1$. Because the kernel $K_1(v, v_1)$ thereby satisfies condition (2.3) for some $r \in (1, 2]$, upon applying Lemma 2.2 with $p = 2$ we see that assertion (2.9) holds. □

2.3. **Compactness of the Gain Operators.** The remainder of this paper establishes that the gain operators $\mathcal{K}_2$ and $\mathcal{K}_3$ are compact from $L^2(aM\mathrm{d}v)$ to $L^2(aM\mathrm{d}v)$. In this section we reduce the proof to technical lemmas that will be proved in later sections.

Theorem 2.2. *Let the collision kernel b have the factored form (1.5) and satisfy conditions (1.6). Then for $j = 2, 3$*

$$\mathcal{K}_j : L^2(aM\mathrm{d}v) \to L^2(aM\mathrm{d}v) \quad \textit{is compact}\,. \tag{2.14}$$

Proof. We see from definition (1.12) of $\mathcal{K}_2$ and $\mathcal{K}_3$ and the factored form (1.5) of b that

$$\begin{aligned} \mathcal{K}_2 f(v) &= \frac{1}{a(v)} \iint_{\mathbb{S}^{D-1} \times \mathbb{R}^D} f(v') \, \hat{b}(\omega \cdot n) \, |v_1 - v|^\beta \, \mathrm{d}\omega \, M_1 \, \mathrm{d}v_1 \,, \\ \mathcal{K}_3 f(v) &= \frac{1}{a(v)} \iint_{\mathbb{S}^{D-1} \times \mathbb{R}^D} f(v_1') \, \hat{b}(\omega \cdot n) \, |v_1 - v|^\beta \, \mathrm{d}\omega \, M_1 \, \mathrm{d}v_1 \,. \end{aligned} \tag{2.15}$$

We employ expressions for the kernels of these operators similar to those introduced by Grad [10]. Because v', v_1' and $\hat{b}$ are even functions of ω, the integrals over ω in (2.15) are just twice the integrals over the region $\omega \cdot n > 0$. It then follows from (1.3) that in this region we have the identities

$$
\begin{aligned}
&|v_1 - v|^2 = |v_1' - v|^2 + |v' - v|^2 \,, \qquad (v_1' - v)\cdot(v' - v) = 0 \,, \\
&\omega \cdot n = \frac{(v' - v)\cdot(v_1 - v)}{|v' - v|\,|v_1 - v|} = \frac{|v' - v|}{\sqrt{|v_1' - v|^2 + |v' - v|^2}} \,, \\
&|v_1|^2 = |v|^2 + 2v\cdot(v_1' - v) + 2v\cdot(v' - v) + |v_1' - v|^2 + |v' - v|^2 \,.
\end{aligned}
\tag{2.16}
$$

From these identities it can be shown that the gain operators $\mathcal{K}_2$ and $\mathcal{K}_3$ have the forms

$$
\mathcal{K}_2 f(v) = \int_{\mathbb{R}^D} K_2(v, v')\, f(v')\, a' M' \mathrm{d}v' \,, \qquad \mathcal{K}_3 f(v) = \int_{\mathbb{R}^D} K_2(v, v_1')\, f(v_1')\, a_1' M_1' \mathrm{d}v_1' \,,
\tag{2.17}
$$

where, off their diagonals, the symmetric kernels $K_2(v, v')$ and $K_3(v, v_1')$ are given by

$$
\begin{aligned}
K_2(v, v') &= \frac{2}{a(v)\,a(v')} \int_{y \perp (v'-v)} \frac{\left(|y|^2 + |v' - v|^2\right)^{\frac{\beta}{2}}}{|v' - v|^{D-1}}\, e^{-\frac{1}{2}|y|^2 - y\cdot w'}\, \hat{b}\!\left(\frac{|v' - v|}{\sqrt{|y|^2 + |v' - v|^2}}\right) \mathrm{d}y \,, \\
K_3(v, v_1') &= \frac{2}{a(v)\,a(v_1')} \int_{z \perp (v_1'-v)} \frac{\left(|v_1' - v|^2 + |z|^2\right)^{\frac{\beta}{2}}}{|z|^{D-1}}\, e^{-\frac{1}{2}|z|^2 - z\cdot w_1'}\, \hat{b}\!\left(\frac{|z|}{\sqrt{|v_1' - v|^2 + |z|^2}}\right) \mathrm{d}z \,,
\end{aligned}
\tag{2.18}
$$

with the vectors $w'(v, v')$ and $w_1'(v, v_1')$ defined by

$$
w' = \frac{|v|^2 - v'\cdot v}{|v' - v|^2}\, v' + \frac{|v'|^2 - v'\cdot v}{|v' - v|^2}\, v \,, \qquad w_1' = \frac{|v|^2 - v_1'\cdot v}{|v_1' - v|^2}\, v_1' + \frac{|v_1'|^2 - v_1'\cdot v}{|v_1' - v|^2}\, v \,.
\tag{2.19}
$$

These vectors have minimum norm on the lines $\{v + s(v' - v) \;:\; s \in \mathbb{R}\}$ and $\{v + s(v_1' - v) \;:\; s \in \mathbb{R}\}$ respectively. In (2.18) the variables of integration y and z are identified as $y = v_1' - v$ and $z = v' - v$.

The compactness of $\mathcal{K}_2$ and $\mathcal{K}_3$ is established by showing that they are each the limit of a sequence of compact operators. We construct the approximating sequences by truncations. Specifically, for every $R > 0$ and $\epsilon > 0$ we construct the operators $\mathcal{K}_2^{\epsilon,R}$ and $\mathcal{K}_3^{\epsilon,R}$ by replacing the kernels in (2.17) by the doubly truncated symmetric kernels that are respectively given by

$$
\begin{aligned}
K_2^{\epsilon,R}(v, v') &= \mathbf{1}_{\{|v|<R\}}\, \mathbf{1}_{\{|v'|<R\}}\, \frac{2}{a(v)\,a(v')} \int_{y \perp (v'-v)} \frac{\left(|y|^2 + |v' - v|^2\right)^{\frac{\beta}{2}}}{|v' - v|^{D-1}}\, e^{-\frac{1}{2}|y|^2 - y\cdot w'} \\
&\qquad\qquad \times \hat{b}\!\left(\frac{|v' - v|}{\sqrt{|y|^2 + |v' - v|^2}}\right) \mathbf{1}_{\{|v'-v|>\epsilon|v_1-v|\}}\, \mathrm{d}y \,, \\
K_3^{\epsilon,R}(v, v_1') &= \mathbf{1}_{\{|v|<R\}}\, \mathbf{1}_{\{|v_1'|<R\}}\, \frac{2}{a(v)\,a(v_1')} \int_{z \perp (v_1'-v)} \frac{\left(|v_1' - v|^2 + |z|^2\right)^{\frac{\beta}{2}}}{|z|^{D-1}}\, e^{-\frac{1}{2}|z|^2 - z\cdot w_1'} \\
&\qquad\qquad \times \hat{b}\!\left(\frac{|z|}{\sqrt{|v_1' - v|^2 + |z|^2}}\right) \mathbf{1}_{\{|v_1'-v|>\epsilon|v_1-v|\}}\, \mathbf{1}_{\{|z|>\epsilon|v_1-v|\}}\, \mathrm{d}z \,,
\end{aligned}
\tag{2.20}
$$

where $\mathbf{1}_S$ denotes the indicator function of a set S. The fact that the operators $\mathcal{K}_2^{\epsilon,R}$ and $\mathcal{K}_3^{\epsilon,R}$ are compact over $L^2(aM\mathrm{d}v)$ for every $R > 0$ and $\epsilon > 0$ will follow from Lemma 2.4, which is stated and proved in Section 2.4. The fact that the operators $\mathcal{K}_2^{\epsilon,R}$ and $\mathcal{K}_3^{\epsilon,R}$ approximate $\mathcal{K}_2$ and $\mathcal{K}_3$ as $\epsilon \to 0$ and $R \to \infty$ will follow from Lemma 2.5, which is stated and proved in Section 2.5. Because the compact operators are a closed subspace of $B(L^2)$, we thereby obtain the compactness of $\mathcal{K}_2$ and $\mathcal{K}_3$. □

Remark. When this result is combined with the compactness of the loss operator $\mathcal{K}_1$ over $L^2(aM\mathrm{d}v)$ provided by Theorem 2.1, and the boundedness of $\mathcal{K}_1$, $\mathcal{K}_2$, and $\mathcal{K}_3$ over $L^p(aM\mathrm{d}v)$ for every $p \in [1,\infty)$ provided by Lemma 2.1 then our Main Theorem follows by a standard interpolation argument.

2.4. **Compactness of the Approximating Operators.** In this section we establish the compactness of the approximating operators $\mathcal{K}_2^{\epsilon,R}$ and $\mathcal{K}_3^{\epsilon,R}$. We have the following.

Lemma 2.4. *Let $\mathcal{K}_2^{\epsilon,R}$ and $\mathcal{K}_3^{\epsilon,R}$ be the operators with kernels $K_2^{\epsilon,R}(v,v')$ and $K_3^{\epsilon,R}(v,v_1')$ respectively given by (2.20). Then*

(a) $\mathcal{K}_2^{\epsilon,R} : L^2(aM\mathrm{d}v) \to L^2(aM\mathrm{d}v)$ *is compact;*
(b) $\mathcal{K}_3^{\epsilon,R} : L^2(aM\mathrm{d}v) \to L^2(aM\mathrm{d}v)$ *is compact.*

Proof. The normalization (1.9) and a change of variables yields

$$1 = \int_{\mathbb{S}^{D-1}} \hat{b}(\omega\cdot n)\,\mathrm{d}\omega = 2\big|\mathbb{S}^{D-2}\big| \int_0^{\frac{\pi}{2}} \hat{b}(\cos(\theta))\,\sin(\theta)^{D-2}\,\mathrm{d}\theta\,. \tag{2.21}$$

We will use this fact to bound the kernels $K_2^{\epsilon,R}(v,v')$ and $K_3^{\epsilon,R}(v,v_1')$ in such a way that the compactness of the operators $\mathcal{K}_2^{\epsilon,R}$ and $\mathcal{K}_3^{\epsilon,R}$ will follow from Lemma 2.2.

We first establish assertion (a). By employing the pointwise bound $e^{-\frac{1}{2}|y|^2-y\cdot w'} \le e^{\frac{1}{2}|w'|^2}$ on the integrand in definition of $K_2^{\epsilon,R}(v,v')$ given by (2.20), we reduce the resulting bounding integral to a single radial integral over $|y|$. By introducing the change of variables

$$\cos(\theta) = \frac{|v'-v|}{\sqrt{|y|^2+|v'-v|^2}}\,, \qquad \frac{\mathrm{d}\theta}{\cos(\theta)\sin(\theta)} = \frac{\mathrm{d}|y|}{|y|}\,, \tag{2.22}$$

we may use (2.21) to express the resulting pointwise bound as

$$\begin{aligned} K_2^{\epsilon,R}(v,v') &\le \mathbf{1}_{\{|v|<R\}}\,\mathbf{1}_{\{|v'|<R\}}\,\frac{2|v'-v|^\beta}{a(v)\,a(v')}\,e^{\frac{1}{2}|w'|^2}\,\big|\mathbb{S}^{D-2}\big| \\ &\qquad \times \int_0^{\frac{\pi}{2}} \cos(\theta)^{-\beta}\,\hat{b}(\cos(\theta))\left(\frac{\sin(\theta)}{\cos(\theta)}\right)^{D-1} \mathbf{1}_{\{\cos(\theta)>\epsilon\}}\,\frac{\mathrm{d}\theta}{\cos(\theta)\sin(\theta)} \\ &\le \mathbf{1}_{\{|v|<R\}}\,\mathbf{1}_{\{|v'|<R\}}\,\frac{|v'-v|^\beta}{a(v)\,a(v')}\,e^{\frac{1}{2}|w'|^2}\,\frac{1}{\epsilon^{D+\beta}}\,. \end{aligned}$$

Because $|w'| \le |v|$, we therefore have the pointwise bounds

$$0 \le K_2^{\epsilon,R}(v,v') \le \frac{|v'-v|^\beta}{a(v)\,a(v')}\,\frac{e^{\frac{1}{2}R^2}}{\epsilon^{D+\beta}}\,.$$

Up to a constant factor, this upper bound has the same form as $K_1(v,v_1)$. Following the proof of Theorem 2.1, there exists $r \in (1,2]$ such that $\beta r \in (-D,\infty)$ and

$$\left(\int_{\mathbb{R}^D} |K_2^{\epsilon,R}(v,v')|^r\,a'M'\,\mathrm{d}v'\right)^{\frac{1}{r}} \le \frac{\overline{C}_r^{\,\frac{1}{r}}}{\underline{C}_a}\,\frac{e^{\frac{1}{2}R^2}}{\epsilon^{D+\beta}} < \infty\,, \quad \text{for every } v \in \mathbb{R}^D\,,$$

where $\overline{C}_r$ and $\underline{C}_a$ are the same constants appearing in the proof of Theorem 2.1. Because the symmetric kernel $K_2^{\epsilon,R}(v,v')$ thereby satisfies condition (2.3) for some $r \in (1,2]$, upon applying Lemma 2.2 with $p = 2$ we see that assertion (a) holds.

We now establish (b), which asserts the compactness of $\mathcal{K}_3^{\epsilon,R} : L^2(aM\mathrm{d}v) \to L^2(aM\mathrm{d}v)$ in a similar way. By employing the pointwise bound $e^{-\frac{1}{2}|y|^2-y\cdot w_1'} \le e^{\frac{1}{2}|w_1'|^2}$ on the integrand in definition of $K_3^{\epsilon,R}(v,v_1')$

given by (2.20), we reduce the resulting bounding integral to a single radial integral over $|z|$. By introducing the change of variables

$$\sin(\theta) = \frac{|v_1' - v|}{\sqrt{|v_1' - v|^2 + |z|^2}}\,, \qquad -\frac{\mathrm{d}\theta}{\cos(\theta)\sin(\theta)} = \frac{\mathrm{d}|z|}{|z|}\,, \tag{2.23}$$

we may use (2.21) to express the resulting pointwise bound as

$$\begin{aligned}
K_3^{\epsilon,R}(v,v_1') &\le \mathbf{1}_{\{|v|<R\}}\,\mathbf{1}_{\{|v_1'|<R\}}\,\frac{2|v_1'-v|^\beta}{a(v)\,a(v_1')}\,e^{\frac12|w_1'|^2}\left|\mathbb{S}^{D-2}\right| \\
&\qquad \times \int_0^{\frac{\pi}{2}} \sin(\theta)^{-\beta}\,\hat b(\cos(\theta))\,\mathbf{1}_{\{\cos(\theta)>\epsilon\}}\,\mathbf{1}_{\{\sin(\theta)>\epsilon\}}\,\frac{\mathrm{d}\theta}{\cos(\theta)\sin(\theta)} \\
&\le \mathbf{1}_{\{|v|<R\}}\,\mathbf{1}_{\{|v_1'|<R\}}\,\frac{|v_1'-v|^\beta}{a(v)\,a(v_1')}\,e^{\frac12|w_1'|^2}\,\frac{1}{\epsilon^{\max\{D+\beta,1\}}}\,.
\end{aligned}$$

Because $|w_1'| \le |v|$, we therefore have the pointwise bounds

$$0 \le K_3^{\epsilon,R}(v,v_1') \le \frac{|v_1'-v|^\beta}{a(v)\,a(v_1')}\,\frac{e^{\frac12 R^2}}{\epsilon^{\max\{D+\beta,1\}}}\,.$$

This upper bound has the same form as the upper bound we obtained for $K_2(v,v')^{\epsilon,R}$. By arguing as we did to establish (a), we see that assertion (b) also follows from Lemma 2.2. □

2.5. **Convergence of the Approximating Operators.** In this section we show that the remainder operators $\mathcal{K}_2 - \mathcal{K}_2^{\epsilon,R}$ and $\mathcal{K}_3 - \mathcal{K}_3^{\epsilon,R}$ can be made arbitrarily small. Their kernels have the form

$$\begin{aligned}
K_2(v,v') - K_2^{\epsilon,R}(v,v') &= \overline{K}_2^{\epsilon}(v,v') + \widetilde{K}_2^{\epsilon,R}(v,v')\,, \\
K_3(v,v_1') - K_3^{\epsilon,R}(v,v_1') &= \overline{K}_3^{\epsilon}(v,v_1') + \widetilde{K}_3^{\epsilon,R}(v,v_1')\,,
\end{aligned}$$

where

$$\begin{aligned}
\overline{K}_2^{\epsilon}(v,v') &= \frac{2|v'-v|^{-(D-1)}}{a(v)\,a(v')}\int_{y\perp(v'-v)} |v_1-v|^\beta\,e^{-\frac12|y|^2-y\cdot w'}\,\hat b\!\left(\frac{|v'-v|}{\sqrt{|y|^2+|v'-v|^2}}\right)\mathbf{1}_{\{|v'-v|\le\epsilon|v_1-v|\}}\,\mathrm{d}y\,, \\
\widetilde{K}_2^{\epsilon,R}(v,v') &= \left(1-\mathbf{1}_{\{|v|<R\}}\,\mathbf{1}_{\{|v'|<R\}}\right)\frac{2|v'-v|^{-(D-1)}}{a(v)\,a(v')} \\
&\qquad \times \int_{y\perp(v'-v)} |v_1-v|^\beta\,e^{-\frac12|y|^2-y\cdot w'}\,\hat b\!\left(\frac{|v'-v|}{\sqrt{|y|^2+|v'-v|^2}}\right)\mathbf{1}_{\{|v'-v|>\epsilon|v_1-v|\}}\,\mathrm{d}y\,, \\
\overline{K}_3^{\epsilon}(v,v_1') &= \frac{2}{a(v)\,a(v_1')}\int_{z\perp(v_1'-v)} \frac{|v_1-v|^\beta}{|z|^{D-1}}\,e^{-\frac12|z|^2-z\cdot w_1'}\,\hat b\!\left(\frac{|z|}{\sqrt{|z|^2+|v_1'-v|^2}}\right) \\
&\qquad \times\left(1-\mathbf{1}_{\{|v_1'-v|>\epsilon|v_1-v|\}}\,\mathbf{1}_{\{|z|>\epsilon|v_1-v|\}}\right)\mathrm{d}z \\
\widetilde{K}_3^{\epsilon,R}(v,v_1') &= \left(1-\mathbf{1}_{\{|v|<R\}}\,\mathbf{1}_{\{|v_1'|<R\}}\right)\frac{2}{a(v)\,a(v_1')} \\
&\qquad \times \int_{z\perp(v_1'-v)} \frac{|v_1-v|^\beta}{|z|^{D-1}}\,e^{-\frac12|z|^2-z\cdot w_1'}\,\hat b\!\left(\frac{|z|}{\sqrt{|v_1'-v|^2+|z|^2}}\right)\mathbf{1}_{\{|v_1'-v|>\epsilon|v_1-v|\}}\,\mathbf{1}_{\{|z|>\epsilon|v_1-v|\}}\,\mathrm{d}z\,.
\end{aligned} \tag{2.24}$$

Let $\overline{\mathcal{K}}_2^\epsilon$, $\widetilde{\mathcal{K}}_2^{\epsilon,R}$, $\overline{\mathcal{K}}_3^\epsilon$, and $\widetilde{\mathcal{K}}_3^{\epsilon,R}$ denote the operators with the kernels $\overline{K}_2^\epsilon(v,v')$, $\widetilde{K}_2^{\epsilon,R}(v,v')$, $\overline{K}_3^\epsilon(v,v_1')$, and $\widetilde{K}_3^{\epsilon,R}(v,v_1')$ respectively. In the following lemma we show that

$$\left\|\overline{\mathcal{K}}_2^\epsilon\right\|_{B(L^2)} \xrightarrow{\epsilon\downarrow 0} 0\,, \qquad \left\|\widetilde{\mathcal{K}}_2^{\epsilon,R}\right\|_{B(L^2)} \xrightarrow{R\uparrow\infty} 0\,, \qquad \left\|\overline{\mathcal{K}}_3^\epsilon\right\|_{B(L^2)} \xrightarrow{\epsilon\downarrow 0} 0\,, \qquad \left\|\widetilde{\mathcal{K}}_3^{\epsilon,R}\right\|_{B(L^2)} \xrightarrow{R\uparrow\infty} 0\,.$$

Its statement and proof employs the $L^2(aM\mathrm{d}v)$ inner product, which we denote

$$\langle g,h\rangle \triangleq \int_{\mathbb{R}^D} g\,h\,aM\mathrm{d}v\,.$$

Lemma 2.5. *For every $\eta>0$ there exists $\epsilon_0>0$ such that for every $g,\ f\in L^2(aM\mathrm{d}v)$ and $\epsilon\in(0,\epsilon_0)$ we have*

(i) $\left|\left\langle g,\overline{\mathcal{K}}_2^\epsilon f\right\rangle\right| \leq \eta\,\|g\|_{L^2(aM\mathrm{d}v)}\,\|f\|_{L^2(aM\mathrm{d}v)}\,,$

(ii) $\left|\left\langle g,\overline{\mathcal{K}}_3^\epsilon f\right\rangle\right| \leq \eta\,\|g\|_{L^2(aM\mathrm{d}v)}\,\|f\|_{L^2(aM\mathrm{d}v)}\,.$

Moreover, for every $\epsilon>0$ there exists $R_\epsilon>0$ such that for every $g,\ f\in L^2(aM\mathrm{d}v)$ and $R>R_\epsilon$ we have

(iii) $\left|\left\langle g,\widetilde{\mathcal{K}}_2^{\epsilon,R} f\right\rangle\right| \leq \eta\,\|g\|_{L^2(aM\mathrm{d}v)}\,\|f\|_{L^2(aM\mathrm{d}v)}\,,$

(iv) $\left|\left\langle g,\widetilde{\mathcal{K}}_3^{\epsilon,R} f\right\rangle\right| \leq \eta\,\|g\|_{L^2(aM\mathrm{d}v)}\,\|f\|_{L^2(aM\mathrm{d}v)}\,.$

Proof. (i) Define the nonnegative measure $\mathrm{d}\mu = b(\omega,v_1-v)\,\mathrm{d}\omega\,M_1\,\mathrm{d}v_1\,M\,\mathrm{d}v$. Then

$$\begin{aligned}\left|\left\langle g,\overline{\mathcal{K}}_2^\epsilon f\right\rangle\right| &\leq \iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} |g(v)|\,|f(v')|\,\mathbf{1}_{\{|v'-v|\leq\epsilon|v_1-v|\}}\,\mathrm{d}\mu\\ &\leq \left(\iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} |f(v')|^2\,\mathbf{1}_{\{|v'-v|\leq\epsilon|v_1-v|\}}\,\mathrm{d}\mu\right)^{\frac12}\,\|g\|_{L^2(aM\mathrm{d}v)}\,.\end{aligned}\tag{2.25}$$

In order to estimate the first factor on the right-hand side of the above inequality, we use the change of variables $(v,v_1)\mapsto(v',v_1')$ and the symmetries of the measure $\mathrm{d}\mu$ to obtain

$$\begin{aligned}\iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} |f(v')|^2\,\mathbf{1}_{\{|v'-v|\leq\epsilon|v_1-v|\}}\,\mathrm{d}\mu &= \iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} |f(v)|^2\,\mathbf{1}_{\{|v'-v|\leq\epsilon|v_1-v|\}}\,\mathrm{d}\mu\\ &= \int_{\mathbb{R}^D} |f(v)|^2 a^\epsilon(v)\,M\,\mathrm{d}v\,,\end{aligned}\tag{2.26}$$

where because $\mathrm{d}\mu = |v_1-v|^\beta\,\hat{b}(\omega\cdot n)\,\mathrm{d}\omega\,M_1\,\mathrm{d}v_1\,M\,\mathrm{d}v$, we can express $a^\epsilon(v)$ as

$$a^\epsilon(v) = \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} |v_1-v|^\beta\,\hat{b}(\omega\cdot n)\,\mathbf{1}_{\{|v'-v|\leq\epsilon|v_1-v|\}}\,\mathrm{d}\omega\,M_1\,\mathrm{d}v_1\,.\tag{2.27}$$

By assumption (1.6) that $\hat{b}(\omega\cdot n)\in L^1(\mathrm{d}\omega)$, we have for every $\eta>0$, there exists $\epsilon_0>0$ such that

$$\int_{\mathbb{S}^{D-1}} \hat{b}(\omega\cdot n)\,\mathbf{1}_{\{|v'-v|\leq\epsilon|v_1-v|\}}\,\mathrm{d}\omega = \int_{\mathbb{S}^{D-1}} \hat{b}(\omega\cdot n)\,\mathbf{1}_{\{|\omega\cdot n|\leq\epsilon\}}\,\mathrm{d}\omega \leq \eta^2 \qquad \text{for every } \epsilon\in(0,\epsilon_0)\,.$$

We then see from (2.27) that for every $\epsilon\in(0,\epsilon_0)$ we have

$$a^\epsilon(v) \leq \eta^2\int_{\mathbb{R}^D} |v_1-v|^\beta\,M_1\,\mathrm{d}v_1 = \eta^2 a(v)\,,$$

which by (2.26) implies that

$$\iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} |f(v')|^2\,\mathbf{1}_{\{|v'-v|\leq\epsilon|v_1-v|\}}\,\mathrm{d}\mu \leq \eta^2\,\|f\|_{L^2(aM\mathrm{d}v)}^2\,.$$

Upon placing this bound into (2.25), we establish (i).

(ii) The proof of (ii) is similar to that of (i). From (2.24), we see that

$$\left|\left\langle g, \overline{\mathcal{K}}_3^{\epsilon} f\right\rangle\right| \leq \iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} |g(v)|\,|f(v_1')|\,\left(1 - \mathbf{1}_{\{|v'-v|>\epsilon|v_1-v|\}}\,\mathbf{1}_{\{|v_1'-v|>\epsilon|v_1-v|\}}\right)\mathrm{d}\mu \leq I_1^{\epsilon} + I_2^{\epsilon}\,, \tag{2.28}$$

where

$$\begin{aligned} I_1^{\epsilon} &= \iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} |g(v)|\,|f(v_1')|\,\mathbf{1}_{\{|v'-v|\leq\epsilon|v_1-v|\}}\,\mathrm{d}\mu\,, \\ I_2^{\epsilon} &= \iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} |g(v)|\,|f(v_1')|\,\mathbf{1}_{\{|v_1'-v|\leq\epsilon|v_1-v|\}}\,\mathrm{d}\mu\,. \end{aligned} \tag{2.29}$$

We use the Schwarz inequality and the symmetries of $\mathrm{d}\mu$ to estimate I_1^{ϵ} as

$$\begin{aligned} I_1^{\epsilon} &\leq \left(\iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} |g(v)|^2\,\mathbf{1}_{|v'-v|\leq\epsilon|v_1-v|}\,\mathrm{d}\mu\right)^{\frac{1}{2}}\left(\iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} |f(v_1')|^2\,\mathrm{d}\mu\right)^{\frac{1}{2}} \\ &= \left(\int_{\mathbb{R}^D} |g(v)|^2\,a^{\epsilon}(v)\,M\mathrm{d}v\right)^{\frac{1}{2}}\|f\|_{L^2(aM\mathrm{d}v)}\,, \end{aligned}$$

where $a^{\epsilon}(v)$ is defined by (2.27). By arguing as in the proof of (i), there exists $\epsilon_1 > 0$ such that for every $\epsilon \in (0, \epsilon_1)$, we have $a^{\epsilon}(v) \leq \frac{1}{4}\eta^2 a(v)$. Therefore, for every $\epsilon \in (0, \epsilon_1)$, we have

$$I_1^{\epsilon} \leq \tfrac{1}{2}\,\eta\,\|g\|_{L^2(aM\mathrm{d}v)}\,\|f\|_{L^2(aM\mathrm{d}v)}\,. \tag{2.30}$$

The estimate for I_2^{ϵ} is similar to that for I_1^{ϵ}. The smallness of I_2^{ϵ} comes from the fact that there exists $\epsilon_2 > 0$ such that for every $\epsilon \in (0, \epsilon_2)$ we have

$$\int_{\mathbb{S}^{D-1}} \hat{b}(\omega\cdot n)\,\mathbf{1}_{\{|v_1'-v|\leq\epsilon|v_1-v|\}}\,\mathrm{d}\omega = \int_{\mathbb{S}^{D-1}} \hat{b}(\omega\cdot n)\,\mathbf{1}_{\{1-|\omega\cdot n|^2\leq\epsilon^2\}}\,\mathrm{d}\omega \leq \tfrac{1}{4}\eta^2\,,$$

which implies that

$$I_2^{\epsilon} \leq \tfrac{1}{2}\,\eta\,\|g\|_{L^2(aM\mathrm{d}v)}\,\|f\|_{L^2(aM\mathrm{d}v)}\,. \tag{2.31}$$

Upon setting $\epsilon_0 = \min\{\epsilon_1, \epsilon_2\}$ and placing bounds (2.30) and (2.31) into (2.28), we establish (ii).

(iii) Next, we estimate $\left|\langle g, \widetilde{\mathcal{K}}_2^{\epsilon,R} f\rangle\right|$ with ϵ fixed.

$$\left|\left\langle g, \widetilde{\mathcal{K}}_2^{\epsilon,R} f\right\rangle\right| \leq \iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} |g(v)|\,|f(v')|\,\left(1 - \mathbf{1}_{\{|v|<R\}}\mathbf{1}_{\{|v'|<R\}}\right)\,\mathbf{1}_{\{|v'-v|>\epsilon|v_1-v|\}}\,\mathrm{d}\mu \leq I_3^{\epsilon,R} + I_4^{\epsilon,R}\,, \tag{2.32}$$

where

$$\begin{aligned} I_3^{\epsilon,R} &= \iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} |g(v)|\,|f(v')|\,\mathbf{1}_{\{|v|\geq R\}}\,\mathbf{1}_{\{|v'-v|>\epsilon|v_1-v|\}}\,\mathrm{d}\mu\,, \\ I_4^{\epsilon,R} &= \iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} |g(v)|\,|f(v')|\,\mathbf{1}_{\{|v'|\geq R\}}\,\mathbf{1}_{\{|v'-v|>\epsilon|v_1-v|\}}\,\mathrm{d}\mu\,. \end{aligned} \tag{2.33}$$

By the change of variables $(v, v') \mapsto (v', v)$, it is clear that bounding $I_4^{\epsilon,R}$ is equivalent to bounding $I_3^{\epsilon,R}$. Therefore, we will only show how to bound $I_3^{\epsilon,R}$.

For every $m > 0$, we can bound $I_3^{\epsilon,R}$ as

$$I_3^{\epsilon,R} \leq J_1^m + J_2^{\epsilon,R,m}\,, \tag{2.34}$$

where

$$
\begin{aligned}
J_1^m &= \iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} |g(v)|\,|f(v')|\,\mathbf{1}_{\{|v_1|\geq m\}}\,\mathrm{d}\mu\,,\\
J_2^{\epsilon,R,m} &= \iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} |g(v)|\,|f(v')|\,\mathbf{1}_{\{|v'-v|>\epsilon|v_1-v|\}}\,\mathbf{1}_{\{|v|\geq R\}}\,\mathbf{1}_{\{|v_1|<m\}}\,\mathrm{d}\mu\,.
\end{aligned}
\tag{2.35}
$$

We bound J_1^m and $J_2^{\epsilon,R.m}$ separately.

We use the Schwarz inequality, the normalization (1.9), and the fact that $M_1\mathbf{1}_{\{|v_1|\geq m\}} \leq e^{-\frac{1}{4}m^2}\sqrt{M_1}$ to estimate J_1^m as

$$
\begin{aligned}
J_1^m &\leq \left(\iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} |g(v)|^2\,\mathbf{1}_{\{|v_1|\geq m\}}\,\mathrm{d}\mu\right)^{\frac{1}{2}} \|f\|_{L^2(aM\mathrm{d}v)}\\
&= \left(\int_{\mathbb{R}^D} |g(v)|^2 \left(\int_{\mathbb{R}^D} |v_1-v|^\beta\,\mathbf{1}_{\{|v_1|\geq m\}}\,M_1\,\mathrm{d}v_1\right) M\,\mathrm{d}v\right)^{\frac{1}{2}} \|f\|_{L^2(aM\mathrm{d}v)}\\
&\leq e^{-\frac{1}{4}m^2}\left(\int_{\mathbb{R}^D} |g(v)|^2 \left(\int_{\mathbb{R}^D} |v_1-v|^\beta\,\sqrt{M_1}\,\mathrm{d}v_1\right) M\,\mathrm{d}v\right)^{\frac{1}{2}} \|f\|_{L^2(aM\mathrm{d}v)}\,.
\end{aligned}
\tag{2.36}
$$

By Lemma 2.3 there exists a positive constant C_1 such that

$$
\int_{\mathbb{R}^D} |v_1-v|^\beta\,\sqrt{M_1}\,\mathrm{d}v_1 \leq C_1^2 a(v)\,.
$$

Therefore, estimate (2.36) yields

$$
J_1^m \leq C_1 e^{-\frac{1}{4}m^2}\,\|g\|_{L^2(aM\mathrm{d}v)}\,\|f\|_{L^2(aM\mathrm{d}v)}\,.
$$

If we choose m large enough so that $C_1 e^{-\frac{1}{4}m^2} < \frac{1}{4}\eta$, then

$$
J_1^m \leq \tfrac{1}{4}\eta\,\|g\|_{L^2(aM\mathrm{d}v)}\,\|f\|_{L^2(aM\mathrm{d}v)}\,.
\tag{2.37}
$$

Next, we show that for every fixed ϵ and m we can make $J_2^{\epsilon,R,m}$ arbitrarily small by taking R large enough. First observe that in the set $\{|v'-v|>\epsilon|v_1-v|\}$ we have

$$
|v_1'-v_1| = |v'-v| > \epsilon|v_1-v|\,,
$$

which implies that $|v_1'| > \epsilon|v_1-v| - |v_1|$. We can thereby choose R large enough so that in the set $\{|v'-v|>\epsilon|v_1-v|\}\cap\{|v|\geq R\}\cap\{|v_1|<m\}$ we have

$$
|v_1'| > \epsilon|v_1-v| - |v_1| \geq \epsilon|v| - (1+\epsilon)|v_1| \geq \epsilon R - (1+\epsilon)m > \tfrac{1}{2}\epsilon R\,.
$$

For every such R we use the fact that $\sqrt{M_1'M_1}\,\mathbf{1}_{\{|v_1'|>\frac{\epsilon R}{2}\}} \leq e^{-\frac{(\epsilon R)^2}{32}}\,e^{-\frac{1}{8}|v_1'|^2}$ to estimate $J_2^{\epsilon,R,m}$ as

$$
\begin{aligned}
J_2^{\epsilon,R,m} &= \iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} \left(|g(v)|\sqrt{M}\right)\left(|f(v')|\sqrt{M'}\right) \mathbf{1}_{\{|v'-v|>\epsilon|v_1-v|\}}\,\mathbf{1}_{\{|v|\geq R\}}\,\mathbf{1}_{\{|v_1|<m\}}\,\frac{\mathrm{d}\mu}{\sqrt{M'M}}\\
&\leq \iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} \left(|g(v)|\sqrt{M}\right)\left(|f(v')|\sqrt{M'}\right) \mathbf{1}_{\{|v_1'|>\frac{\epsilon R}{2}\}}\,\mathbf{1}_{\{|v_1|<m\}}\,\frac{\mathrm{d}\mu}{\sqrt{M'M}}\\
&= \iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} \left(|g(v)|\sqrt{M}\right)\left(|f(v')|\sqrt{M'}\right) \mathbf{1}_{\{|v_1'|>\frac{\epsilon R}{2}\}}\,\mathbf{1}_{\{|v_1|<m\}}\,\sqrt{M_1'M_1}\,b\,\mathrm{d}\omega\,\mathrm{d}v_1\,\mathrm{d}v\\
&\leq e^{-\frac{(\epsilon R)^2}{32}} \iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} \left(|g(v)|\sqrt{M}\right)\left(|f(v')|\sqrt{M'}\right) \mathbf{1}_{\{|v_1|<m\}}\,e^{-\frac{1}{8}|v_1'|^2}\,b\,\mathrm{d}\omega\,\mathrm{d}v_1\,\mathrm{d}v\,.
\end{aligned}
\tag{2.38}
$$

The Schwarz inequality, the change of variable $(v_1', v') \mapsto (v_1, v)$, and the normalization (1.9) give

$$
\begin{aligned}
J_2^{\epsilon,R,m} &\leq e^{-\frac{(\epsilon R)^2}{32}} \left(\iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} |g(v)|^2 \, M \, \mathbf{1}_{\{|v_1|<m\}} \, b \, \mathrm{d}\omega \, \mathrm{d}v_1 \, \mathrm{d}v \right)^{\frac{1}{2}} \\
&\quad \times \left(\iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} |f(v')|^2 M' \, e^{-\frac{1}{4}|v_1'|^2} \, b \, \mathrm{d}\omega \, \mathrm{d}v_1 \, \mathrm{d}v \right)^{\frac{1}{2}} \\
&= e^{-\frac{(\epsilon R)^2}{32}} \left(\int_{\mathbb{R}^D} |g(v)|^2 \left(\int_{\mathbb{R}^D} |v_1 - v|^\beta \, \mathbf{1}_{\{|v_1|<m\}} \, \mathrm{d}v_1 \right) M \, \mathrm{d}v \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{\mathbb{R}^D} |f(v)|^2 \left(\int_{\mathbb{R}^D} |v_1 - v|^\beta \, e^{-\frac{1}{4}|v_1|^2} \, \mathrm{d}v_1 \right) M \, \mathrm{d}v \right)^{\frac{1}{2}} .
\end{aligned}
\tag{2.39}
$$

By Lemma 2.3 there exists a positive constant C_2 such that

$$
\int_{\mathbb{R}^D} |v_1 - v|^\beta \, \mathbf{1}_{\{|v_1|<m\}} \, \mathrm{d}v_1 \leq C_2 \, a(v) \,, \qquad \int_{\mathbb{R}^D} |v_1 - v|^\beta \, e^{-\frac{1}{4}|v_1|^2} \, \mathrm{d}v_1 \leq C_2 \, a(v) \,.
$$

Therefore, estimate (2.39) yields

$$
J_2^{\epsilon,R,m} \leq C_2 \, e^{-\frac{(\epsilon R)^2}{32}} \, \|g\|_{L^2(aM\mathrm{d}v)} \, \|f\|_{L^2(aM\mathrm{d}v)} \,.
$$

If we choose R large enough so that $C_2 e^{-\frac{(\epsilon R)^2}{32}} < \frac{1}{4}\,\eta$, then

$$
J_2^{\epsilon,R,m} \leq \tfrac{1}{4}\,\eta \, \|g\|_{L^2(aM\mathrm{d}v)} \, \|f\|_{L^2(aM\mathrm{d}v)} \,. \tag{2.40}
$$

By combining estimates (2.37) and (2.40) into (2.34), we bound $I_3^{\epsilon,R}$ as

$$
I_3^{\epsilon,R} \leq \tfrac{1}{2}\,\eta \, \|g\|_{L^2(aM\mathrm{d}v)} \, \|f\|_{L^2(aM\mathrm{d}v)} \,. \tag{2.41}
$$

The bound on $I_4^{\epsilon,R}$ is identical to that for $I_3^{\epsilon,R}$. It therefore follows from (2.32) that for every $\epsilon > 0$ there exists $R_\epsilon > 0$ such that (iii) holds for every $R > R_\epsilon$.

(iv) The proof of (iv) closely follows that of (iii). First, we have

$$
\begin{aligned}
\left| \left\langle g, \widetilde{\mathcal{K}}_3^{\epsilon,R} f \right\rangle \right| &\leq \iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} |g(v)| \, |f(v_1')| \, \left(1 - \mathbf{1}_{\{|v|<R\}} \mathbf{1}_{\{|v_1'|<R\}} \right) \mathbf{1}_{\{|v_1'-v|>\epsilon|v_1-v|\}} \, \mathbf{1}_{\{|z|>\epsilon|v_1-v|\}} \, \mathrm{d}\mu \\
&\leq I_5^{\epsilon,R} + I_6^{\epsilon,R} \,,
\end{aligned}
\tag{2.42}
$$

where

$$
\begin{aligned}
I_5^{\epsilon,R} &= \iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} |g(v)| \, |f(v_1')| \, \mathbf{1}_{\{|v|\geq R\}} \mathbf{1}_{\{|v_1'-v|>\epsilon|v_1-v|\}} \, \mathrm{d}\mu \,, \\
I_6^{\epsilon,R} &= \iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} |g(v)| \, |f(v_1')| \, \mathbf{1}_{\{|v_1'|\geq R\}} \mathbf{1}_{\{|v_1'-v|>\epsilon|v_1-v|\}} \, \mathrm{d}\mu \,.
\end{aligned}
\tag{2.43}
$$

The change of variables $(v, v_1') \mapsto (v_1', v)$ shows that bounding $I_6^{\epsilon,R}$ is equivalent to bounding $I_5^{\epsilon,R}$. Therefore we only need to bound $I_5^{\epsilon,R}$. As we did in (2.34–2.35), this estimate begins with the bound

$$
\begin{aligned}
I_5^{\epsilon,R} &\leq \iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} |g(v)| \, |f(v_1')| \, \mathbf{1}_{\{|v_1|\geq m\}} \, \mathrm{d}\mu \\
&\quad + \iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} |g(v)| \, |f(v_1')| \, \mathbf{1}_{\{|v_1'-v|>\epsilon|v_1-v|\}} \, \mathbf{1}_{\{|v|\geq R\}} \, \mathbf{1}_{\{|v_1|<m\}} \, \mathrm{d}\mu \,.
\end{aligned}
$$

The above integrals are bounded as in (2.36–2.37) and (2.38–2.40) respectively. Hence, for any $\epsilon > 0$, there exists $R_\epsilon > 0$ such that (iv) holds for every $R > R_\epsilon$. To avoid repetition, we omit these details.

We have now shown that all the remainders associated with the approximations of $\mathcal{K}_2$ and $\mathcal{K}_3$ can be made arbitrarily small. This completes the proof of Lemma 2.5, and thereby finishes the proof of Theorem 2.2. By the remark at the end of Section 2.3, this also establishes our Main Theorem. □

Remark. The Main Theorem remains true if one replaces the assumption that the collision kernel b has the factored form (1.5-1.6) with the assumption that b satisfies the bounds

$$b(\omega, v_1 - v) \leq \hat{b}(\omega \cdot n)\left(|v_1 - v|^{\alpha} + |v_1 - v|^{\beta}\right), \qquad a(v) \geq \hat{a}\,(1 + |v|)^{\beta},$$

for some $\beta > \alpha > -D$, $\hat{b} \in L^1(\mathrm{d}\omega)$ and $\hat{a} > 0$, where the attenuation coefficient a is given by (1.11). Because we do not know of any additional physical collision kernels (ones derived from a classical intermolecular potential) that would be included by this generalization, we do not present its proof. The proof is similar to the one given above, only longer because there are more terms to estimate.

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