LOCAL WELL-POSEDNESS OF A GHOST EFFECT SYSTEM

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A   . We establish the local well-posedness result for the Cauchy problem of a ghost effect system from gas dynamics that derives from kinetic theory. We show that this system has a unique classical solution for a finite time for all initial data whose deviations from nonzero background values lie in Sobolev spaces of sufficiently high order and such that its initial temperature is positive everywhere.

1. I

In this paper we prove the local well-posedness of a ghost effect system (cf. Sone [21]). This system is non-classical in the sense that it cannot be derived from the compressible Navier-Stokes system. It describes certain gas dynamical flows that are induced by temperature variations and can be derived from kinetic equations by the Hilbert expansion method [21]. Maxwell was the first to study thermal-induced flows [16]. He derived a correction to the Navier-Stokes stress tensor that depends on derivatives of the temperature. However, he just studied regimes in which the effect of this correction entered only through boundary conditions. Kogan, Galkin, and Fridlender [8] subsequently pointed out that in certain regimes with strong temperature variations the correction of Maxwell enters into the dynamical description of the gas at leading order in the interior of the domain. Such regimes can arise in certain geometries when the gas is confined by stationary walls held at different uniform temperatures. We refer the reader to [6, 7, 16, 18, 21, 22] and references therein for more information, including descriptions of devices that operate in these regimes. In such regimes the classical heat-conduction equation fails to correctly describe the temperature field of the gas. Indeed, corrections derived from kinetic equations must be included to accommodate this phenomenon [1, 2, 18, 19, 20, 21, 22, 23, 24, 25]. The moniker "ghost effect" for such systems was coined by Sone [20, 21, 22].

Ghost effect systems, which are formally derived to describe regimes in which the compressible Navier-Stokes system is incomplete, are physically relevant. The objective of this paper is to provide a well-posedness result for one such system as a first step toward the development of rigorous mathematical theories related to these regimes. We will do so over $\mathbb{R}^d$ for any $d \geq 2$ because at this point we do not have a satisfactory theory of boundary conditions for domains with boundary. Our result plays a role in the investigation of low Mach number limits of a dispersive Navier-Stokes system [10, 11, 12].

The ghost effect system we consider describes the evolution of the density $\rho(t, x)$, velocity $u(t, x)$, temperature $\theta(t, x)$, and pressure field $P(t, x)$ of a γ-law gas as a function of time $t \in \mathbb{R}^+$ and position $x \in \mathbb{R}^d$. Let $D \geq d$ be the dimension of the underlying microscopic physics. The system has the form

$$
\begin{aligned}
&\nabla_x(\rho\theta) = 0\,,\\
&\partial_t\rho + \nabla_x\cdot(\rho u) = 0\,,\\
&\partial_t(\rho u) + \nabla_x\cdot(\rho u\otimes u) + \nabla_x P = \nabla_x\cdot\Sigma + \nabla_x\cdot\tilde{\Sigma}\,,\\
&\partial_t(c_{\mathrm{v}}\rho\theta) + \nabla_x\cdot(\gamma c_{\mathrm{v}}\rho\theta u) = -\nabla_x\cdot q\,,
\end{aligned}
\tag{1.1}
$$

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where Σ is the viscous stress, $\tilde{\Sigma}$ is the thermal stress, $c_{\rm v}$ is the specific heat capacity at constant volume, γ is the adiabatic exponent, and q is the heat flux. Here $c_{\rm v} \geq \frac{D}{2}$ and $\gamma > 1$ are constants while Σ, $\tilde{\Sigma}$ and q are related to the fluid variables ρ, u, and θ through the constitutive relations

$$\begin{aligned}
\Sigma &= \mu(\theta)\left(\nabla_x u + (\nabla_x u)^T - \tfrac{2}{D}(\nabla_x \cdot u)I\right),\\
\tilde{\Sigma} &= \tau_1(\rho,\theta)\left(\nabla_x^2\theta - \tfrac{1}{D}(\Delta_x\theta)I\right) + \tau_2(\rho,\theta)\left(\nabla_x\theta\otimes\nabla_x\theta - \tfrac{1}{D}|\nabla_x\theta|^2 I\right)\\
&\qquad + \tau_3(\rho,\theta)\left(\nabla_x\rho\otimes\nabla_x\theta + \nabla_x\theta\otimes\nabla_x\rho - \tfrac{2}{D}\nabla_x\rho\cdot\nabla_x\theta I\right),\\
q &= -\gamma c_{\rm v}\kappa(\theta)\,\nabla_x\theta\,,
\end{aligned}$$

where $\mu(\theta) > 0$ is the coefficient of shear viscosity, $\gamma c_{\rm v}\kappa(\theta) > 0$ is the coefficient of thermal conductivity, and $\tau_j(\rho,\theta)$ for $j = 1, 2, 3$ are transport coefficients that arise from kinetic theory. Our unusual normalization of the coefficient of thermal conductivity here will lead to a simplification shortly. We remark that it is the presence of the thermal stress $\tilde{\Sigma}$ that is the main source of difficulty in our analysis.

We will impose the boundary conditions that there exist positive constants $\bar{\rho}$ and $\bar{\theta}$ such that

$$\rho \to \bar{\rho} \quad \text{and} \quad \theta \to \bar{\theta} \qquad \text{as} \quad |x| \to \infty. \tag{1.2}$$

Notice that the first equation in (1.1) implies that $\rho\theta$ is a function of t only. However, our boundary conditions imply that $\rho\theta \to \bar{\rho}\bar{\theta}$ as $|x| \to \infty$, whereby $\rho\theta$ is independent of t. Without loss of generality, we can set $\rho\theta = 1$. We then use this relation to eliminate ρ from (1.1). The resulting system in (θ, u, P) has the form

$$\begin{aligned}
&\partial_t\theta + u\cdot\nabla_x\theta = \theta\,\nabla_x\cdot\left[\kappa(\theta)\nabla_x\theta\right],\\
&\frac{1}{\theta}\left(\partial_t u + u\cdot\nabla_x u\right) + \nabla_x P = \nabla_x\cdot\Sigma + \nabla_x\cdot\tilde{\Sigma}\,,\\
&\nabla_x\cdot\left[u - \kappa(\theta)\nabla_x\theta\right] = 0\,,
\end{aligned}\tag{1.3}$$

where

$$\begin{aligned}
\Sigma &= \mu(\theta)\left(\nabla_x u + (\nabla_x u)^T - \tfrac{2}{D}(\nabla_x\cdot u)I\right),\\
\tilde{\Sigma} &= \hat{\tau}_1(\theta)\left(\nabla_x^2\theta - \tfrac{1}{D}(\Delta_x\theta)I\right) + \hat{\tau}_2(\theta)\left(\nabla_x\theta\otimes\nabla_x\theta - \tfrac{1}{D}|\nabla_x\theta|^2 I\right),
\end{aligned}$$

with

$$\hat{\tau}_1(\theta) = \tau_1\!\left(\frac{1}{\theta},\theta\right), \qquad \hat{\tau}_2(\theta) = \tau_2\!\left(\frac{1}{\theta},\theta\right) - \tfrac{2}{\theta^2}\tau_3\!\left(\frac{1}{\theta},\theta\right).$$

We will establish well-posedness for the reduced system (1.3) subject to the boundary conditions

$$\theta \to \bar{\theta} \quad \text{and} \quad u \to 0 \qquad \text{as } |x| \to \infty. \tag{1.4}$$

and the initial conditions

$$(\theta, u)\big|_{t=0} = (\theta^{in}, u^{in})\,, \tag{1.5}$$

where the data (θ^{in}, u^{in}) are consistent with the boundary conditions (1.4) and satisfy the constraints

$$\theta^{in} > 0\,, \qquad \text{and} \qquad \nabla_x\cdot\left[u^{in} - \kappa(\theta^{in})\nabla_x\theta^{in}\right] = 0\,. \tag{1.6}$$

By setting $\rho = 1/\theta$ we will then establish well-posedness for system (1.1).

While the third equation in system (1.3) shows that the system does not describe incompressible flow, it is a constraint that plays a role similar to that played by the incompressibility condition for the incompressible Navier-Stokes system. Indeed, the pressure P in the motion equation plays the role of a Lagrangian multiplier by which the constraint given by the third equation is maintained. Indeed, system (1.3) formally reduces to an incompressible Navier-Stokes system when $\theta - \bar{\theta}$ is small. However there are important differences between these systems. For one, the constraint in system

(1.3) is nonlinear. The most important difference is the presence of the term $\nabla_x \cdot \tilde{\Sigma}$ in the motion equation, which will be the main source of difficulty in our analysis. This is because $\nabla_x \cdot \tilde{\Sigma}$ includes third-order derivatives of θ, which prevents a direct application of integration by parts to obtain a closed energy inequality for system (1.3). To overcome this difficulty, we observe that $\nabla_x \cdot \tilde{\Sigma}$ can be written as

$$\begin{aligned} \nabla_x \cdot \tilde{\Sigma} = {}& \nabla_x\Big(\nabla_x \cdot [\hat{\tau}_1(\theta)\nabla_x\theta]\Big) - \nabla_x\left(\tfrac{1}{D}\hat{\tau}_1(\theta)\,\Delta_x\theta\right) \\ & - \nabla_x \cdot \left(\hat{\tau}_1'(\theta)\,\nabla_x\theta\otimes\nabla_x\theta\right) + \nabla_x \cdot \left(\hat{\tau}_2(\theta)\,\left(\nabla_x\theta\otimes\nabla_x\theta - \tfrac{1}{D}|\nabla_x\theta|^2 I\right)\right). \end{aligned} \tag{1.7}$$

Notice that the first and second terms on the right-hand side of (1.7) are gradients while the third and fourth terms are second-order in θ. The key observation here is that the gradient terms can be incorporated into the pressure term to produce a new pressure term $\nabla_x\tilde{p}$ where

$$\tilde{p} = P - \nabla_x \cdot [\hat{\tau}_1(\theta)\nabla_x\theta] + \tfrac{1}{D}\hat{\tau}_1(\theta)\,\Delta_x\theta\,. \tag{1.8}$$

By introducing this new pressure, we decrease the order of the perturbation in the motion equation to second order in θ. We refer the reader to Sone [21] who uses this same observation to analyze the structure of the stationary system. The observation goes back to Maxwell [16] who used it to explain why the thermal stress would not effect the dynamics of incompressible flows away from boundaries. Here we use it to motivate a reformulation of system (1.3) for which we can obtain a closed energy estimate from the dissipation of both θ and u.

The main result of this paper establishes the local well-posedness of system (1.3), and consequently of system (1.1), as follows.

Main Theorem. *Let the transport coefficients μ, κ, $\hat{\tau}_1$, and $\hat{\tau}_2$ appearing in system (1.3) be smooth functions over $\mathbb{R}_+$ with $\mu > 0$ and $\kappa > 0$. Let $\overline{\theta} > 0$ and $s > d/2 + 1$. Let the initial data (θ^{in}, u^{in}) satisfy the constraints (1.6) such that*

$$\theta^{in} - \overline{\theta} \in H^{s+1}(\mathbb{R}^d)\,, \qquad u^{in} \in H^s(\mathbb{R}^d)\,. \tag{1.9}$$

Then there exists $T > 0$ such that system (1.3-1.5) has a unique solution (θ, u) with

$$\begin{aligned} \theta - \overline{\theta} &\in C([0,T]; H^{s+1}(\mathbb{R}^d)) \cap L^2([0,T]; H^{s+2}(\mathbb{R}^d)) \cap C^\infty((0,T)\times\mathbb{R}^d)\,, \\ u &\in C([0,T]; H^{s}(\mathbb{R}^d)) \cap L^2([0,T]; H^{s+1}(\mathbb{R}^d)) \cap C^\infty((0,T)\times\mathbb{R}^d)\,, \\ \nabla_x P &\in C([0,T]; H^{s-2}(\mathbb{R}^d)) \cap C^\infty((0,T)\times\mathbb{R}^d)\,. \end{aligned} \tag{1.10}$$

Moreover, T depends only on $\|\theta^{in} - \overline{\theta}\|_{H^{s+1}(\mathbb{R}^d)}$, $\|u^{in}\|_{H^s(\mathbb{R}^d)}$, and $\lambda_0 = \inf\{\theta^{in}(x) \,:\, x \in \mathbb{R}^d\} > 0$.

The proof of this theorem will be given in the next section. Here we mention that this result leaves many open questions. For starters, the evident smoothing of the dynamics indicates the result should extend to larger classes of initial data. Given appropriate boundary conditions for system (1.3), the above result should also have extensions to domains with boundaries. In that setting it would be natural to seek global classical solutions that are small perturbations of certain stationary solutions. One could also try to prove similar theorems for ghost-effect systems that arise from more general gases than the γ-law gases considered here — for example, for systems that arise from general ideal gases. Finally, given the similarities of system (1.3) with incompressible Navier-Stokes systems, it is natural to ask if it has a Leray-like theory of global weak solutions.

2. L W -P

In this section we establish the local well-posedness of system (1.3) asserted by our Main Theorem. The existence is established by an iterative argument. Both the convergence of the iterates to a solution and the uniqueness of that solution are consequences of an associated energy estimate that we obtain for a reformulation of system (1.3). Finally, we establish the regularity asserted in our Main Theorem.

2.1. **Reformulation.** In order to obtain the energy estimate, we reformulate system (1.3) in terms of the new velocity variable

$$v = u - \kappa(\theta)\nabla_x\theta\,. \tag{2.1}$$

In our reformulation we will use the notation

$$\Sigma_\theta(w) := \mu(\theta)\left(\nabla_x w + (\nabla_x w)^T - \tfrac{2}{D}(\nabla_x\cdot w)I\right)\,. \tag{2.2}$$

We will show that system (1.3) expressed in terms of (θ, v) has the form

$$\begin{aligned} \partial_t\theta + v\cdot\nabla_x\theta &= \nabla_x\cdot\left[\theta\kappa(\theta)\nabla_x\theta\right] - 2\kappa(\theta)|\nabla_x\theta|^2\,,\\ \partial_t v + v\cdot\nabla_x v + \theta\nabla_x p &= \theta\nabla_x\cdot\Sigma_\theta(v) + F_1(\theta,\nabla_x\theta,\nabla_x v) + F_2(\theta, v,\nabla_x\theta,\nabla_x^2\theta)\,,\\ \nabla_x\cdot v &= 0\,, \end{aligned} \tag{2.3}$$

where the specific forms of p, F_1, and F_2 will be given below. The new pressure term $\nabla_x p$ is formed by combining the original pressure term $\nabla_x P$ in (1.3) with other gradient terms that arise during the calculation.

The derivation of the motion equation in (2.3) begins with the momentum local conservation law, which because $\rho = 1/\theta$ is

$$\partial_t\left(\frac{u}{\theta}\right) + \nabla_x\cdot\left(\frac{u\otimes u}{\theta}\right) + \nabla_x P = \nabla_x\cdot\Sigma_\theta(u) + \nabla_x\cdot\tilde{\Sigma}\,. \tag{2.4}$$

We will now use (2.1) to re-express this in terms of v. First, we see that

$$\partial_t\left(\frac{u}{\theta}\right) = \partial_t\left(\frac{v}{\theta}\right) + \partial_t\left(\frac{\kappa(\theta)}{\theta}\,\nabla_x\theta\right) = \partial_t\left(\frac{v}{\theta}\right) + \nabla_x(\partial_t K_1(\theta))\,, \tag{2.5}$$

where $K_1(\theta)$ satisfies $K_1'(\theta) = \kappa(\theta)/\theta$. The term $\nabla_x(\partial_t K_1(\theta))$ above can be combined with $\nabla_x P$ in (2.4) by redefining the pressure.

Second, from (2.1) and the fact that $\nabla_x\cdot v = 0$ we obtain

$$\begin{aligned} \nabla_x\cdot\left(\frac{u\otimes u}{\theta}\right) &= \nabla_x\cdot\left(\frac{(v+\kappa(\theta)\nabla_x\theta)\otimes(v+\kappa(\theta)\nabla_x\theta)}{\theta}\right)\\ &= \nabla_x\cdot\left(\frac{v\otimes v}{\theta}\right) + \nabla_x\cdot\left(\frac{\kappa(\theta)}{\theta}\,(v\otimes\nabla_x\theta + \nabla_x\theta\otimes v)\right) + \nabla_x\cdot\left(\frac{\kappa(\theta)^2}{\theta}\,\nabla_x\theta\otimes\nabla_x\theta\right)\\ &= \nabla_x\cdot\left(\frac{v\otimes v}{\theta}\right) + v\cdot\nabla_x^2 K_1(\theta) + v\Delta_x K_1(\theta) + \nabla_x K_1(\theta)\cdot\nabla_x v + \nabla_x\cdot\left(\frac{\kappa(\theta)^2}{\theta}\,\nabla_x\theta\otimes\nabla_x\theta\right)\,. \end{aligned} \tag{2.6}$$

Third, let the function $K_2(\theta)$ satisfy $K_2'(\theta) = \kappa(\theta)$. Then from (2.1) and (2.2) we see that

$$\begin{aligned} \nabla_x\cdot\Sigma_\theta(u) &= \nabla_x\cdot\Sigma_\theta(v) + \nabla_x\cdot\Sigma_\theta(\kappa(\theta)\nabla_x\theta)\\ &= \nabla_x\cdot\Sigma_\theta(v) + 2\,\nabla_x\cdot\left[\mu(\theta)\left(\nabla_x^2 K_2(\theta) - \tfrac{1}{D}\Delta_x K_2(\theta) I\right)\right]\\ &= \nabla_x\cdot\Sigma_\theta(v) + 2\,\nabla_x\Big(\nabla_x\cdot\left[\mu(\theta)\nabla_x K_2(\theta)\right]\Big) - 2\,\nabla_x\cdot\Big[\kappa(\theta)\mu'(\theta)\nabla_x\theta\otimes\nabla_x\theta\Big]\\ &\quad - \nabla_x\left(\tfrac{2}{D}\mu(\theta)\Delta_x K_2(\theta)\right)\,. \end{aligned} \tag{2.7}$$

The two gradient terms above, $2\nabla_x\Big(\nabla_x\cdot[\mu(\theta)\nabla_x K_2(\theta)]\Big)$ and $\nabla_x\left(\frac{2}{D}\mu(\theta)\Delta_x K_2(\theta)\right)$, can also be absorbed into the pressure.

Next, for the term $\nabla_x \cdot \tilde{\Sigma}$, we have

$$
\begin{aligned}
\nabla_x \cdot \tilde{\Sigma} &= \nabla_x \cdot \left[\hat{\tau}_1(\theta)\left(\nabla_x^2\theta - \tfrac{1}{D}(\Delta_x\theta)I\right)\right] + \nabla_x \cdot \left[\hat{\tau}_2(\theta)\left(\nabla_x\theta \otimes \nabla_x\theta - \tfrac{1}{D}|\nabla_x\theta|^2 I\right)\right] \\
&= \nabla_x\Big(\nabla_x \cdot [\hat{\tau}_1(\theta)\nabla_x\theta]\Big) - \nabla_x \cdot \left[\hat{\tau}_1'(\theta)\nabla_x\theta \otimes \nabla_x\theta\right] - \nabla_x\left(\tfrac{1}{D}\hat{\tau}_1(\theta)\Delta_x\theta\right) \\
&\quad + \nabla_x \cdot \left[\hat{\tau}_2(\theta)\nabla_x\theta \otimes \nabla_x\theta\right] - \nabla_x\left(\tfrac{1}{D}\hat{\tau}_2(\theta)|\nabla_x\theta|^2\right) .
\end{aligned}
\tag{2.8}
$$

The three gradient terms above, $\nabla_x\Big(\nabla_x \cdot [\hat{\tau}_1(\theta)\nabla_x\theta]\Big)$, $\nabla_x\left(\frac{1}{D}\hat{\tau}_1(\theta)\Delta_x\theta\right)$, and $\nabla_x\left(\frac{1}{D}\hat{\tau}_2(\theta)|\nabla_x\theta|^2\right)$ can also be absorbed into the pressure.

By the first and third equations in (2.3), we have

$$
\begin{aligned}
\partial_t\left(\frac{v}{\theta}\right) + \nabla_x \cdot \left(\frac{v \otimes v}{\theta}\right) &= \frac{1}{\theta}(\partial_t v + v \cdot \nabla_x v) - \frac{v}{\theta^2}(\partial_t\theta + v \cdot \nabla_x\theta) \\
&= \frac{1}{\theta}(\partial_t v + v \cdot \nabla_x v) - \frac{v}{\theta^2}\left(\Delta_x K_3(\theta) - 2\kappa(\theta)|\nabla_x\theta|^2\right) ,
\end{aligned}
\tag{2.9}
$$

where $K_3(\theta)$ satisfies $K_3'(\theta) = \theta\kappa(\theta)$.

By collecting (2.5-2.9) we see that system (1.3) can be expressed as

$$
\begin{aligned}
&\partial_t\theta + v \cdot \nabla_x\theta = \nabla_x \cdot [\theta\kappa(\theta)\nabla_x\theta] - 2\kappa(\theta)|\nabla_x\theta|^2 , \\
&\partial_t v + v \cdot \nabla_x v + \theta\nabla_x p = \theta\nabla_x \cdot \Sigma_\theta(v) + F_1(\theta, \nabla_x\theta, \nabla_x v) + F_2(v, \theta, \nabla_x\theta, \nabla_x^2\theta) , \\
&\nabla_x \cdot v = 0 ,
\end{aligned}
\tag{2.10}
$$

where

$$
\begin{aligned}
p &= P + \partial_t K_1(\theta) - 2\nabla_x \cdot [\mu(\theta)\nabla_x K_2(\theta)] + \tfrac{2}{D}\mu(\theta)\Delta_x K_2(\theta) \\
&\quad - \nabla_x \cdot [\hat{\tau}_1(\theta)\nabla_x\theta] + \tfrac{1}{D}\hat{\tau}_1(\theta)\Delta_x\theta + \tfrac{1}{D}\hat{\tau}_2(\theta)|\nabla_x\theta|^2 , \\
F_1 &= -\theta\nabla_x K_1(\theta) \cdot \nabla_x v = -\kappa(\theta)\nabla_x\theta \cdot \nabla_x v , \\
F_2 &= \frac{v}{\theta}\left(\Delta_x K_3(\theta) - 2\kappa(\theta)|\nabla_x\theta|^2\right) - \theta v \cdot \nabla_x^2 K_1(\theta) - \theta v\Delta_x K_1(\theta) - \theta\nabla_x \cdot [K_4(\theta)\nabla_x\theta \otimes \nabla_x\theta] ,
\end{aligned}
\tag{2.11}
$$

with

$$
\nabla_x K_1(\theta) = \frac{\kappa(\theta)}{\theta}\nabla_x\theta , \qquad \nabla_x K_2(\theta) = \kappa(\theta)\nabla_x\theta , \qquad \nabla_x K_3(\theta) = \theta\kappa(\theta)\nabla_x\theta ,
$$
$$
K_4(\theta) = 2\kappa(\theta)\mu'(\theta) + \hat{\tau}_1'(\theta) - \hat{\tau}_2(\theta) - \frac{\kappa(\theta)^2}{\theta} .
$$

The boundary conditions (1.4) become

$$
\theta \to \overline{\theta} \quad \text{and} \quad v \to 0 \qquad \text{as } |x| \to \infty,
\tag{2.12}
$$

while the initial conditions (1.5) become

$$
(\theta, v)\Big|_{t=0} = (\theta^{in}, v^{in}) ,
\tag{2.13}
$$

where $v^{in} = u^{in} - \kappa(\theta^{in})\nabla_x\theta^{in}$. The initial data (θ^{in}, v^{in}) are consistent with the boundary conditions (2.12) and by (1.6) satisfy the constraints

$$
\theta^{in} > 0 , \qquad \text{and} \qquad \nabla_x \cdot v^{in} = 0 .
\tag{2.14}
$$

2.2. **Energy Estimate.** We will construct an approximating sequence by iteration using a linearized version of system (2.10) that has the form

$$
\begin{aligned}
\partial_t\theta + w\cdot\nabla_x\theta &= \nabla_x\cdot[\eta\kappa(\eta)\nabla_x\theta] - 2\kappa(\eta)\nabla_x\eta\cdot\nabla_x\theta + B_1(t,x)\,,\\
\partial_t v + w\cdot\nabla_x v + \eta\nabla_x p &= \eta\nabla_x\cdot\Sigma_\eta(v) + B_2(t,x)\,,\\
\nabla_x\cdot v &= 0\,,\\
(\theta,v)|_{t=0} &= (\theta^{in},v^{in})\,,
\end{aligned}
\tag{2.15}
$$

where (w,η) are given functions and (B_1,B_2) are given forcing terms.

Notation. We always use $C(I;X)$ to denote the space of continuous functions over an interval I into a topological space X. When it is clear from the context what is meant, L^p will denote either $L^p(\mathbb{R}^d)$ or $L^p(\mathbb{R}^d;\mathbb{R}^d)$ for any $p\in[1,\infty]$, while H^s will denote either $H^s(\mathbb{R}^d)$ or $H^s(\mathbb{R}^d;\mathbb{R}^d)$ for any $s\in\mathbb{R}$.

Lemma 2.1. *Let $s>\frac{d}{2}+1$. Suppose there exists T, M, λ_0, and $\overline{\theta}>0$ such that*

$$
\eta-\overline{\theta}\in C([0,T];H^{s+1})\,,\qquad w\in C([0,T];H^s)\,,
$$

$$
\sup_{t\in[0,T]}\left\{\|\eta(t)-\overline{\theta}\|_{H^{s+1}}\right\}<M\,,\qquad \sup_{t\in[0,T]}\left\{\|w(t)\|_{H^s}\right\}<M\,,
\tag{2.16}
$$

$$
\lambda_0=\inf\left\{\eta(t,x)\;:\;(t,x)\in[0,T]\times\mathbb{R}^d\right\}>0\,,\qquad \nabla_x\cdot w=0\,,
\tag{2.17}
$$

$$
B_1\in L^2([0,T];H^s)\,,\qquad B_2\in L^2([0,T];H^{s-1})\,.
$$

Then (2.15) has a unique solution $(\theta,v,\nabla_x p)$ such that

$$
\theta-\overline{\theta}\in C([0,T];H^{s+1})\,,\qquad v\in C([0,T];H^s)\,,\qquad \nabla_x p\in C([0,T];H^{s-1})\,.
$$

Moreover, the following energy inequality holds:

$$
\begin{aligned}
\sup_{t\in[0,T]}\|\theta(t)-\overline{\theta}\|^2_{H^{s+1}} + c_0\int_0^T\|\nabla_x\theta\|^2_{H^{s+1}(t)}\mathrm{d}t &\le e^{G(M)T}\left(\|\theta^{in}-\overline{\theta}\|^2_{H^{s+1}}+G(M)\int_0^T\|B_1(t)\|^2_{L^2}\,\mathrm{d}t\right),\\
\sup_{t\in[0,T]}\|v\|^2_{H^s}+c_0\int_0^T\|\nabla_x v(t)\|^2_{H^s}\,\mathrm{d}t &\le e^{G(M)T}\left(\|v^{in}\|^2_{H^s}+G(M)\int_0^T\|B_2(t)\|^2_{H^{s-1}}\,\mathrm{d}t\right),
\end{aligned}
\tag{2.18}
$$

$$
\begin{aligned}
\sup_{t\in[0,T]}\|\nabla_x p\|^2_{H^{s-1}} \le G(M)\,e^{G(M)T}&\left(\|\nabla_x v^{in}\|^2_{H^{s-1}}+G(M)\int_0^T\|B_2(t)\|^2_{H^{s-1}}\,\mathrm{d}t\right)\\
&+G(M)\sup_{t\in[0,T]}\|B_2(t)\|^2_{H^{s-1}}\,.
\end{aligned}
\tag{2.19}
$$

where c_0 depends only on λ_0, that is, the lower bound of η, and $G(\cdot)$ is an increasing function of its argument that is determined by λ_0 and the functional forms of κ and μ.

Proof. Following the classical theory of parabolic equations, we first prove the energy inequalities (2.18) and (2.19). Toward this end, define

$$
\Lambda_m := (I-\Delta_x)^{m/2}\,,
$$

for any integer m and recall the commutator estimate [5] when $m>d/2$:

$$
\|\Lambda_m(fg)-f\Lambda_m g\|_{L^2}\le C_m\left(\|\nabla_x f\|_{L^\infty}\|g\|_{H^{m-1}}+\|g\|_{L^\infty}\|f\|_{H^m}\right)\,,
\tag{2.20}
$$

for any $f\in H^m$, $g\in H^{m-1}\cap L^\infty$.

The L^2-estimate of $\theta - \overline{\theta}$ is obtained by multiplying the θ-equation in (1.3) by $\theta - \overline{\theta}$ and integrate over $\mathbb{R}^d$. Integration by parts then shows

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta - \overline{\theta}\|_{L^2}^2 + 2c_{0,1}\|\nabla_x\theta\|_{L^2}^2 \le G_0(M)\,\|\theta - \overline{\theta}\|_{L^2}^2 + \|B_1\|_{L^2}^2\,,$$

where $2c_{0,1} > 0$ is the lower bound of $\eta\kappa(\eta)$, which depends only on λ_0. By the Gronwall inequality we have

$$\sup_{t\in[0,T]}\|\theta - \overline{\theta}\|_{L^2}^2 + c_{0,1}\int_0^T \|\nabla_x\theta(t)\|_{L^2}^2\,\mathrm{d}t \le e^{G_0(M)\,T}\left(\|\theta^{in} - \overline{\theta}\|_{L^2}^2 + \int_0^T \|B_1(t)\|_{L^2}^2\,\mathrm{d}t\right). \tag{2.21}$$

To obtain the estimate for θ, apply Λ_s to the θ-equation in (2.15), which gives

$$\partial_t(\Lambda_s\theta) + w\cdot\nabla_x(\Lambda_s\theta) = \nabla_x\cdot(\eta\kappa(\eta)\nabla_x(\Lambda_s\theta)) - 2\kappa(\eta)\nabla_x\eta\cdot\nabla_x(\Lambda_s\theta) + \Lambda_s B_1 + R_1 + R_2 + R_3\,, \tag{2.22}$$

where

$$R_1 = -[\Lambda_s,\, w]\cdot\nabla_x\theta\,, \qquad R_2 = \nabla_x\cdot\Big([\eta\kappa(\eta),\, \Lambda_s]\nabla_x\theta\Big), \qquad R_3 = -2[\Lambda_s,\, \kappa(\eta)\nabla_x\eta]\cdot\nabla_x\theta\,.$$

Here we use $[A, B]$ to denote the commutator operator $AB - BA$. To avoid confusion, we will only use brackets to denote commutators in the remainder of this proof. Upon multiplying (2.22) by $\Delta_x\Lambda_s\theta$ and integrating by parts, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_x\theta\|_{H^s}^2 + 2c_{0,1}\|\nabla_x\theta\|_{H^{s+1}}^2 \le{}& \|w\|_{L^\infty}\,\|\nabla_x\theta\|_{H^s}\,\|\Delta_x\theta\|_{H^s}\\ &+ \Big(2\|\kappa(\eta)\nabla_x\eta\|_{L^\infty} + \|\nabla_x(\eta\kappa(\eta))\|_{L^\infty}\Big)\|\nabla_x\theta\|_{H^s}\|\Delta_x\theta\|_{H^s}\\ &+ \|B_1\|_{H^s}\,\|\Delta_x\theta\|_{H^s} + \Big(\|R_1\|_{L^2} + \|R_2\|_{L^2} + \|R_3\|_{L^2}\Big)\|\Delta_x\theta\|_{H^s}\,.\end{aligned} \tag{2.23}$$

By the definitions of (R_1, R_2, R_3) and the commutator estimate (2.20), we have

$$\begin{aligned}\|R_1\|_{L^2} &= \|[\Lambda_s, w]\cdot\nabla_x\theta\|_{L^2} \le C_{s,1}\Big(\|\nabla_x w\|_{L^\infty}\,\|\nabla_x\theta\|_{H^{s-1}} + \|w\|_{H^s}\,\|\nabla_x\theta\|_{L^\infty}\Big) \le C_{s,2}\,M\,\|\nabla_x\theta\|_{H^s}\,,\\ \|R_2\|_{L^2} &= \|\nabla_x\cdot([\eta\kappa(\eta),\, \Lambda_s]\nabla_x\theta)\|_{L^2} \le \|[\nabla_x(\eta\kappa(\eta)),\, \Lambda_s]\cdot\nabla_x\theta\|_{L^2} + \|[\eta\kappa(\eta),\, \Lambda_s]\Delta_x\theta\|_{L^2}\\ &\le C_{s,3}\Big(\|\nabla_x(\eta\kappa(\eta))\|_{L^\infty}\,\|\nabla_x\theta\|_{H^{s-1}} + \|\nabla_x(\eta\kappa(\eta))\|_{H^s}\,\|\nabla_x\theta\|_{L^\infty}\Big)\\ &\quad + C_{s,4}\Big(\|\eta\kappa(\eta)\|_{L^\infty}\,\|\Delta_x\theta\|_{H^{s-1}} + \|\eta\kappa(\eta) - \overline{\theta}\kappa(\overline{\theta})\|_{H^s}\,\|\Delta_x\theta\|_{L^\infty}\Big) \le C_{s,5}\,G_1(M)\,\|\nabla_x\theta\|_{H^s}\,,\\ \|R_3\|_{L^2} &= \|2[\Lambda_s,\, \kappa(\eta)\nabla_x\eta]\cdot\nabla_x\theta\|_{L^2}\\ &\le C_{s,6}\Big(\|\nabla_x(\kappa(\eta)\nabla_x\eta)\|_{L^\infty}\,\|\nabla_x\theta\|_{H^{s-1}} + \|\kappa(\eta)\nabla_x\eta\|_{H^s}\,\|\nabla_x\theta\|_{L^\infty}\Big) \le C_{s,7}\,G_2(M)\,\|\nabla_x\theta\|_{H^s}\,,\end{aligned}$$

where $C_{s,k}$ for $1 \le k \le 7$ depend only on s while $G_1(\cdot)$ and $G_2(\cdot)$ are positive increasing functions of their arguments that are determined by λ_0 and the functional form of κ. By plugging these estimates for (R_1, R_2, R_3) into (2.23) and using the Cauchy-Schwartz inequality, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_x\theta\|_{H^s}^2 + c_{0,1}\|\Delta_x\theta\|_{H^s}^2 \le G_3(M)\,\|\nabla_x\theta\|_{H^s}^2 + C\|B_1\|_{H^s}\,,$$

where C depends only on c_0, which is determined by λ_0, the lower bound of η. The Gronwall inequality together with the L^2-estimate (2.21) then yield

$$\sup_{t\in[0,T]}\|\theta(t) - \overline{\theta}\|_{H^{s+1}}^2 + c_{0,1}\int_0^T\|\nabla_x\theta(t)\|_{H^{s+1}}^2\,\mathrm{d}t \le e^{G_3(M)T}\left(\|\theta^{in} - \overline{\theta}\|_{H^{s+1}}^2 + C\int_0^T\|B_1(t)\|_{H^s}^2\,\mathrm{d}t\right), \tag{2.24}$$

where $G_3(M)$ is determined by $G_1(M)$ and $G_2(M)$ and $c_{0,1}$, C depend only on λ_0.

The estimate of $\|v\|_{H^s}$ follows a similar line of argument. First, upon multiplying the v-equation in (2.15) by v and integrating over $\mathbb{R}^d$, one obtains the L^2-estimate of v as

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_{L^2}^2 + 2c_{0,2}\|\nabla_x v\|_{L^2}^2 \le G_4(M)\left(\|v\|_{L^2}^2 + \|\nabla_x p\|_{L^2}^2\right) + \|B_2\|_{L^2}^2 ,$$

where $c_{0,2}$ depends only on λ_0 and $G_4(\cdot)$ is an increasing function of its argument and is given by λ_0 and the functional form of μ.

To obtain the higher-derivative estimates, we apply Λ_{s-1} to the v-equation in (2.15), multiply the resulting equation for $\Lambda_{s-1}v$ by $\Delta_x\Lambda_{s-1}v$, and integrate over $\mathbb{R}^d$. The energy inequality then shows

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_x v\|_{H^{s-1}}^2 + 2c_{0,2}\|\Delta_x v\|_{H^{s-1}}^2 \le G_5(M)\left(\|\nabla_x v\|_{H^{s-1}}^2 + \|\nabla_x p\|_{H^{s-1}}^2\right) + C\|B_2\|_{H^{s-1}}^2 , \tag{2.25}$$

where $c_{0,2}$, C depend only on λ_0 and $G_5(\cdot)$ is an increasing function of its argument which is given by λ_0 and the functional form of μ.

To close estimate (2.25), one needs to estimate $\nabla_x p$. The equation for $\nabla_x p$ is

$$\nabla_x\cdot(\eta\nabla_x p) = \nabla_x\cdot F_3 , \tag{2.26}$$

where

$$F_3 = \eta\nabla_x\cdot\Sigma_\eta(v) - w\cdot\nabla_x v + B_2(t,x) .$$

Multiply (2.26) by p and integrate over $\mathbb{R}^d$. An integration by parts then yields

$$\|\nabla_x p\|_{L^2} \le \tfrac{1}{\lambda_0}\|F_3\|_{L^2} \le G_6(M)\|\nabla_x v\|_{H^{s-1}} + \tfrac{1}{\lambda_0}\|B_2\|_{L^2} ,$$

where $G_6(\cdot)$ is an increasing function in its argument that is determined by λ_0 and the functional forms of κ and μ. To bound the high-order norms of $\nabla_x p$, we consider the cases $s > \frac{d}{2}+2$ and $\frac{d}{2}+1 < s \le \frac{d}{2}+2$ separately.

For the case when $s > \frac{d}{2}+2$, we apply $\partial_i\Lambda_{s-2}$ to (2.26) for $i = 1,2,\cdots,d$, multiply the resulting equation by $\partial_i\Lambda_{s-2}p$, and integrate over $\mathbb{R}^d$. By integration by parts, we have

$$\begin{aligned}
\lambda_0\|\nabla_x\partial_i\Lambda_{s-2}p\|_{L^2}^2 &\le \left|\int_{\mathbb{R}^d}\nabla_x(\partial_i\Lambda_{s-2}p)\cdot\partial_i\Lambda_{s-2}F_3\,\mathrm{d}x\right| + \left|\int_{\mathbb{R}^d}\nabla_x(\partial_i\Lambda_{s-2}p)\cdot[\eta,\,\partial_i\Lambda_{s-2}]\nabla_x p\,\mathrm{d}x\right| \\
&\le \left|\int_{\mathbb{R}^d}\nabla_x(\partial_i\Lambda_{s-2}p)\cdot\partial_i\Lambda_{s-2}\left(\eta\nabla_x\cdot\Sigma_\eta(v)\right)\mathrm{d}x\right| + \left|\int_{\mathbb{R}^d}\nabla_x(\partial_i\Lambda_{s-2}p)\cdot\partial_i\Lambda_{s-2}B_2\,\mathrm{d}x\right| \\
&\quad + \left|\int_{\mathbb{R}^d}\nabla_x(\partial_i\Lambda_{s-2}p)\cdot\partial_i\Lambda_{s-2}(w\cdot\nabla_x v)\,\mathrm{d}x\right| + \left|\int_{\mathbb{R}^d}\nabla_x(\partial_i\Lambda_{s-2}p)\cdot[\eta,\,\partial_i\Lambda_{s-2}]\nabla_x p\,\mathrm{d}x\right| .
\end{aligned} \tag{2.27}$$

The bound for the first term on the right-hand side of (2.27) is

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d}\nabla_x(\partial_i\Lambda_{s-2}p)\cdot\partial_i\Lambda_{s-2}\left(\eta\nabla_x\cdot\Sigma_\eta(v)\right)\mathrm{d}x\right| = \left|\int_{\mathbb{R}^d}\left(\partial_i^2\Lambda_{s-2}p\right)\,\Lambda_{s-2}\nabla_x\cdot\left(\eta\nabla_x\cdot\Sigma_\eta(v)\right)\mathrm{d}x\right| \\
&\le \|\Delta_x p\|_{H^{s-2}}\left\|\eta\nabla_x\mu(\eta)\cdot\left(\nabla_x v + (\nabla_x v)^T\right)\right\|_{H^{s-1}} + \|\Delta_x p\|_{H^{s-2}}\|\nabla_x(\eta\mu(\eta))\cdot\Delta_x v\|_{H^{s-2}} \\
&\le \|\Delta_x p\|_{H^{s-2}}\left(G_7(M)\,\|\nabla_x v\|_{H^{s-1}} + \sum_{k=1}^{d}\|\partial_k\left(\nabla_x(\eta\mu(\eta))\cdot\partial_k v\right)\|_{H^{s-2}} + \sum_{k=1}^{d}\|(\partial_k\nabla_x(\eta\mu(\eta))\cdot\partial_k v)\|_{H^{s-2}}\right) \\
&\le \|\Delta_x p\|_{H^{s-2}}\left(G_7(M)\,\|\nabla_x v\|_{H^{s-1}} + \|\nabla_x(\eta\mu(\eta))\cdot\nabla_x v\|_{H^{s-1}}\right) \\
&\quad + \|\Delta_x p\|_{H^{s-2}}\|\nabla_x\nabla_x\left(\eta\mu(\eta)\right)\|_{L^\infty}\|\nabla_x v\|_{H^{s-2}} + \|\Delta_x p\|_{H^{s-2}}\|\nabla_x v\|_{L^\infty}\|\nabla_x(\eta\mu(\eta))\|_{H^{s-1}} \\
&\le G_8(M)\,\|\nabla_x p\|_{H^{s-1}}\|\nabla_x v\|_{H^{s-1}} ,
\end{aligned} \tag{2.28}$$

where $G_7(\cdot)$ and $G_8(\cdot)$ are increasing functions that are determined by the functional form of μ.

The second and third terms on the right-hand side of (2.27) containing B_2 and $w \cdot \nabla_x v$ are bounded directly as

$$\left| \int_{\mathbb{R}^d} \nabla_x(\partial_i \Lambda_{s-2} p) \cdot \partial_i \Lambda_{s-2} B_2 \, \mathrm{d}x \right| \le \|\nabla_x p\|_{H^{s-1}} \|B_2\|_{H^{s-1}} , \tag{2.29}$$

$$\begin{aligned} \left| \int_{\mathbb{R}^d} \nabla_x(\partial_i \Lambda_{s-2} p) \cdot \partial_i \Lambda_{s-2}(w \cdot \nabla_x v) \, \mathrm{d}x \right| &\le \|\nabla_x p\|_{H^{s-1}} \|w \cdot \nabla_x v\|_{H^{s-1}} \\ &\le G_9(M) \|\nabla_x p\|_{H^{s-1}} \|\nabla_x v\|_{H^{s-1}} . \end{aligned} \tag{2.30}$$

The last term on the right-hand side of (2.27) has the bound

$$\begin{aligned} \left| \int_{\mathbb{R}^d} \nabla_x(\partial_i \Lambda_{s-2} p) \cdot [\eta, \, \partial_i \Lambda_{s-2}] \nabla_x p \, \mathrm{d}x \right| &\le \|\nabla_x p\|_{H^{s-1}} \, \|[\eta, \, \partial_i \Lambda_{s-2}] \nabla_x p\|_{L^2} \\ &\le \|\nabla_x p\|_{H^{s-1}} \|\eta\|_{H^{s-1}} \|\nabla_x p\|_{H^{s-2}} , \end{aligned} \tag{2.31}$$

where we applied the commutator estimate (2.20) and the Sobolev embedding $H^{s-2}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ for $s > \frac{d}{2} + 2$. By the interpolation $\|\nabla_x p\|_{H^{s-2}} \le \epsilon \|\nabla_x p\|_{H^{s-1}} + C_\epsilon \|\nabla_x p\|_{L^2}$ for any $\epsilon > 0$, we can choose appropriate $\epsilon = \epsilon(M)$ such that

$$\begin{aligned} \left| \int_{\mathbb{R}^d} \nabla_x(\partial_i \Lambda_{s-2} p) \cdot [\eta, \, \partial_i \Lambda_{s-2}] \nabla_x p \, \mathrm{d}x \right| &\le \tfrac{\lambda_0}{2} \|\nabla_x p\|^2_{H^{s-1}} + G_{10}(M) \|\nabla_x p\|^2_{L^2} \\ &\le \tfrac{\lambda_0}{2} \|\nabla_x p\|^2_{H^{s-1}} + G_{11}(M) \|\nabla_x v\|^2_{H^{s-1}} + G_{12}(M) \|B_2\|^2_{L^2} . \end{aligned}$$

Therefore, summing all $i = 1, 2, \cdots, d$ and applying Cauchy-Schwartz inequality, there exist $G_{13}(\cdot)$ and $G_{14}(\cdot)$, which depend only on λ_0 and the functional form of μ, such that

$$\|\nabla_x p\|_{H^{s-1}} \le G_{13}(M) \|\nabla_x v\|_{H^{s-1}} + G_{14}(M) \|B_2\|_{H^{s-1}} . \tag{2.32}$$

For the case when $\frac{d}{2} + 1 < s \le \frac{d}{2} + 2$, apply D^α_x to equation (2.26), multiply $D^\alpha_x p$ to the resulting equation, and integrate both sides over $\mathbb{R}^d$. Here α is a multi-index such that $|\alpha| = s - 1$. By integration by parts, we have

$$\lambda_0 \|D^\alpha_x \nabla_x p\|^2_{L^2} \le \left| \int_{\mathbb{R}^d} D^\alpha_x \nabla_x p \cdot D^\alpha_x F_3 \, \mathrm{d}x \right| + \left| \int_{\mathbb{R}^d} D^\alpha_x \nabla_x p \cdot [\eta, \, D^\alpha_x] \nabla_x p \, \mathrm{d}x \right| . \tag{2.33}$$

Estimates for the first term containing F_3 is similar as in (2.28), (2.29), and (2.30), which gives

$$\left| \int_{\mathbb{R}^d} D^\alpha_x \nabla_x p \cdot D^\alpha_x F_3 \, \mathrm{d}x \right| \le \|\nabla_x p\|_{H^{s-1}} \left(G_8(M) \|\nabla_x v\|_{H^{s-1}} + \|B_2\|_{H^{s-1}} + G_9(M) \|\nabla_x v\|_{H^{s-1}} \right) . \tag{2.34}$$

The second term on the right-hand side of (2.33) is bounded as

$$\left| \int_{\mathbb{R}^d} D^\alpha_x \nabla_x p \cdot [\eta, \, D^\alpha_x] \nabla_x p \, \mathrm{d}x \right| \le \|\nabla_x p\|_{H^{s-1}} \, \left\| [\eta, \, D^\alpha_x] \nabla_x p \right\|_{L^2} . \tag{2.35}$$

To bound the commutator term $\left\| [\eta, \, D^\alpha_x] \nabla_x p \right\|_{L^2}$, we have

$$[\eta, \, D^\alpha_x] \nabla_x p = \sum_{\substack{|\alpha_1| + |\alpha_2| = |\alpha| = s-1, \\ |\alpha_1| \ge 1}} (D^{\alpha_1}_x \eta) \, (D^{\alpha_2}_x \nabla_x p) ,$$

where for those terms satisfying $|\alpha_1| = 1$, the bounds are

$$\left\| (D^{\alpha_1}_x \eta) \, (D^{\alpha_2}_x \nabla_x p) \right\|_{L^2} \le \|\nabla_x \eta\|_{L^\infty} \|D^{|\alpha_2|}_x \nabla_x p\|_{L^2} \le C \|\nabla_x \eta\|_{H^{s-1}} \|\nabla_x p\|_{H^{s-2}} , \tag{2.36}$$

with C being a generic constant that only depends on d.

For those terms when $|\alpha_1| \geq 2$ (if there exists any, i.e., $d \geq 3$), one has

$$
\begin{aligned}
\left\|(D_x^{\alpha_1}\eta)\ (D_x^{\alpha_2}\nabla_x p)\right\|_{L^2} &\leq \left\|D_x^{\alpha_1}\eta\right\|_{L^{\frac{2d}{d-2(s-|\alpha_1|)}}} \left\|D_x^{\alpha_2}\nabla_x p\right\|_{L^{\frac{d}{s-|\alpha_1|}}} \leq C\left\|D_x^{\alpha_1}\eta\right\|_{H^{s-|\alpha_1|}} \left\|D_x^{\alpha_2}\nabla_x p\right\|_{H^{\frac{d}{2}+|\alpha_1|-s}} \\
&\leq C\left\|D_x^{\alpha_1}\eta\right\|_{H^{s-|\alpha_1|}} \|\nabla_x p\|_{H^{\frac{d}{2}+|\alpha_1|+|\alpha_2|-s}} \leq C\left\|D_x^{\alpha_1}\eta\right\|_{H^{s-|\alpha_1|}} \|\nabla_x p\|_{H^{\frac{d}{2}-1}} \\
&\leq C\,\|\eta\|_{H^s}\ \|\nabla_x p\|_{H^{s-2}}\ ,
\end{aligned}
\tag{2.37}
$$

by the assumption that $\frac{d}{2}+1 < s \leq \frac{d}{2}+2$. Here we have applied the Sobolev inequality associated with the embedding $H^m(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ where $q = \frac{2d}{d-2m}$ and $m \leq d/2$. Once again C is a generic constant that only depends on d. By combining (2.36) and (2.37) we have

$$
\left\|[\eta,\ D_x^{\alpha}]\nabla_x p\right\|_{L^2} \leq C\,\|\eta\|_{H^s}\ \|\nabla_x p\|_{H^{s-2}}\ ,
$$

where C depends only on d. Therefore, the bound in (2.35) now becomes

$$
\left|\int_{\mathbb{R}^d} D_x^{\alpha}\nabla_x p\cdot[\eta,\ D_x^{\alpha}]\nabla_x p\,\mathrm{d}x\right| \leq C\|\nabla_x p\|_{H^{s-1}}\ \|\eta\|_{H^s}\ \|\nabla_x p\|_{H^{s-2}}\ ,
$$

which is exactly of the form as in (2.31). Thus following the same argument using interpolation together with (2.34), we have the same estimate for the case $\frac{d}{2}+1 < s \leq \frac{d}{2}+2$ as for the case $s > \frac{d}{2}+2$, which shows there exist $G_{15}(\cdot)$ and $G_{16}(\cdot)$ depending only on λ_0 and $\mu(\cdot)$ such that

$$
\|\nabla_x p\|_{H^{s-1}} \leq G_{15}(M)\,\|\nabla_x v\|_{H^{s-1}} + G_{16}(M)\,\|B_2\|_{H^{s-1}}\,,
\tag{2.38}
$$

for $s > \frac{d}{2}+1$.

Upon plugging (2.38) into (2.25) we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_x v\|^2_{H^{s-1}} + 2c_0\|\Delta_x v\|^2_{H^{s-1}} \leq G_{17}(M)\,\|\nabla_x v\|^2_{H^{s-1}} + G_{18}(M)\|B_2\|^2_{H^{s-1}}\,,
$$

which by the Gronwall inequality implies

$$
\sup_{t\in[0,T]}\|\nabla_x v\|^2_{H^{s-1}} + c_0\int_0^T \|\Delta_x v(t)\|^2_{H^{s-1}}\,\mathrm{d}t \leq e^{G_{17}(M)\,T}\left(\|\nabla_x v^{in}\|^2_{H^{s-1}} + G_{18}(M)\int_0^T \|B_2(t)\|^2_{H^{s-1}}\,\mathrm{d}t\right),
\tag{2.39}
$$

where the specific forms of $G_{17}(\cdot)$ and $G_{18}(\cdot)$ depend only on λ_0 and the functional forms of μ and κ. By (2.38), we also have the bound for p as

$$
\begin{aligned}
\sup_{t\in[0,T]}\|\nabla_x p\|^2_{H^{s-1}} \leq G_{15}(M)\,e^{G_{17}(M)\,T}&\left(\|\nabla_x v^{in}\|^2_{H^{s-1}} + G_{18}(M)\int_0^T \|B_2(t)\|^2_{H^{s-1}}\,\mathrm{d}t\right) \\
&+ G_{16}(M)\sup_{t\in[0,T]}\|B_2(t)\|^2_{H^{s-1}}\,.
\end{aligned}
\tag{2.40}
$$

By setting $c_0 = \min\{c_{0,1}, c_{0,2}, \lambda_0\}$, $G(M) = \max_{0\leq k\leq 18}\{G_k(M)\}$, and collecting estimates (2.24), (2.39), and (2.40), we have

$$
\sup_{t\in[0,T]}\|\theta(t)-\bar{\theta}\|^2_{H^{s+1}} + c_0\int_0^T \|\nabla_x\theta\|^2_{H^{s+1}(t)}\mathrm{d}t \leq e^{G(M)T}\left(\|\theta^{in}-\bar{\theta}\|^2_{H^{s+1}} + G(M)\int_0^T \|B_1(t)\|^2_{L^2}\,\mathrm{d}t\right),
$$

$$
\sup_{t\in[0,T]}\|v\|^2_{H^s} + c_0\int_0^T \|\nabla_x v(t)\|^2_{H^s}\,\mathrm{d}t \leq e^{G(M)\,T}\left(\|v^{in}\|^2_{H^s} + G(M)\int_0^T \|B_2(t)\|^2_{H^{s-1}}\,\mathrm{d}t\right),
$$

$$
\sup_{t\in[0,T]}\|\nabla_x p\|^2_{H^{s-1}} \leq G(M)\,e^{G(M)\,T}\left(\|\nabla_x v^{in}\|^2_{H^{s-1}} + G(M)\int_0^T \|B_2(t)\|^2_{H^{s-1}}\,\mathrm{d}t\right) + G(M)\sup_{t\in[0,T]}\|B_2(t)\|^2_{H^{s-1}}\,.
$$

We have thereby proved the bounds (2.18) and (2.19) for solutions $(\theta, v, \nabla_x p)$ of the linear system (2.15). One then obtains the uniqueness of the solution by this a prior bound.

To show the existence of the solution to the linear system (2.15), we first notice that the two equations in the system are decoupled. The θ-equation is linear and strictly parabolic. Therefore, the classical theory for parabolic equations [9] guarantees the existence in θ. The existence in v is obtained using a standard technique for non-homogeneous incompressible systems (see [3] for example), namely solving the v-equation with respect to $\{\eta\nabla_x p\}$ and applying the divergence operator in the resulting equation we arrive at an elliptic equation in p

$$\nabla_x\cdot(\eta\nabla_x p)=\nabla_x\cdot\Big(\eta\nabla_x\cdot\Sigma_\eta(v)+B_2(t,x)-w\cdot\nabla_x v\Big)\,, \tag{2.41}$$

which yields, in turn a linear ordinary differential equation with respect to v in a Banach space. The regularity assumptions (2.16), (2.17) guarantee the existence and uniqueness of a solution of this ordinary differential equation. Using this solution we can solve for $\nabla_x p$. This completes the proof of existence, and therefore the proof of Lemma 2.1. □

2.3. **Existence and Uniqueness.** We now use the energy estimates (2.18) and (2.19) to construct an approximating sequence $\{(\theta_n, v_n, \nabla_x p_n)\}_{n=0}^\infty$ and show it converges to the solution $(\theta, v, \nabla_x p)$ of system (2.10).

We initialize our approximating sequence as follows. We first let θ_0 and v_0 be given by

$$\theta_0=\theta^{in}\,,\qquad v_0=v^{in}=u^{in}-\kappa(\theta^{in})\nabla_x\theta^{in}\,, \tag{2.42}$$

Because (θ^{in}, u^{in}) satisfies (1.9), we see that

$$\theta_0-\overline{\theta}\in H^{s+1}(\mathbb{R}^d)\,,\qquad v_0\in H^s(\mathbb{R}^d)\qquad \text{for some } s>\tfrac{d}{2}+1 \text{ and } \overline{\theta}>0\,. \tag{2.43}$$

Because (θ^{in}, u^{in}) satisfies (1.6), we see moreover that

$$\theta_0\ge\lambda_0=\inf\{\theta^{in}(x)\,:\,x\in\mathbb{R}^d\}>0\,,\qquad \nabla_x\cdot v_0=0\,. \tag{2.44}$$

We then let $\nabla_x p_0$ be the unique solution in $H^{s-1}(\mathbb{R}^d)$ to

$$\nabla_x\cdot[\theta_0\nabla_x p_0]=\nabla_x\cdot\Big[\theta_0\nabla_x\cdot\Sigma_{\theta_0}(v_0)+F_1(\theta_0,\nabla_x\theta_0,\nabla_x v_0)+F_2(v_0,\theta_0,\nabla_x\theta_0,\nabla_x^2\theta_0)-v_0\cdot\nabla_x v_0\Big]\,. \tag{2.45}$$

The initial approximate $(\theta_0, v_0, \nabla_x p_0)$ is thereby time independent.

Given $(\theta_n, v_n, \nabla_x p_n)$ for some $n\in\mathbb{N}$, we will define $(\theta_{n+1}, v_{n+1}, \nabla_x p_{n+1})$ to be the solution of the system

$$\begin{aligned}\partial_t\theta_{n+1}+v_n\cdot\nabla_x\theta_{n+1}&=\nabla_x\cdot[\theta_n\kappa(\theta_n)\nabla_x\theta_{n+1}]-2\kappa(\theta_n)\nabla_x\theta_n\cdot\nabla_x\theta_{n+1}\,,\\ \partial_t v_{n+1}+v_n\cdot\nabla_x v_{n+1}+\theta_n\nabla_x p_{n+1}&=\theta_n\nabla_x\cdot\Sigma_{\theta_n}(v_{n+1})+F_1^n+F_2^n\,,\\ \nabla_x\cdot v_{n+1}&=0\,,\\ (\theta_{n+1},v_{n+1})|_{t=0}&=(\theta^{in},v^{in})\,,\end{aligned} \tag{2.46}$$

where

$$F_1^n=F_1(\theta_n,\nabla_x\theta_n,\nabla_x v_n)\,,\qquad F_2^n=F_2(v_n,\theta_n,\nabla_x\theta_n,\nabla_x^2\theta_n)\,. \tag{2.47}$$

Here $\Sigma_{\theta_n}(v_n)$, F_1, and F_2 are defined in (2.2) and (2.11). The existence of $(\theta_{n+1}, v_{n+1}, \nabla_x p_{n+1})$ will follow from Lemma 2.1 once we establish that $(\theta_n, v_n, \nabla_x p_n)$ satisfies the necessary hypotheses.

Lemma 2.2. *Let $s>\frac{d}{2}+1$ and $\overline{\theta}>0$ as in the Main Theorem. Let*

$$M=2\max\Big\{\|\theta^{in}-\overline{\theta}\|_{H^{s+1}}\,,\ \|v^{in}\|_{H^s}\Big\}\,. \tag{2.48}$$

Then there exists $T>0$ such that the sequence $\{(\theta_n, v_n, \nabla_x p_n)\}_{n=0}^\infty$ defined above exists with each iterate satisfying

$$\theta_n-\overline{\theta}\in C([0,T];H^{s+1})\,,\qquad v_n\in C([0,T];H^s)\,,\qquad \nabla_x p_n\in C([0,T];H^{s-1})\,, \tag{2.49}$$

the norm bounds

$$\sup_{t\in[0,T]}\Big\{\|\theta_n(t)-\bar\theta\|_{H^{s+1}}\Big\} < M\,,\qquad \sup_{t\in[0,T]}\Big\{\|v_n(t)\|_{H^s}\Big\} < M\,,\qquad \sup_{t\in[0,T]}\Big\{\|\nabla_x p_n(t)\|_{H^{s-1}}\Big\} < \tilde G(M)\,,\tag{2.50}$$

and the constraints

$$\inf\Big\{\theta_n(t,x)\,:\,(t,x)\in[0,T]\times\mathbb{R}^d\Big\}\ge\lambda_0>0\,,\qquad \nabla_x\cdot v_n=0\,,\tag{2.51}$$

where $\tilde G(\cdot)$ is an increasing function of its argument that is independent of n.

Proof. Because it is time independent, it is clear that the initial approximate $(\theta_0, v_0, \nabla_x p_0)$ given by (2.42) and (2.45) satisfies (2.49), (2.50), and (2.51) for every $T>0$.

Now suppose that for some $n\in\mathbb{N}$ the approximate $(\theta_n, v_n, \nabla_x p_n)$ satisfies (2.49), (2.50), and (2.51) for some $T>0$. Then by Lemma 2.1 the approximate $(\theta_{n+1}, v_{n+1}, \nabla_x p_{n+1})$ governed by (2.46) exists with

$$\theta_{n+1}-\bar\theta\in C([0,T];H^{s+1})\,,\qquad v_{n+1}\in C([0,T];H^s)\,,\qquad \nabla_x p_{n+1}\in C([0,T];H^{s-1})\,.$$

Moreover, it satisfies the energy inequalities

$$\sup_{t\in[0,T]}\|\theta_{n+1}(t)-\bar\theta\|^2_{H^{s+1}}+c_0\int_0^T\|\nabla_x\theta_{n+1}(t)\|^2_{H^{s+1}}\,\mathrm{d}t\le e^{G(M)T}\left(\left\|\theta^{in}-\bar\theta\right\|^2_{H^{s+1}}+T\,G(M)\right),\tag{2.52}$$

$$\sup_{t\in[0,T]}\|v_{n+1}(t)\|^2_{H^s}+c_0\int_0^T\|\nabla_x v_{n+1}(t)\|^2_{H^s}\,\mathrm{d}t\le e^{G(M)T}\left(\left\|v^{in}\right\|^2_{H^s}+T\,G(M)\right),\tag{2.53}$$

$$\|\nabla_x p_{n+1}\|_{H^{s-1}}\le G(M)+G(M)\,e^{G(M)\,T}\left(\left\|v^{in}\right\|^2_{H^s}+T\,G(M)\right),\tag{2.54}$$

where c_0 and C depend only on λ_0, and $G(\cdot)$ is an increasing function of its argument that is independent of n. It is clear that by picking T small enough we can insure that $(\theta_{n+1}, v_{n+1}, \nabla_x p_{n+1})$ satisfies (2.50). The choice of T is solely determined by λ_0 and M. In particular, it is independent of n. Finally, a direct application of the classical maximum principle for strictly parabolic equations (cf. [4] for example) shows that θ_{n+1} satisfies the lower bound in (2.51). □

Based on the uniform bound (2.50) of $(\theta_n, v_n, \nabla_x p_n)$, we employ the standard high-low argument (cf. [13]) to show the convergence of $\{(\theta_n, v_n, \nabla_x p_n)\}_{n=0}^\infty$ to the solution $(\theta, v, \nabla_x p)$ of system (2.10).

Lemma 2.3. *Let $\{(\theta_n, v_n, \nabla_x p_n)\}_{n=0}^\infty$ be the sequence constructed in Lemma 2.2 with $(\theta_0, v_0, \nabla_x p_0)$ being defined by (2.42) and (2.45). Then for any $0\le s'<s$, $1\le s''<s$, there exists*

$$\begin{aligned}\theta-\bar\theta&\in L^\infty([0,T];H^{s+1})\cap L^2([0,T];H^{s+2})\cap C([0,T];H^{s'+1})\,,\\ v&\in L^\infty([0,T];H^{s})\cap L^2([0,T];H^{s+1})\cap C([0,T];H^{s'})\,,\\ \nabla_x p&\in L^\infty([0,T];H^{s-1})\cap C([0,T];H^{s''-1})\,,\end{aligned}\tag{2.55}$$

such that

$$\begin{aligned}\theta_n&\to\theta &&\text{in}\quad C([0,T];H^{s'+1})\,,\\ v_n&\to v &&\text{in}\quad C([0,T];H^{s'})\,,\\ \nabla_x p_n&\to\nabla_x p &&\text{in}\quad C([0,T];H^{s''-1})\,.\end{aligned}\tag{2.56}$$

and $(\theta, v, \nabla_x p)$ is the unique classical solution to system (1.3).

Proof. We show the convergence of the sequence $(\theta_n, v_n, \nabla_x p_n)$ in $L^2(\mathbb{R}^d)$ using the equations satisfied by $(\theta_n, v_n, \nabla_x p_n)$ and the boundedness of their high-order norms. To this end, consider the system for

$$(\tilde{\theta}_n,\ \tilde{v}_n,\ \nabla_x \tilde{p}_n) = (\theta_{n+1} - \theta_n,\ v_{n+1} - v_n,\ \nabla_x p_{n+1} - \nabla_x p_n)\,,$$

which has the form

$$\begin{aligned} &\partial_t \tilde{\theta}_n + v_n \cdot \nabla_x \tilde{\theta}_n = \nabla_x \cdot \Big[\theta_n \kappa(\theta_n) \nabla_x \tilde{\theta}_n\Big] - 2\kappa(\theta_n) \nabla_x \theta_n \cdot \nabla_x \tilde{\theta}_n + R_1(\theta_n, \theta_{n-1}, v_n, v_{n-1})\,, \\ &\partial_t \tilde{v}_n + v_n \cdot \nabla_x \tilde{v}_n + \theta_n \nabla_x \tilde{p}_n = \theta_n \nabla_x \cdot \Sigma_{\theta_n}(\tilde{v}_n) + \Big(F_1^n - F_1^{n-1}\Big) + \Big(F_2^n - F_2^{n-1}\Big) + R_2(\theta_n, \theta_{n-1}, v_n, v_{n-1})\,, \\ &\nabla_x \cdot \tilde{v}_n = 0\,, \end{aligned} \tag{2.57}$$

where

$$\begin{aligned} R_1 &= -(v_n - v_{n-1}) \cdot \nabla_x \theta_n + \nabla_x \cdot [(\theta_n \kappa(\theta_n) - \theta_{n-1} \kappa(\theta_{n-1})) \nabla_x \theta_n] \\ &\quad - 2\Big(\kappa(\theta_n) \nabla_x \theta_n - \kappa(\theta_{n-1}) \nabla_x \theta_{n-1}\Big) \cdot \nabla_x \theta_n\,, \\ R_2 &= -(v_n - v_{n-1}) \cdot \nabla_x v_n - (\theta_n - \theta_{n-1}) \nabla_x p_n \\ &\quad + \Big(\theta_n \nabla_x \cdot \Sigma_{\theta_n}(v_n) - \theta_{n-1} \nabla_x \cdot \Sigma_{\theta_{n-1}}(v_{n-1})\Big)\,. \end{aligned} \tag{2.58}$$

By using the uniform bounds (2.50), one has

$$\begin{aligned} \int_0^T \|R_1(t)\|_{H^1}^2 \,\mathrm{d}t &\le G_{19}(M)\, T \sup_{t\in[0,T]} \Big(\|v_n - v_{n-1}\|_{H^1}^2 + \|\theta_n - \theta_{n-1}\|_{H^2}^2\Big)\,, \\ \int_0^T \|R_2(t)\|_{L^2}^2 \,\mathrm{d}t &\le G_{20}(M)\, T \sup_{t\in[0,T]} \Big(\|v_n - v_{n-1}\|_{H^1}^2 + \|\theta_n - \theta_{n-1}\|_{H^1}^2\Big)\,, \\ \int_0^T \|F_1^n - F_1^{n-1}\|_{L^2}^2 \,\mathrm{d}t &\le G_{21}(M)\, T \sup_{t\in[0,T]} \Big(\|v_n - v_{n-1}\|_{H^1}^2 + \|\theta_n - \theta_{n-1}\|_{H^1}^2\Big)\,, \\ \int_0^T \|F_2^n - F_2^{n-1}\|_{L^2}^2 \,\mathrm{d}t &\le G_{22}(M)\, T \sup_{t\in[0,T]} \Big(\|v_n - v_{n-1}\|_{L^2}^2 + \|\theta_n - \theta_{n-1}\|_{H^2}^2\Big)\,. \end{aligned}$$

By the energy estimate (2.18) for the linear system, we have

$$\begin{aligned} &\sup_{t\in[0,T]} \Big(\|\tilde{\theta}_n(t)\|_{H^2}^2 + \|\tilde{v}_n(t)\|_{H^1}^2\Big) \\ &\le G_{23}(M)\, e^{G(M)T} \int_0^T \Big(\|R_1(t)\|_{H^1}^2 + \|R_2(t)\|_{L^2}^2 + \|F_1^n - F_1^{n-1}\|_{L^2}^2 + \|F_2^n - F_2^{n-1}\|_{L^2}^2\Big)\,\mathrm{d}t \\ &\le G_{24}(M)\, T\, e^{G(M)T} \sup_{t\in[0,T]} \Big(\|\tilde{\theta}_{n-1}\|_{H^2}^2 + \|\tilde{v}_{n-1}\|_{H^1}^2\Big)\,. \end{aligned}$$

Now we choose T small enough such that $G_{23}(M)\, T\, e^{G(M)T} < 1$, which makes $\{\theta_n\}_{n=0}^\infty$, $\{v_n\}_{n=0}^\infty$ Cauchy sequences in $C([0,T]; H^2)$ and $C([0,T]; H^1)$ respectively. Therefore, there exist $\theta - \theta \in C([0,T], H^2)$ and $v \in C([0,T]; H^1)$ such that

$$\begin{aligned} \theta_n &\to \theta \quad \text{in} \quad C([0,T]; H^2)\,, \\ v_n &\to v \quad \text{in} \quad C([0,T]; H^1)\,. \end{aligned}$$

By the uniform bounds (2.50) and interpolation, we have for any $0 \le s' < s$,

$$\begin{aligned} \theta_n &\to \theta \quad \text{in} \quad C([0,T]; H^{s'+1})\,, \\ v_n &\to v \quad \text{in} \quad C([0,T]; H^{s'})\,. \end{aligned} \tag{2.59}$$

The equation for $\nabla_x p_n$ is of the form

$$\nabla_x \cdot (\theta_n \nabla_x p_n) = \nabla_x \cdot (\theta_n \nabla_x \cdot \Sigma_{\theta_n}(v_{n+1}) - v_n \cdot \nabla_x v_{n+1} + F_1^n(t,x) + F_2^n(t,x)) \triangleq \nabla_x \cdot Q_n \,, \tag{2.60}$$

It follows from (2.59) that Q_n is a Cauchy sequence in $C([0,T];H^{s'-1})$ for any $1 \le s' < s$. By repeating the energy estimate for $\nabla_x p_n$, one can see that $\nabla_x p_n$ is also a Cauchy sequence in $C([0,T];H^{s'-1})$. Therefore, there exists $\nabla_x p \in C([0,T];H^{s'-1})$ such that

$$\nabla_x p_n \to \nabla_x p \quad \text{in} \quad C([0,T];H^{s'-1}) \,, \tag{2.61}$$

for any $1 \le s' < s$. This completes the proof the convergence (2.56). Note that since $s > \frac{d}{2} + 1$, the approximating system (2.46) converges to the system (2.10) and $(\theta, v, \nabla_x p)$ is a classical solution.

The uniqueness of $(\theta, u, \nabla_x p)$ is guaranteed by the energy estimate. Suppose we have two solutions $(\theta, v, \nabla_x p)$ and $(\hat\theta, \hat v, \nabla_x \hat p)$. Consider the system satisfied by their difference $(\theta - \hat\theta, v - \hat v, \nabla_x p - \nabla_x \hat p)$ which is

$$\begin{aligned}
\partial_t \left(\theta - \hat\theta\right) + v \cdot \nabla_x \left(\theta - \hat\theta\right) &= \nabla_x \cdot \left(\theta\kappa(\theta)\nabla_x \left(\theta - \hat\theta\right)\right) - 2\kappa(\theta)\nabla_x\theta \cdot \nabla_x \left(\theta - \hat\theta\right) + R_1(\theta, \hat\theta, v, \hat v)\,,\\
\partial_t (v - \hat v) + v \cdot \nabla_x (v - \hat v) + \theta \nabla_x (p - \hat p) &= \theta \nabla_x \cdot \Sigma_\theta (v - \hat v) + \left(F_1(\theta, \nabla_x\theta, \nabla_x v) - F_1(\hat\theta, \nabla_x\hat\theta, \nabla_x \hat v)\right)\\
&\quad + \left(F_2(\theta, v, \nabla_x\theta, \nabla_x^2\theta) - F_2(\hat\theta, \hat v, \nabla_x\hat\theta, \nabla_x^2\hat\theta)\right) + R_2(\theta, \theta, v, \hat v)\,,\\
\nabla_x \cdot (v - \hat v) &= 0\,,
\end{aligned}$$

Then the uniqueness is obtained by applying the linear estimate (2.18) as we have done to prove that $\{\theta_n\}_{n=0}^\infty$, $\{v_n\}_{n=0}^\infty$ are Cauchy sequences. We thereby finish the proof of the existence and uniqueness of the solution to system (2.10). □

2.4. **Regularity.** To complete the proof of the Main Theorem, we show in the following lemma that the regularity of $(\theta, u, \nabla_x p)$ can be improved compared to that given by Lemma 2.3. The proof uses a method that is classically applied to hyperbolic and parabolic systems (cf. [13, 26] for examples).

We first show in the following lemma that the lifespan of the solution does not shrink when one considers higher regularity solution.

Lemma 2.4. *Let $s > \frac{d}{2} + 1$. Suppose $(\theta,\ u, \nabla_x p)$ is the classical solution to system (2.10) obtained in Lemma 2.3 such that*

$$\begin{aligned}
\theta - \overline{\theta} &\in C([0,T];H^{s+1}) \cap L^2([0,T];H^{s+2})\,,\\
u &\in C([0,T];H^s(\mathbb{R}^d)) \cap L^2([0,T];H^{s+1})\,,\\
\nabla_x p &\in C([0,T];H^{s-1})\,.
\end{aligned} \tag{2.62}$$

If in addition the initial data $(\theta^{in},\ u^{in},\ \nabla_x p^{in})$ satisfy

$$\theta^{in} - \overline{\theta} \in H^{s_1+1}\,, \qquad u^{in} \in H^s\,,$$

for any $s_1 \ge s$. Then on the same time interval $[0,T]$ we have

$$\begin{aligned}
\theta - \overline{\theta} &\in C([0,T];H^{s_1+1}) \cap L^2([0,T];H^{s_1+2})\,,\\
u &\in C([0,T];H^{s_1}) \cap L^2([0,T];H^{s_1+1})\,,\\
\nabla_x p &\in C([0,T];H^{s_1-1})\,.
\end{aligned} \tag{2.63}$$

Proof. We just need to show that (2.63) holds for $s_1 = s + 1$. We work on the reformulated system (2.10) with variables $(\theta,\, v,\, \nabla_x p)$. Apply $\partial_i = \partial_{x_i}$ to system (2.10) for any $1 \leq i \leq d$ and rearrange terms in the resulting system. This yields

$$\begin{aligned} \partial_t(\partial_i\theta) + v\cdot\nabla_x(\partial_i\theta) &= \nabla_x\cdot[\theta\kappa(\theta)\nabla_x(\partial_i\theta)] - 2\kappa(\theta)\nabla_x\theta\cdot\nabla_x(\partial_i\theta) + J_1(t,x)\,,\\ \partial_t(\partial_i v) + v\cdot\nabla_x(\partial_i v) + \theta\nabla_x(\partial_i p) &= \theta\nabla_x\cdot\Sigma_\theta(\partial_i v) + J_2(t,x)\,,\\ \nabla_x\cdot(\partial_i v) &= 0\,, \end{aligned} \tag{2.64}$$

with the initial data

$$\partial_i\theta^{in} \in H^{s+1}\,, \qquad \partial_i v^{in} \in H^s\,.$$

Here the inhomogeneous terms J_1 and J_2 have the form

$$\begin{aligned} J_1 &= -\partial_i v\cdot\nabla_x\theta + \nabla_x\cdot[\partial_i(\theta\kappa(\theta))\nabla_x\theta] - 2\partial_i(\kappa(\theta)\nabla_x\theta)\cdot\nabla_x\theta\,,\\ J_2 &= -\partial_i v\cdot\nabla_x v - (\partial_i\theta)\nabla_x p + (\partial_i\theta)\nabla_x\cdot\Sigma_\theta(v) + \partial_i F_1 + \partial_i F_2\\ &\quad + \theta\nabla_x\cdot\left[(\partial_i\mu(\theta))\left(\nabla_x v + (\nabla_x v)^T - \tfrac{2}{D}(\nabla_x\cdot v)I\right)\right]\,, \end{aligned}$$

where F_1, F_2 are defined in (2.11). By the regularity assumption (2.62), there exists $0 < M < \infty$ such that

$$\theta - \overline{\theta} \in C([0,T];H^{s+1})\,, \qquad v \in C([0,T];H^s)\,, \qquad \nabla_x\cdot v = 0\,,$$

$$\sup_{t\in[0,T]}\left\{\|\theta-\overline{\theta}\|_{H^{s+1}}(t)\,,\ \|w\|_{H^s}(t)\right\} < M\,, \qquad \inf_{[0,T]\times\mathbb{R}^d}\theta \geq \lambda_0 > 0\,.$$

$$J_1 \in L^2([0,T];H^s)\,, \qquad J_2 \in L^2([0,T];H^{s-1})\,.$$

Therefore, Lemma 2.1 applies to system (2.64) and this proves the assertion (2.63). □

Now we prove the additional regularity of the solution $(\theta,\, u, \nabla_x p)$ due to dissipation.

Lemma 2.5. *Let $(\theta, v, \nabla_x p)$ be the unique classical solution to system (1.3) as obtained in Lemma 2.3. Then we have in addition*

$$\begin{aligned} \theta - \overline{\theta} &\in C([0,T];H^{s+1}) \cap C^\infty((0,T)\times\mathbb{R}^d)\,,\\ u &\in C([0,T];H^s) \cap C^\infty((0,T)\times\mathbb{R}^d)\,,\\ \nabla_x p &\in C([0,T];H^{s-1}) \cap C^\infty((0,T)\times\mathbb{R}^d)\,. \end{aligned}$$

Proof. The proof has three steps. First we show that

$$\begin{aligned} \theta_n - \overline{\theta} &\to \theta - \overline{\theta} &&\text{in}\quad C([0,T];H^{s+1}_w)\,,\\ u_n &\to u &&\text{in}\quad C([0,T];H^s_w)\,, \end{aligned} \tag{2.65}$$

where X_w indicates that the Banach space X is equipped with its weak topology. By (2.56), for every $\phi_1 \in H^{-s'-1}$ and $\phi_2 \in H^{-s'}$ with $0 \leq s' < s$, we have

$$\begin{aligned} \langle\phi_1, \theta_n - \overline{\theta}\rangle(t) &\to \langle\phi_1, \theta - \overline{\theta}\rangle(t) &&\text{in}\quad C([0,T];\mathbb{R})\,,\\ \langle\phi_2, u_n\rangle(t) &\to \langle\phi_2, u\rangle(t) &&\text{in}\quad C([0,T];\mathbb{R})\,, \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the usual L^2 inner product. Because H^{s_2} is dense in H^{s_1} for any $s_1 < s_2$ and because (θ_n, u_n), (θ, u) are uniformly bounded in $H^{s+1}\times H^s$, the convergence (2.65) is then established by a density argument.

Next, we show that $(\theta - \overline{\theta})(t)$ and $u(t)$ are right continuous at $t = 0$ in H^{s+1} and H^s respectively. By (2.65) and the energy bound (2.52) for (θ_n, u_n), we have

$$\begin{aligned}\overline{\lim_{t\downarrow 0}}\, \|(\theta - \overline{\theta})(t)\|_{H^{s+1}} &\le \overline{\lim_{t\downarrow 0}}\, \underline{\lim_{n\to\infty}}\, \|(\theta_n - \overline{\theta})(t)\|_{H^{s+1}} \le \overline{\lim_{t\downarrow 0}} \left(\sup_n \|(\theta_n - \overline{\theta})(t)\|_{H^{s+1}} \right) \\ &\le \|\theta^{in} - \overline{\theta}\|_{H^{s+1}} .\end{aligned}$$

$$\overline{\lim_{t\downarrow 0}}\, \|u(t)\|_{H^s} \le \overline{\lim_{t\downarrow 0}}\, \underline{\lim_{n\to\infty}}\, \|u_n(t)\|_{H^s} \le \overline{\lim_{t\downarrow 0}} \left(\sup_n \|u_n(t)\|_{H^s} \right) \le \|u^{in}\|_{H^s} .$$

Meanwhile, the weak continuity (2.65) of $(\theta - \overline{\theta})$ and $u(t)$ at $t = 0$ shows that

$$\|\theta^{in} - \overline{\theta}\|_{H^{s+1}} \le \underline{\lim_{t\downarrow 0}}\, \|(\theta - \overline{\theta})(t)\|_{H^{s+1}} , \quad \|u^{in}\|_{H^s} \le \underline{\lim_{t\downarrow 0}}\, \|u(t)\|_{H^s} .$$

Thus we obtain the strong right-continuity of $(\theta - \overline{\theta})(t)$ in the topology of H^{s+1} and $u(t)$ in H^s at $t = 0$.

To show the continuity in time for all $t \in [0, T]$, we use the parabolic regularization in (2.55) where $\theta - \overline{\theta} \in L^2([0, T]; H^{s+2})$, $u \in L^2([0, T]; H^{s+1})$. This shows for almost every $t_0 \in [0, T]$, one has

$$(\theta - \overline{\theta})(t_0) \in H^{s+2} , \quad u(t_0) \in H^{s+1} .$$

We can choose such $t_0 > 0$ as a new initial time. By Lemma 2.3, there exists a unique classical solution $(\tilde{\theta},\ \tilde{u},\ \nabla_x \tilde{p})$ to system (1.3) such that for any $0 \le s' < s$, $1 \le s'' < s$,

$$\begin{aligned}\tilde{\theta} - \overline{\theta} &\in C([t_0, T]; H^{s'+2}) \cap L^2([t_0, T]; H^{s'+3}) , \\ u &\in C([t_0, T]; H^{s'+1}) \cap L^2([t_0, T]; H^{s'+2}) , \\ \nabla_x \tilde{p} &\in C([t_0, T]; H^{s''}) ,\end{aligned}$$

where T is the same as in Lemma 2.3. By the uniqueness guaranteed by Lemma 2.3, we have $(\tilde{\theta},\ \tilde{u},\ \nabla_x \tilde{p}) = (\theta,\ u,\ \nabla_x p)$ on $[t_0, T] \times \mathbb{R}^d$. Therefore, the pointwise continuity in time of the solution is improved to

$$(\theta - \overline{\theta})(t) \in H^{s'+2}(\mathbb{R}^d) , \quad u(t) \in H^{s'+1} , \quad \nabla_x p(t) \in H^{s''} , \quad \forall t \in (0, T) ,$$

for any $0 \le s' < s$, $1 \le s'' < s$. Together with the continuity at $t = 0$, this particularly implies that

$$\theta - \overline{\theta} \in C([0, T]; H^{s+1}) , \quad u \in C([0, T]; H^s) , \quad \nabla_x p \in C([0, T]; H^{s-1}) .$$

By iterating the process of increasing the space regularity of $(\theta,\ u, \nabla_x p)$ at each time, we have that

$$(\theta - \overline{\theta},\ u,\ \nabla_x p)(t) \in C^\infty(\mathbb{R}^d) , \quad \forall t \in (0, T) .$$

By system (1.3), the pointwise time regularity of the solution is determined by its space regularity. Therefore,

$$(\theta - \overline{\theta},\ u,\ \nabla_x p) \in C^\infty((0, T) \times \mathbb{R}^d) ,$$

which completes the proof of this lemma, thereby the proof of the Main Theorem. □

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