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Estimation of Transport Coefficients in Re-entrant Factory Models

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We develop continuum models of re-entrant factory production systems that treat the flow of products in analogy to traffic flow. Specifically, we model the dynamics of material flow through a re-entrant factory via a parabolic conservation law describing the product density and flux in the factory. We first extract the transport coefficients, in particular, velocity and diffusion coefficients of the particles in the production system using discrete event simulation (DES). Since PDE - conservation laws are successfully used for modeling the dynamical behavior of product flow in manufacturing systems, we model the manufacturing system using a diffusive partial differential equation (PDE). The specifics of the production process enter into the velocity and diffusion coefficient of the conservation law. The resulting nonlinear parabolic conservation law model allows fast and accurate simulations.

Keywords: supply chains; re-entrant factory; conservation law

1. Introduction

In recent years, factories and production systems have become larger and more complicated. For this reason, a research endeavor has been initiated to find time and cost efficient ways of production for such supply chains. Moreover, a wide range of traffic flow theories and models have been developed to find the most effective means of production. Our goal in this paper is to present a computationally efficient way to simulate the production systems.

Understanding the behavior of large supply chains under different polices and scenarios is a major issue for many businesses today. In large factories, no experiments can be done involving whole supply chains. Therefore, simulation models are developed, which substitute for the real environment. Especially in recent years, fast scalable simulations of production flows in a supply chain have become a very important research topic. The long term goal of a supply chain simulation is to optimize and control the production across the whole supply chain. Since most production deals with individual parts and the processes that these parts undergo, discrete event simulators would be the regular method of choice for accurate simulations.

While discrete event simulators have been highly successful to simulate single factories, they are computationally too expensive to simulate even a moderately complicated supply chain. Also, they are not scalable to a full supply chain (2).

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Alternative models that endow supply chain nodes with fixed production capacities and fixed lead times are not accurate enough since they do not take into account the fact that capacitated system respond nonlinearly to increases in demand close to the limit of the production capacity.

The approach followed in this paper is to extract TPTs (Throughput Times) and WIP (Work in Progress) levels from a discrete event simulation, replacing actual observations of physical system for the purposes of this work, and then to transform them into transport coefficients of a macroscopic conservation law. Using discrete event simulations we estimate the transport coefficients (V-velocity and D-diffusion). Then, we are able to solve the diffusive Partial Differential Equation (PDE). The PDE model that we use in this project is the conservation law given as follows:

$$\partial_t \rho(x,t) + \partial_x F(x,t) = 0, \quad F(x,t) = V\rho(x,t) - D\partial_x \rho(x,t) \tag{1}$$

where x and t are space and time variables respectively. Here, $\rho(x,t)$ denotes the density of the particles and $F(x,t)$ denotes the flux, both of which are used to model the dynamics of material flow through the re-entrant factory. The velocity and diffusion coefficients are extracted from observations of the system (replaced by a discrete event simulation model for the purposes of this paper). Since PDE models are amenable to optimization and control (10), we basically merge the randomness and optimization by using both DES and PDE models.

We base our theory on the assumption that we have the following type of data available. The production process consists of M stages. a_n^m denotes the time at which the lot number n arrives at stage m. Thus, the basic input of our macroscopic model consists of a table of arrival times a_n^m, $n = 1 : N$, $m = 1 : M$. From this table, the throughput times τ_n^m for each stage as well as WIP in any part of the supply chain can be easily computed. For instance, given the arrival times a_n^m the WIP in the m-th supplier $W_m(t)$ is computed as

$$W_m(t) = \sum_n H(t - a_n^m) - H(t - a_n^{m+1})$$

where H denotes the usual Heaviside function. As lots/particles go through the stages, we compute their velocities and variances depending on observed parameters. Since each particle has a different processing time at each stage, it has a varying velocity through out the system, so this causes a variance in velocity for each particle. According to fluid dynamics theory, variance results in diffusion. This results in computing TPTs (Throughput Times), means, and variances depending on a chosen set of state parameters. Assumed data form the times that lots have passed point in the production process. At the same time we record some macroscopic quantity (such as the WIP) of the system at each of the arrival times a_n^m, giving a statistical distribution of throughput times, parameterized by this macroscopic quantity. The velocity V and the diffusion coefficient D in (1) are then related to the mean and variance of this distribution. Here, the goal is to form a compact model which reorders itself to optimization. PDE model has this important advantage, it is fast and optimization can be applied to obtain more accurate results.

The contents of the rest of the paper are as follows: In Section 2, we present theoretical background of simulations and how to relate TPTs and WIPs to velocity and diffusion coefficients. In Section 3, we explain the parameter extraction and how we compute the TPT (Throughput Time) distributions and their means

and variances from the given data. In Section 4, we describe a re-entrant system structure and apply the theories that we cover in this paper on a model problem; also we explain how to interpret the dependence of the observed data. Finally, in Section, 5 we present numerical results comparing the discrete event simulation model (2) to the PDE model.

2. Background of Simulation Models

2.1 *Discrete Event Simulation (DES) Models*

In the discrete event simulation, each item to be produced is modeled individually, i.e. the arrival time of an item is recorded at a certain stage and then the item is passed to the next stage after the processing time has elapsed. The processing time that it takes at a stage is known as the throughput time (TPT) and denoted as τ_n^m. TPT is chosen randomly from a given distribution replacing actual observations of the system for the purpose of this project. Here, a chain of M stages $(s_1, .., s_M)$ is considered; a_n^m denotes the time at which the lot number n arrives at stage s_m and e_n^m denotes the time at which the lot number n exits stage s_m (and arrives at stage s_{m+1}). So, the difference between the exit time and the arrival time of a lot at a particular stage is known as throughput time of that lot at that stage and given by

$$\tau_n^m = e_n^m - a_n^m \ .$$

The arrival and exit times are computed via the law given in (2) as follows:

$$(a)\ e_n^m = a_n^m + \tau_n^m, \quad (b)\ d\mathcal{P}\{\tau_n^m = r\} = \mathcal{T}_m(r, a_n^m)\ dr, \quad (c)\ a_n^{m+1} = e_n^m \qquad (2)$$

Here $\mathcal{T}_m(r, a)$ denotes the time dependent distribution of throughput times of stage s_m and $\mathcal{P}$ denotes the probability distribution of the processing time. So, τ_n^m is chosen randomly according to (2)(b). The throughput time distribution $\mathcal{T}_m$ is usually dependent on the total number $W_m(t)$ of lots handled by stage s_m at time t, the so called Work in Progress (WIP) (3).

While discrete event simulators have been highly successful to simulate single factories, they are computationally too expensive to simulate even a moderately complicated supply chain. As the complexity of the factory increases with more buffers, machines, and parts, time to program and simulate the discrete event simulation (DES) grows nonlinearly (8). Also, DES models are not scalable to a full supply chain. Detailed information on all features of DES can be found in (4), (5).

2.2 *Rate Equations (Fluid Models)*

Fluid models come from traffic theory and were introduced by Newell (9) to approximately solve queueing problems. Fluid models do not model parts individually in a queue, they consider the length of a queue $q(t)$ as a continuous variable whose rate of change is given by:

$$\frac{d}{dt}q(t) = \lambda(t) - \mu(t) \qquad (3)$$

where λ is the arrival rate and μ is the processing rate of the queue.

If the number of lots considered in a given time interval is very large, it is computationally preferable to replace the model (2) by a model where the WIP "$W_m(t)$" is a continuous function. This leads to a class of models often referred to as fluid models in the supply chain literature (1), (6). In these continuous product models, the primary variables are the WIP "$W_m(t)$" and the fluxes "$\lambda_m(t)$" from stage s_{m-1} to stage s_m. A random throughput time function $\tau(t)$ is computed from the WIP same as before. The rate of change of WIP in front of machine m is given through the influx to that machine λ_m minus its outflux μ_m. A differential $W_m(t)$ equation is then given by the conservation law in (4).

$$\frac{d}{dt}W_m(t) = \lambda_m(t) - \mu_m(t), \quad \lambda_{m+1}(t) = \mu_m(t) \tag{4}$$

Here, λ and μ represent influx and outflux respectively. Moreover, in (5), the fluxes are given according to (2) by

$$\mu_m(t) = \int \delta(s + \tau(s) - t)\lambda_m(s)\ ds\ . \tag{5}$$

One of the most important disadvantages of fluid models is that they do not handle stochasticity well; i.e., in Equation (3), if $\lambda(t)$ and $\mu(t)$ are mean rates, then this is a fully deterministic system, stochasticity is not modeled at all. Otherwise; if $\lambda(t)$ and $\mu(t)$ are stochastic processes, then it allows us to do some theoretical analysis, but reduces the advantages of a continuum model as a simulation tool (8).

2.3 *Partial Differential Equation (PDE) Models*

PDE models are actually continuum limit of fluid models. PDE models do have several advantages; i.e., they are scalable, more detailed results can be found out as compared to fluid models, and most important of all, they are amenable to optimization and control (10).

There are two different types of PDE modeling: Heuristic modeling and direct modeling from observation. In the heuristic modeling; on average there exists a functional relation between TPT and WIP, this represents a generalization of the clearing functions introduced by Graves et al. to clearing distributions (11). In the direct modeling; clearing function is replaced by a probability distribution function, which is obtained from the observed data. In this paper, we use direct modeling in our PDE model. After we obtain the data (randomly generated times) from DES, we extract the transport coefficients. We can then solve the PDE given in (1). Our ultimate goal is to predict and optimize the behavior of a system from observation without knowing details about model; i.e. we always want to know how the system behaves even though the features of the system is changed. The PDE model that we use in this project is the conservation law given in Equation (1).

A conservation law is a relation asserting that a specific quantity is conserved, for example; conservation of energy, conservation of momentum, conservation of electron number...etc. Basically, for a quantity, to be conserved means that whatever enters to the system has to come out of that system after some time.

Furthermore, a conservation law can be defined as a partial differential equation that expresses the fact that some physical quantity is locally conserved in a fluid or other continuous physical system, such as energy, momentum, or the quantity

of fluid itself. Conservation law in the integral form is given by

$$\partial_t \int_a^b \rho(x,t)dx = In\ Flux\ -\ Out\ Flux \tag{6}$$

where $[a,b] \in \Re$. Here, $\rho(x,t)$ is the conserved variable, it is the density of the products with units [parts/space] in the system. The integral in Equation (6) gives the number of products per time between $x = a$ and $x = b$.

Flux is given by constitutive relation such that $F(\rho(x,t)) = F(x,t)$ is the flux function of $\rho(x,t)$. Here, x is defined to be the completion variable, $x = 0$ denotes the start of a product into the factory and $x = 1$ denotes the end of a product. So, in the closed interval: $x \in [0,1]$. The total number of products in the system can be found by taking the integral of density of products $\rho(x,t)$ over the stage variable x from 0 to 1. We get the total WIP $W(t)$ as a function of time as follows

$$W(t) = \int_0^1 \rho(x,t)dx. \tag{7}$$

We use an upwinding scheme to discretize the PDE which is given by Equation (8).

$$\rho_j(t+k) = \rho_j(t) - \frac{k}{h}[F_j(t) - F_{j-1}(t)] \tag{8}$$

where k is the time step, h is the space step, and $(j \geq 1)$; besides, ρ is the density of the products, F is the flux, and $j = 1, ..., J$ are space grid points. Discretized flux function is given by:

$$F_j(t) = V_j\rho_j(t) - D_j\frac{\rho_{j+1}(t) - \rho_j(t)}{h}$$

where V_j and D_j are the varying velocity and diffusion coefficients with respect to space.

The PDE that we aim to solve here is an initial boundary value problem.

$$\partial_t\rho + \partial_x F = 0, \quad F = V\rho - D\partial_x\rho$$

Since initially there is nothing in the system, the initial condition $\rho^I(x)$ for the density of the products at $t = 0$ is given by

$$\rho^I(x) = \rho(x,0) = 0.$$

Besides, we have two (left and right) boundary conditions; namely, $F^{B_1}(t)$ and $F^{B_2}(t)$ for the flux function $F(x,t)$ given as follows:

i) At $x = 0$; $F^{B_1}(t) = F(0,t) = \lambda(t)$ (averaged influx that we get out of 100 DES runs).

ii) At $x = L$; $F^{B_2}(t) = F(L,t) = V(L,t)(\rho(L,t))$, where L is chosen large enough on position (x)-axis such that $L > 1$. So, after $x = 1$, we decay the diffusion coefficient D_j linearly up to 0 and at $x = L$, we get $D(L,t) = 0$. Hence, in the flux function $F(L,t)$, the second term with the diffusion coefficient is set equal to 0.

CFL (Courant, Friedrichs, Levy) Condition

Due to the finite traveling speed of waves hyperbolic partial differential equations have a finite physical domain of dependence. The full numerical domain of

dependence must contain the physical domain of dependence.

The domain of dependence of a hyperbolic partial differential equation (PDE) for a given point in the problem domain is that portion of the problem domain that influences the value of the solution at the given point. Similarly, the domain of dependence of an explicit finite difference scheme for a given mesh point is the set of mesh points that affect the value of the approximate solution at the given mesh point. The CFL condition, named for its originators Courant, Friedrichs, and Levy, requires that the domain of dependence of the PDE must lie within the domain of dependence of the finite difference scheme for each mesh point of an explicit finite difference scheme for a hyperbolic PDE. Any explicit finite difference scheme that violates the CFL condition is necessarily unstable, but satisfying the CFL condition does not necessarily guarantee stability. In other words, the CFL condition is necessary for stability, but not sufficient (7). The CFL condition is formulated as follows:

$$\left|\frac{k}{h}V_{max}(t)\right| \leq 1 \tag{9}$$

where $k = \Delta t$ and $h = \Delta x$ are the step sizes in time and space respectively, $V_{max}(t)$ is the maximum of all occurring velocities in the system at time t.

3. Parameter Extraction

In this section, we describe how to compute the throughput time probability distribution in the kinetic model $\mathcal{T}$ as well as its mean and variance depending on a macroscopic state variable from the given observation, discrete event experiments. Also, we explain how to compute the transport coefficients (V and D).

As we run a simulation where N parts pass through M stages $s_1, ..., s_M$, we record the times a_n^m when part number n arrives at stage m and finally at the exit, which corresponds to stage $M+1$. At the same time, we record some macroscopic variables Z_n^m at the times a_n^m.

$Z = (Z_1, ..., Z_K)$ is in general a vector of K components. They can be variables like total WIP, or the WIP in front and the WIP behind the part, or just station index m itself. We eventually form two tables:

$$\begin{pmatrix} a(1,1) & a(1,2) & ... \\ a(2,1) & a(2,2) & ... \\ . & . & ... \\ . & . & a(N,M+1) \end{pmatrix} \begin{pmatrix} Z(1,1) & Z(1,2) & ... \\ Z(2,1) & Z(2,2) & ... \\ . & . & ... \\ . & . & Z(N,M) \end{pmatrix} \tag{10}$$

where there are different possibilities that the macroscopic variable Z might depend on. $Z_n^m = m$ if we want to make the velocity dependent on space only, or $Z_n^m = W(a_n^m)$ for dependence on the total WIP only, or $Z_n^m = (W_{<m}(a_n^m), W_{>m}(a_n^m))$ for dependence on the WIP in front and behind the part. From one of these possibilities, we compute a distribution for the velocities, dependent on the global variables Z. We map the work stations $s_1, ..., s_M$ on a stage interval $[0, X]$ where s_m corresponds to the interval (x_{m-1}, x_m) of length λ with $x_m = m\lambda$, $m = 0, ..., M$, $\lambda = \frac{X}{M}$. Then, the throughput time of lot n entering s_m is given by $\tau_n^m = a_n^{m+1} - a_n^m$, where $m = 1, ...M$ and $n = 1, ..., N$. We divide the K dimensional space of the macroscopic variables Z_n^m into cells $C(\mathbf{k})$ where $\mathbf{k} = (k_1, ..., k_K)$ and then we define the

indicator function $\chi_{\mathbf{k}}(Z)$ by

$$\chi_{\mathbf{k}}(Z) = \begin{cases} 1 \ for \ Z \in C(\mathbf{k}) \\ 0 \ else \end{cases}$$

The collected data give a discrete probability distribution, for the throughput times τ_n^m, dependent on the macroscopic state variable Z, of the form

$$\mathcal{T}(t, x, Z) = \frac{\sum_n \delta(t - \tau_{nm})\chi_{\mathbf{k}}(Z_{nm})}{\sum_n \chi_{\mathbf{k}}(Z_{nm})}$$

for $Z \in C(\mathbf{k})$ and $x \in [x_{m-1}, x_m)$, with a mean $\langle\tau\rangle(x, Z)$ and a variance $\sigma^2(x, Z)$ given by $\langle\tau\rangle(x, Z) = \frac{\sum_n \tau_{nm}\chi_{\mathbf{k}}(Z_{nm})}{\sum_n \chi_{\mathbf{k}}(Z_{nm})}$ and $\sigma^2(x, Z) = \frac{\sum_n \tau_{nm}^2\chi_{\mathbf{k}}(Z_{nm})}{\sum_n \chi_{\mathbf{k}}(Z_{nm})} - \langle\tau\rangle(x, Z)^2$ for $Z \in C_{\mathbf{k}},\ x \in [x_{m-1}, x_m)$.

According to the fluid dynamics theory, the mean and the variance of the distribution $\mathcal{T}$ are related to the average velocity coefficient $V(x, Z)$ and the diffusion coefficient $D(x, Z)$ in a fluid dynamics approximation of the form

$$\partial_t \rho + \partial_x F = 0, \quad F = V\rho - D\partial_x \rho \tag{11}$$

where $\rho = \rho(x, t)$ represents density of the products dependent on space and time and $F(\rho(x, t))$ represents the product flow (flux).

Using the theory developed in part **??**, we relate the velocity (V) and the diffusion (D) coefficients to the mean and variance of the distribution $\mathcal{T}$ respectively via the formulas given in (12):

$$V = \frac{\lambda}{\langle\tau\rangle_m}, \quad D = \frac{\lambda^2\Omega}{\langle\tau\rangle_m} \tag{12}$$

where λ denotes the length of space interval from one stage to the other and $\Omega = \frac{\sigma_m^2}{\langle\tau\rangle_m^2}$ denotes the variation coefficient, which is a measure of stochasticity. As Ω gets large, the stochasticity increases; and as Ω gets smaller, the system becomes more deterministic. The diffusion coefficient is derived from particle model by long time averaging, which is known as "Chapman - Enskog expansion". The proof and the details of the derivations of the formulas can be found in (3).

Note that the PDE given in (11) represents in general a nonlinear problem since the transport coefficients V and D depend on the state variable Z which, in turn, will depend on the density ρ.

4. A Model Problem

In this part, we briefly describe a re-entrant system structure and apply the theories that we covered so far on a model problem. Moreover, we explain how to compute work in progress (WIP), downstream WIP, upstream WIP, where different types of service rules i.e., FIFO, PULL, PUSH can be used in the experiments.

The block diagram of the system structure consist of a generator, buffers, machines, and an exit processor. The structure of the re-entrant manufacturing system can be seen in Figure 1. Here; G, B, M, and E stand for Generator, Buffer, Machine, and Exit respectively. In the first loop, particles change their attribute from 1 to 2 at the last machine M_N and then the particles start being processed in the second loop. When the particles have attribute 2, they exit the system after they

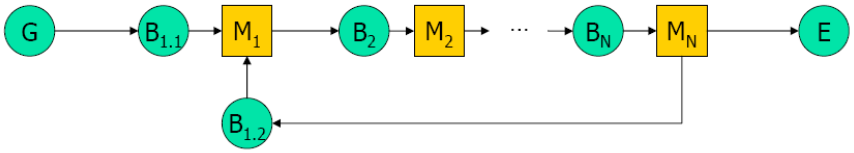

Figure 1. The structure of a re-entrant manufacturing system

reach the last machine M_N. The system designed in this project is FIFO, namely: First In First Out. However, it can also be transformed into a Pull or Push policy by altering the features of the buffers $B_{1.1}$ and $B_{1.2}$.

The goal is to solve the PDE model (11) after extracting the transport coefficients V and D from the observed data (a discrete event simulation for the purpose of this project). A major issue is how to parameterize the observed throughput times, i.e. how to choose the macroscopic state variable Z_n^m. This choice will in general depend on the structure under consideration and will significantly influence the approximation quality of the fluid dynamic model (11). There is of course an infinite number of possible choices. We will consider three possibilities for the purposes of this paper.

If the service rule in the system, depicted in Figure 1 is of 'PULL' type, i.e. if parts on their second pass through the system always have priority (parts that have passed the re-entrant loop), it is obvious that only the lots downstream in the production process will influence the velocity of an individual lot. In this case, we would choose the macroscopic state as the downstream (right) WIP given by

$$Z_n^m = \sum_{j=1}^{N} H(a_n^m - a_j^m) - H(a_n^m - a_j^{M+1}) \ , \tag{13}$$

Here, formula (13) counts the lots which have entered stage m at time a_n^m and have not yet arrived at the last stage M. In the fluid dynamic model given in (11), the downstream WIP is computed accordingly as

$$Z(x,\rho) = \int_x^1 \rho(y,t)\ dy \ .$$

Conversely, if the service rule is of a 'PUSH' type, i.e. if parts on their first pass through the system always have priority (parts that have not yet arrived at the re-entrant loop), the upstream (left) WIP should be used and given by

$$Z_n^m = \sum_{j=1}^{N} H(a_n^m - a_j^1) - H(a_n^m - a_j^m), \quad Z(x,\rho) = \int_0^x \rho(y,t)\ dy \ . \tag{14}$$

Here, formula (14) counts the lots which have entered the first stage at time a_j^1 and have not yet arrived at stage m.

If, on the other hand, the service rule is FIFO (first in first out), then it is probably wise to use the total WIP as parameter, since velocities depend on the upstream as well as on the downstream WIP. This would give

$$Z_n^m = \sum_{j=1}^{N} H(a_n^m - a_j^1) - H(a_n^m - a_j^M), \quad Z(\rho) = \int_0^1 \rho(y,t)\ dy \ . \tag{15}$$

Therefore, as discussed earlier, transport coefficients $V(x, Z_\rho)$ and $D(x, Z_\rho)$ depend on the macroscopic variable Z, which might be downstream (right) WIP, upstream (left) WIP, or total WIP given by

$$Z(x,\rho) = \int_x^1 \rho(y,t)\ dy, \quad Z(x,\rho) = \int_0^x \rho(y,t)\ dy, \quad Z(\rho) = \int_0^1 \rho(y,t)\ dy\ .$$

respectively.

5. Numerical Results

We carry out a numerical experiment to demonstrate the agreement of PDE and discrete event simulation (DES) models. The purpose of this comparison is to substantiate the prediction of the transient (non-steady state) behavior of DES with the PDE model according to the influx that we get out of the averaged DES models. In the simulations, we compute the transport coefficients (V and D) depending on the total WIP and since the total WIP is chosen as a macroscopic state variable (Z), we use FIFO (first in first out) as the service rule in the system.

Firstly, in the DES (χ-Chi Simulation), we consider a system with 300 lots. We start with an empty system and then generate data for the estimation of the transport coefficients (V and D) by running the system in quasi steady state under six different loads. Using 20 identical machines with a mean throughput time $\Delta T_{service} = 2$ and two loops through the system (see Figure 1), we compute that the system has an effective capacity of $\kappa = \frac{1}{2\Delta T_{service}} = \frac{1}{4}$. So, a constant influx $l > \kappa$ will on average produce a temporary buildup of queues since we have a finite number of lots.

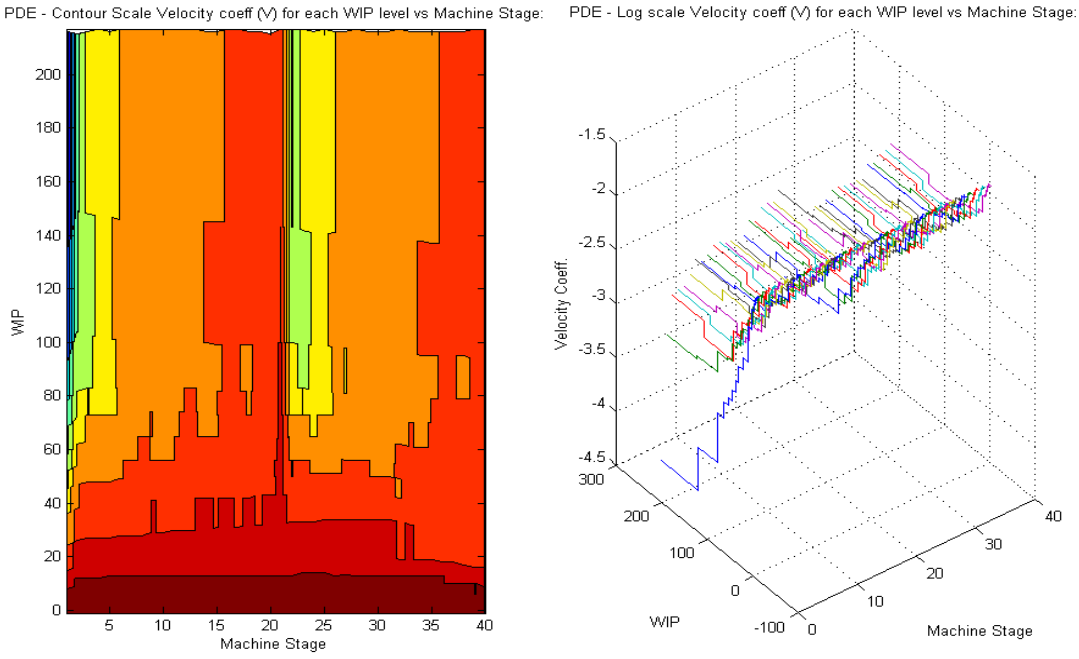

Figure 2. Velocity coefficients vs Machine Stage vs WIP in PDE model

Velocity (V) coefficients can be seen in Figure 2 both in contour and log scale plots as functions of WIP for any stage. Moreover, diffusion (D) coefficients can

be seen in Figure 3 as functions of WIP for any stage.

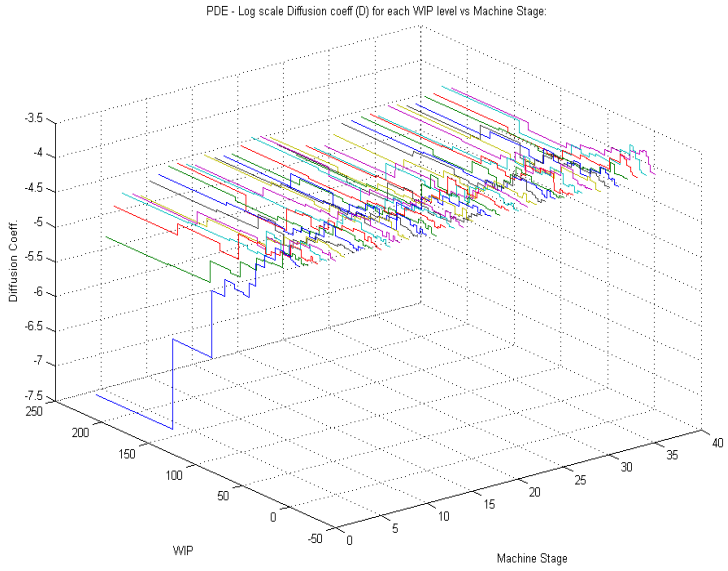

Figure 3. Diffusion coefficients vs Machine Stage vs WIP in PDE model

In the observed data that we get out of discrete event experiments, we use the following utilizations: 200%, 110%, 100%, 90%, 50%, 20%. Lots arrive randomly in mean intervals of length $\Delta T_{arrival} = \frac{2\Delta T_{service}}{p}$, and we use the efficiency capacity values of $p = 2,\ 1.1,\ 1,\ 0.9,\ 0.5,\ 0.2$ to calculate the mean interarrival time lengths that we use in DES runs. Each case is repeated ten times using different seeds for the random number generator in the discrete event simulator - χ(Chi). This produces a total of $10 \times 6 \times 300 = 18000$ data points.

We then execute the DES model by generating a total of 1000 lots with a varying influx such that the first one third of the lots produce an average influx of 0.3κ, i.e. lots arrive in intervals with mean $\frac{10}{3\kappa}$, the second one third of the lots have an influx of 0.8κ, and the last one third of the lots have an influx of 0.3κ. So, using these data we observe both ramp up and ramp down situations in our comparison for PDE model versus DES model. We run the DES model 100 times and average over these 100 simulation results. The averaged influx, computed out of 100 simulations,

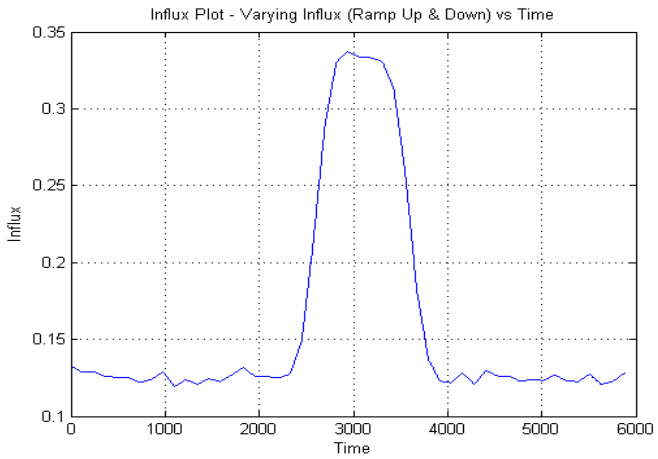

Figure 4. Varying Influx (Ramp Up and Ramp Down)

can be seen in Figure 4.

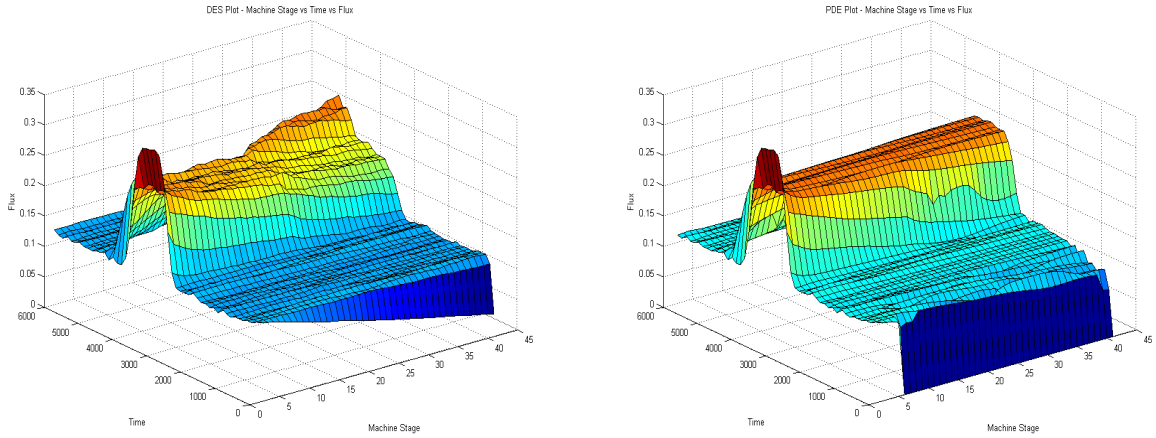

Figure 5. 3D Comparison of Fluxes for DES and PDE models

Finally, it is time to solve the PDE given in (1) using upwinding scheme where the flux is given as $F = V\rho - D\partial_x\rho$, with the velocity ($V$) and the diffusion ($D$) coefficients extracted from the 'observed data' as outlined in Section 3. With the PDE model, we predict the transient behavior of the DES model according to the influx given in Figure 4, which we get from the average of 100 DES model runs. The comparisons of all DES and PDE fluxes as functions of time at each machine stage can be seen in $3D$ surface plots in Figure 5. Similarly, the WIP levels of DES and PDE models are compared as functions of time at each machine stage in $3D$ surface plots in Figure 6.

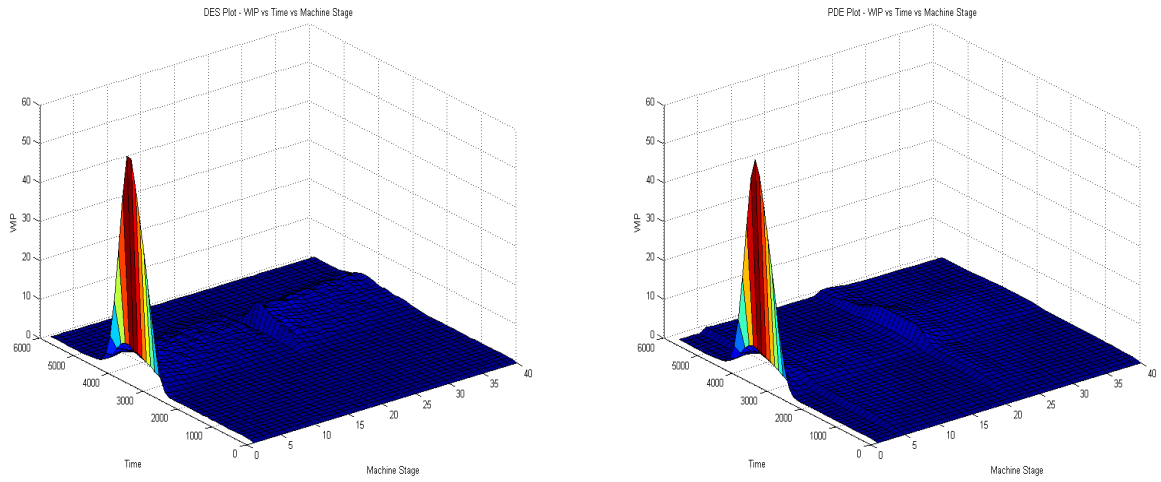

Figure 6. 3D Comparison of WIP Levels for DES and PDE models

As we see in Figures 5 and 6, although the WIP levels and flux values of PDE and DES models at each stage are not exactly equal, they are close enough to each other to substantiate the agreement of PDE and DES models.

In Figure 7, the WIP levels of PDE and DES models are compared at time steps 25 and 35 which can be seen in $2D$ plots. In the simulations, each time step corresponds to 120 mesh units depending on what time unit is chosen; i.e. minutes, hours, or days...etc.

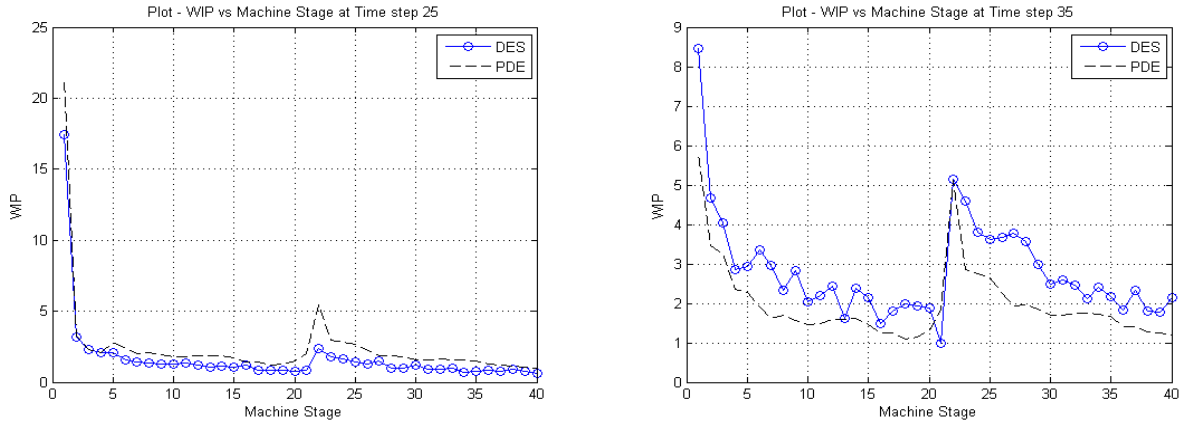

Figure 7. 2D WIP Level Comparison of PDE and DES models at Time Steps: 25 & 35

Moreover, in Figure 8, the comparisons of PDE and DES fluxes at machine stages 10 and 30 can be seen in $2D$ plots where the agreement of PDE and DES models is demonstrated more easily. Machine stages 10 and 30 also correspond to the same machine (10-th machine) in the system since the re-entrant loop starts at machine 20. One can tell from Figure 8 that the agreement between DES and PDE fluxes is better at the machine stage 10 as compared to the machine stage 20, because stage 10 is closer to the first stage where we start the simulations with the same influx for both DES and PDE models.

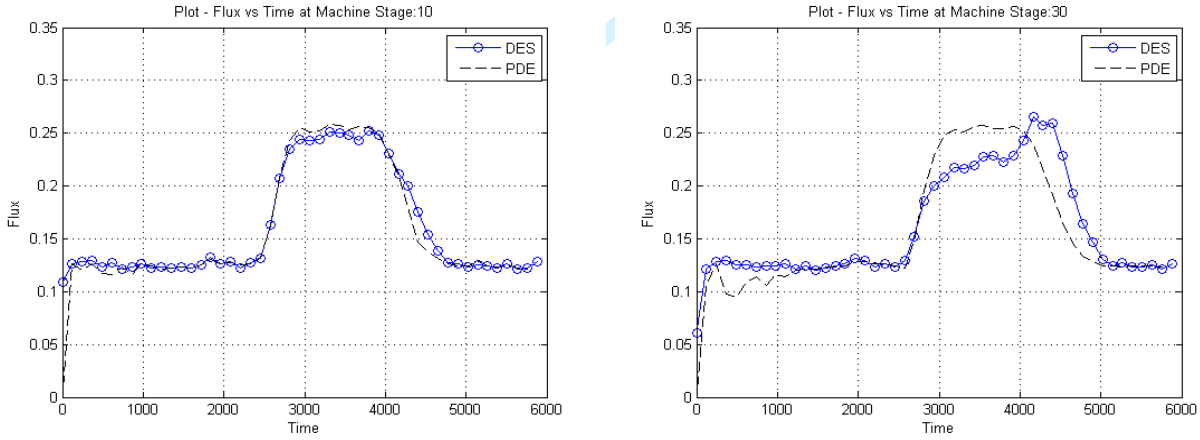

Figure 8. 2D Flux Comparison of PDE and DES models at Machine Stages: 10 & 30

6. Conclusion

In this paper, we have presented a computationally efficient way to simulate the production systems. Our goal was to solve the PDE model (1) after extracting the transport coefficients (V and D) from the observed data which we got out of a discrete event (Chi-0.8) simulation for the purpose of this paper. With the PDE model, using MATLAB program, we predicted the transient behavior of the DES model according to the averaged influx which we get out of 100 discrete event experiments. Finally, we compared the PDE and DES models in terms of WIP levels and fluxes at each machine stage. We conclude that two models agree with each other.

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