ERROR ESTIMATES FOR GAUSSIAN BEAM SUPERPOSITIONS August 6, 2010

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Abstract. Gaussian beams are asymptotically valid high frequency solutions to hyperbolic partial differential equations, concentrated on a single curve through the physical domain. They can also be extended to some dispersive wave equations, such as the Schrödinger equation. Superpositions of Gaussian beams provide a powerful tool to generate more general high frequency solutions that are not necessarily concentrated on a single curve. This work is concerned with the accuracy of Gaussian beam superpositions in terms of the wavelength ε. We present a systematic construction of Gaussian beam superpositions for all strictly hyperbolic and Schrödinger equations subject to highly oscillatory initial data of the form $Ae^{i\Phi/\varepsilon}$. Through a careful estimate of an oscillatory integral operator, we prove that the k-th order Gaussian beam superposition converges to the original wave field at a rate proportional to $\varepsilon^{k/2}$ in the appropriate norm dictated by the well-posedness estimate. In particular, we prove that the Gaussian beam superposition converges at this rate for the acoustic wave equation in the standard, ε-scaled, energy norm and for the Schrödinger equation in the L^2 norm. The obtained results are valid for any number of spatial dimensions and are unaffected by the presence of caustics. We present a numerical study of convergence for the constant coefficient acoustic wave equation in $\mathbb{R}^2$ to analyze the sharpness of the theoretical results.

Key words. high-frequency wave propagation, error estimates, Gaussian beams

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1. Introduction.

Simulation of wave propagation is expensive when the frequency of the waves is high. In this case, a large number of grid points is needed to resolve the wave oscillations and the computational cost to maintain constant accuracy grows algebraically with the frequency. At sufficiently high frequencies, therefore, direct simulation is no longer feasible.

Instead, one can use high frequency asymptotic models for wave propagation. The most popular approach is geometrical optics [7, 31] which is obtained in the limit when the frequency tends to infinity. It is also known as the WKB method or ray-tracing. The solution of the partial differential equation (PDE) is assumed to be of the form

$$a(t, y)e^{i\phi(t,y)/\varepsilon}, \tag{1.1}$$

where $1/\varepsilon$ is the large high frequency parameter, a is the amplitude of the solution, and ϕ is the phase. The phase and amplitude are independent of the frequency and vary on a much coarser scale than the full wave solution. They can therefore be computed at a computational cost independent of the frequency. However, the geometrical optics approximation breaks down at caustics, where rays concentrate and the predicted amplitude is unbounded [24, 19]. The consideration of difficulties caused by caustics, beginning with Keller in [16] and Maslov and Fedoriuk (see [25]), led to the development of the theory of Fourier integral operators, e.g., as given by Hörmander in [10].

Gaussian beams form another high frequency asymptotic model which is closely related to geometrical optics. However, unlike geometrical optics, Gaussian beams

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are valid at caustics. For Gaussian beams, the solution is also assumed to be of the geometrical optics form (1.1), but a Gaussian beam is a localized solution that concentrates close to a standard ray of geometrical optics in space-time. Although the phase function is real-valued along the central ray, Gaussian beams have a complex-valued phase functions off their central ray. The imaginary part of the phase is chosen such that the solution decays exponentially away from the central ray, maintaining a Gaussian-shaped profile. To form a Gaussian beam solution, we first pick a ray and solve a system of ordinary differential equations (ODEs) along it to find the values of the phase, its first and second order derivatives and the amplitude on the ray. To define the phase and amplitude away from this ray to all of space-time, we extend them using a Taylor expansion. Heuristically speaking, along each ray we propagate information about the phase and amplitude and their derivatives that allows us to reconstruct the wave field locally in a Gaussian envelope. The existence of Gaussian beam solutions has been know since sometime in the 1960's, first in connection with lasers, see Babič and Buldyrev [2]. Later, they were used in the analysis of propagation of singularities in partial differential equations by Hörmander [11] and Ralston [30].

In this article, we are interested in accuracy of Gaussian beam solutions to m-th order linear, strictly hyperbolic PDEs with highly oscillatory initial data of the type

$$\begin{aligned} Pu = 0, \qquad & (t,y) \in (0,T] \times \mathbb{R}^n, \\ \partial_t^\ell u(0,y) = \varepsilon^{-\ell} \sum_{j=0}^{N} \varepsilon^j B_{\ell,j}(y) e^{i\Phi(y)/\varepsilon} \ , \qquad & \ell = 0,\dots,m-1 \ , \end{aligned} \tag{1.2}$$

where the strictly hyperbolic operator, P, is defined in Section 2.1, the phase, $\Phi(y)$, is smooth and real valued and the amplitudes, $B_{\ell,j}(y)$, are smooth with $\mathrm{supp}\{B_{\ell,j}\} \subseteq K_0$, for some compact subset K_0 of $\mathbb{R}^n$. Furthermore, we will assume that $|\nabla\Phi(y)|$ is bounded away from zero on K_0. As a special case, we include the acoustic wave equation,

$$\begin{gathered} u_{tt} - c(y)^2 \triangle u = 0, \qquad (t,y) \in (0,T] \times \mathbb{R}^n \ , \\ u(0,y) = \sum_{j=0}^{N} \varepsilon^j A_j(y) e^{i\Phi(y)/\varepsilon} \ , \quad \text{and} \quad u_t(0,y) = \frac{1}{\varepsilon}\sum_{j=0}^{N} \varepsilon^j B_j(y) e^{i\Psi(y)/\varepsilon} \ . \end{gathered} \tag{1.3}$$

Note that for the acoustic wave equation, each of the initial conditions has a different phase function. This can also be done in the hyperbolic PDE (1.2), but for the sake of simplicity we will only use different phase functions for the acoustic wave equation. As in the strictly hyperbolic PDE case, we will assume that the magnitude of the gradients of both phase functions are bounded away from zero on K_0.

We also treat the dispersive Schrödinger equation,

$$\begin{gathered} -i\varepsilon u_t - \frac{\varepsilon^2}{2}\triangle u + V(y)u = 0, \qquad (t,y) \in (0,T] \times \mathbb{R}^n, \\ u(0,y) = \sum_{j=0}^{N} \varepsilon^j A_j(y) e^{i\Phi(y)/\varepsilon} \ . \end{gathered} \tag{1.4}$$

For the Schrödinger equation, the asymptotic parameter ε appears in the equation and there is no need to assume the bound on $|\nabla\Phi|$ that is necessary for strictly hyperbolic equations.

Since these partial differential equations are linear, it is a natural extension to consider sums of Gaussian beams to represent more general high frequency solutions that are not necessarily concentrated on a single ray. This idea was first introduced by Babič and Pankratova in [3] and was later proposed as a method for wave propagation by Popov in [28]. The sum, or rather the integral superposition, of Gaussian beams in the simplest first order form can be written as

$$u_{GB}(t,y) = \left(\frac{1}{2\pi\varepsilon}\right)^{\frac{n}{2}} \int_{K_0} a(t;z) e^{i\phi(t,y-x(t;z);z)/\varepsilon} dz \ , \tag{1.5}$$

where K_0 is a compact subset of $\mathbb{R}^n$ and the phase that defines the Gaussian beam is given by

$$\phi(t,y;z) = \phi_0(t;z) + y \cdot p(t;z) + y \cdot \frac{1}{2} M(t;z) y \ . \tag{1.6}$$

The real vector $p(t;z)$ is the direction of wave propagation and the matrix $M(t;z)$ has a positive definite imaginary part and it gives Gaussian beams their profile. Extensions of the above superposition are possible in several directions, including using higher order Gaussian beams in the superposition and using a sum of several superpositions to approximate the different modes of wave propagation. Higher order Gaussian beams are created by using an asymptotic series for the amplitude and using higher order Taylor expansions to describe the phase and the amplitude functions, see (2.10) and (2.11). Also, for higher order beams, a cutoff function (2.8) is necessary to avoid spurious growth away from the center ray. Superpositions with higher order Gaussian beams have an improved asymptotic convergence rate. For m-th order strictly hyperbolic PDEs, which have m pieces of initial data, we use m different Gaussian beam superpositions chosen in such a way so that their sum approximates the initial data. Each of these superpositions corresponds to one of the m distinct modes of wave propagation.

Accuracy studies for a Gaussian beam solution $u_{\rm GB}$ have traditionally focused on how well it asymptotically satisfies the PDE, i.e. the size of the norm of $Pu_{\rm GB}$ in terms of ε. The question of determining the error of the Gaussian beam superposition compared to the exact solution was thought to be a rather difficult problem decades ago, see the conclusion section of the review article by Babič and Popov [4]. However, some progress on estimates of the error has been made in the past few years. This accuracy study was initiated by Tanushev in [32], where a convergence rate was obtained for the initial data. Some earlier results on this were also established by Klimeš in [18]. The part of the error that is due to the Taylor expansion off the central ray was considered by Motamed and Runborg in [27] for the Helmholtz equation. Liu and Ralston [22, 23] gave rigorous convergence rates in terms of ε for the acoustic wave equation in the energy norm and for the Schrödinger equation in the L^2 norm. However, the error estimates they obtained depend on the number of space dimensions in the presence of caustics, since the projected Hamiltonian flow to physical space becomes singular at caustics. The superpositions in (1.5) can also be carried out in full phase space, i.e. over both z and p in (1.6), as shown in [22, 23]. In this formulation Hamiltonian flow is regular and there are no caustics, so one expects to obtain a dimensionally independent error estimate. This has been confirmed for the wave equation by Bougacha, Akian and Alexandre in [5] for the case of initial data based on the Fourier–Bros–Iaglonitzer (FBI) transform. However, from a computational stand point, the full phase space formulation is far more expensive. A more detailed discussion about previous results is given in Section 3.

Built upon these recent advances, together with a newly developed non-squeezing argument in Lemma 4.2, we are able to provide a definite answer to the question of accuracy for Gaussian beam superposition solutions. More precisely, we obtain dimensionally independent estimates for the superposition in physical space for general m-th order strictly hyperbolic PDEs and the Schrödinger equation. Our main result is the following theorem.

THEOREM 1.1. *If u is the solution to the strictly hyperbolic m-th order PDE (1.2) and u_k is the k-th order Gaussian beam superposition given in Section 2.3, we have the following estimate:*

$$\varepsilon^{m-1} \sum_{\ell=0}^{m-1} \left\| \partial_t^\ell [u(t,\cdot) - u_k(t,\cdot)] \right\|_{H^{m-\ell-1}} \le C(T)\varepsilon^{k/2} \ .$$

Furthermore, if u is the solution to the wave equation (1.3) and u_k is the k-th order Gaussian beam superposition given in Section 2.4, we have the following estimate in the scaled energy norm (3.2),

$$||u(t,\cdot) - u_k(t,\cdot)||_E \le C(T)\varepsilon^{k/2} \ .$$

Finally, if u is the solution to Schrödinger's equation (1.4) and u_k is the k-th order Gaussian beam superposition given in Section 2.5, we have the following L^2 norm estimate:

$$||u(t,\cdot) - u_k(t,\cdot)||_{L^2} \le C(T)\varepsilon^{k/2} \ .$$

At present there is considerable interest in using numerical methods based on superpositions of beams to resolve high frequency waves near caustics, which began in the 1980's with numerical methods for wave propagation in [28, 15, 6] and more specifically in geophysical applications in [17, 8, 9]. Recent work in this direction includes simulations of gravity waves [34], of the semiclassical Schrödinger equation [14, 20], and of acoustic wave equations [26, 32]. Numerical techniques based on both Lagrangian and Eulerian formulations of the problem have been devised [14, 21, 20, 26]. A numerical approach for general high frequency initial data closely related to the FBI transform, but avoiding the superposition over all of phase space, is presented in [29] for the Schrödinger equation. Numerical approaches for treating general high frequency initial data for superposition over physical space were considered in [33, 1] for the wave equation. Our theoretical results show that the numerical solutions found in these papers will be accurate when $\varepsilon \ll 1$.

To test the sharpness of the theoretical convergence rates, we present a short numerical study in the case of the acoustic wave equation with constant sound speed. Our numerical results indicate that the theoretical rates are sharp for even order k, but similar to the result in [27], we observe a gain of convergence by a factor of $\varepsilon^{1/2}$ for odd order k, which suggests that the actual convergence rate is $\mathcal{O}(\varepsilon^{\lceil k/2 \rceil})$.

The paper is organized as follows: Section 2 introduces Gaussian beams and their superpositions for m-th order strictly hyperbolic equations. As a special case of $m = 2$, we look at the classical acoustic wave equation. Furthermore, we construct Gaussian beams for the Schrödinger equation. We give a brief review of existing results on the accuracy of Gaussian beam superpositions in Section 3. Section 4 is devoted to error estimates for Gaussian beam superpositions. Detailed norm estimates of the oscillatory operators are given in Section 5. Numerical validation of our results is finally presented in Section 6.

2. Construction of Gaussian Beams. In this section, we go through the construction of Gaussian beam superpositions for strictly hyperbolic PDEs with the acoustic wave equation as a detailed example. We also construct Gaussian beams for the Schrödinger equation.

2.1. Hyperbolic Equations. Let $P = P_m + L$ be a linear strictly hyperbolic m-th order partial differential operator (PDO) with

$$P_m = \partial_t^m + \sum_{j=0}^{m-1} \left(\sum_{|\beta|=m-j} g_\beta(t,y) \partial_y^\beta \right) \partial_t^j \ , \tag{2.1}$$

and L a differential operator of order $m-1$. The principal symbol of P, denoted by $\sigma_m(t,y,\tau,p)$, is defined by the formal relationship $P_m = \sigma_m(t,y,-i\partial_t,-i\partial_y)$. Following [13, 30], we make the assumptions:

1. The coefficients $g_\beta(t,y)$ are smooth functions, uniformly bounded in t and y along with all their derivatives.
2. For $|p| \neq 0$, the principal symbol $\sigma_m(t,y,\tau,p)$ has m distinct real roots, when it is considered as a polynomial in τ.
3. These roots are uniformly simple in the sense that

$$\left| \frac{\partial \sigma_m(t,y,\tau,p)}{\partial \tau} \right| \geq c_0 |p|^{m-1} \quad \text{whenever} \quad \sigma_m(t,y,\tau,p) = 0. \tag{2.2}$$

We consider a null bicharacteristic $(t(s), x(s), \tau(s), p(s))$ associated with the principal symbol σ_m, defined by the Hamiltonian system of ODEs:

$$\dot{t} = \frac{\partial \sigma_m}{\partial \tau} \ , \qquad \dot{x} = \frac{\partial \sigma_m}{\partial p} \ , \qquad \dot{\tau} = -\frac{\partial \sigma_m}{\partial t} \ , \qquad \dot{p} = -\frac{\partial \sigma_m}{\partial y} \ , \tag{2.3}$$

and initial conditions $(t(0), x(0), \tau(0), p(0))$ such that $\sigma_m(t(0), x(0), \tau(0), p(0)) = 0$, with $p(0) \neq 0$. Note that for fixed $t(0)$, $x(0)$ and $p(0)$, we have m distinct choices for $\tau(0)$, equal to the m distinct real roots of $\sigma_m(t(0), x(0), \tau, p(0))$. These choices for $\tau(0)$ give m distinct waves that travel in different directions. The curve $(t(s), x(s))$ in physical space is the space-time ray that Gaussian beams are concentrated near. For a proof that the Gaussian beam construction is only possible near this ray, we refer the reader to [30].

The following lemma summarizes some results related to the Hamiltonian flow above.

LEMMA 2.1. *Let $(t(s), x(s), \tau(s), p(s))$ be a null bicharacteristic of the Hamiltonian flow (2.3) with initial data such that $|p(0)| \neq 0$ and $\sigma_m(t(0), x(0), \tau(0), p(0)) = 0$. Without loss of generality, assume that the parametrization is taken so that $\dot{t}(0) \geq 0$. Then for $s \in [0, \infty)$, we have*

1. $\sigma_m(s) = \sigma_m(t(s), x(s), \tau(s), p(s)) = 0$,
2. *$t(s) - t(0)$ is strictly increasing and*

$$t(s) - t(0) \geq C_1 \begin{cases} s \ , & m = 1 \ , \\ \log\left(1 + C_2 |p(0)|^{m-1} \ s\right) \ , & m > 1 \ , \end{cases}$$

where C_1 and C_2 are independent of s and initial data.

3. *There is a constant λ independent of s and initial data such that*

$$|p(s)| \geq |p(0)| e^{-\lambda(t(s)-t(0))}. \tag{2.4}$$

Proof. We compute

$$\begin{aligned}\dot\sigma_m(s) &= \frac{\partial\sigma_m}{\partial t}\dot t + \frac{\partial\sigma_m}{\partial y}\cdot\dot x + \frac{\partial\sigma_m}{\partial\tau}\dot\tau + \frac{\partial\sigma_m}{\partial p}\cdot\dot p\\ &= \frac{\partial\sigma_m}{\partial t}\frac{\partial\sigma_m}{\partial\tau} + \frac{\partial\sigma_m}{\partial y}\cdot\frac{\partial\sigma_m}{\partial p} - \frac{\partial\sigma_m}{\partial\tau}\frac{\partial\sigma_m}{\partial t} - \frac{\partial\sigma_m}{\partial p}\cdot\frac{\partial\sigma_m}{\partial y} = 0,\end{aligned}$$

and since $\sigma_m(0) = 0$, we have that $\sigma_m(s) = 0$, proving the first point.

We observe that since the coefficient of τ^m in σ_m has magnitude 1, $|p(s_0)| = 0$ implies that $\tau(s_0) = 0$ on the bicharacteristic. Thus, if $|p(s_0)| = 0$ then $\dot p(s_0) = \frac{\partial\sigma_m}{\partial y}(s_0) = 0$. Hence, we have that for all $s \in [0,\infty)$, $|p(s)| \neq 0$ by uniqueness for solutions of ODEs and $|p(0)| \neq 0$. Recall that for a polynomial with distinct root, both the polynomial and its derivative cannot vanish at the same point. Since $|p(s)| \neq 0$ and strict hyperbolicity imply that $\sigma_m(t(s), x(s), \tau, p(s))$, as a polynomial of τ, has distinct roots and $\sigma_m = 0$ on the null bicharacteristic, we have that $\frac{\partial\sigma_m}{\partial\tau}(s) \neq 0$. Continuity implies that $\dot t(s) = \frac{\partial\sigma_m}{\partial\tau}(s)$ never changes sign and, by the choice in parametrization, we have that $\dot t(s) > 0$. Hence, $t(s) - t(0)$ is strictly increasing for all $s \in [0,\infty)$.

We next show that there is a constant C such that $|\tau(s)| \leq C|p(s)|$ for all s and initial data. This is obviously true if $\tau = 0$. When $\tau \neq 0$, let $\xi = p/\tau$ and observe that, by homogeneity,

$$0 = \sigma_m(t, y, \tau, p) = \tau^m\sigma_m(t, y, 1, \xi)\ ,$$

on the null bicharacteristic. Hence ξ is a root of the polynomial equation

$$1 + \sum_{|\beta|=1}^{m} g_\beta(t, y)\xi^\beta = 0\ .$$

Since the coefficients $g_\beta(t, y)$ are uniformly bounded it follows that there is a constant such that $|\xi| \geq C > 0$, which proves that $|\tau(s)| \leq C|p(s)|$ for all s.

We can now bound $\dot p(s)$ as follows.

$$\begin{aligned}|\dot p| &= \left|\frac{\partial\sigma_m}{\partial y}\right| = \left|\sum_{j=0}^{m-1}\left(\sum_{|\beta|=m-j}\partial_y g_\beta(t, y)p^\beta\right)\tau^j\right|\\ &\leq C\sum_{j=0}^{m-1}\sum_{|\beta|=m-j}|p|^{|\beta|}|\tau|^j = C\sum_{j=0}^{m-1}\sum_{|\beta|=m-j}|p|^m|\xi|^{-j} \leq c_1|p|^m\ ,\end{aligned}$$

for some constant c_1 independent of s and initial data. Let $\lambda = c_1/c_0$ where c_0 is the constant in (2.2). Moreover, set $\tilde t(s) = t(s) - t(0) \geq 0$ and note that $\dot{\tilde t}(s) = \dot t(s) > 0$. Then,

$$\begin{aligned}\frac{d}{ds}|p(s)|^2 e^{2\lambda\tilde t(s)} &= 2\,\dot p\cdot p\ e^{2\lambda\tilde t} + 2\lambda\dot t|p|^2 e^{2\lambda\tilde t} \geq -2|\dot p||p|e^{2\lambda\tilde t} + 2\lambda\frac{\partial\sigma_m}{\partial\tau}|p|^2 e^{2\lambda\tilde t}\\ &\geq -2c_1|p|^{m+1}e^{2\lambda\tilde t} + 2\lambda c_0|p|^{m+1}e^{2\lambda\tilde t} = 0\ .\end{aligned}$$

Consequently, $|p(s)|^2 e^{2\lambda\tilde t(s)} \geq |p(0)|^2$ and (2.4) follows. With $\lambda_m = (m-1)\lambda$, we obtain

$$\dot t = \frac{\partial\sigma_m(t, y, \tau, p)}{\partial\tau} \geq c_0|p|^{m-1} \geq c_0|p(0)|^{m-1}e^{-\lambda_m\tilde t(s)}\ .$$

It follows that $\tilde{t}(s) \geq c_0 s$ for $m = 1$, since $\lambda_1 = 0$. For $m > 1$,

$$sc_0|p(0)|^{m-1} \leq \int_0^s \dot{t}(s')e^{\lambda_m \tilde{t}(s')}ds' = \int_0^s \dot{\tilde{t}}(s')e^{\lambda_m \tilde{t}(s')}ds'$$
$$= \int_0^{\tilde{t}(s)} e^{\lambda_m \theta} d\theta = \frac{e^{\lambda_m \tilde{t}(s)} - 1}{\lambda_m} \ .$$

This proves the stated logarithmic growth of $\tilde{t}(s)$ for $m > 1$. □

An immediate consequence of this lemma is that we can use t to parametrize the Hamiltonian flow instead of s.

2.2. Single Gaussian Beams. In this section, we will employ the notation $y_0 = t$ and $p_0 = \tau$ to simplify the equations. In this notation, $\tilde{y} = (y_0, y) = (y_0, y_1, \dots, y_n)$ is an $n+1$ dimensional vector and similarly $\tilde{p} = (p_0, p)$. The Hamiltonian flow can then be simply written as

$$\dot{\tilde{x}}(s) = \partial_{\tilde{p}}\sigma_m(\tilde{x}(s), \tilde{p}(s)) \ , \qquad \dot{\tilde{p}}(s) = -\partial_{\tilde{y}}\sigma_m(\tilde{x}(s), \tilde{p}(s)) \ .$$

We proceed by considering functions of the form

$$v(\tilde{y}) = \sum_{j=0}^{J} \varepsilon^j a_j(\tilde{y} - \tilde{x}(s)) e^{i\phi(\tilde{y} - \tilde{x}(s))/\varepsilon} \ . \tag{2.5}$$

Applying the operator P to this ansatz and collecting terms containing the same power of ε, we have

$$Pv(\tilde{y}) = \left(\sum_{r=-m}^{J} \varepsilon^r c_r(\tilde{y}) \right) e^{i\phi(\tilde{y} - \tilde{x}(s))/\varepsilon} \ .$$

In order for v to be an asymptotic solution to $Pu = 0$, we have to make each term in the sum above small in terms of ε. First, we will make the c_r's vanish on the space-time ray along with their derivatives. We will accomplish this by choosing ϕ and a_j as Taylor polynomials in $\tilde{y}$ about the ray $\tilde{x}(s)$. In other words, we will choose the values of ϕ and a_j, their first, second, etc derivatives on $\tilde{x}(s)$. With a slight abuse of notation we employ the shorthand $\partial_{\tilde{y}}^{\alpha}\phi(s)$ for $\partial_{\tilde{y}}^{\alpha}\phi(\tilde{x}(s))$ and similar for derivatives of a_j. For the phase on the ray we also use the special notation $\phi_0(s) := \phi(\tilde{x}(s))$.

We first show that $c_{-m}(\tilde{x}(s)) = \partial_{\tilde{y}} c_{-m}(\tilde{x}(s)) = 0$ if we take $\partial_{\tilde{y}}\phi(s) = \tilde{p}(s)$. Indeed, observing that $c_{-m}(\tilde{y}) = \sigma_m(\tilde{y}, \partial_{\tilde{y}}\phi) a_0(\tilde{y} - \tilde{x}(s))$ we then immediately have that $c_{-m}(\tilde{x}(s)) = 0$ since $(\tilde{x}(s), \tilde{p}(s))$ is a null bicharacteristic, and, hence, $\sigma_m(\tilde{x}(s), \tilde{p}(s)) = 0$. Moreover, differentiating c_{-m} in $\tilde{y}$ and evaluating on the ray, we have by the definition of the Hamiltonian flow,

$$\partial_{\tilde{y}} c_{-m}(\tilde{x}(s)) = (\partial_{\tilde{y}}\sigma_m + (\partial_{\tilde{y}}^2\phi)\partial_{\tilde{p}}\sigma_m)a_0 + \sigma_m \partial_{\tilde{y}} a_0 = \left(-\dot{\tilde{p}}(s) + (\partial_{\tilde{y}}^2\phi)\ \dot{\tilde{x}}(s)\right) a_0$$
$$= \left(-\dot{\tilde{p}}(s) + \frac{d}{ds}(\partial_{\tilde{y}}\phi)\right) a_0 = \left(-\dot{\tilde{p}}(s) + \frac{d}{ds}(\tilde{p})\right) a_0 = 0 \ .$$

We also obtain an ODE for the phase on the ray,

$$\dot{\phi}_0(s) = \partial_{\tilde{y}}\phi(s) \cdot \dot{\tilde{x}}(s) = \partial_{\tilde{p}}\sigma_m(\tilde{x}(s), \tilde{p}(s)) \cdot \tilde{p}(s) \ ,$$

by the chain rule. Looking at the second order derivatives of c_{-m} on the ray and setting them equal to 0, one next obtains the following equation for the Hessian of ϕ:

$$\dot{\tilde{M}}(s) = -\tilde{A} - \tilde{M}\tilde{B} - \tilde{B}^{\mathsf{T}}\tilde{M} - \tilde{M}\tilde{C}\tilde{M} \ , \tag{2.6}$$

with

$$\tilde{M}(s) = \frac{\partial^2 \phi}{\partial\tilde{y}\partial\tilde{y}}(s) \ , \qquad \tilde{A}(s) = \frac{\partial^2 \sigma_m}{\partial\tilde{y}\partial\tilde{y}}(\tilde{x}(s),\tilde{p}(s)) \ ,$$
$$\tilde{B}(s) = \frac{\partial^2 \sigma_m}{\partial\tilde{p}\partial\tilde{y}}(\tilde{x}(s),\tilde{p}(s)) \ , \qquad \tilde{C}(s) = \frac{\partial^2 \sigma_m}{\partial\tilde{p}\partial\tilde{p}}(\tilde{x}(s),\tilde{p}(s)) \ .$$

This equation is non-linear and it is not immediately clear that it will have a solution for all s. However, noting that this equation is a Ricatti matrix equation for $\tilde{M}$, we have the following result:

LEMMA 2.2 (Ralston '82, [30]). *Suppose that $\tilde{M}(0)$ is chosen so that the $n \times n$ sub-matrix $\tilde{M}_{j,\ell}(0)$, for $1 \le j,\ell \le n$, has a positive definite imaginary part, $\tilde{M}(0)\dot{\tilde{x}}(0) = \dot{\tilde{p}}(0)$, and $\tilde{M}(0) = \tilde{M}^{\mathsf{T}}(0)$. Then, the solution of the Ricatti equation (2.6), $\tilde{M}(s)$, subject to the initial condition $\tilde{M}(0)$, will be such that the $n \times n$ sub-matrix $\tilde{M}_{j,\ell}(s)$, for $1 \le j,\ell \le n$, will have a positive definite imaginary part and $\tilde{M}(s) = \tilde{M}^{\mathsf{T}}(s)$ for all s.*

Indeed, we choose $\tilde{M}(0)$ to satisfy the conditions of this lemma and this is the most crucial step in the construction of Gaussian beams. The effect of this result is two fold: first, the symmetry of $\tilde{M}$ allows us to define the second order derivatives of ϕ for all s and second, it provides the localization for v near the ray, since $|e^{i\phi/\varepsilon}|$ will have a Gaussian profile in directions orthogonal to $\dot{\tilde{x}}(s)$ as $\dot{\tilde{x}}_0(s) \neq 0$ by Lemma 2.1. The name "Gaussian beams" also stems from this fact. For details and the proof, we point the reader to [30].

Continuing in this fashion, we can look at the 3-rd and higher order derivatives of c_{-m}. Setting each of them to zero gives ODEs for the third and higher order derivatives of ϕ on the ray. Each of these equations is linear and will have solutions for all s. We note that $\partial_{\tilde{y}}^{\alpha}\phi(s)$ depends only on σ_m and $\partial_{\tilde{y}}^{\beta}\phi(s)$ for $|\beta| \le |\alpha|$, so that these equations can be solved sequentially.

By differentiating c_r for $r > -m$ and setting these derivatives to zero on the ray, we obtain ODEs for the values of $\partial_{\tilde{y}}^{\tilde{\alpha}} a_j(s)$ on the ray. Each of these equations is complicated, but linear, and so its solution will exist for all s. We remark that these equations are also solved sequentially, and the equations for $\partial_{\tilde{y}}^{\alpha} a_j(s)$ depend only on the values of $\partial_{\tilde{y}}^{\beta}\phi(s)$ for $|\beta| \le |\alpha| + 2(j+1)$ and $\partial_{\tilde{y}}^{\beta} a_r(s)$ for $r \le j$ and $|\beta| \le |\alpha| + 2(j-r)$. Finally, we note that a careful examination of the dependence of the solutions of these ODEs on the initial data shows that if all of the required $\partial_{\tilde{y}}^{\beta} a_r(0)$ are equal to zero for $r \le j$, then $\partial_{\tilde{y}}^{\alpha} a_j(s)$ will be equal to zero for all s.

For a given set of initial parameters for all of the ODEs and a particular root for $\tau(0)$, the values of $\partial_{\tilde{y}}^{\alpha}\phi(s)$ and $\partial_{\tilde{y}}^{\alpha} a_j(s)$ are determined. Thus, we can define $\phi(\tilde{y}-\tilde{x}(s))$ and $a_j(\tilde{y}-\tilde{x}(s))$ as Taylor polynomials. Recalling that y_0 and t are related, for a fixed $t \in [0,T]$ we find $\hat{s}$ such that $y_0(\hat{s}) = t$. Lemma 2.1 guarantees that $\hat{s}$ exists and is unique. Then, writing $\partial_y^{\alpha}\phi(t) = \partial_y^{\alpha}\phi(\hat{s}(t))$, etc., and returning to the previous (t,y)

notation, we write $\phi(t,y)$ and $a_j(t,y)$ as Taylor polynomials:

$$\phi(t,y) = \sum_{|\beta|=0}^{k+1} \frac{1}{\beta!}\phi_\beta(t)y^\beta \equiv \phi_0(t) + y\cdot p(t) + y\cdot\frac{1}{2}M(t)y + \sum_{|\beta|=3}^{k+1}\frac{1}{\beta!}\phi_\beta(t)y^\beta\ ,$$
$$a_j(t,y) = \sum_{|\beta|=0}^{k-2j-1}\frac{1}{\beta!}a_{j,\beta}(t)y^\beta\ . \tag{2.7}$$

One last point needs to be addressed before we can define the Gaussian beam. Since $x(t)$ and $p(t)$ are real and if $\phi_0(0)$ is real, then the imaginary part of $\phi(t,y)$ will be a positive quadratic plus higher order terms about $x(t)$. Thus, we must only construct the Gaussian beam in a domain on which the quadratic part is dominant. To this end, we use a cutoff function $\rho_\eta \in C^\infty(\mathbb{R}^n;\mathbb{R})$ with cutoff radius $0<\eta\le\infty$ satisfying,

$$\rho_\eta(z)\ge 0 \quad \text{and} \quad \rho_\eta(z) = \begin{cases} \begin{array}{l} 1 \text{ for } |z|\le\eta \\ 0 \text{ for } |z|\ge 2\eta \end{array} & \text{for } 0<\eta<\infty \\ \\ 1 & \text{for } \eta=\infty \end{cases}\ . \tag{2.8}$$

Now, by choosing $\eta>0$ sufficiently small, we can ensure that on the support of $\rho_\eta(y-x(t))$, $\Im\phi(t,y-x(t)) > \delta|y-x(t)|^2$ for $t\in[0,T]$. However, note that for first order beams the imaginary part of the phase is quadratic with no higher order terms, so that the cutoff is unnecessary. Thus, to include this case we let the cutoff function be defined for $\eta=\infty$ by $\rho_\infty\equiv 1$. We are now ready to define the k-th order Gaussian beam $v_k(t,y)$ as:

$$v_k(t,y) = \sum_{j=0}^{\lceil k/2\rceil-1} \varepsilon^j\rho_\eta(y-x(t))a_j(t,y-x(t))e^{i\phi(t,y-x(t))/\varepsilon}\ .$$

2.3. Superpositions of Gaussian Beams. In the previous section, we constructed Gaussian beam solutions that satisfy $Pu=0$ in an asymptotic sense (see Theorem 3.2 for a precise statement), without too much concern for the values of the solution at $t=0$. The initial value problem that a single Gaussian beam $v_k(t,y)$ approximates has initial data that is simply given by its values (and the values of its time derivatives) at $t=0$. While they resemble the initial conditions for (1.2), they are quite different since for example the phase, ϕ, for v_k is complex valued and v_k is concentrated in y.

Our goal is to create an asymptotically valid solution to the PDE (1.2), thus we must also consider the m distinct pieces of initial data given in the form of time derivatives of u at $t=0$. To generate solutions based on Gaussian beams that approximate the initial data for (1.2), we exploit the linearity properties of P. That is, we use the fact that a linear combination of two Gaussian beams, with different initial parameters, will also be an asymptotic solution to $Pu=0$, since each Gaussian beam is itself an asymptotic solution. Building on this idea, let us assume that a family of Gaussian beams can be indexed by a parameter $z\in K_0$. We will use the notation $x_\ell(t;z)$, $p_\ell(t;z)$, $\phi_\ell(t,y-x(t;z);z)$, etc, to denote the dependence of these quantities on the indexing parameter z and on the m choices for $\tau(0;z)$ denoted by $\ell=0,\dots,m-1$. Thus, we will write the k-th order Gaussian beam $v_{k,\ell}(t,y;z)$. Note that the cutoff radius η may vary with z and ℓ between beams, however, as the beam superposition is taken over the compact set K_0 and $0 \le \ell \le m-1$, there is a minimum value for η that will work for all beams in the superposition. Thus, we form the k-th order superposition solution $u_k(t,y)$ as

$$u_k(t,y) = \sum_{\ell=0}^{m-1} \left(\frac{1}{2\pi\varepsilon}\right)^{\frac{n}{2}} \int_{K_0} v_{k,\ell}(t,y;z)dz \ , \tag{2.9}$$

where the phase and amplitude that define the Gaussian beam,

$$v_{k,\ell}(t,y;z) = \sum_{j=0}^{\lceil k/2\rceil-1} \varepsilon^j \rho_\eta(y - x_\ell(t;z)) a_{\ell,j}(t, y - x_\ell(t;z);z) e^{i\phi_\ell(t,y-x_\ell(t;z);z)/\varepsilon} \ ,$$

are given by

$$\phi_\ell(t,y;z) = \phi_{\ell,0}(t;z) + y\cdot p_\ell(t;z) + y \cdot \frac{1}{2}M_\ell(t;z)y + \sum_{|\beta|=3}^{k+1} \frac{1}{\beta!}\phi_{\ell,\beta}(t;z)y^\beta \ , \tag{2.10}$$

$$a_{\ell,j}(t,y;z) = \sum_{|\beta|=0}^{k-2j-1} \frac{1}{\beta!} a_{\ell,j,\beta}(t;z)y^\beta \ . \tag{2.11}$$

We remind the reader that each $v_{k,\ell}(t,y;z)$ requires initial values for the ray and all of the amplitude and phase Taylor coefficients. The appropriate choice of these initial values will make $u_k(0,y)$ asymptotically converge the initial conditions in (1.2). The first step is to choose the origin of the rays and the initial coefficients of ϕ_ℓ up to order $k+1$. Letting the ray begin at a point z and expanding $\Phi(y)$ in a Taylor series about this point,

$$\Phi(y) = \sum_{|\beta|=0}^{k+1} \frac{1}{\beta!}\Phi_\beta(z)(y-z)^\beta + \text{error} \ ,$$

we initialize the ray and Gaussian beam phase associated with τ_ℓ as

$$\begin{aligned} &x_\ell(0;z) = z \ , && p_\ell(0;z) = \nabla_y \Phi(z) \ , \\ &M_\ell(0;z) = \partial_y^2 \Phi(z) + i\ \mathrm{Id}_{n\times n} \ , && \phi_{\ell,\beta}(0;z) = \Phi_\beta(z) \quad |\beta| = 0 \ , \quad |\beta| = 3,\dots,k+1 \ . \end{aligned}$$

Determining the initial coefficients for the amplitudes involves a more complicated procedure. As with the phase, we expand all of the amplitude functions in Taylor series,

$$B_{\ell,j}(y) = \sum_{|\beta|=0}^{k-2j-1} \frac{1}{\beta!} B_{\ell,j,\beta}(z)(y-z)^\beta + \text{error} \ .$$

Next, we look at the time derivatives of u_k at $t=0$, to the lowest order in ε. We equate the coefficients of $(y-z)$ to the the corresponding terms in the Taylor expansions of $B_{\ell,j}$ and recalling that $\partial_t \phi_\ell(0,0;z) = \tau_\ell(0)$, we obtain the following $m \times m$ system of linear equations:

$$\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ (i\tau_1)^\ell & \cdots & (i\tau_m)^\ell \\ \vdots & & \vdots \\ (i\tau_1)^{m-1} & \cdots & (i\tau_m)^{m-1} \end{bmatrix} \begin{bmatrix} a_{0,0,\beta} \\ \vdots \\ a_{\ell,0,\beta} \\ \vdots \\ a_{m-1,0,\beta} \end{bmatrix} = \begin{bmatrix} B_{0,0,\beta} \\ \vdots \\ B_{\ell,0,\beta} \\ \vdots \\ B_{m-1,0,\beta} \end{bmatrix} .$$

Since the τ_ℓ are distinct, this Vandermonde matrix is invertible, so the solution will give the initial coefficients for the first amplitude for each of the m Gaussian beams. If we proceed with the next orders in ε, we would obtain the same $m \times m$ linear system for $[a_{0,j,\beta}, \ldots, a_{m-1,j,\beta}]^\mathsf{T}$, except that the right hand side will not only depend on the Taylor coefficients $B_{\ell,j,\beta}$, but also on previously computed $a_{\ell,q,\gamma}$, $q < j$, coefficients and their time derivatives. Thus, all of the necessary initial coefficients for each of the m Gaussian beams can be computed sequentially. Summarizing this construction, we have that at $t = 0$ and $\ell = 0, \ldots, m-1$,

$$\partial_t^\ell u_k(0,y) = \left(\frac{1}{2\pi\varepsilon}\right)^{\frac{n}{2}} \int_{K_0} \varepsilon^{-\ell} \sum_{j=0}^{N} \varepsilon^j \rho_\eta(y-z) b_{\ell,j}(y) e^{i\phi(y)/\varepsilon} dz + \mathcal{O}(\varepsilon^\infty) \ , \tag{2.12}$$

where $\phi(y)$ is the Taylor expansion of $\Phi(y) + i|y-z|^2/2$ to order $k+1$ and each $b_{\ell,j}$ is the same as the Taylor expansion of $B_{\ell,j}$ up to order $k-2j-1$. The $\mathcal{O}(\varepsilon^\infty)$ term is present because some of the time derivatives fall on the cutoff function $\rho_\eta(y - x_\ell(t;z))$. The contributions of such terms decays exponentially as $\varepsilon \to 0$, since the derivatives of $\rho_\eta(y - x_\ell(t;z))$ are compactly supported and vanish near $y = z$.

REMARK 2.1. *For ease of notation and exposition, in (1.2) we have taken the phase Φ to be the same for all of the m initial data, however, this is not a requirement. We can form m different Gaussian beam superpositions that each satisfy one of these m conditions with a specific phase and the rest with zero initial data. Then, summing these m solutions we obtain a more general solution of (1.2) with m different phase functions for each of initial data piece. We will provide a more in depth example of how to do this for the wave equation.*

REMARK 2.2. *In the initialization of $M_\ell(0;z)$ we take its imaginary part to be given by $i\,\mathrm{Id}_{n\times n}$ for simplicity. All of the results in this paper can be carried out if we instead took the imaginary part to be given by $i\gamma\,\mathrm{Id}_{n\times n}$, for some constant $\gamma > 0$, and adjusted the normalization constant in (2.9) appropriately. However, it is important to note that the constants throughout this paper will depend on γ and that in general, we expect that increasing γ will increase the evolution error in the Gaussian beams and that decreasing γ will increase the error in approximating the initial data.*

This completes the construction of the Gaussian beam superpositions u_k for the initial value problem (1.2) for general m-th order strictly hyperbolic operators. From this point, we will assume that the parameter η is chosen as $\eta = \infty$ for the first order superposition u_1 and that for higher order superpositions it is taken small enough (and independent of z and ℓ) to make $\Im\phi_\ell(t, y - x_\ell(t;z); z) > \delta|y - x_\ell(t;z)|^2$ for $t \in [0,T]$ and $y \in \mathrm{supp}\{\rho_\eta(y - x_\ell(t;z))\}$. Lemma 4.3 shows that this is always possible.

2.4. The Wave Equation. As an example of the construction of beams described in Section 2.1 we will consider the wave equation (1.3) with sound speed $c(y)$ in more detail. In terms of the previous section, the wave operator, $\square$, is a 2-nd order strictly hyperbolic PDO with principal symbol $\sigma_2(t,y,\tau,p) = -\tau^2 + c(y)^2|p|^2$. Thus, the null bicharacteristics satisfy the system of ODEs:

$$\begin{aligned} \dot{t}(s) &= \frac{\partial\sigma_2}{\partial\tau} = -2\tau(s) \ , & \dot{\tau}(s) &= -\frac{\partial\sigma_2}{\partial t} = 0 \ , \\ \dot{x}(s) &= \frac{\partial\sigma_2}{\partial p} = 2c(x(s))^2 p(s) \ , & \dot{p}(s) &= -\frac{\partial\sigma_2}{\partial y} = -2|p(s)|^2 c(x(s))\nabla c(x(s)) \ , \end{aligned}$$

with $\sigma_2(t(s), x(s), \tau(s), p(s)) = 0$. Thus, after picking the initial values $t(0)$, $x(0)$ and $p(0) \neq 0$, we have two choices for $\tau(0) = \pm c(x(0))|p(0)|$. Let us for now pick the bicharacteristics that correspond to "+", and note that since $\tau(s) = \tau(0) = c(x(0))|p(0)|$, $t = -2c(x(0))|p(0)|s + t(0)$. Using this equivalence between t and s, we will write all of the Gaussian beam quantities (such as $x(s)$, $p(s)$, etc) as function of t (e.g. $x(t)$, $p(t)$, etc). Similarly, we can write down the ODEs for the higher order derivatives of the phase and the amplitudes. Furthermore, since σ_2 and its derivatives vanish on the ray, we have a relationship between the t and y derivatives of the phase on the ray and thus we can substitute all of the $\partial^\alpha_{\{t,y\}}\phi(t)$ in the ODEs with expressions containing only $\partial^\alpha_y \phi(t)$ terms. A similar statement holds for the amplitudes from the transport equations. From here on, we will assume that this substitution has been carried out, and that all of the ODEs are written in terms of y derivatives only with t as the independent variable, with $t(0) = 0$. Then, the first several ODEs will be:

$$\begin{aligned}
\dot{\phi}_0(t) &= 0 \ , \\
\dot{x}(t) &= -c(x(t))p(t)/|p(t)| \ , \\
\dot{p}(t) &= |p(t)|\nabla c(x(t)) \ , \\
\dot{M}(t) &= -A(t) - M(t)B(t) - B^{\mathsf{T}}(t)M(t) - M(t)C(t)M(t) \ , \\
\dot{a_0}(t) &= a_0(t)\left(-\frac{p(t)}{2|p(t)|}\cdot\frac{\partial c}{\partial y}(x(t)) - \frac{p(t)\cdot M(t)p(t)}{2|p(t)|^3} + \frac{c(x(t))\mathrm{Tr}[M(t)]}{2|p(t)|}\right) \ ,
\end{aligned}$$

where

$$A(t) = -|p(t)|\frac{\partial^2 c}{\partial y^2}(x(t)) \ , \qquad B(t) = -\frac{p(t)}{|p(t)|}\otimes\frac{\partial c}{\partial y}(x(t)) \ ,$$
$$C(t) = -\frac{c(x(t))}{|p(t)|}\left(\mathrm{Id}_{n\times n} - \frac{p(t)\otimes p(t)}{|p(t)|^2}\right) \ .$$

Following [32, 22], we will now generate initial values for the ODEs, to approximate the initial conditions for the wave equation. We begin by expanding the first phase function $\Phi(y)$ and amplitudes $A_j(y)$ in a Taylor series about a point $z \in K_0$,

$$\begin{aligned}
\Phi(y) &= \sum_{|\beta|=0}^{k+1} \frac{1}{\beta!}\Phi_\beta(z)(y-z)^\beta + \text{error} \\
A_j(y) &= \sum_{|\beta|=0}^{k-2j-1} \frac{1}{\beta!}A_{j,\beta}(z)(y-z)^\beta + \text{error} \ .
\end{aligned}$$

Then, we set the Gaussian beam parameters (note the sign of $\tau(0)$),

$$\begin{aligned}
&x(0;z) = z \ , && p(0;z) = \nabla_y\Phi(z) \ , \\
&\tau(0;z) = c(z)|p(0;z)| \ , && M(0;z) = \partial_y^2\Phi(z) + i\ \mathrm{Id}_{n\times n} \ , \\
&\phi_\beta(0;z) = \Phi_\beta(z) \quad |\beta| = 0 \ , \ \ |\beta| = 3,\dots,k+1 \ , && a_{j,\beta}(0;z) = A_{j,\beta}(z) \ .
\end{aligned}$$

Using this set of initial parameters, we generate a Gaussian beam solution $v_k^{\Phi^+}(t,y;z)$, where we have emphasized that this Gaussian beam is associated with the initial phase, Φ, and the "+" sign for $\tau(0)$. Noting that $v_k^{\Phi^+}(-t,y;z)$ will also be a Gaussian beam for the wave equation, we form the first half of the Gaussian beam superposition:

$$u_k^{<1>}(t,y) = \left(\frac{1}{2\pi\varepsilon}\right)^{\frac{n}{2}} \int_{K_0} \frac{1}{2}\left(v_k^{\Phi^+}(t,y;z) + v_k^{\Phi^+}(-t,y;z)\right) dz \ .$$

At $t = 0$,

$$u_k^{<1>}(0,y) = \left(\frac{1}{2\pi\varepsilon}\right)^{\frac{n}{2}} \int_{K_0} v_k^{\Phi^+}(0,y;z)dz \quad \text{and} \quad (\partial_t u_k^{<1>})(0,y) = 0 ,$$

and we note that,

$$v_k^{\Phi^+}(0,y;z) = \sum_{j=0}^{\lceil k/2\rceil -1} \varepsilon^j \rho_\eta(y-z) a_j(0,y-z;z) e^{i\phi(0,y-z;z)/\varepsilon} ,$$

with a_j the Taylor expansion of A_j up to order $k-2j-1$ and ϕ the Taylor expansion of $\Phi + i|y-z|^2/2$ up to order $k+1$. In terms of the previous section, we have generated two Gaussian beam superposition solutions, for $\tau_0 = +c(z)|\xi|$ and $\tau_1 = -c(z)|\xi|$ with initial amplitude coefficients satisfying the linear 2×2 systems given in Section 2.3, but we have simplified the calculations by noting that $v^{\Phi^-}(t,y;z) = v^{\Phi^+}(-t,y;z)$.

In a similar fashion, we generate the Gaussian beam solutions $v_k^{\Psi^+}(t,y;z)$ using the Taylor expansions of Ψ and B_j, with initial parameters for the Gaussian beam

$$\begin{aligned}
x(0;z) &= z , & p(0;z) &= \nabla_y \Psi(z) ,\\
\tau(0;z) &= c(z)|p(0;z)| , & M(0;z) &= \partial_y^2 \Psi(z) + i\,\mathrm{Id}_{n\times n} ,\\
\phi_\beta(0;z) &= \Psi_\beta(z) \quad |\beta| = 0 ,\ |\beta| \geq 3 , && \\
a_{0,\beta}(0;z) &= \partial_y^\beta \left(\frac{B_0(y)}{i\tau(0;z)}\right)\Bigg|_{y=z} , && \\
a_{j,\beta}(0;z) &= \partial_y^\beta \left(\frac{B_j(y) - (\partial_t a_{j-1})(0,y)}{i\tau(0;z)}\right)\Bigg|_{y=z} , && \text{for} \quad j > 0 .
\end{aligned}$$

Note that the above initial conditions can be computed since all of the equations are evaluated sequentially and $\tau(0;z)$ never vanishes. Now, forming the second part of the superposition,

$$u_k^{<2>}(t,y) = \left(\frac{1}{2\pi\varepsilon}\right)^{\frac{n}{2}} \int_{K_0} \frac{1}{2}\left(v_k^{\Psi^+}(t,y;z) - v_k^{\Psi^+}(-t,y;z)\right) dz ,$$

we have that

$$u_k^{<2>}(0,y) = 0 \quad \text{and} \quad (\partial_t u_k^{<2>})(0,y) = \left(\frac{1}{2\pi\varepsilon}\right)^{\frac{n}{2}} \int_{K_0} \partial_t v_k^{\Psi^+}(0,y;z)dz ,$$

and we note that,

$$\partial_t v_k^{\Psi^+}(0,y;z) = \varepsilon^{-1} \sum_{j=0}^{\lceil k/2\rceil -1} \varepsilon^j \rho_\eta(y-z) b_j(0,y-z;z) e^{i\phi(0,y-z;z)/\varepsilon}$$

with b_j the Taylor expansion of B_j up to order $k-2j-1$ and ϕ the Taylor expansion of $\Psi + i|y-z|^2/2$ up to order $k+1$. As for $u^{<1>}$, we have generated two Gaussian beam superposition solutions, for $\tau_0 = +c(z)|\xi|$ and $\tau_1 = -c(z)|\xi|$, with initial amplitude coefficients satisfy the linear 2×2 systems given in Section 2.3. Thus, the full Gaussian beam superposition solution is given by

$$u_k(t,y) = u_k^{<1>}(t,y) + u_k^{<2>}(t,y) .$$

2.5. The Schrödinger Equation. The construction of Gaussian beams for hyperbolic equations can be extended to the Schrödinger equation by replacing the operator P with a semiclassical operator P^ε. Then, we can similarly construct asymptotic solutions to $P^\varepsilon u = 0$ as $\varepsilon \to 0$. In this section, we review the construction presented in [23] for the Schrödinger equation (1.4) with a smooth external potential $V(y)$. Note that the small parameter ε represents the fast space and time scale introduced in the equation, as well as the typical wavelength of oscillations of the initial data.

We recall that the k-th order Gaussian beam solutions are of the form

$$v_k(t,y;z) = \rho_\eta(y - x(t;z)) \sum_{j=0}^{\lceil k/2\rceil -1} \varepsilon^j a_j(t, y - x(t;z); z) e^{i\phi(t,y-x(t;z);z)/\varepsilon},$$

where the phase and amplitudes are given in (2.10) and (2.11) and ρ_η is the cutoff function (2.8) with η chosen to ensure $\Im(\phi(t, y - x(t;z); z)) > \delta|y - x(t;z)|^2$ on the support of ρ_η for $t \in [0,T]$. Note that for first order beams the cutoff is unnecessary, in which case we take $\eta = \infty$ and $\rho_\infty \equiv 1$. Furthermore, for the Schrödinger equation there is only one choice for $\tau(0;z)$, and therefore the subindex "ℓ" has been suppressed.

For the Schrödinger equation the bicharacteristics $(x(t;z), p(t;z))$ satisfy the the following Hamiltonian system:

$$\begin{aligned} \dot{x} &= p \ , & x(0;z) &= z \ , \\ \dot{p} &= -\nabla_y V, & p(0;z) &= \nabla_y \Phi(z) \ . \end{aligned}$$

The equations for these phase and amplitude Taylor coefficients are derived recursively, starting with the phase and then progressing through the amplitudes. At each stage (phase function, leading amplitude, next amplitude, etc.), we have to derive the Taylor coefficients up to sufficiently high order before passing to the next function in the expansion.

The equations for phase coefficients along the bicharacteristic curve are given by

$$\begin{aligned} \dot{\phi}_0 &= \frac{|p|^2}{2} - V(x(s)) \ , \\ \dot{M} &= -M^2 - \partial_y^2 V(x(s)) \ , \\ \dot{\phi}_\beta &= -\sum_{|\gamma|=2}^{|\beta|} \frac{(\beta-1)!}{(\gamma-1)!(\beta-\gamma)!}\phi_\gamma \phi_{\beta-\gamma+2} - \partial_y^\beta V, \quad |\beta| = 3, \dots, k+1 \ . \end{aligned}$$

The amplitude coefficients are obtained by recursively solving transport equations for $a_{j,\beta}$ with $|\beta| \le k - 2j - 1$, starting from

$$\dot{a}_{0,0} = -\frac{1}{2} a_{0,0} \mathrm{Tr}(M(t;z)).$$

These equations when equipped with the following initial data

$$\begin{aligned} \phi_0(0;z) &= \Phi_0(z) \ , & M(0;z) &= \partial_y^2 \Phi(z) + i \ \mathrm{Id}_{n\times n} \ , \\ \phi_\beta(0;z) &= \Phi_\beta(z), \quad |\beta| = 3, \dots, k+1 \ , & a_{j,\beta}(0;z) &= A_{j,\beta}(z) \ , \end{aligned}$$

have global in time solution, thus $v_k(t,y;z)$ is well-defined for all $0 \le t \le T$.

The k-th order Gaussian beam superposition is formed as

$$u_k(t,y) = \left(\frac{1}{2\pi\varepsilon}\right)^{\frac{n}{2}} \int_{K_0} v_k(t,y;z)dz \ .$$

As in the case of strictly hyperbolic PDEs, we will assume that the parameter η is chosen as $\eta = \infty$ for the first order superposition u_1 and that for higher order superpositions it is taken small enough (and independent of z) to make $\Im\phi(t, y - x(t;z);z) > \delta|y - x(t;z)|^2$ for $t \in [0,T]$ and $y \in \mathrm{supp}\{\rho_\eta(y - x(t;z))\}$. Again, Lemma 4.3 ensures that this can be done. Furthermore, we note that Remark 2.2 concerning the initial choice of the imaginary part of $M(0;z)$ also applies to the superposition for the Schrödinger equation.

3. Previous Results. In this section, we review some of the known error estimates for Gaussian beams as well as some useful results for the PDEs that we are considering. The corner stone of our error estimates are the wellposedness estimates for each PDE. Since they are crucial to our analysis we summarize them here.

THEOREM 3.1. *The generic wellposedness estimate*

$$\|u(t,\cdot)\|_S \le \|u(0,\cdot)\|_S + C\varepsilon^q \int_0^t \|\Theta[u](\tau,\cdot)\|_{L^2} d\tau \ , \tag{3.1}$$

applies to

- *the m-th order strictly hyperbolic PDE (1.2) with $\Theta = P$, $q = 0$ and $\|\cdot\|_S$ the Sobolev space–time norm,*

$$\sum_{\ell=0}^{m-1} \left\|\partial_t^\ell u(t,\cdot)\right\|_{H^{m-\ell-1}} \ ,$$

  where H^s is the Sobolev s-norm ($H^0 = L^2$),
- *the wave equation (1.3) with $\Theta = \Box$, $q = 1$ and $\|\cdot\|_S$ the ε-scaled energy norm,*

$$||u(t,\cdot)||_E := \left(\frac{\varepsilon^2}{2}\int_{\mathbb{R}^n} \frac{|u_t|^2}{c(y)^2} + |\nabla u|^2 dy\right)^{1/2} \ , \tag{3.2}$$

- *and the Schrödinger equation (1.4) with $\Theta = P^\varepsilon$, $q = -1$ and $\|\cdot\|_S$ the standard L^2 norm.*

Proof. The results for the wave and Schrödinger equation are standard and can be found in most books on PDEs. The result for m-th order equations is a bit more technical to prove and appears in Section 23.2 of [13] (Lemma 23.2.1). □

REMARK 3.1. *Since the wave equation is a second order strictly hyperbolic PDE, we have two distinct well-posedness estimates in terms of two different norms. Furthermore, we note that $\|\cdot\|_E$ is only a norm over the class of functions that tend to zero at infinity, which we are considering here.*

When Theorem 3.1 is applied to the difference between the Gaussian beam superposition, u_k, and the true solution, u, for any one of the PDEs that we are considering, we obtain the following estimate for $t \in [0,T]$,

$$\|u_k(t,\cdot) - u(t,\cdot)\|_S \le \|u_k(0,\cdot) - u(0,\cdot)\|_S + C\varepsilon^q \int_0^t \|\Theta[u_k](\tau,\cdot)\|_{L^2} d\tau \ , \tag{3.3}$$

with the appropriate choices for Θ, q and $\|\cdot\|_S$. We will refer to the first term on the right hand side as the error in approximating the initial data or the initial data error and to the second term as the evolution error.

Looking first at the evolution error, the most basic error estimate that we can have is how well single Gaussian beams satisfy the PDE, which was proved in [30]:

THEOREM 3.2 (Ralston '82, [30]). *The Gaussian beams $v_k(t, y)$ constructed in Section 2.1 for the strictly hyperbolic linear m-th order PDO, P, satisfy*

$$\sup_{t\in[0,T]} ||Pv_k(t,\cdot)||_{H^s} \leq C(T)\varepsilon^{-m-s+k/2+n/4+1} .$$

Thus, if we compare a single Gaussian beam to a solution with initial data that is exactly the same as the Gaussian beam at $t = 0$, we will indeed have an asymptotically valid solution of $Pu = 0$, provided that the beam order k is sufficiently high. Unfortunately, if this estimate is applied directly to the Gaussian beam superposition and the errors from each beam are simply added together, the estimate will not be sharp and will only guarantee asymptotic convergence for high order Gaussian beams even for the wave equation [32].

To obtain better error estimates, we have to do a careful analysis of the evolution error. In [22, 23], the construction of beam superpositions by level set methods was presented for both the acoustic wave equation and the Schrödinger equation, respectively. The construction is based on Gaussian beams in physical space, with superpositions carried over sub-domains moving with the Hamiltonian flow. For these equations the authors prove that the right hand side of the PDE, $\Theta[u_k]$ can be expressed in the form,

$$\Theta[u_k] = \varepsilon^{k/2-q}\sum_{j=1}^{J}\varepsilon^{r_j}(\mathcal{Q}_{\alpha_j,g_j,\eta}f_j)(t,y) + \mathcal{O}(\varepsilon^{\infty}) , \tag{3.4}$$

where $r_j \geq 0$, $f_j \in L^2(K_0)$ and $\mathcal{Q}_{\alpha_j,g_j,\eta} : L^2(K_0) \mapsto L^\infty([0,T]; L^2(\mathbb{R}^n))$ belongs to a class of oscillatory integral operators defined in Section 4, equation (4.1). They furthermore make a direct estimate of the underlying oscillatory integrals and are able to prove the following norm bound on the $\mathcal{Q}_{\alpha_j,g_j,\eta}$ operators:

THEOREM 3.3 (Liu/Ralston '09, '10, [22, 23]). *The operators $\mathcal{Q}_{\alpha,g,\eta}$ satisfy the following operator norm bound:*

$$\sup_{t\in[0,T]} ||\mathcal{Q}_{\alpha,g,\eta}||_{L^2} \leq C(T)\varepsilon^{-\gamma} .$$

with $\gamma = (n-1)/4$ for the acoustic wave equation and $\gamma = n/4$ for the Schrödinger equation.

Here γ represents the possible damage done by caustics in the solution, and it depends on the number of space dimensions. This can happen since the projected Hamiltonian flow to physical space is not necessarily regular. The exponent $1/4$ in the case of the wave equation is a gain from the fact that the level set of the phase represents the wave front, see [22] for details.

Before we can obtain a convergence result for the Gaussian beam superposition, we also have to consider the initial data error, which can be accomplished with the following approximation theorem proved in [32].

THEOREM 3.4 (Tanushev '08, [32]). *Let $\Phi \in C^\infty(\mathbb{R}^n)$ be a real-valued function and $A_0 \in C_0^\infty(\mathbb{R}^n)$. Define*

$$u(y) = A_0(y)e^{i\Phi(y)/\varepsilon} ,$$
$$u_k(y) = \left(\frac{1}{2\pi\varepsilon}\right)^{\frac{n}{2}}\int_{\mathbb{R}^n}\rho_\eta(y-z)T^z_{k-1}[A_0](y)e^{i(T^z_{k+1}[\Phi](y)-|y-z|^2/2)/\varepsilon}dz ,$$

where $T_j^z[f](y)$ denotes the j-th order Taylor polynomial of f about z as a function of y. Then for some constant C,

$$\|u_k - u\|_{L^2} \leq C\varepsilon^{k/2} .$$

Thus, if the Gaussian beams are initialized as in Section 2.4 or Section 2.5 and we apply Theorem 3.4 to the terms in $\nabla(u_k - u)$ and $\partial_t(u_k - u)$ or to the terms in $(u_k - u)$, we have for the wave equation or the Schrödinger equation, respectively,

$$\|u_k(0,\cdot) - u(0,\cdot)\|_E \leq C\varepsilon^{k/2} \qquad \text{or} \qquad \|u_k(0,\cdot) - u(0,\cdot)\|_{L^2} \leq C\varepsilon^{k/2} .$$

Now, combining the Gaussian beam error estimate (3.3) with the evolution error estimate in Theorem 3.3 and the initial data error estimate above, gives the following results proved in [22, 23]:

Theorem 3.5 (Liu/Ralston '09, '10, [22, 23]). *The error in u_k, the k-th order Gaussian beam superposition solution compared to the exact solution u can be estimated as*

$$||u(t,\cdot) - u_k(t,\cdot)||_S \leq C(T)\varepsilon^{\frac{k}{2}-\gamma} ,$$

where $S = E$, $\gamma = (n-1)/4$ for the wave equation and $S = L^2$, $\gamma = n/4$ for the Schrödinger equation.

We also mention the result in [5] where the beam superposition is taken over the whole phase space, i.e. integration is over both z and p in (1.5). In this setting, the FBI transform is used to represent the initial data as a superposition of Gaussian beams. Since the FBI transform is defined for any L^2 function, the approach has the advantage of treating general high frequency initial data that is not necessarily in the form of $Ae^{i\Phi/\varepsilon}$, as we have assumed here in (1.2), (1.3) and (1.4). However, the superposition over the full phase space has the drawback, especially for numerical applications, that the integrals are over domains in $\mathbb{R}^{2n}$, rather than just $\mathbb{R}^n$ as in the case of superpositions over physical space. The authors of [5] consider the initial-boundary value problem for the wave equation. They obtain the same type of representation for $\Box u$ as in (3.4) but with a different type of oscillatory integral operators $\mathcal{Q}_{\alpha_j,g_j,\eta}$. For their operators, the authors prove Theorem 3.3 with $\gamma = 0$ and then in the same way as above they prove Theorem 3.5 with $\gamma = 0$ for the wave equation. Since the Hamiltonian flow is regular in full phase there are no caustics and hence, dimensionally independent estimate can be expected.

Another related result was proved in [27] for the Helmholtz equation. The authors divide the Gaussian beam superposition error into a Taylor expansion part (truncation of the series in (2.7)) and a high frequency approximation part (truncation of the series in (2.5)). They analyze the Taylor expansion part and show that, away from caustics, this error is pointwise of order $\mathcal{O}(\varepsilon^{\lceil k/2 \rceil})$, hence, half an order better than Theorem 3.5 with $\gamma = 0$ when k is odd.

In this paper we note that the steps followed earlier in [22, 23, 5] are quite general and can be applied to any strictly hyperbolic equation as well as any linear dispersive wave equation as long as it is semi-classically rescaled. We show that the form $\Theta[u_k]$ as a sum of terms involving $\mathcal{Q}_{\alpha,g,\eta}$ (as in equation (3.4)) is the same for all of these PDEs and therefore, using the general well-posedness estimate, we will obtain an error estimate for the Gaussian beam superposition. These results are contained within Lemma 4.3 and Lemma 4.4. Furthermore, we show in Theorem 4.6 that the difference between the Gaussian beam superposition and the initial data can be approximated in any Sobolev norm, thereby extending Theorem 3.4 so that it can be used with more general well-posedness estimates. Furthermore, we improve the norm estimate of $\mathcal{Q}_{\alpha,g,\eta}$ given in Theorem 3.3. Instead of estimating the integral directly, we follow the arguments in [5] to relate the estimate of the oscillatory integral to the operator norm, through the use of a dual operator. An essential ingredient in estimating the operator norm is the non-squeezing lemma (Lemma 4.2), which states that the distance between two physical points is comparable to the distance between their Hamiltonian trajectories measured in phase space, even in the presence of caustics. Equipped with this key lemma, we are able to show in Theorem 4.1 that Theorem 3.3 holds also with $\gamma = 0$, i.e. $||\mathcal{Q}_{\alpha,g,\eta}||_{L^2} \leq C(T)$, where $C(T)$ is independent of ε; this is the same result as in [5] for beam superposition over all of phase space. With these results we obtain improved error estimates in terms of ε for the Gaussian beam superposition that are independent of dimension as given in Theorem 1.1.

4. Error Estimates for Gaussian Beams. In this section we prove the asymptotic convergence results for superpositions of Gaussian beams given in our main result, Theorem 1.1. The key to obtaining this result is precise norm estimates in terms of ε of the oscillatory integral operators $\mathcal{Q}_{\alpha_j,g_j,\eta} : L^2(K_0) \mapsto L^\infty([0,T]; L^2(\mathbb{R}^n))$, defined as follows. For a fixed $t \in [0,T]$, a multi-index α, a compact set $K_0 \subset \mathbb{R}^n$, a cutoff function ρ_η (2.8) with cutoff radius $0 < \eta \leq \infty$ and a function $g(t,y;z)$, we let

$$(\mathcal{Q}_{\alpha,g,\eta} w)(t,y) \tag{4.1}$$
$$:= \varepsilon^{-\frac{n+|\alpha|}{2}} \int_{K_0} w(z) g(t,y;z)(y - x(t;z))^\alpha e^{i\phi(t,y-x(t;z);z)/\varepsilon} \rho_\eta(y - x(t;z)) dz \ ,$$

with the functions $g(t,y;z)$, $\phi(t,y;z)$ and $x(t;z)$ satisfying for all $t \in [0,T]$,

- (A1) $x(t;z) \in C^\infty([0,T] \times K_0)$,
- (A2) $\phi(t,y;z), g(t,y;z) \in C^\infty([0,T] \times \mathbb{R}^n \times K_0)$,
- (A3) $\nabla\phi(t,0;z)$ is real and there is a constant C such that for all $z, z' \in K_0$,

$$|\nabla_y \phi(t,0;z) - \nabla_y \phi(t,0;z')| + |x(t;z) - x(t;z')| \geq C|z - z'| \ ,$$

- (A4) for $|y| \leq 2\eta$ (or for all y if $\eta = \infty$), there exists a constant δ such that for all $z, z' \in K_0$,

$$\Im\phi(t,y;z) \geq \delta |y|^2 \ ,$$

- (A5) for any multi-index β, there exists a constant C_β, such that

$$\sup_{\substack{z\in K_0 \\ y\in\mathbb{R}^n}} \left|\partial_y^\beta g(t,y;z)\right| \leq C_\beta \ .$$

With this definition, the following norm estimates of $\mathcal{Q}_{\alpha,g,\eta}$ will be proved in Section 5.

THEOREM 4.1. *Under the assumptions (A1)–(A5),*

$$\sup_{t\in[0,T]} \|\mathcal{Q}_{\alpha,g,\eta}\|_{L^2} \leq C(T).$$

REMARK 4.1. *The assumption of C^∞ smoothness for all functions is made for simplicity to avoid a too technical discussion about precise regularity requirements. In* *this sense, Theorem 4.1 can be sharpened, since it will be true also for less regular functions.*

REMARK 4.2. *If the condition in assumption (A4) is satisfied for all y there is no need for the cutoff function in the definition of the operator $\mathcal{Q}$ in (4.1). We treat this case by taking $\eta = \infty$ and defining $\rho_\infty \equiv 1$. The operators with $\eta = \infty$ are used in the case of first order Gaussian beams.*

4.1. Gaussian Beam Phase. In this section, we show that the Gaussian beam phase ϕ given in (2.10) is an admissible phase for the operators $\mathcal{Q}_{\alpha,g,\eta}$. We begin with a lemma based on the regularity of the Hamiltonian flow map, S_t, stating that the difference $|z - z'|$ is comparable to the *sum* $|p(t;z) - p(t;z')| + |x(t;z) - x(t;z')|$. Note, however, that because of caustics it is not true that $|z - z'|$ is related in this way to either of the individual terms $|p(t;z) - p(t;z')|$ or $|x(t;z) - x(t;z')|$.

LEMMA 4.2 (Non-squeezing lemma). *Let S_t be the Hamiltonian flow map, $(x(t;z), p(t;z)) = S_t(x(0;z), p(0;z))$, and assume that $p(0;z)$ is Lipschitz continuous in $z \in K_0$ for the flow associated with the Schrödinger operator. Additionally, assume that $\inf_{z\in K_0} |p(0;z)| = \delta > 0$ for the flow associated with the strictly hyperbolic PDO. Under these conditions, there exist positive constants c_1 and c_2 depending on T and δ, such that*

$$c_1|z - z'| \le |p(t;z) - p(t;z')| + |x(t;z) - x(t;z')| \le c_2|z - z'| \ , \tag{4.2}$$

for all $z, z' \in K_0$ and $t \in [0,T]$.

Proof. We prove the result for the flows associated with the two types of operators separately, however, we use the common notation,

$$\begin{aligned} Z &= (z, p_0) = (z, p(0;z)) \ , & Z' &= (z', p_0') = (z', p(0;z')) \ , \\ X &= S_t(Z) = (x, p) = (x(t;z), p(t;z)) \ , & X' &= S_t(Z') = (x', p') = (x(t;z'), p(t;z')) \ . \end{aligned}$$

We begin with the flow associated with the Hamiltonian for the Schrödinger operator. Let us introduce the set $\mathcal{K}_0 = \{(z, p(0;z)) : z \in K_0\}$ and note that S_t is invertible with inverse S_{-t} and regular for all t so that

$$\sup_{t\in[0,T]} \sup_{\tilde{Z}\in\mathrm{conv}(\mathcal{K}_0)} \left\| \frac{\partial S_t(\tilde{Z})}{\partial Z} \right\| \le C \ , \qquad \sup_{t\in[0,T]} \sup_{\tilde{X}\in\mathrm{conv}(S_t(\mathcal{K}_0))} \left\| \frac{\partial S_{-t}(\tilde{X})}{\partial X} \right\| \le C \ ,$$

where $\mathrm{conv}(E)$ denotes the convex hull of the set E. Now, noting that

$$X - X' = \int_0^1 \frac{d}{ds} S_t(sZ + (1-s)Z')ds = \int_0^1 \frac{\partial S_t(sZ + (1-s)Z')}{\partial Z}(Z - Z')ds \ , \tag{4.3}$$

and taking ℓ_1 norms, we have

$$|x - x'| + |p - p'| = ||X - X'||_1 \le C||Z - Z'||_1 = C(|z - z'| + |p_0 - p_0'|) \le C'|z - z'| \ ,$$

where we have used the Lipschitz continuity of p_0 in z. This gives the right half of (4.2). By the equivalent of (4.3) for S_{-t}, we have

$$|z - z'| \le |z - z'| + |p_0 - p_0'| = \|Z - Z'\|_1 \le C\|X - X'\|_1 = C(|x - x'| + |p - p'|) \ ,$$

which completes the proof of the lemma for the Hamiltonian flow associated with the Schrödinger operator.

For the flow associated with the strictly hyperbolic operator, we follow the same idea, but we have to be careful near $|p| = 0$. Thus, in addition to $\mathcal{K}_0$, we introduce the sets

$$B_\delta = \{(z,p) \ : \ z \in K_0, \ |p| < \delta\},$$
$$\tilde{\mathcal{K}}_0 = \mathrm{conv}\left(\mathcal{K}_0 \cup B_\delta\right) .$$

Note that S_t is regular away from $|p_0| = 0$ by Lemma 2.1 as $|p(t)| \neq 0$, for all t so

$$\sup_{t\in[0,T]} \sup_{\tilde{Z}\in\tilde{\mathcal{K}}_0\setminus B_{\delta/2}} \left\| \frac{\partial S_t(\tilde{Z})}{\partial Z} \right\| \leq C.$$

Thus, again by (4.3) we obtain,

$$|x-x'|+|p-p'| = ||X-X'||_1 \leq C||Z-Z'||_1 = C(|z-z'|+|p_0-p_0'|) \leq C'|z-z'| \, , \tag{4.4}$$

provided that $\tilde{p}(s) = (1-s)p_0 + sp_0'$ satisfies $\inf_{0\leq s\leq 1} |\tilde{p}(s)| \geq \delta/2$, which guarantees that $sZ + (1-s)Z' \in \tilde{\mathcal{K}}_0 \setminus B_{\delta/2}$ for $0 \leq s \leq 1$. On the other hand, suppose that $\inf_{0\leq s\leq 1} |\tilde{p}(s)| < \delta/2$. We define p^* as the point on the line connecting p_0 to p_0' with smallest norm and let $s^* = \mathrm{argmin}_{0\leq s\leq 1} |\tilde{p}(s)|$ so that $p^* = \tilde{p}(s^*)$. Then

$$\frac{\delta}{2} > |p^*| = |p_0 - s^*(p_0 - p_0')| \geq |p_0| - s^*|p_0 - p_0'| \geq \delta - |p_0 - p_0'|.$$

Hence, $|p_0 - p_0'| \geq \delta/2$. Now, let

$$d = \sup_{t\in[0,T]} \sup_{Z\in\mathcal{K}_0} ||S_t(Z)||_1 < \infty,$$

so that

$$\|X - X'\|_1 \leq \|X\|_1 + \|X'\|_1 \leq 2d \leq 2d\frac{2}{\delta}|p_0 - p_0'| \leq C\|Z - Z'\|_1,$$

which shows that (4.4) holds for all $Z, \ Z' \in \mathcal{K}_0$. This gives the right half of (4.2). For the left half of (4.2), we note that by Lemma 2.1, $\inf_{t\in[0,T],z\in K_0} |p(t;z)| \geq \tilde{\delta} > 0$, so that we can consider the inverse map S_{-t} in exactly the same way as S_t above and show that $||Z - Z'||_1 \leq C||X - X'||_1$. Thus we obtain the left half of (4.2),

$$|z - z'| \leq |z - z'| + |p_0 - p_0'| = ||Z - Z'||_1 \leq C||X - X'||_1 \leq C(|x - x'| + |p - p'|) \ .$$

Thus, the proof of the lemma is complete. □

We are now ready to show that the phase function for a k-th order Gaussian beam is an admissible phase for the operators $\mathcal{Q}_{\alpha,g,\eta}$.

Lemma 4.3. *The rays $x(t;z)$ satisfy (A1) and the phase for a k-th order Gaussian beam (2.10), satisfies assumptions (A2) through (A4) if η is sufficiently small. In the case of $k = 1$, η can take any value in $(0,\infty]$.*

Proof. The smoothness assumptions (A1) and (A2) follow from smoothness of initial data and smoothness of the coefficients in the underlying PDE. By definition, $\nabla_y \phi(t,0;z) = p(t;z)$ and (A3) follows from the non-squeezing lemma, Lemma 4.2. Finally, since the lower order terms of ϕ are real,

$$\Im\phi(t,y;z) = y \cdot (\Im M(t;z))y + \sum_{|\beta|=3}^{k+1} \frac{1}{\beta!}\Im\phi_\beta(t;z)y^\beta \ .$$

Recalling that by Lemma 2.2, $\Im M(t;z)$ is positive definite, we therefore have that when $|y| \leq 2\eta$,

$$\begin{aligned}
\Im\phi(t,y;z) &\geq C|y|^2 - \sum_{|\beta|=3}^{k+1} \frac{1}{\beta!} \|\Im\phi_\beta(t;z)\|_{L^\infty} |y|^{|\beta|} \\
&\geq C|y|^2 - |y|^2 \sum_{|\beta|=3}^{k+1} \frac{1}{\beta!} \|\Im\phi_\beta(t;z)\|_{L^\infty} (2\eta)^{|\beta|-2} \\
&\geq |y|^2 \left(C - 2\eta \sum_{|\beta|=3}^{k+1} \frac{1}{\beta!} \|\Im\phi_\beta(t;z)\|_{L^\infty} (2\eta)^{|\beta|-3} \right) \\
&\geq \delta(\eta)|y|^2,
\end{aligned}$$

where the constant $\delta(\eta)$ is positive for small enough η and independent of $t \in [0,T]$, since $\phi_\beta(t;z)$ are smooth functions of t. This shows that $\phi(t,y;z)$ satisfies (A4). When $k=1$, there are only quadratic terms in the phase and in this case $\phi(t,y;z)$ will satisfy (A4) for any choice of $\eta \in (0,\infty]$. ☐

4.2. Representations of $P[u_k]$ in Terms of $\mathcal{Q}_{\alpha,g,\eta}$. In this section, we show that several of the intermediate quantities in the proof of Theorem 1.1 can be written as sums involving the operators $\mathcal{Q}_{\alpha,g,\eta}$.

LEMMA 4.4. *The m-th order strictly hyperbolic operator P acting on the Gaussian beam superposition u_k given in Section 2.3 and the semiclassical Schrödinger operator P^ε acting on the superposition given in Section 2.5 can be expressed as a finite sum of the operators $\mathcal{Q}$:*

$$\left.\begin{array}{l} P[u_k] \\ \\ P^\varepsilon[u_k] \end{array}\right\} = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \sum_{j=1}^{J} \varepsilon^{\ell_j} \left(\mathcal{Q}_{\alpha_j,g_j,\eta}\ \chi_{K_0}\right)(t,y) + \mathcal{O}(\varepsilon^\infty), \ \left\{\begin{array}{l} \ell_j \geq k/2 - m + 1 \\ \\ \ell_j \geq k/2 + 1 \end{array}\right. ,$$

with $\chi_{K_0}(z)$ the characteristic function on K_0 and $g_j(t,y;z)$ satisfying (A5).

Proof. For a single Gaussian beam $v_{k,\ell}(t,y;z)$ for the hyperbolic operator, following the discussion in [30] and Section 2.2, we have

$$P[v_{k,\ell}(t,y;z)] = \sum_{r=-m}^{\lceil k/2\rceil -1} \varepsilon^r \rho_\eta(y - x_\ell(t;z)) c_{r,\ell}(t,y;z) e^{i\phi(t,y-x_\ell(t;z);z)/\varepsilon} + E_k(t,y;z)\ ,$$

where E_k contains terms that are multiplied by derivatives of the cutoff function. For first order beams $\eta = \infty$ and $\rho_\infty \equiv 1$ making $E_1(t,y;z) \equiv 0$. For higher order beams, $E_k \equiv 0$ in an η neighborhood of $x_\ell(t;z)$ and since η is small enough so that the imaginary part of ϕ_ℓ is strictly positive for $\eta \leq |y - x_\ell(t;z)| \leq 2\eta$, E_k will decay exponentially as $\varepsilon \to 0$. Thus, $E_k = \mathcal{O}(\varepsilon^\infty)$ for all orders of beams and $t \in [0,T]$.

Each c_r can be expressed in terms of the symbols of P:

$$c_{-m+j,\ell}(t,y;z) = La_{\ell,j-1} + \sigma_m\left(t,y,\partial_t\phi_\ell,\nabla_y\phi_\ell\right) a_{\ell,j} + R_{\ell,j}(t,y;z)\ ,$$

where $a_{\ell,j} \equiv 0$ for $j \notin [0,\lceil k/2\rceil - 1]$ and L is given by (using the Einstein summation),

$$La = -i\left(\frac{\partial\sigma_m}{\partial\tilde{p}_j}(\tilde{y},\tilde{\nabla}\phi_\ell)\frac{\partial a}{\partial\tilde{y}_j}\right) - \left(\frac{i}{2}\frac{\partial^2\sigma_m}{\partial\tilde{p}_j\tilde{p}_q}(\tilde{y},\tilde{\nabla}_y\phi_\ell)\phi_{\ell,\tilde{y}_j\tilde{y}_q} + \sigma_{m-1}(\tilde{y},\tilde{\nabla}\phi_\ell)\right) a\ ,$$

with $\tilde{\nabla} = (\partial_t, \nabla)$, $\tilde{y} = (t, y)$ and $\tilde{p} = (\tau, p)$. The function $R_{\ell,0}(t, y; z) \equiv 0$ and the functions $R_{\ell,j}(t, y; z)$ for $j > 0$ are complicated functions of the $m-2$ and lower order symbols of P and the functions ϕ_ℓ, $a_{\ell,0}, \dots, a_{\ell,j-1}$ and their derivatives. We note that since $a_{\ell,j}(t, y; z)$ are compactly supported in $z \in K_0$, so are the functions $c_{r,\ell}(t, y; z)$.

By the construction of a Gaussian beam, $c_{r,\ell}$ vanishes up to order $k - 2(r + m) + 1$ on $y = x_\ell(t; z)$. Note that $k - 2(r + m) + 1$ may be negative, in which case, $c_{r,\ell}$ is not necessarily 0 on $y = x_\ell(t; z)$. Thus, by Taylor's remainder formula,

$$c_{r,\ell}(t, y; z) = \sum_{|\alpha|=[k-2(r+m)+2]_+} c_{r,\ell,\alpha}(t, y; z)\ (y - x_\ell(t; z))^\alpha\ ,$$

for some coefficient function $c_{r,\ell,\alpha}(t, y; z)$ that are compactly supported in $z \in K_0$ and $[a]_+ = \max[a, 0]$. For the superposition $u_k(t, y)$, we have

$$\begin{aligned} P[u_k] &= \left(\frac{1}{2\pi\varepsilon}\right)^{\frac{n}{2}} \sum_{\ell=0}^{m-1} \int_{K_0} P[v_{k,\ell}(t, y; z)] dz \\ &= \left(\frac{1}{2\pi\varepsilon}\right)^{\frac{n}{2}} \sum_{\ell=0}^{m-1} \left(\sum_{r=-m}^{\lceil k/2\rceil-1} \int_{K_0} \varepsilon^r \rho_\eta(y - x_\ell(t; z)) c_{r,\ell}(t, y; z) e^{i\phi_\ell(t, y - x_\ell(t;z); z)/\varepsilon} dz \right), \end{aligned}$$

modulo additive terms that are $\mathcal{O}(\varepsilon^\infty)$. Now substituting for $c_{r,\ell}$ and with the help of Lemma 4.3, we can rewrite this expression in terms of the operators $\mathcal{Q}$,

$$\begin{aligned} P[u_k] = \left(\frac{1}{2\pi\varepsilon}\right)^{\frac{n}{2}} \sum_{\ell=0}^{m-1} &\left(\sum_{r=-m}^{\lceil k/2\rceil-1} \left(\sum_{|\alpha|=[k-2(r+m)+2]_+} \int_{K_0} \varepsilon^r \rho_\eta(y - x_\ell(t; z)) \right.\right. \\ &\left.\left. \times\, c_{r,\ell,\alpha}(t, y; z)(y - x_\ell(t; z))^\alpha e^{i\phi_\ell(t, y - x_\ell(t;z); z)/\varepsilon} dz \right)\right) \\ = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \sum_{\ell=0}^{m-1} &\left(\sum_{r=-m}^{\lceil k/2\rceil-1} \left(\sum_{|\alpha|=[k-2(r+m)+2]_+} \varepsilon^{r+|\alpha|/2} \left(\mathcal{Q}_{\alpha, c_{r,\ell,\alpha}, \eta}\ \chi_{K_0}\right)(t, y) \right)\right), \end{aligned}$$

modulo additive terms that are $\mathcal{O}(\varepsilon^\infty)$ and where $\chi_{K_0}(z)$ is the characteristic function on K_0 and the functions $c_{r,\ell,\alpha}(t, y; z)$ satisfy condition (A5) since the coefficients of P, $\phi_{\ell,\beta}$ and $a_{\ell,j,\beta}$ are all smooth.

To simplify the notation, let $j = 1, \dots, J < \infty$ enumerate all of the combination of ℓ, r, and α in the triple sum above and rewrite the sums as

$$P[u_k] = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \sum_{j=1}^{J} \varepsilon^{\ell_j} \left(\mathcal{Q}_{\alpha_j, g_j, \eta}\ \chi_{K_0}\right)(t, y) + \mathcal{O}(\varepsilon^\infty)\ ,$$

with $\ell_j \geq (k/2 - m + 1)$ and $g_j = c_{r_j,\ell_j,\alpha_j}$. Thus, we have the desired result for P.

Now, for each Gaussian beam $v_k(t, y; z)$ for the Schrödinger equation defined in Section 2.5, following [23], we compute

$$P^\varepsilon[v_k] = \rho_\eta P^\varepsilon \left[\sum_{j=0}^{\lceil k/2\rceil-1} \varepsilon^j a_j e^{i\phi/\varepsilon} \right] + E_k(t, y; z)$$

where again $E_k = \mathcal{O}(\varepsilon^\infty)$. Note that

$$e^{-i\phi/\varepsilon} P^\varepsilon[a e^{i\phi/\varepsilon}] = aG(t,y) - i\varepsilon La - \frac{\varepsilon^2}{2}\triangle a \ ,$$

with

$$G = \partial_t \phi + \frac{1}{2}|\nabla_y \phi|^2 + V(y) \qquad \text{and} \qquad L = \partial_t + \nabla_y \phi \cdot \nabla_y + \frac{1}{2}\triangle_y \phi \ .$$

Thus, we have

$$P^\varepsilon\left[\sum_{j=0}^{\lceil k/2\rceil -1} \varepsilon^j a_j e^{i\phi/\varepsilon}\right] = \sum_{j=0}^{\lceil k/2\rceil -1} \varepsilon^j \left[a_j G - i\varepsilon L a_j - \frac{\varepsilon}{2}\triangle_y a_j\right] e^{i\phi/\varepsilon} = \sum_{r=0}^{\lceil k/2\rceil +1} \varepsilon^r d_r e^{i\phi/\varepsilon} \ ,$$

where for convenience $a_j \equiv 0$ for $j \notin [0, \lceil k/2\rceil - 1]$ and

$$d_r = a_r G - iLa_{r-1} - \frac{1}{2}\triangle_y a_{r-2} \ , \qquad r = 0, \dots, \lceil k/2\rceil + 1 \ .$$

By construction of the phase coefficients in Section 2.5, we have that on $y = x(t;z)$, G vanishes to order $k+1$ and by construction of the amplitude coefficients, the quantity $-iLa_{r-1} - \frac{1}{2}\triangle_y a_{r-2}$ vanishes to order $k-2r+1$. Thus, d_r vanishes to order $k-2r+1$ on $y = x(t;z)$, where again we remark that if $k-2r+1$ is negative, d_r does not vanish on $y = x(t;z)$. By Taylor's remainder formula, we have

$$d_r(t,y;z) = \sum_{|\alpha| = [k+2-2r]_+} d_{r,\alpha}(t,y;z)\ (y - x(t;z))^\alpha \ ,$$

with $d_{r,\alpha}(t,y;z)$ compactly supported in $z \in K_0$. Hence,

$$P^\varepsilon[v_k] = \rho_\eta \sum_{r=0}^{\lceil k/2\rceil +1} \varepsilon^r \left(\sum_{|\alpha| = [k+2-2r]_+} d_{r,\alpha}(y - x(t;z))^\alpha\right) e^{i\phi/\varepsilon} + \mathcal{O}(\varepsilon^\infty) \ .$$

Using Lemma 4.3, for the superposition, we have

$$\begin{aligned} P^\varepsilon[u_k] &= \left(\frac{1}{2\pi\varepsilon}\right)^{\frac{n}{2}} \int_{K_0} P^\varepsilon[v_k] dz \\ &= \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \sum_{r=0}^{\lceil k/2\rceil -1} \sum_{|\alpha| = [k+2-2r]_+} \varepsilon^{r+|\alpha|/2} (\mathcal{Q}_{\alpha, d_{r,\alpha}, \eta} \chi_{K_0})(t,y) + \mathcal{O}(\varepsilon^\infty) \ . \end{aligned}$$

Again, we let $j = 1, \dots, J < \infty$ enumerate all of the combination of r and α in the double sum above. Thus,

$$P^\varepsilon[u_k] = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \sum_{j=1}^{J} \varepsilon^{\ell_j} \left(\mathcal{Q}_{\alpha_j, g_j, \eta}\ \chi_{K_0}\right)(t,y) + \mathcal{O}(\varepsilon^\infty) \ ,$$

with $\ell_j \geq (k/2+1)$ and $g_j = d_{r_j, \alpha_j}$. Thus, the proof is complete. □

4.3. Initial Data Errors. In addition to establishing the role of the $\mathcal{Q}$-operators and estimating their norms, we need to consider the convergence of the Gaussian beam superposition to the initial data to be able to prove Theorem 1.1. We start with the following lemma.

LEMMA 4.5. *Let $\Phi \in C^\infty(\mathbb{R}^n)$ be a real-valued function and $A_j \in C_0^\infty(\mathbb{R}^n)$. With $k \geq 1$, define*

$$u(y) = \sum_{j=0}^{N} \varepsilon^j A_j(y) e^{i\Phi(y)/\varepsilon},$$
$$u_k(y) = \left(\frac{1}{2\pi\varepsilon}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \rho_\eta(y-z) \sum_{j=0}^{N} \varepsilon^j a_j(y-z;z) e^{i\phi(y-z;z)/\varepsilon - |y-z|^2/2\varepsilon} dz \ ,$$

where $\phi(y-z;z)$ is the $k+1$ order Taylor series of Φ about z, $a_j(y-z;z)$ is the same as the Taylor series of A_j about z up to order $k-2j-1$ (but may differ in higher order terms) and ρ_η is the cutoff function (2.8) with $0 < \eta \leq \infty$. Then for some constant C_s,

$$\|u_k - u\|_{H^s} \leq C_s \varepsilon^{\frac{k}{2}-s} \ .$$

Proof. The proof of this lemma is based on a discussion in [32]. We first assume that $\eta < \infty$. Looking at each of the terms in the sum above separately, the estimate depends on how well

$$\varepsilon^j A_j(y) e^{i\Phi(y)/\varepsilon} = \left(\frac{1}{2\pi\varepsilon}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \varepsilon^j A_j(y) e^{i\Phi(y)/\varepsilon - |y-z|^2/2\varepsilon} dz$$

is approximated by

$$\left(\frac{1}{2\pi\varepsilon}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \varepsilon^j \rho_\eta(y-z) a_j(y-z;z) e^{i\phi(y-z;z)/\varepsilon - |y-z|^2/2\varepsilon} dz \ .$$

Since we want to estimate the difference between these two functions in a Sobolev norm, we differentiate each of the above expression in y:

$$\begin{aligned}
&\partial_y^\beta \left[\left(\frac{1}{2\pi\varepsilon}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \varepsilon^j A_j(y) e^{i\Phi(y)/\varepsilon - |y-z|^2/2\varepsilon} dz \right] \\
&= \left(\frac{1}{2\pi\varepsilon}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \sum_{\delta_1+\gamma=\beta} \varepsilon^j C_{\delta_1,\gamma}\, \partial_y^{\delta_1} [A_j]\, \partial_y^{\gamma} \left[e^{i\Phi/\varepsilon - |y-z|^2/2\varepsilon} \right] dz \\
&= \left(\frac{1}{2\pi\varepsilon}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \sum_{\delta_1+\gamma=\beta} C_{\delta_1,\gamma}\, \partial_y^{\delta_1} [A_j] \sum_{\ell=1}^{|\gamma|} \sum_{\substack{|\gamma_1|,\ldots,|\gamma_\ell|\geq 1 \\ \gamma_1+\ldots+\gamma_\ell=\gamma}} C_\ell^{\{\gamma_\ell\}} \varepsilon^{j-\ell} \\
&\qquad\qquad \left(\prod_{m=1}^{\ell} \partial_y^{\gamma_m} \left[i\Phi - |y-z|^2/2 \right] \right) e^{i\Phi/\varepsilon - |y-z|^2/2\varepsilon} dz \ ,
\end{aligned}$$

and

$$\partial_y^\beta \left[\left(\frac{1}{2\pi\varepsilon}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \varepsilon^j \rho_\eta(y-z) a_j(y-z;z) e^{i\phi(y-z;z)/\varepsilon - |y-z|^2/2\varepsilon} dz \right]$$

$$= \left(\frac{1}{2\pi\varepsilon}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \sum_{\delta_1+\delta_2+\gamma=\beta} \varepsilon^j C_{\delta_1,\delta_2,\gamma}\, \partial_y^{\delta_2}[\rho_\eta]\, \partial_y^{\delta_1}[a_j]\, \partial_y^\gamma \left[e^{i\phi/\varepsilon - |y-z|^2/2\varepsilon}\right] dz$$

$$= \left(\frac{1}{2\pi\varepsilon}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \rho_\eta \sum_{\delta_1+\gamma=\beta} C_{\delta_1,\gamma}\, \partial_y^{\delta_1}[a_j] \sum_{\ell=1}^{|\gamma|} \sum_{\substack{|\gamma_1|,\dots,|\gamma_\ell|\ge 1\\ \gamma_1+\dots+\gamma_\ell=\gamma}} C_\ell^{\{\gamma_\ell\}} \varepsilon^{j-\ell}$$

$$\left(\prod_{m=1}^{\ell} \partial_y^{\gamma_m}\left[i\phi - |y-z|^2/2\right]\right) e^{i\phi/\varepsilon - |y-z|^2/2\varepsilon} dz$$

$$+ \left(\frac{1}{2\pi\varepsilon}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \sum_{\substack{\delta_1+\delta_2+\gamma=\beta\\ |\delta_2|\ge 1}} \varepsilon^j C_{\delta_1,\delta_2,\gamma}\, \partial_y^{\delta_2}[\rho_\eta]\, \partial_y^{\delta_1}[a_j]\, \partial_y^\gamma \left[e^{i\phi/\varepsilon - |y-z|^2/2\varepsilon}\right] dz\ ,$$

where $C_{\delta_1,\gamma}$ and $C_{\delta_1,\delta_2,\gamma}$ are combinatorial coefficients. We have used the formula, for $|\gamma| \ge 1$:

$$\partial_y^\gamma \left[e^{f/\varepsilon}\right] = \sum_{\ell=1}^{|\gamma|} \sum_{\substack{|\gamma_1|,\dots,|\gamma_\ell|\ge 1\\ \gamma_1+\dots+\gamma_\ell=\gamma}} C_\ell^{\{\gamma_\ell\}} \varepsilon^{-\ell} \left(\prod_{m=1}^{\ell} \partial_y^{\gamma_m}[f]\right) e^{f/\varepsilon}$$

with $C_\ell^{\{\gamma_\ell\}}$ some combinatorial coefficients depending on $\ell,\ \gamma_1,\ \dots,\ \gamma_\ell$, and for $|\gamma| = 0$, we define

$$\sum_{\ell=1}^{|\gamma|} \sum_{\substack{|\gamma_1|,\dots,|\gamma_\ell|\ge 1\\ \gamma_1+\dots+\gamma_\ell=\gamma}} C_\ell^{\{\gamma_\ell\}} \varepsilon^{-\ell} \left(\prod_{m=1}^{\ell} \partial_y^{\gamma_m}[f]\right) e^{f/\varepsilon} \equiv e^{f/\varepsilon}\ .$$

This formula can be proved by induction, but we omit the proof for brevity.

Since terms involving derivatives of the cutoff function $\rho_\eta(y-z)$ vanish in a neighborhood of $z = y$ and the integrand is compactly supported in y and z, they will be $\mathcal{O}(\varepsilon^\infty)$ in the L^2 norm and do not contribute to the estimate. For the other terms, we use the properties of Taylor series to obtain the desired estimate. Recalling that $i\phi(y-z;z) - |y-z|^2/2$ is the $k+1$ order Taylor series of $i\Phi(y) - |y-z|^2/2$ about z and $a_j(y-z;z)$ agrees with the Taylor series of $A_j(y)$ about z up to order $k-2j-1$,

$$C_\ell^{\{\gamma_\ell\}} \partial_y^{\delta_1}[a_j(y-z;z)] \left(\prod_{m=1}^{\ell} \partial_y^{\gamma_m}\left[i\phi(y-z;z) - |y-z|^2/2\right]\right)$$

will agree with the Taylor series of

$$C_\ell^{\{\gamma_\ell\}} \partial_y^{\delta_1}[A_j(y)] \left(\prod_{m=1}^{\ell} \partial_y^{\gamma_m}\left[i\Phi(y) - |y-z|^2/2\right]\right)\ ,$$

up to order

$$\min\left[k-2j-1-|\delta_1|, k+1-\max_{1\le m\le \ell}|\gamma_m|\right] \ge k-2j-1-|\delta_1|-|\gamma|+\ell$$
$$= k-2j-1-|\beta|+\ell\ .$$

We have used the fact that $\max_m[\gamma_m] \leq |\gamma|-\ell+1$. Thus, we have that $\|\partial_y^\beta(u_k-u)\|_{L^2}$ can be estimated by a sum of terms of the form

$$\varepsilon^{j-\ell}\left\| \left(\frac{1}{2\pi\varepsilon}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} \Big[B_\ell(y)e^{i\Phi(y)/\varepsilon-|y-z|^2/2\varepsilon} - \rho_\eta(y-z)b_\ell(y-z;z)e^{i\phi(y-z;z)/\varepsilon-|y-z|^2/2\varepsilon}\Big]dz\right\|_{L^2}, \qquad (4.5)$$

where $\ell \leq |\beta|$, $|B_\ell - b_\ell| = \mathcal{O}(|y-z|^{k-2j-|\beta|+\ell})$ and $|\Phi-\phi| = \mathcal{O}(|y-z|^{k+2})$. Now, the proof of Theorem 2.1 in [32] can be applied directly to (4.5) to obtain the estimate,

$$\|\partial_y^\beta(u_k-u)\|_{L^2} \leq \sum_{j=0}^N \varepsilon^{j-\ell}C\varepsilon^{\frac{k}{2}-j-\frac{|\beta|}{2}+\frac{\ell}{2}} \leq C\varepsilon^{\frac{k}{2}-|\beta|} .$$

Thus, we have the result for $\eta<\infty$. The extension for $\rho_\infty \equiv 1$ follows directly, since the cutoff ρ_ν for $\nu<\infty$ introduces $\mathcal{O}(\varepsilon^\infty)$ errors in the L^2 norm:

$$\left\|\left(\frac{1}{2\pi\varepsilon}\right)^{\frac{n}{2}}\int_{\mathbb{R}^n}\left[1-\rho_\eta(y-z)\right]b_\ell(y-z;z)e^{i\phi(y-z;z)/\varepsilon-|y-z|^2/2\varepsilon}dz\right\|_{L^2} = \mathcal{O}(\varepsilon^\infty) ,$$

as $1-\rho_\nu$ vanishes in a neighborhood $z=y$ and the integrand is compactly supported in z. ◻

Using Lemma 4.5, we can estimate the asymptotic convergence rate of the superposition solution to the initial data.

Theorem 4.6. *For the Gaussian beam superposition, u_k, given in Section 2.3 and the solution, u, to the strictly hyperbolic PDE (1.2), we have*

$$\left\|\partial_t^\ell u_k(0,\cdot) - \partial_t^\ell u(0,\cdot)\right\|_{H^s} \leq C_{\ell,s}\varepsilon^{\frac{k}{2}-\ell-s} ,$$

for some constant $C_{\ell,s}$ and $0\leq \ell \leq m-1$.

Similarly, for the Gaussian beam superposition, u_k, given in Section 2.4 and the solution, u, to the wave equation (1.3), we have

$$\|u_k(0,\cdot)-u(0,\cdot)\|_E \leq C\varepsilon^{\frac{k}{2}} .$$

Furthermore, for the superposition, u_k, given in Section 2.5 and the solution, u, to the Schrödinger equation (1.4), we have

$$\|u_k(0,\cdot)-u(0,\cdot)\|_{L^2} \leq C\varepsilon^{\frac{k}{2}} .$$

Proof. The proof of this theorem for hyperbolic PDEs follows directly from Lemma 4.5, since for each power of ε, $\partial_t^\ell u_k(0,y)$ and $\partial_t^\ell u(0,y)$ given in (2.12) and (1.2), respectively, are exactly in the assumed form in the Lemma 4.5.

The result for the wave equation follows after noting that

$$\|u_k(0,\cdot)-u(0,\cdot)\|_E \leq C\varepsilon\left(\|u_k(0,\cdot)-u(0,\cdot)\|_{H^1} + \|\partial_t u_k(0,\cdot)-\partial_t u(0,\cdot)\|_{L^2}\right) .$$

Similarly, the result for the Schrödinger equation follows directly from the definition of the u_k and u at $t=0$ in Section 2.5. ◻

4.4. Proof of Theorem 1.1. We prove the results for each type of PDE separately. For the strictly hyperbolic m-th order PDE (1.2), applying the well-posedness estimate given in Theorem 3.1 to the difference between the true solution u and the k-th order Gaussian beam superposition, u_k, defined in Section 2.3, we obtain for $t \in [0,T]$,

$$\begin{aligned}\sum_{\ell=0}^{m-1} \left\|\partial_t^\ell [u(t,\cdot) - u_k(t,\cdot)]\right\|_{H^{m-\ell-1}} \\ \leq C(T)\left(\sum_{\ell=0}^{m-1} \left\|\partial_t^\ell [u(0,\cdot) - u_k(0,\cdot)]\right\|_{H^{m-\ell-1}} + \int_0^T \|P[u_k](\tau,\cdot)\|_{L^2}\, d\tau\right) .\end{aligned}$$

The first term of the right hand side, which represents the difference in the initial data, can be estimated by Theorem 4.6 and the second term, which represents the evolution error, can be estimated by Lemma 4.4 to obtain

$$\begin{aligned}\sum_{\ell=0}^{m-1} \left\|\partial_t^\ell [u(t,\cdot) - u_k(t,\cdot)]\right\|_{H^{m-\ell-1}} \\ \leq C(T)\left(\varepsilon^{\frac{k}{2}-m+1} + \sum_{j=1}^{J} \varepsilon^{\ell_j} \sup_{t\in[0,T]} \left\|\mathcal{Q}_{\alpha_j,g_j,\eta}\right\|_{L^2}\right) + \mathcal{O}(\varepsilon^\infty) ,\end{aligned}$$

with $\ell_j \geq (k/2 - m + 1)$. Thus, using Theorem 4.1, we obtain

$$\sum_{\ell=0}^{m-1} \left\|\partial_t^\ell [u(t,\cdot) - u_k(t,\cdot)]\right\|_{H^{m-\ell-1}} \leq C(T)\varepsilon^{\frac{k}{2}-m+1} ,$$

which completes the proof for strictly hyperbolic PDEs:

$$\varepsilon^{m-1} \sum_{\ell=0}^{m-1} \left\|\partial_t^\ell [u(t,\cdot) - u_k(t,\cdot)]\right\|_{H^{m-\ell-1}} \leq C(T)\varepsilon^{\frac{k}{2}} .$$

The rescaling by ε^{m-1} is convenient here, since it is exactly balances the rate at which the above norm of initial data for the PDE (1.2) goes to infinity as $\varepsilon \to 0$.

Since the wave equation is a second order strictly hyperbolic PDE, applying the estimate for strictly hyperbolic PDEs to the solution u of the wave equation (1.3) and the k-th order Gaussian beam superposition, u_k, defined in Section 2.4, we obtain for $t \in [0,T]$,

$$\|u(t,\cdot) - u_k(t,\cdot)\|_E \leq \varepsilon \sum_{\ell=0}^{1} \left\|\partial_t^\ell [u(t,\cdot) - u_k(t,\cdot)]\right\|_{H^{1-\ell}} \leq C(T)\varepsilon^{\frac{k}{2}} ,$$

which completes the proof of Theorem 1.1 for the wave equation.

For the Schrödinger equation (1.4), applying the well-posedness estimate given in Theorem 3.1 to the difference between the true solution u and the k-th order Gaussian beam superposition, u_k, defined in Section 2.5, we obtain for $t \in [0,T]$,

$$\|u_k(t,\cdot) - u(t,\cdot)\|_{L^2} \leq \|u_k(0,\cdot) - u(0,\cdot)\|_{L^2} + \frac{1}{\varepsilon}\int_0^T \|P^\varepsilon[u_k](\tau,\cdot)\|_{L^2} d\tau .$$

The initial data part of the right hand side can be estimated by Theorem 4.6 to obtain $\|u(0,\cdot)-u_k(0,\cdot)\|_{L^2} \leq C\varepsilon^{\frac{k}{2}}$. With the help of Lemma 4.4, we can estimate the second part of the right hand side as

$$\frac{1}{\varepsilon}\int_0^T \|P^\varepsilon[u_k](\tau,\cdot)\|_{L^2} d\tau \leq \sum_{j=1}^{J} \varepsilon^{\ell_j - 1} \sup_{t\in[0,T]} \|\mathcal{Q}_{\alpha_j,g_j,\eta}\|_{L^2} + \mathcal{O}(\varepsilon^\infty) \ ,$$

with $\ell_j \geq (k/2+1)$. Again, using Theorem 4.1 and combining, we obtain,

$$\|u_k(t,\cdot) - u(t,\cdot)\|_{L^2} \leq C\varepsilon^{\frac{k}{2}} + \varepsilon^{\frac{k}{2}} \sum_{j=1}^{J} C_\alpha(T) + \mathcal{O}(\varepsilon^\infty) \leq C(T)\varepsilon^{\frac{k}{2}} \ .$$

Thus, the proof of Theorem 1.1 is complete.

5. Norm Estimates of $\mathcal{Q}_{\alpha,g,\eta}$.

In this section we prove Theorem 4.1. We follow the ideas in [5] to relate the estimate of the oscillatory integral to the operator norm, through the use of a dual operator. A key ingredient in estimating the operator norm is the non-squeezing lemma (Lemma 4.2), which allows us to obtain a dimensionally independent estimates for the oscillatory integral operator.

5.1. Operator Norm Estimates of $\mathcal{Q}_{\alpha,g,\eta}$.

We let $\mathcal{Q}^*_{\alpha,g,\eta}$ be the dual operator and consider the squared expression,

$$\begin{aligned}
&(\mathcal{Q}^*_{\alpha,g,\eta}\mathcal{Q}_{\alpha,g,\eta}u)(t,z) \\
&\quad = \varepsilon^{-n-|\alpha|}\int_{\mathbb{R}^n\times K_0} u(z')e^{i\phi(t,y-x(t;z);z')/\varepsilon}\overline{e^{i\phi(t,y-x(t;z);z)/\varepsilon}}g(t,y;z')\overline{g(t,y;z)} \\
&\qquad\qquad \times (y-x(t;z'))^\alpha (y-x(t;z))^\alpha \rho_\eta(y-x(t;z'))\rho_\eta(y-x(t;z))dydz' \\
&\quad := \varepsilon^{-n-|\alpha|}\int_{K_0} I^\varepsilon_{\alpha,g}(t,z,z')u(z')dz',
\end{aligned} \tag{5.1}$$

where

$$\begin{aligned}
I^\varepsilon_{\alpha,g}(t,z,z') &= \int_{\mathbb{R}^n} e^{i\phi(t,y-x(t;z);z')/\varepsilon}\overline{e^{i\phi(t,y-x(t;z);z)/\varepsilon}}g(t,y;z')\overline{g(t,y;z)} \\
&\qquad \times (y-x(t;z'))^\alpha (y-x(t;z))^\alpha \rho_\eta(y-x(t;z'))\rho_\eta(y-x(t;z))dy \\
&= \int_{\mathbb{R}^n} e^{i\psi(t,y,z,z')/\varepsilon} g(t,y+\bar{x};z')\overline{g(t,y+\bar{x};z)} \\
&\qquad \times (y-\Delta x)^\alpha \,(y+\Delta x)^\alpha \,\rho_\eta\,(y-\Delta x)\,\rho_\eta\,(y+\Delta x)\,dy \ ,
\end{aligned}$$

after a change of variables and

$$\begin{aligned}
\bar{x} = \bar{x}(t,z,z') &:= \frac{x(t;z)+x(t;z')}{2} \ , \\
\Delta x = \Delta x(t,z,z') &:= \frac{x(t;z)-x(t;z')}{2} \ , \\
\psi(t,y,z,z') &:= \phi(t,y+\Delta x;z') - \overline{\phi(t;y-\Delta x;z)} \ .
\end{aligned}$$

This symmetrization will simplify expressions later on.

Recall Schur's lemma:

LEMMA 5.1 (Schur). *For integrable kernels $K(x,y)$,*

$$\left\|\int K(x,y)u(x)dx\right\|_{L^2}^2 \le \left(\sup_x \int |K(x,y)|dy\right)\left(\sup_y \int |K(x,y)|dx\right)||u||_{L^2}^2 \ .$$

Using Schur's lemma, we can now deduce that

$$\begin{aligned}||\mathcal{Q}_{\alpha,g,\eta}||_{L^2}^2 &= \sup_{w\in L_2(K_0)} \frac{\langle w, \mathcal{Q}^*_{\alpha,g,\eta}\mathcal{Q}_{\alpha,g,\eta}w\rangle}{||w||_{L^2}^2} \le \sup_{w\in L_2(K_0)} \frac{||\mathcal{Q}^*_{\alpha,g,\eta}\mathcal{Q}_{\alpha,g,\eta}w||_{L^2}}{||w||_{L^2}} \\ &\le \varepsilon^{-n-|\alpha|}\left(\sup_{z\in K_0}\int_{K_0}|I^\varepsilon_{\alpha,g}(t,z,z')|dz'\right)^{\frac{1}{2}}\left(\sup_{z'\in K_0}\int_{K_0}|I^\varepsilon_{\alpha,g}(t,z,z')|dz\right)^{\frac{1}{2}} \\ &\le \varepsilon^{-n-|\alpha|}\left(\sup_{z\in K_0}\int_{K_0}|I^\varepsilon_{\alpha,g}(t,z,z')|dz'\right)\end{aligned}$$

upon noting that $|I^\varepsilon_{\alpha,g}(t,z,z')| = |I^\varepsilon_{\alpha,g}(t,z',z)|$.

Before continuing, we need some utility results.

5.1.1. Utility results. We will prove a few general results that will be useful in the proof of Theorem 4.1.

LEMMA 5.2 (Phase estimate). *Let η be the same as in assumption (A4). Then, under the assumptions (A2)–(A4), $t \in [0,T]$, and y such that $|y \pm \Delta x| \le 2\eta$ (or all y if $\eta = \infty$), we have:*

- *For all $z, z' \in K_0$, there exists a constant δ independent of t such that*

$$\Im\psi(t,y,z,z') \ge \frac{1}{2}\delta\left[|y+\Delta x|^2 + |y-\Delta x|^2\right] = \delta|y|^2 + \frac{1}{4}\delta|x(t;z) - x(t;z')|^2 \ .$$

- *For $|x(t;z) - x(t;z')| \le \theta|z-z'|$,*

$$\inf_{y\in\Omega(t,\mu)} |\nabla_y\psi(t,y,z,z')| \ge C(\theta,\mu)|z-z'| \ ,$$

where $\Omega(t,\mu) = \{y : |y-\Delta x| \le 2\mu$ and $|y+\Delta x| \le 2\mu\}$ and $C(\theta,\mu)$ is independent of t and positive if θ and $\mu < \eta$ are sufficiently small.

Proof. By assumption (A4), there exists a constant δ independent of t such that

$$\begin{aligned}\Im\psi(t,y,z,z') &= \Im\phi(t,y+\Delta x;z') + \Im\phi(t,y-\Delta x;z) \ge \delta\left(|y+\Delta x|^2 + |y-\Delta x|\right) \\ &= \delta\left[\left|y+\frac{x-x'}{2}\right|^2 + \left|y-\frac{x-x'}{2}\right|^2\right] = 2\delta|y|^2 + \frac{1}{2}\delta|x-x'|^2 \ .\end{aligned}$$

For convenience, we divide by $1/2$ to eliminate the factor in front of $\delta|y|^2$.

For the second result, we first note that on $\Omega(t,\mu)$,

$$|y| = \left|\frac{1}{2}y - \frac{1}{2}\Delta x + \frac{1}{2}y + \frac{1}{2}\Delta x\right| \le \frac{1}{2}\left(|y-\Delta x| + |y+\Delta x|\right) \le 2\mu \ .$$

Also, by the Fundamental Theorem of Calculus and the smoothness assumption (A2), for $y \in \Omega(t,\mu)$ we have,

$$\begin{aligned}&|(\nabla_y\phi(t,y;z') - \nabla_y\phi(t,0;z')) - (\nabla_y\phi(t,y;z) - \nabla_y\phi(t,0;z))| \\ &\quad = \left|\int_0^1 \left[\partial_y^2\phi(t,sy;z') - \partial_y^2\phi(t,sy;z)\right]yds\right| \le C|z-z'||y| \le C_1\mu|z-z'| \ ,\end{aligned}$$

with C_1 independent of $t \in [0,T]$. Similarly, with $y \in \Omega(t,\mu)$ and C_2 independent of $t \in [0,T]$,

$$|\nabla_y \phi(t, y \pm \Delta x; z) - \nabla_y \phi(t, y; z)| \le C_2 |\Delta x| \ .$$

Using these estimates and assumption (A3), we have

$$\begin{aligned}
|\nabla_y \psi(t,y,z,z')| &\ge |\Re \nabla_y \psi(t,y,z,z')| \\
&= |\Re\nabla_y\phi(t,y+\Delta x;z') - \Re\nabla_y\phi(t,y-\Delta x;z)| \\
&= |\Re\nabla_y\phi(t,0;z') - \Re\nabla_y\phi(t,0;z) - \Re\nabla_y\phi(t,0;z') + \Re\nabla_y\phi(t,0;z) \\
&\quad - \Re\nabla_y\phi(t,y;z') + \Re\nabla_y\phi(t,y;z) + \Re\nabla_y\phi(t,y;z') - \Re\nabla_y\phi(t,y;z) \\
&\quad + \Re\nabla_y\phi(t,y+\Delta x;z') - \Re\nabla_y\phi(t,y-\Delta x;z)| \\
&\ge C|z-z'| - |\Delta x| - 2C_2|\Delta x| - C_1\mu|z-z'| \\
&\ge C|z-z'| - (2C_2+1)\theta|z-z'| - C_1\mu|z-z'| \\
&\ge C(\theta,\mu)|z-z'| \ ,
\end{aligned}$$

where $C(\theta,\mu)$ is independent of $t \in [0,T]$ and positive if θ and μ are small enough. ◻

Next we have a version of the non-stationary phase lemma.

LEMMA 5.3 (Non-stationary phase lemma). *Suppose that $u(y;\zeta) \in C_c^\infty(D \times Z)$ and $\psi(y;\zeta) \in C^\infty(D \times Z)$, where D and Z are compact sets. If $\nabla_y\psi$ never vanishes in $D \times Z$, then for any $K = 0,1,\ldots,$*

$$\left|\int_D u(y;\zeta)e^{i\psi(y;\zeta)/\varepsilon}dy\right| \le C_K\varepsilon^K \sum_{|\alpha|\le K}\int_D \frac{|\partial^\alpha u(y;\zeta)|}{|\nabla_y\psi(y;\zeta)|^{2K-|\alpha|}} e^{-\Im\psi(y;\zeta)/\varepsilon}dy \ ,$$

where C_K is a constant independent of ζ.

Proof. This is a classical result. A proof can be obtained by modifying the proof of Lemma 7.7.1 of [12]. However, we omit the details for the sake of brevity. ◻

With this lemma in hand we can estimate $|I^\varepsilon_{\alpha,g}(t,z,z')|$.

LEMMA 5.4. *Under the assumptions (A2)–(A5), for any $K = 0,1,\ldots,$ fixed $0 < \mu < \eta \le \infty$, $s > 0$ and $t \in [0,T]$, there are constants C_K and C_s independent of t such that*

$$|I^\varepsilon_{\alpha,g}(t,z,z')| \le C_K\varepsilon^{n/2+|\alpha|}\frac{\exp\left(-\frac{\delta|\Delta x|^2}{\varepsilon}\right)}{1+\inf_{y\in\Omega(t,\mu)}|\nabla_y\psi(t,y,z,z')/\sqrt{\varepsilon}|^K} + C_s\varepsilon^s \ , \qquad (5.2)$$

where $\Omega(t,\mu) = \{y : |y-\Delta x| \le 2\mu$ and $|y+\Delta x| \le 2\mu\} \subseteq \{y : |y| \le 2\mu\}$ is a compact set.

Proof. By the definition of $I^\varepsilon_{\alpha,g}(t,z,z')$, we have

$$\begin{aligned}
I^\varepsilon_{\alpha,g}(t,z,z') &= \int_{\mathbb{R}^n} e^{i\psi(t,y,z,z')/\varepsilon} g(t,y+\bar{x};z')\overline{g(t,y+\bar{x};z)} \\
&\qquad\qquad \times (y-\Delta x)^\alpha(y+\Delta x)^\alpha\rho_\eta(y+\Delta x)\rho_\eta(y-\Delta x)dy \\
&= \int_{\Omega(t,\mu)} e^{i\psi(t,y,z,z')/\varepsilon} g(t,y+\bar{x};z')\overline{g(t,y+\bar{x};z)} \\
&\qquad\qquad \times (y-\Delta x)^\alpha(y+\Delta x)^\alpha\rho_\eta(y+\Delta x)\rho_\eta(y-\Delta x)dy \\
&\quad + \int_{\Omega(t,\eta)\setminus\Omega(t,\mu)} e^{i\psi(t,y,z,z')/\varepsilon} g(t,y+\bar{x};z')\overline{g(t,y+\bar{x};z)} \\
&\qquad\qquad \times (y-\Delta x)^\alpha(y+\Delta x)^\alpha\rho_\eta(y+\Delta x)\rho_\eta(y-\Delta x)dy \\
&=: I_1 + I_2.
\end{aligned}$$

The integral I_1 will correspond to the first part of the right hand side of the estimate in the lemma and I_2 to the second part. We begin estimating I_1. By Lemma 5.2 and (A5), for a fixed t, we compute,

$$|I_1| \le C \int_{\Omega(t,\mu)} |y-\Delta x|^{|\alpha|} |y+\Delta x|^{|\alpha|} e^{-\delta(|y-\Delta x|^2+|y+\Delta x|^2)/\varepsilon} dy \ .$$

Now, using the estimate $s^p e^{-as^2} \le (p/e)^{p/2} a^{-p/2} e^{-as^2/2}$, with $p=|\alpha|$, $a=\delta/\varepsilon$ and $s=|y-\Delta x|$ or $|y+\Delta x|$, and continuing the estimate of I_1, we have for a constant, C, independent of t, z and z',

$$\begin{aligned} |I_1| &\le C \left(\frac{\varepsilon}{\delta}\right)^{|\alpha|} \int_{\Omega(t,\mu)} e^{-\frac{\delta}{2\varepsilon}(|y+\Delta x|^2+|y-\Delta x|^2)}\ dy \\ &\le C \left(\frac{\varepsilon}{\delta}\right)^{|\alpha|} \int_{\Omega(t,\mu)} e^{-\frac{\delta}{\varepsilon}|y|^2-\frac{\delta}{\varepsilon}|\Delta x|^2}\ dy \le C\varepsilon^{n/2+|\alpha|} e^{-\frac{\delta}{\varepsilon}|\Delta x|^2} \ . \end{aligned}$$

Thus, we have proved the needed estimate for I_1 for the case $K=0$ as well as the the case $K>0$ when $\inf_{y\in\Omega(t,\mu)} |\nabla_y \psi(t,y,z,z')| = 0$. Therefore, in the remainder of the proof we will consider the case $K \neq 0$ and $\inf_{y\in\Omega(t,\mu)} |\nabla_y \psi(t,y,z,z')| \neq 0$. In this case, Lemma 5.3 can be applied to I_1 with $\zeta = (t,z,z') \in [0,T]\times K_0 \times K_0$ to give,

$$\begin{aligned} |I_1| &\le C_K \varepsilon^K \sum_{|\beta|\le K} \int_{\Omega(t,\mu)} \frac{\left|\partial_y^\beta \left[(y-\Delta x)^\alpha (y+\Delta x)^\alpha g'\overline{g}\rho_\eta^+\rho_\eta^-\right]\right|}{|\nabla_y\psi(t,y,z,z')|^{2K-|\beta|}} e^{-\Im\psi(t,y,z,z')/\varepsilon} dy \\ &\le C_K \sum_{|\beta|\le K} \left(\frac{\varepsilon^{|\beta|/2}}{\inf_{y\in\Omega(t,\mu)} |\nabla_y \psi/\sqrt{\varepsilon}|^{2K-|\beta|}} \right. \\ &\qquad\qquad \left. \times \int_{\Omega(t,\mu)} \left|\partial_y^\beta \left[(y-\Delta x)^\alpha (y+\Delta x)^\alpha g'\overline{g}\rho_\eta^+\rho_\eta^-\right]\right| e^{-\Im\psi/\varepsilon} dy \right) \\ &\le C_K \sum_{|\beta|\le K} \frac{\varepsilon^{|\beta|/2}}{\nu(t,z,z')^{2K-|\beta|}} \left(\sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_1\le 2\alpha}} \int_{\Omega(t,\mu)} \left|\partial_y^{\beta_1} \left[(y-\Delta x)^\alpha (y+\Delta x)^\alpha\right]\right| \right. \\ &\qquad\qquad \left. \times \left|\partial_y^{\beta_2}\left[g'\overline{g}\rho_\eta^+\rho_\eta^-\right]\right| e^{-\Im\psi/\varepsilon} dy \right) , \end{aligned}$$

where $\rho_\eta^\pm = \rho_\eta(y\pm\Delta x)$, $\nu(t,z,z') = \inf_{y\in\Omega(t,\mu)} |\nabla_y\psi(t,y,z,z')/\sqrt{\varepsilon}|$ and C_K is independent of t, z and z'. By assumption (A5) and since ρ_η is uniformly smooth and t, z, z' vary in a compact set, $\left|\partial_y^{\beta_2}\left[g'\overline{g}\rho_\eta^+\rho_\eta^-\right]\right|$ can be bounded by a constant independent of y, t, z and z'. We estimate the other term as follows,

$$\begin{aligned} \left|\partial_y^{\beta_1}\left[(y-\Delta x)^\alpha (y+\Delta x)^\alpha\right]\right| &\le C \sum_{\substack{\beta_{11}+\beta_{12}=\beta_1 \\ \beta_{11},\beta_{12}\le\alpha}} \left|(y-\Delta x)^{\alpha-\beta_{11}} (y+\Delta x)^{\alpha-\beta_{12}}\right| \\ &\le C \sum_{\substack{\beta_{11}+\beta_{12}=\beta_1 \\ \beta_{11},\beta_{12}\le\alpha}} |y-\Delta x|^{|\alpha|-|\beta_{11}|}\ |y+\Delta x|^{|\alpha|-|\beta_{12}|} \ . \end{aligned}$$

Now, using the same argument as for the $K = 0$ case, we have

$$\begin{aligned}\int_{\Omega(t,\mu)} &\left|\partial_y^{\beta_1}\left[(y-\Delta x)^\alpha (y+\Delta x)^\alpha\right]\right| \left|\partial_y^{\beta_2}\left[g'\overline{g}\rho_\eta^+\rho_\eta^-\right]\right| e^{-\Im\psi/\varepsilon} dy \\ &\le C \sum_{\substack{\beta_{11}+\beta_{12}=\beta_1 \\ \beta_{11},\beta_{12}\le\alpha}} \int_{\Omega(t,\mu)} |y-\Delta x|^{|\alpha|-|\beta_{11}|}\,|y+\Delta x|^{|\alpha|-|\beta_{12}|} e^{-\Im\psi/\varepsilon} dy \\ &\le C(\beta_2)\varepsilon^{\frac{n+|\alpha|-|\beta_{11}|+|\alpha|-|\beta_{12}|}{2}} e^{-\frac{\delta}{2\varepsilon}|\Delta x|^2} = C(\beta_2)\varepsilon^{n/2+|\alpha|-|\beta_1|/2} e^{-\frac{\delta}{\varepsilon}|\Delta x|^2} ,\end{aligned}$$

and consequently,

$$\begin{aligned}|I_1| &\le C_K \sum_{|\beta|\le K} \frac{\varepsilon^{|\beta|/2}}{\nu(t,z,z')^{2K-|\beta|}} \sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_1\le 2\alpha}} C(\beta_2)\varepsilon^{n/2+|\alpha|-|\beta_1|/2} e^{-\frac{\delta}{\varepsilon}|\Delta x|^2} \\ &\le C_K \varepsilon^{n/2+|\alpha|} e^{-\frac{\delta}{\varepsilon}|\Delta x|^2} \sum_{|\beta|\le K} \frac{1}{\nu(t,z,z')^{2K-|\beta|}} .\end{aligned}$$

Using the fact that $|I_1|$ will be bounded by the minimum of the $K = 0$ and $K > 0$ estimates, we have

$$|I_1| \le C\varepsilon^{n/2+|\alpha|} e^{-\frac{\delta}{\varepsilon}|\Delta x|^2} \min\left[1, \sum_{|\beta|\le K} \frac{1}{\nu(t,z,z')^{2K-|\beta|}}\right] .$$

Noting that for positive a, b, and c,

$$\min[a, b+c] \le \min[a,b] + \min[a,c] \qquad \text{and} \qquad \min[1, 1/a] \le 2/(1+a) ,$$

we have,

$$\begin{aligned}\min\left[1, \sum_{|\beta|\le K} \frac{1}{\nu(t,z,z')^{2K-|\beta|}}\right] &\le \sum_{|\beta|\le K} \min\left[1, \frac{1}{\nu(t,z,z')^{2K-|\beta|}}\right] \\ &\le \sum_{|\beta|\le K} \frac{2}{1+\nu(t,z,z')^{2K-|\beta|}} \le C_K \frac{1}{1+\nu(t,z,z')^K} .\end{aligned}$$

This shows the I_1 contribution to the estimate (5.2). It remains to show the smallness of I_2. Indeed, since either $|y + \Delta x| > \mu$ or $|y - \Delta x| > \mu$ on $\Omega(t, \eta) \setminus \Omega(t, \mu)$, we get in the same way as for I_1 in the $K = 0$ case,

$$|I_2| \le C\left(\frac{\varepsilon}{\delta}\right)^{|\alpha|} \int_{\Omega(t,\eta)\setminus\Omega(t,\mu)} e^{-\frac{\delta}{2\varepsilon}(|y+\Delta x|^2+|y-\Delta x|^2)}\, dy \le C\left(\frac{\varepsilon}{\delta}\right)^{|\alpha|+\frac{n}{2}} e^{-\frac{\delta\mu}{2\varepsilon}} \le C_s\varepsilon^s ,$$

for any $s > 0$ with C_s independent of t. This concludes the proof of the lemma. □

5.1.2. Proof of Theorem 4.1. We now have all of the ingredients to complete the proof of Theorem 4.1. We fix $t \in [0, T]$ and start with the estimate,

$$||\mathcal{Q}_{\alpha,g,\eta}||_{L^2}^2 \le \varepsilon^{-n-|\alpha|}\left(\sup_z \int_{K_0} |I^\varepsilon_{\alpha,g}(t,z,z')|dz'\right) ,$$

derived in the beginning of Section 5.1 and we turn our attention to estimating the integral of $|I^{\varepsilon}_{\alpha,g}(t,z,z')|$.

We will use the shorthand notation $I^{\varepsilon}_{D}(t,z,z') \equiv \chi_D(z,z')I^{\varepsilon}_{\alpha,g}(t,z,z')$, where $\chi_D(z,z')$ is the characteristic function on a domain $D \subseteq K_0 \times K_0$. We will consider two disjoint subsets of $K_0 \times K_0$ given by

$$D_1(t,\theta) = \{(z,z') : |x(t;z) - x(t;z')| \geq \theta|z-z'|\}$$
$$D_2(t,\theta) = \{(z,z') : |x(t;z) - x(t;z')| < \theta|z-z'|\} \ ,$$

where θ is the small parameter θ in Lemma 5.2. Note that $D_1 \cup D_2 = K_0 \times K_0$ and $D_1 \cap D_2 = \emptyset$. Thus,

$$\int_{K_0} \left|I^{\varepsilon}_{\alpha,g}(t,z,z')\right| dz' = \int_{K_0} \left|I^{\varepsilon}_{D_1}(t,z,z')\right| dz' + \int_{K_0} \left|I^{\varepsilon}_{D_2}(t,z,z')\right| dz' \ .$$

The set D_1 corresponds to the non-caustic region of the solution. There, we estimate $I^{\varepsilon}_{D_1}(t,z,z')$ by taking $K = 0$ and $s = n + |\alpha|$ in Lemma 5.4.

$$\begin{aligned}
\int_{K_0} \left|I^{\varepsilon}_{D_1}(t,z,z')\right| dz' &\leq C\varepsilon^{n/2+|\alpha|} \int_{K_0} e^{-\frac{\delta}{4\varepsilon}|x(t;z)-x(t;z')|^2} + \varepsilon^{n/2} dz' \\
&\leq C\varepsilon^{n/2+|\alpha|} \int_{K_0} e^{-\frac{\delta\theta^2}{4\varepsilon}|z-z'|^2} dz' + C|K_0|\varepsilon^{n+|\alpha|} \\
&\leq C\varepsilon^{n/2+|\alpha|} \int_0^{\infty} s^{n-1} e^{-\frac{\delta\theta^2}{4\varepsilon}s^2} ds + C\varepsilon^{n+|\alpha|} \\
&\leq C\varepsilon^{n+|\alpha|} \ .
\end{aligned}$$

The set D_2 corresponds to the region near caustics of the solution. On D_2, we estimate $I^{\varepsilon}_{D_2}(t,z,z')$ using Lemma 5.4 again where we pick μ small enough to allow us to use also Lemma 5.2. Letting $R = \sup_{z,z' \in K_0} |z - z'| < \infty$ be the diameter of K_0 and $s = n + |\alpha|$, we compute

$$\begin{aligned}
\int_{K_0} \left|I^{\varepsilon}_{D_2}(t,z,z')\right| dz' &\leq C\varepsilon^{\frac{n}{2}+|\alpha|} \int_{K_0} \left(\frac{e^{-\frac{\delta}{4\varepsilon}|x(t;z)-x(t;z')|^2}}{1 + \inf_{y\in\Omega(t,\mu)} |\nabla_y \psi(t,y,z,z')/\sqrt{\varepsilon}|^K} + \varepsilon^{\frac{n}{2}} \right) dz' \\
&\leq C\varepsilon^{\frac{n}{2}+|\alpha|} \int_{K_0} \frac{1}{1 + \left(\frac{C(\theta,\mu)|z-z'|}{\sqrt{\varepsilon}}\right)^K} dz' + C\varepsilon^{n+|\alpha|} \\
&\leq C\varepsilon^{\frac{n}{2}+|\alpha|} \int_0^R \frac{1}{1 + (C(\theta,\mu)s/\sqrt{\varepsilon})^K} s^{n-1} ds + C\varepsilon^{n+|\alpha|} \\
&\leq C\varepsilon^{n+|\alpha|} \ ,
\end{aligned}$$

if we take $K = n + 1$.

Since all of the constants are independent of the fixed $t \in [0,T]$, by putting all of these estimate together we obtain

$$||\mathcal{Q}_{\alpha,g,\eta}||^2_{L^2} \leq \varepsilon^{-n-|\alpha|} \left(\sup_{z\in K_0} \int_{K_0} |I^{\varepsilon}_{\alpha,g}(t,z,z')| dz' \right) \leq C \ ,$$

for all $t \in [0,T]$, which proves the theorem.

6. Numerical study of convergence. In this section, we perform numerical convergence analyses to study the sharpness of the of the theoretical estimates in this paper for the constant coefficient wave equation with sound speed $c(y) = 1$. The ODEs that define the Gaussian beams are solved numerically using an explicit Runge-Kutta $(4, 5)$ method (MATLAB's ode45). We use the fast Fourier transform to obtain the "exact solution" and use it to determine the error in the Gaussian beam solution. When the norms require it, we compute derivatives via analytical forms rather than numerical differentiation.

6.1. Single Gaussian Beams. First, we study the convergence rate for a single Gaussian beam to show the sharpness of the estimate in Theorem 3.2 proved in [30]. For 2D, Theorem 3.2 establishes that for a single Gaussian beam $v_k(t, y)$,

$$\|\Box v_k(t, y)\|_{L^2} \le C\varepsilon^{k/2-1/2} \ ,$$

for $t \in [0, T]$. Using the well-posedness estimate for the wave equation, we obtain the rescaled energy norm estimate

$$||v_k(t, \cdot) - u(t, \cdot)||_E \le ||v_k(0, \cdot) - u(0, \cdot)||_E + T\,\varepsilon \sup_{t\in[0,T]} ||\Box v_k(t, \cdot)||_{L^2} \ ,$$

for $t \in [0, T]$, where u is the exact solution to the wave equation. Taking the initial conditions for u to be the same as the 1-st, 2-nd and 3-rd order Gaussian beams at $t = 0$ (modulo the cutoff function), we obtain the asymptotic error estimate,

$$\|v_k(t, \cdot) - u(t, \cdot)\|_E \le C\varepsilon^{k/2+1/2} \ .$$

To test the sharpness of this estimate, we investigate the numerical convergence as follows. Let the Gaussian beam parameters for the 1-st order Gaussian beam be given by

$$x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} , \qquad p(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} , \qquad \tau(0) = +1 \ ,$$

$$\phi_0(0) = 0 \ , \qquad M(0) = \begin{bmatrix} i & 0 \\ 0 & 2+i \end{bmatrix} , \qquad a_{0,0}(0) = 1 \ .$$

The additional coefficients necessary for the Taylor polynomials of phase and amplitudes for 2-nd and 3-rd order Gaussian beams are all initially taken to be 0. Note that even though the phase and amplitudes for 1-st, 2-nd and 3-rd order beams are the same at $t = 0$, their time derivatives will not be, as the ODEs for the higher order coefficients are inhomogeneous. Thus, since the initial data for the exact solution, u, matches the initial data for the Gaussian beam, u depends on the order of the beam. With this choice of parameters, we generate the 1-st, 2-nd and 3-rd order Gaussian beam solution at $t = \{0.5, 1\}$. For 2-nd and 3-rd order beams we additionally use a cutoff function with $\eta = 1/10$. At each t, we compute the rescaled energy norm of the difference $v_k - u$. The asymptotic convergence as $\varepsilon \to 0$ is shown in Figure 6.1. We draw the attention of the reader to the following features of the plots in Figure 6.1:

1. For 1-st order beam: The error decays as ε^1.
2. For 2-nd and 3-rd order beams: The error for larger ε is dominated by the error induced by the cutoff function. In this region, the error decays exponentially fast. As ε gets smaller, the error decays as $\varepsilon^{3/2}$ for the 2-nd order beam and ε^2 for the 3-rd order beam.

The numerical results agree with the estimate given in [30] and, thus, the estimate obtained using Theorem 3.2 for single Gaussian beams is sharp.

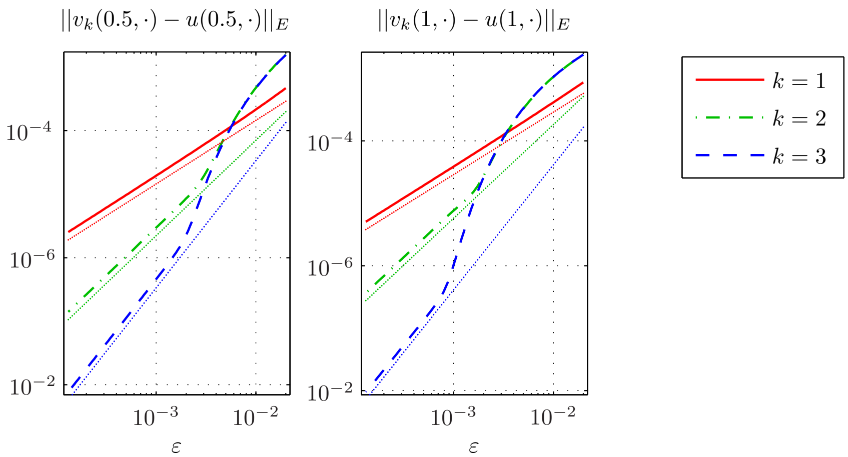

FIG. 6.1. *Single Gaussian beams: Asymptotic behavior of $v_k - u$ in the rescaled energy norm at $t = \{0.5, 1\}$ for $k = \{1, 2, 3\}$ order beams. The results are shown on log-log plots along with $c_1\varepsilon^1$, $c_2\varepsilon^{3/2}$, and $c_3\varepsilon^2$ to help with the interpretation of the asymptotic behavior. The asymptotic behavior agrees with the analytical estimates: 1-st order beam is $\mathcal{O}(\varepsilon^1)$, 2-nd order beam is $\mathcal{O}(\varepsilon^{3/2})$, and 3-rd order beam is $\mathcal{O}(\varepsilon^2)$.*

6.2. Cusp Caustic. We consider an example in 2D that develops a cusp caustic. The initial data for u at $t = 0$ is given by

$$u(0, y) = e^{-10|y|^2} e^{i(-y_1+y_2^2)/\varepsilon} .$$

Thus, the initial phase and amplitudes are given by

$$\begin{aligned} \Phi(y) &= -y_1 + y_2^2 \\ A_0(y) &= e^{-10|y|^2} \\ A_1(y) &= 0 . \end{aligned}$$

For the initial data for $u_t(0, y)$, we take

$$u_t(0, y) = \left(\frac{i}{\varepsilon}\Phi_t(y)\,[A_0(y) + \varepsilon A_1(y)] + [A_{0,t}(y) + \varepsilon A_{1,t}(y)]\right) e^{i\Phi(y)/\varepsilon} ,$$

where the Φ_t, $A_{0,t}$ and $A_{1,t}$ are obtained from the Gaussian beam ODEs and various y derivatives of Φ, A_0 and A_1. Specifically, we take $\Phi_t(y) = +|\nabla_y \Phi(y)|$ so that waves propagate in the positive y_1 direction. As was shown in [32], this particular example develops a cusp caustic at $t = 0.5$ and two fold caustics for $t > 0.5$.

To form the Gaussian beam superposition solutions, we take the initial Taylor coefficients to be

$$\begin{aligned} x(0; z) &= \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} , & p(0; z) &= \begin{bmatrix} -1 \\ 2z_2 \end{bmatrix} , \\ \tau(0; z) &= +|p(0; z)| , & \phi_0(0; z) &= -z_1 + z_2^2 , \\ M(0; z) &= \begin{bmatrix} i & 0 \\ 0 & 2+i \end{bmatrix} , & \phi_\beta(0; z) &= 0,\ |\beta| = 3, 4 , \\ a_{0,\beta}(0; z) &= \partial_y^\beta A_0(z),\ |\beta| = 0, 1, 2 , & a_{1,0} &= 0 , \end{aligned}$$

where only the necessary parameters are used for each of the 1-st, 2-nd and 3-rd order beams. For 2-nd and 3-rd order beams, we additionally use a cutoff function with $\eta = 1/10$. We propagate the Gaussian beam solutions to $t = \{0, 0.25, 0.5, 0.75, 1\}$. At each t, we compute the rescaled energy norm of the difference $u_k - u$. The asymptotic convergence as $\varepsilon \to 0$ is shown in Figure 6.2.

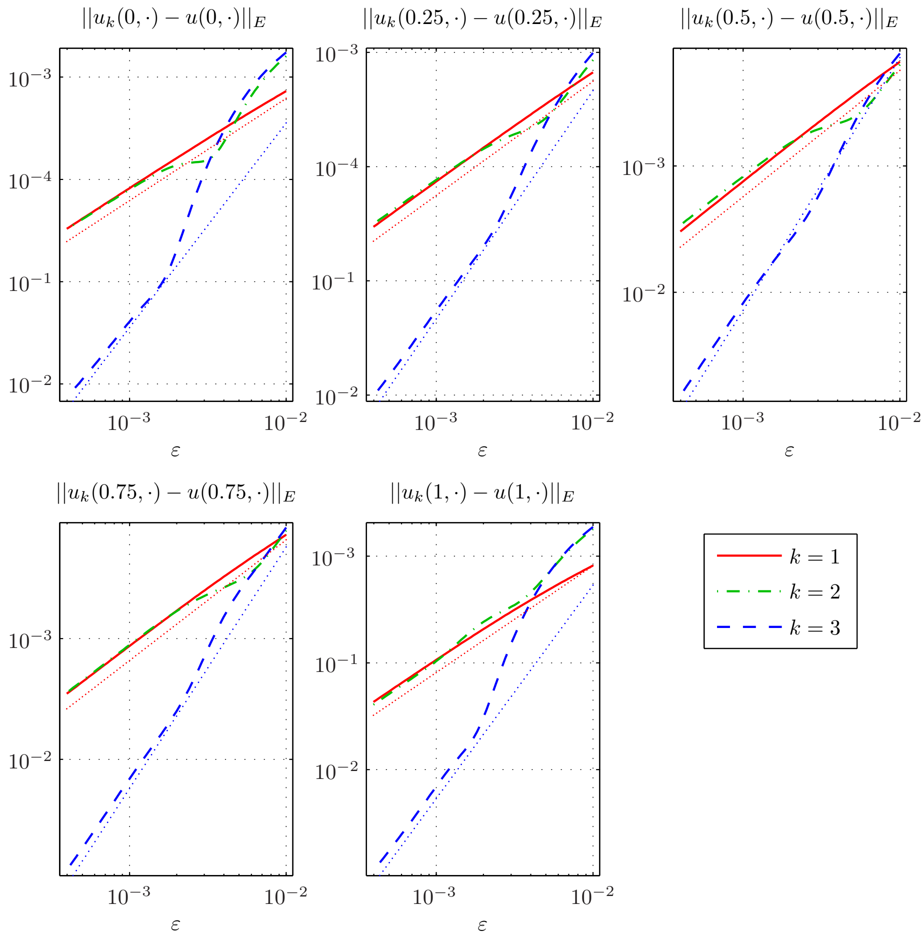

FIG. 6.2. *Superpositions of Gaussian beams for cusp caustic: Asymptotic behavior of* $u_k - u$ *in the rescaled energy norm at* $t = \{0, 0.25, 0.5, 0.75, 1\}$ *for* $k = \{1, 2, 3\}$ *order Gaussian beam superpositions. The results are shown on log-log plots along with* $c_1\varepsilon^1$ *and* $c_2\varepsilon^2$ *to help with the interpretation of the asymptotic behavior.*

We draw the attention of the reader to the following features in the plots:

1. At all t:
   (a) The asymptotic behavior of 1-st and 2-nd order solutions is the same and of order $\mathcal{O}(\varepsilon^1)$.
   (b) The asymptotic behavior of 3-rd order solution is of order $\mathcal{O}(\varepsilon^2)$.
   (c) For large value of ε the error for 2-nd and 3-rd order solutions is dominated by the error induced by the cutoff function. In this region the error decays exponentially.
   (d) For midrange values of ε, 2-nd order solutions experience a fortuitous error cancellation. This is due to the cutoff function. Changing the

cutoff radius η shifts this region.

2. At $t = 0.5$ and $t > 0.5$, the asymptotic behavior of the error is unaffected by the cusp and fold caustics, respectively.

For this example we note that the convergence rate for odd beams $k \in \{1, 3\}$ is in fact half an order better than our theoretical estimates. This improvement was also observed and proved for a simplified setting in [27], where the analysis shows that the gain is due to error cancellations between adjacent beams; it is therefore not present for single Gaussian beams. The result in [27] is for the Helmholtz equation and only concerned the pointwise error away from caustics. Here the numerical results indicate that the same improvement appears for the wave equation in the energy norm even when caustics are present. We conjecture that this gain of convergence is due to error cancellations also here and that it is present for all Gaussian beam superpositions with beams of odd order k. Moreover, we conjecture that that the theoretical result is sharp for beams of even order k, giving us an optimal error estimate in the energy norm of $\mathcal{O}(\varepsilon^{\lceil k/2 \rceil})$ for all k.

7. Concluding Remarks. Gaussian beams are asymptotically valid high frequency solutions to strictly hyperbolic PDEs concentrated on a single curve through the physical domain. They can also be constructed for the Schrödinger equation. Superpositions of Gaussian beams provide a powerful tool to generate more general high frequency solutions. In this work, we establish error estimates of the Gaussian beam superposition for all strictly hyperbolic PDEs and the Schrödinger equation. Our study gives the surprising conclusion that even if the superposition is done over physical space, the error is still independent of the number of dimension and of the presence of caustics. Thus, we improve upon earlier results by Liu and Ralston [22, 23].

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