An asymptotic preserving scheme for the Vlasov-Poisson-Fokker-Planck system in the high field regime *

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Abstract

The Vlasov-Poisson-Fokker-Planck system under the high field scaling describes the Brownian motion of a large system of particles in a surrounding bath where both collision and field effects (electrical or gravitational) are dominant. Numerically solving this system becomes challenging due to the stiff collision term and stiff nonlinear transport term with respect to the high field. We present a class of Asymptotic-Preserving scheme which is efficient in the high field regime, namely, large time steps and coarse meshes can be used, yet the high field limit is still captured. The idea is to combine the two stiff terms and treat them implicitly. Thanks to the linearity of the collision term, using the discretization described in [18], we only need to invert a symmetric matrix. This method can be easily extend to high dimensions. The method is shown to be positive, stable, mass and asymptotic preserving. Numerical experiments validate its efficiency in both kinetic and high field regimes including mixing regimes.

1 Introduction

The Vlasov-Poisson-Fokker-Planck (VPFP) system is the kinetic description of the Brownian motion of a large system of particles in a surrounding bath. For example, in electrostatic plasma, when the interactions between the electrons and a surrounding bath through Coulomb force are taken into account, the time evolution of the electron distribution function $f:\ (t,x,v)\in\mathbb{R}_+\times\mathbb{R}^{\mathbb{N}}\times\mathbb{R}^{\mathbb{N}}\to\mathbb{R}_+$ solves the VPFP system, under the action of a self-consistent potential ϕ:

$$\begin{cases} \partial_t f + v\cdot\nabla_x f - \dfrac{q}{m_e}\nabla_x\phi\cdot\nabla_v f = \dfrac{1}{\tau_e}\mathcal{L}_{\mathcal{FP}}(f), & \text{(1.1a)}\\ -\triangle_x\,\phi = \dfrac{q}{\epsilon_0}(\rho - h(x)), & \text{(1.1b)} \end{cases}$$

where ϵ_0 is the vacuum permittivity, q and m_e are elementary charge and mass of the electrons, and τ_e is the relaxation time due to the collisions of the particles with the surrounding bath.

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The function $h(x)$ is a given positive background charge, and one can assume the *global neutrality relation*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f^0(x,v)dxdv = \int_{\mathbb{R}^N} h(x)dx. \tag{1.2}$$

$\rho(t,x)$ is the density of electrons given by

$$\rho(t,x) = \int_{\mathbb{R}^N} f(t,x,v)dv.$$

$\mathcal{L}_{\mathcal{FP}}(f)$ is the Fokker-Planck operator

$$\mathcal{L}_{\mathcal{FP}}(f) = \nabla_v \cdot (vf + \mu_e \nabla_v f),$$

where $\sqrt{\mu_e} = \sqrt{\frac{k_B T_{th}}{m_e}}$ is the thermal velocity, k_B is the Planck constant, and T_{th} is the temperature of the bath. Two important physical quantities that characterize the particle system are the mean free path $l_e = \sqrt{\mu_e}\tau_e$, which is the average distance traveled by a particle between two successive collisions, and the Debye length $\Lambda = \sqrt{\frac{\epsilon_0 k_B T_{th}}{q^2 \mathcal{N}}}$, which is the typical distance over which significant charge separation can occur. Here $\mathcal{N}$ is the typical value for the concentration of the particles. Another application of the VPFP system is in galaxies where massive particles interacting through gravitational force. The main difference is that the force is attractive, so in (1.1b) we have $\triangle_x \phi = \frac{q}{\epsilon_0}(\rho - h(x))$ instead, and the physical meanings of the constants are different.

The existence and uniqueness of the weak and classical solutions of the VPFP and related systems have been well studied. Degond [7] first showed the existence of a global-in-time smooth solution for the Vlasov-Fokker-Planck equation in one and two space dimensions in electrostatic case, and also proved the convergence of the solution to that of the Vlasov-Poisson equations when the diffusion coefficient goes to zero. Later on, Bouchut [1] [2] extended the result to three dimensions when the electric field was coupled through a Poisson equation, and the results were given in both electrostatic and gravitational cases. Zheng and Majda [28] gave the existence of a global weak solution of the VPFP system from a new prospect, where by allowing the initial data to be measure-valued, it includes some physically interesting case such as electron sheets. For more results, one can refer to [3], [4], [25].

If the mean free path of the electrons is much smaller than the Debye length, then system (1.1) can be written in the dimensionless form as

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \dfrac{1}{\epsilon}\nabla_x \phi \cdot \nabla_v f = \dfrac{1}{\epsilon}\mathcal{P}_{non}(f), & \text{(1.3a)} \\ -\triangle_x \phi = \rho - h, & \text{(1.3b)} \end{cases}$$

where $\epsilon = \left(\frac{l_e}{\Lambda}\right)^2$, the ratio between the mean free path and the Debye length, and $\mathcal{P}_{non}$ is the nondimensionalized Fokker-Planck operator:

$$\mathcal{P}_{non} = \nabla_v \cdot (vf_\epsilon + \nabla_v f_\epsilon).$$

Under this scaling, the limiting process $\epsilon \to 0$ is the so-called high-field limit which is different from the low-field limit (or named as parabolic limit), in which the diffusion dominates the behavior, see [23] for example. The high field limit was first introduced in [22] in which it gave a fluid approximation to the semiconductor Boltzmann equation for high electric fields. Later some numerical simulations of this kinetic model and high-field model were performed in [5].

Now one can formally derive the limit equation. First integrating (1.3a) over $\mathbb{R}^N$, one gets

$$\partial_t \rho + \nabla_x \cdot j = 0, \tag{1.4}$$

where $j = \int_{\mathbb{R}^{\mathbb{N}}} v f(t,x,v) dv$. Then multiplying (1.3a) by v and integrating over $\mathbb{R}_v$ to get

$$\epsilon(\partial_t j + \nabla_x \cdot q) + \rho \nabla_x \phi + j = 0, \tag{1.5}$$

where $q = \int_{\mathbb{R}^{\mathbb{N}}} v \otimes v f(t,x,v) dv$. Let $\epsilon \to 0$ in (1.5), one obtains

$$j = -\rho \nabla_x \phi. \tag{1.6}$$

Then plugging it into (1.4) to get the high field limit equation

$$\begin{cases} \partial_t \rho - \nabla_x \cdot (\rho \nabla_x \phi) = 0, & \text{(1.7a)} \\ -\triangle_x \phi = \rho - h(x). & \text{(1.7b)} \end{cases}$$

This formal analysis can be made rigorous. In [21], it was first proved that in one dimension (for both space and velocity) the solution of (1.1) converges to (1.7) when $\epsilon \to 0$. It was also shown that the limit system has a smooth global-in-time solution in electrostatic case and a local-in-time solution in gravitational case. The results were extended to multidimension in the electrostatic case in [10].

Efforts have been devoted to numerically solving the VPFP system, see for instance, [14], [15], [26], [27]. All these schemes use a particle method, random or deterministic, to treat the convective part and deal with the Fokker-Planck operator by reconstructing the distribution function via the field-free Fokker-Planck kernel. These methods are efficient but only have first order accuracy. Another approach was given in [24] using a finite difference method, with implicit time discretization. Although this method is free of the constrain $\triangle t \sim \triangle v^2$, it has to invert a nonsymmetric matrix which is the main difficulty in higher dimension.

Unlike the previous works which intend to capture the behavior of the Vlasov-Poisson system such as Landau damping when the diffusion effect is rather weak, our goal is to develop a scheme that is efficient in the high field regime. The numerical difficulties arise in two ways. The first one is the stiff coefficient in the forcing term containing the electric potential. An explicit method would require that $\triangle t \sim \min(\triangle x,\ \epsilon \triangle v)$ which becomes too expensive when ϵ is small. The other one is the diffusive nature of the Fokker-Planck operator, which poses the constrain $\triangle t \sim O(\epsilon \triangle v^2)$. Instead of treating the forcing term and Fokker-Planck operator separately, our idea is to combine both stiff terms and propose a time implicit method to overcome these two difficulties simultaneously. The combined term still has the form of a Fokker-Planck operator, with a Maxwellian that depends on $\nabla_x \phi$, and it is treated implicitly in the same way as [18] so that only a *symmetric* tri-diagonal matrix has to be inverted. This induces an Asymptotic Preserving(AP) method, as characterized by Jin in [16]. See [17] for a review. It allows large time steps and coarse meshes in the regime $\epsilon \ll 1$. This method can be extended to higher dimension directly.

The rest of the paper is organized as follows. In section 2 we give the first order scheme and prove some properties of it such as positivity, stability, mass and asymptotic preservation. A second order scheme is given at the end of this section. Section 3 is devoted to numerically validate the properties of the scheme. By comparing with the explicit scheme on a resolved mesh, we show that our scheme is efficient in capturing the high field limit. At last, some concluding remarks are given in section 4.

2 An AP scheme for the VPFP system

A standard explicit scheme for the VPFP system requires time step $\triangle t \sim \min(\epsilon \triangle v^2, \triangle x)$ due to the stiffness of the forcing term and collision term contained $\frac{1}{\epsilon}$ and diffusive nature of the Fokker-Planck operator. In order to avoid this constraint, we propose the following scheme which is based on an implicit treatment of the combined stiff terms.

We first combine the two stiff terms in (1.3a), $\frac{1}{\epsilon}\nabla_x\phi_\epsilon \cdot \nabla_v f$ and $\frac{1}{\epsilon}\mathcal{P}_{non}$. In this way, we will not change the property of Fokker-Planck operator, but can treat the two stiff terms simultaneously.

An equivalent form of the VPFP system reads

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \dfrac{1}{\epsilon}\mathcal{P}(f), & \text{(2.1a)} \\ -\triangle_x \phi = \rho - h, & \text{(2.1b)} \end{cases}$$

where

$$\mathcal{P}(f) = \nabla_v \cdot \left[e^{-\frac{|v+\nabla_x\phi|^2}{2}} \nabla_v \left(e^{\frac{|v+\nabla_x\phi|^2}{2}} f \right) \right]. \tag{2.2}$$

In this form, one can introduce

$$M = e^{-\frac{|v+\nabla_x\phi|^2}{2}}, \tag{2.3}$$

and one will see that formally f goes to

$$f_{eq} = \frac{\rho}{(2\pi)^{\frac{N}{2}}} M = \frac{\rho}{(2\pi)^{\frac{N}{2}}} e^{-\frac{|v+\nabla_x\phi|^2}{2}}$$

when pushing ϵ to 0. This is the so-called "local Maxwellian", and it is easy to check that the limit ρ indeed solves the high field limit equation (1.7), see for example [10].

2.1 The first order scheme

The time discretization of the first order scheme reads

$$\frac{f^{n+1} - f^n}{\triangle t} + v \cdot \nabla_x f^n = \frac{1}{\epsilon} P(f^{n+1}), \tag{2.4}$$

where the operator P is the discrete version of operator $\mathcal{P}$, and it is treated in the same way as [18]. In fact, there are several methods about how to discretize the Fokker-Planck operator, such as [6], [8], and [19]. Here we choose the method of [18] because it gives a symmetric matrix which is not only easy to invert, but also has some good properties such as negative definiteness. From now on, denote $f(x_i, v_j, t^n)$ by $f^n_{i,j}$, where $0 \le i \le N_x$, $0 \le j \le N_v$, and N_x and N_v are the number of mesh points in x and v directions respectively. We briefly state the discretization as follows. Let

$$\tilde{P}(g) = \frac{1}{\sqrt{M}} \nabla_v \cdot \left(M \nabla_x \left(\frac{g}{\sqrt{M}} \right) \right), \tag{2.5}$$

then it relates to P as

$$P(f) = \sqrt{M}\tilde{P}\left(\frac{f}{\sqrt{M}} \right). \tag{2.6}$$

The discretization of $\tilde{P}$ is straightforward, and the one dimensional version takes the form

$$\begin{aligned}(\tilde{P}g)_j &= \frac{1}{\triangle v^2\sqrt{M_j}}\left(\sqrt{M_jM_{j+1}}\left[\left(\frac{g}{\sqrt{M}}\right)_{j+1}-\left(\frac{g}{\sqrt{M}}\right)_j\right]-\sqrt{M_jM_{j-1}}\left[\left(\frac{g}{\sqrt{M}}\right)_j-\left(\frac{g}{\sqrt{M}}\right)_{j-1}\right]\right)\\ &= \frac{1}{\triangle v^2}\left(g_{j+1}-\frac{\sqrt{M_{j+1}}+\sqrt{M_{j-1}}}{\sqrt{M_j}}g_j+g_{j-1}\right). \qquad (2.7)\end{aligned}$$

Similarly, one can extend it to higher dimension with no extra efforts. Therefore (2.4) becomes

$$\frac{f^{n+1}-f^n}{\triangle t}+v\cdot\nabla_x f^n=\frac{1}{\epsilon}\sqrt{M^{n+1}}\tilde{P}\left(\frac{f^{n+1}}{\sqrt{M^{n+1}}}\right). \qquad (2.8)$$

Now we can summarize the algorithm for the first order method. Given f^n, ρ^n and ϕ^n at time t^n.

- **Step 1.** Approximate the transport term $v\cdot\nabla_x f^n$ in (2.8) by a first order upwind method or second order high resolution method.
- **Step 2.** Sum (2.8) over discrete v, note that the right hand side will be zero (see(2.9)), so ρ^{n+1} can be obtained explicitly in this step.
- **Step 3.** Solve (1.3b) by any Poisson solver, say, fast Fourier transform for periodic case, to get ϕ^{n+1}. Then calculate M^{n+1} via (2.3).
- **Step 4.** Plug M^{n+1} into (2.8), one ends up with a linear system for f^{n+1}, invert the system by the conjugate gradient method to get f^{n+1}.

2.2 Some properties of the scheme

In this section, we show that in one space dimension, the first order scheme has some good properties under the hyperbolic CFL condition, which is not restrictive at all.

Mass conservation

The original system preserves mass, so it is desirable to have this property numerically. Observe that

$$\begin{aligned}&\sum_j\sqrt{M_j}\tilde{P}\left(\frac{f}{\sqrt{M}}\right)_j\\ =\ &\frac{1}{\triangle v^2}\left(\sum_j\sqrt{M_jM_{j+1}}\left[\left(\frac{f}{M}\right)_{j+1}-\left(\frac{f}{M}\right)_j\right]-\sum_j\sqrt{M_{j-1}M_j}\left[\left(\frac{f}{M}\right)_{j-1}-\left(\frac{f}{M}\right)_j\right]\right)\\ =\ &0, \qquad (2.9)\end{aligned}$$

the conservation of mass follows if a conservative scheme is used for the convection term $v\partial_x f$.

Positivity preservation

Plugging (2.7) into (2.8) and with the upwind discretization on $\partial_x f$, the first order scheme reads

$$\begin{aligned}&\frac{f_{i,j}^{n+1}-f_{i,j}^n}{\Delta t}+\max(v_j,\ 0)\frac{f_{i,j}^n-f_{i-1,j}^n}{\triangle x}+\min(v_j,\ 0)\frac{f_{i+1,j}^n-f_{i,j}^n}{\triangle x}\\ =\ &\frac{\sqrt{M_{i,j}^{n+1}}}{\epsilon\triangle v^2}\left(\sqrt{M_{i,j+1}^{n+1}}\left[\left(\frac{f}{M}\right)_{i,j+1}^{n+1}-\left(\frac{f}{M}\right)_{i,j}^{n+1}\right]+\sqrt{M_{i,j-1}^{n+1}}\left[\left(\frac{f}{M}\right)_{i,j-1}^{n+1}-\left(\frac{f}{M}\right)_{i,j}^{n+1}\right]\right). \qquad (2.10)\end{aligned}$$

We use the maximum principle argument. If at time t^n, $f^n_{i,j}$ is positive for all $0 \le i \le N_x$, $0 \le j \le N_v$, and assume $\left(\frac{f}{M}\right)^{n+1}_{k,l} = \min_{i,j}\left(\frac{f}{M}\right)^{n+1}_{i,j}$ where $0 \le k \le N_x$, $0 \le l \le N_v$. Then from (2.10), one has

$$\begin{aligned} f^{n+1}_{k,l} &= f^n_{k,l}\left(1 - v_l^+\frac{\triangle t}{\triangle x} + v_l^-\frac{\triangle t}{\triangle x}\right) + v_l^+\frac{\triangle t}{\triangle x}f^n_{k-1,l} - v_l^-\frac{\triangle t}{\triangle x}f^n_{k+1,l} \\ &\quad + \frac{\sqrt{M^{n+1}_{k,l}}}{\epsilon \triangle v^2}\left(\sqrt{M^{n+1}_{k,l+1}}\left[\left(\frac{f}{M}\right)^{n+1}_{k,l+1} - \left(\frac{f}{M}\right)^{n+1}_{k,l}\right] + \sqrt{M^{n+1}_{k,l-1}}\left[\left(\frac{f}{M}\right)^{n+1}_{k,l-1} - \left(\frac{f}{M}\right)^{n+1}_{k,l}\right]\right), \end{aligned}$$

where $v_l^+ = \max(v_l, 0) \le 0$, $v_l^- = \min(v_l, 0) \le 0$. Under the CFL condition $\max_j |v_j|\frac{\triangle t}{\triangle x} \le 1$, it is easy to see that the right hand side of the above expression is positive. Note that M is always positive, so $\min_{i,j}\left(\frac{f}{M}\right)^{n+1}_{i,j}$ is positive, which implies that $f^{n+1}_{i,j}$ is positive for all $0 \le i \le N_x$, $0 \le j \le N_v$.

Stability

Having the properties of positivity and mass conservation, stability directly follows. Consider l^1 norm $\| f^n \|_{l^1} = \sum_{i,j} |f^n_{i,j}|$, then one has

$$\| f^{n+1} \|_{l^1} = \sum_{i,j} |f^{n+1}_{i,j}| = \sum_{i,j} f^{n+1}_{i,j} = \sum_{i,j} f^n_{i,j} = \| f^n \|_{l^1}, \tag{2.11}$$

where the second equality comes from positivity, and the third equality is a consequence of the mass conservation.

Asymptotic preservation

Following the idea in [11], define the discrete entropy

$$H^n_{i,j} = \sum_j f_{i,j} \log\left(\frac{f}{M}\right)^n_{i,j}, \tag{2.12}$$

where $H_{i,j} = H(x_i, v_j)$, and for the time being we will omit the subscript i and superscript n without any ambiguity. Then it is not hard to show the following inequality:

$$\begin{aligned} &\sum_j P(f_j)\log\left(\frac{f}{M}\right)_j \\ =\ &\sum_j \sqrt{M_j}\tilde{P}\left(\frac{f_j}{\sqrt{M_j}}\right)\log\left(\frac{f}{M}\right)_j \\ =\ &\frac{1}{\triangle v^2}\sum_j \sqrt{M_j M_{j+1}}\left[\left(\frac{f}{M}\right)_{j+1} - \left(\frac{f}{M}\right)_j\right]\log\left(\frac{f}{M}\right)_j \\ &\qquad - \frac{1}{\triangle v^2}\sum_j \sqrt{M_j M_{j-1}}\left[\left(\frac{f}{M}\right)_j - \left(\frac{f}{M}\right)_{j-1}\right]\log\left(\frac{f}{M}\right)_j \\ =\ &\frac{1}{\triangle v^2}\left(\sum_j \sqrt{M_j M_{j+1}}\left[\left(\frac{f}{M}\right)_{j+1} - \left(\frac{f}{M}\right)_j\right]\left[\log\left(\frac{f}{M}\right)_j - \log\left(\frac{f}{M}\right)_{j+1}\right]\right) \qquad (2.13) \\ \le\ &0. \end{aligned}$$

And from the last equality, every term in the summation is no greater than 0, so

$$\sum_j P(f_j)\log\left(\frac{f}{M}\right)_j = 0 \;\Rightarrow\; \left(\frac{f}{M}\right)_j \text{ is independent of } j,$$

or $f_j = CM_j,\ \forall j$, where C is a function independent of j (or v). And by mass conservation, $C = \frac{\rho}{(2\pi)^{\frac{N}{2}}}$. Then from (2.4), one has

$$\epsilon\left[\sum_j\left(\frac{f_j^{n+1}-f_j^n}{\triangle t}+v\partial_x f_j^n\right)\log\left(\frac{f}{M}\right)_j^{n+1}\right] = \sum_j P(f_j^{n+1})\log\left(\frac{f}{M}\right)_j^{n+1}, \tag{2.14}$$

so $\epsilon\to 0$ implies $\sum_j P(f_j^{n+1})\log\left(\frac{f}{M}\right)_j^{n+1}\to 0$, thus $f_j^{n+1}\to\frac{\rho^{n+1}}{(2\pi)^{\frac{N}{2}}}M_j^{n+1}$, $\forall j$.

Now go back to the scheme (2.8), summation over j gives

$$\frac{\rho^{n+1}-\rho^n}{\triangle t}+\sum_j v_j\cdot\partial_x f_j^n = 0. \tag{2.15}$$

The above argument says f^n converges to $\frac{\rho^n}{(2\pi)^{\frac{N}{2}}}M^n$ as $\epsilon\to 0$, so

$$\begin{aligned}\sum_j v_j\partial_x f_j^n &= \partial_x\left[\sum_j(v_j+\partial_x\phi^n-\partial_x\phi^n)e^{-\frac{|v_j+\partial_x\phi^n|^2}{2}}\frac{\rho^n}{(2\pi)^{\frac{N}{2}}}\right]\\ &= -\partial_x\left[\sum_j\partial_x\phi^n e^{-\frac{|v_j+\partial_x\phi^n|^2}{2}}\frac{\rho^n}{(2\pi)^{\frac{N}{2}}}\right]\\ &= -\partial_x(\rho^n\partial_x\phi^n)\sum_j\frac{1}{(2\pi)^{\frac{N}{2}}}e^{-\frac{|v_j+\partial_x\phi^n|^2}{2}},\end{aligned} \tag{2.16}$$

and one can see that $\sum_j\frac{1}{(2\pi)^{\frac{N}{2}}}e^{-\frac{|v_j+\partial_x\phi^n|^2}{2}}$ approximates 1 with a second order accuracy in v, plugging (2.16) into (2.15) one gets a time consistent semidiscretized form of the limit equation (1.7), thus justifying the correct high field limit in the time discrete case.

Remark 1. In fact, for the space homogeneous case ($\partial_x f = 0$), one can show that the entropy decays from the inequality (2.13). Note that in this case M does not change with time. Multiply (2.4) with $\log\left(\frac{f_j^{n+1}}{M_j}\right)$ and summing over j, and by (2.13) one has

$$\sum_j f_j^{n+1}\log\left(\frac{f_j^{n+1}}{M_j}\right)-\sum_j f_j^n\log\left(\frac{f_j^{n+1}}{M_j}\right) = \frac{1}{\epsilon}\sum_j P(f_j^{n+1})\log(\frac{f_j^{n+1}}{M_j})\le 0,$$

or equivalently,

$$\sum_j f_j^{n+1}\log\left(\frac{f_j^{n+1}}{M_j}\right)-\sum_j f_j^n\log\left(\frac{f_j^n}{M_j}\right)+\sum_j f_j^n\left[\log\left(\frac{f_j^n}{M_j}\right)-\log\left(\frac{f_j^{n+1}}{M_j}\right)\right]\le 0$$

Thus

$$\begin{aligned}\sum_j f_j^{n+1}\log\left(\frac{f_j^{n+1}}{M_j}\right)-\sum_j f_j^n\log\left(\frac{f_j^n}{M_j}\right) &= \sum_j f_j^n\log\left(\frac{f_j^{n+1}}{f_j^n}-1+1\right)\\ &\le \sum_j f_j^n\left(\frac{f_j^{n+1}}{f_j^n}-1\right)=\sum_j(f_j^{n+1}-f_j^n)=0,\end{aligned}$$

where the inequality comes from the inequality $\log(1+x)\le x$, and the last equality is the result of mass conservation.

2.3 A second order scheme

Using backward difference formula for time discretization [12], the second order scheme in one space dimension is given by

$$\frac{3f^{n+1}-4f^n+f^{n-1}}{2\triangle t}+2v\partial_x f^n-v\partial_x f^{n-1}=\frac{1}{\epsilon}\sqrt{M^{n+1}}\tilde{P}(\frac{f^{n+1}}{\sqrt{M^{n+1}}}). \tag{2.17}$$

For space discretization, we use the MUSCL scheme, i.e.,

$$v_j\cdot\partial_x f=v_j\frac{f_{i+\frac{1}{2},j}-f_{i-\frac{1}{2},j}}{\triangle x}, \tag{2.18}$$

and $f_{i+\frac{1}{2},j}$ takes the form

$$v_j>0,\qquad f_{i+\frac{1}{2},j}=f_{i,j}+\frac{1}{2}\phi(\theta_{i+\frac{1}{2}})(f_{i+1,j}-f_{i,j}); \tag{2.19}$$

$$v_j<0,\qquad f_{i+\frac{1}{2},j}=f_{i+1,j}+\frac{1}{2}\phi(\theta_{i+\frac{1}{2}})(f_{i+1,j}-f_{i,j}), \tag{2.20}$$

where $\theta_{i+\frac{1}{2}}$ is the smooth indicator, and ϕ is the slope limiter function, say, the minmod limiter [20]

$$\phi(\theta)=\max\{0,\ \min\{1,\theta\}\}. \tag{2.21}$$

3 Numerical Examples

In order to avoid some difficulties that might be introduced by boundaries, we will consider periodic boundary condition in x-direction. Most of our simulations will be for the electrostatic case, for which a global-in-time smooth solution exists. For the gravitational case where the solution only exists locally in time, we will give one example in the end.

3.1 The order of convergence

This section is devoted to check the order of accuracy of the schemes (2.8) and (2.17). Consider the VPFP system in $1d_x\times 1d_v$. Take the equilibrium initial data

$$\rho^0(x)=\frac{\sqrt{2\pi}}{2}(2+\cos(2\pi x)),\quad f^0(x,v)=\frac{\rho^0(x)}{\sqrt{2\pi}}e^{-\frac{|v+\phi_x^0|^2}{2}}, \tag{3.1}$$

where $x \in [0,1]$, $v \in [-6,6]$. ϕ^0 is the solution to (1.3b) with

$$h(x) = \frac{\sqrt{2\pi}}{1.2661} e^{\cos(2\pi x)}, \tag{3.2}$$

and satisfies the periodic boundary condition $\phi(0) = \phi(1)$.

We take $N_v = 64$ as the number of grid points in $v-$direction, and take space grid points $N_x = 32,\ 64,\ 128,\ 256,\ 512$ respectively. Choose time step $\triangle t = \triangle x/8$ to satisfy the CFL condition $\triangle t \leq \triangle x / \max_j |v_j|$ in transport part. The output time is $T_{max} = 0.125$. Check the relative error in l^1 norm

$$e_{\triangle x} = \max_{t \in (0, T_{max})} \frac{\| f_{\triangle x}(t) - f_{2\triangle x}(t) \|_1}{\| f^0 \|_1},$$

where $f_{\triangle x}$ is the numerical solution calculated from a grid of size $\triangle x$. If $e_{\triangle x} \leq C \triangle x^k$ for all $0 < \triangle x \ll 1$, then the scheme is said to be k-th order accurate.

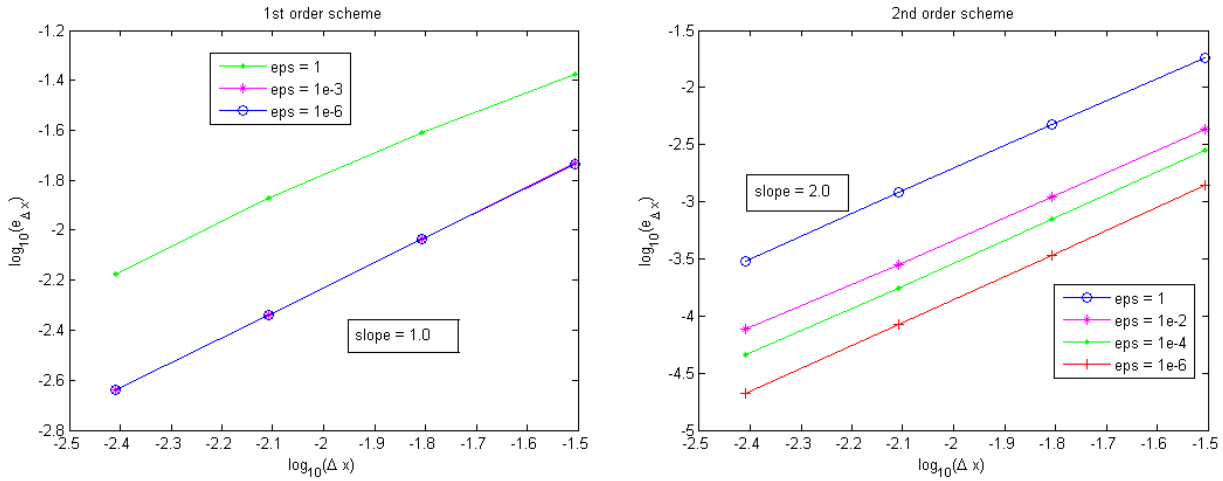

Figure 1: The l^1 errors of the first order scheme (left) and second order scheme (right)

The l^1 error of the first order and second order methods are presented in Figure 1. The order of accuracy is shown to be first and second in space and time uniformly with respect to ϵ. (The error in v is spectrally small, see [18], so it will not contribute much to the errors.)

3.2 The asymptotic preserving property

In this section, we want to show that no matter whether the initial data is in equilibrium, the first order method (2.8) and second order method (2.17) will push f towards the local Maxwellian in one step, and this is exactly the strong AP property defined in [9].

For the equilibrium initial data, we take the same one as in previous section (3.1) . For nonequilibrium initial data we take the following "double peak" function

$$\rho^0(x) = \frac{\sqrt{2\pi}}{2}(2 + \cos(2\pi x)), \quad f^0(x,v) = \frac{\rho^0(x)}{\sqrt{2\pi}} \left(e^{-\frac{|v+1.5|^2}{2}} + e^{-\frac{|v-1.5|^2}{2}} \right), \tag{3.3}$$

and let $h(x) = \frac{5.0132}{1.2661} e^{\cos(2\pi x)}$ which satisfies the neutrality condition. We show the time evolution

of the "distance" between f and equilibrium $M^{eq} = \frac{\rho}{\sqrt{2\pi}} e^{-\frac{|v+\phi_x|^2}{2}}$ with respect to different ϵ.

$$\| f - M^{eq} \|_1 = \sum_{i,j} |f_{i,j} - M^{eq}_{i,j}| \triangle x \triangle v.$$

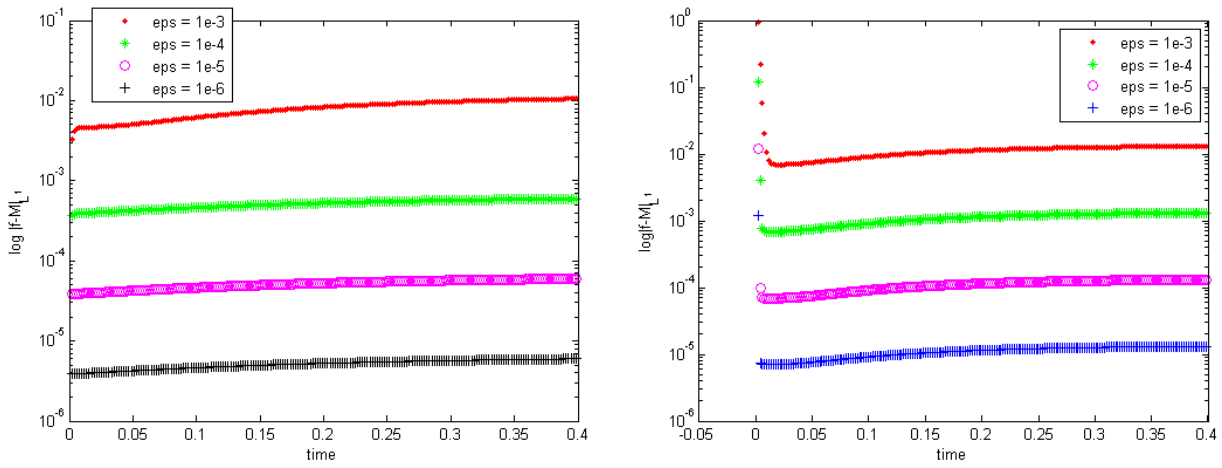

Figure 2: The time evolution of $\| f - M^{eq} \|_1$ for different ϵ with equilibrium initial data (left) and nonequilibrium initial data (right) using the first order scheme. The mesh sizes are $N_v = 64$, $N_x = 64$, $\triangle t = \triangle x/8$.

Figure 2 gives the time evolution of $\| f - M^{eq} \|_1$ for different ϵ, which shows that $f^n - (M^{eq})^n = O(\epsilon)$ for all $n \geq 1$ whether the initial condition is in equilibrium or not. This validates that the first order scheme is indeed AP.

For the second order scheme, we have similar results, see Figure 3.

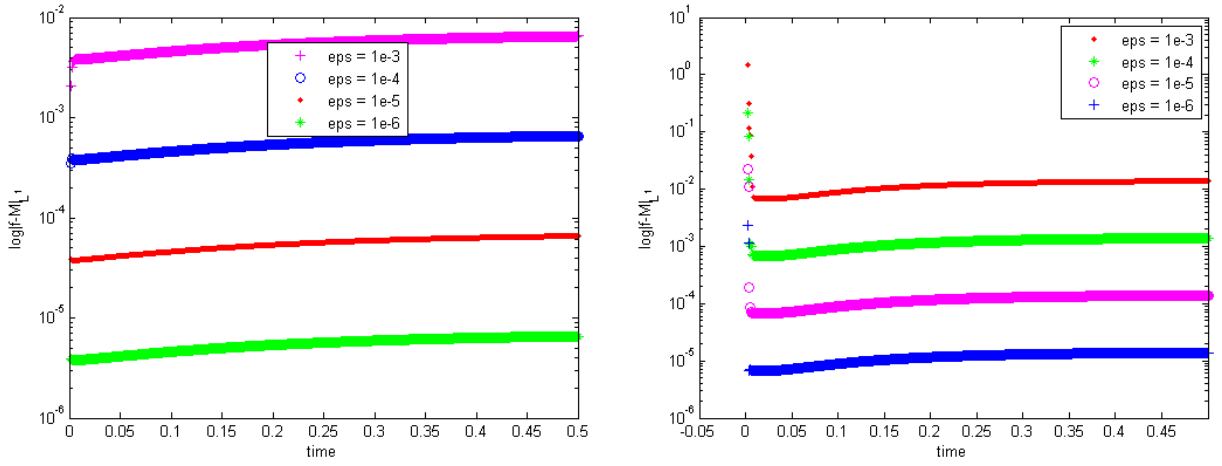

Figure 3: The time evolution of $\| f - M^{eq} \|_1$ for different ϵ with equilibrium initial data (left) and nonequilibrium initial data (right) using the second order scheme. The mesh size are $N_v = 64$, $N_x = 64$, $\triangle t = \triangle x/15$.

3.3 Mixing regimes

Now we test our scheme in mixing regimes, where ϵ varies in space by several orders of magnitude. For example, take ϵ to be

$$\epsilon(x) = \begin{cases} \epsilon_0 + \frac{1}{2}(\tanh(5-10x) + \tanh(5+10x)) & x \le 0.3; \\ \epsilon_0 & x > 0.3, \end{cases}$$

where $\epsilon_0 = 0.001$, so that it contains both the kinetic and high field regimes. The initial data is given by

$$\rho^0(x) = \frac{\sqrt{2\pi}}{6}(2 + \sin(2\pi x)), \quad f^0(x,v) = \frac{\rho(x)}{\sqrt{2\pi}} e^{-\frac{|v+\phi_x^0|^2}{2}}, \tag{3.4}$$

where $x \in [-1,1]$ and ϕ^0 is the solution to (1.3b) with

$$h(x) = \frac{1.6711}{1.2661} e^{\cos(2\pi x)}, \tag{3.5}$$

and satisfies the periodic boundary condition $\phi(-1) = \phi(1)$.

In this test we compare the second order scheme (2.17) with the explicit scheme which uses the second order Runge-Kutta discretization in time and MUSCL scheme for space discretization. In our scheme, we take $N_x = 100$ and $\triangle t = \triangle x/15 = 0.00125$, while in explicit scheme, we take $N_x = 2000$ and $\triangle t = \min\{\frac{\triangle x}{\max|v|}, \epsilon_0 \Delta x, \epsilon_0 \Delta v^2\}/5 = 7.0313\ e-6$. The shape of ρ at three different times are presented in Figure 4, and one can see that our new second order scheme gives a good approximation to the "reference" solution obtained by the explicit method with much smaller mesh size and time step.

3.4 A Riemann problem

Now we apply our second order method to the $1-D$ Riemann problem:

$$\begin{cases} (\rho_l, h_l) = (1/8, 1/2), & 0 \le x < 1/4; \quad (3.6a) \\ (\rho_m, h_m) = (1/2, 1/8), & 1/4 \le x < 3/4; \quad (3.6b) \\ (\rho_r, h_r) = (1/8, 1/2), & 3/4 \le x \le 1. \quad (3.6c) \end{cases}$$

Let ϕ initially be the solution to $-\triangle_x \phi = \rho - h$, and $f = \frac{\rho}{\sqrt{2\pi}} e^{-\frac{|x+\nabla_x\phi|^2}{2}}$. Again periodic boundary condition in x direction is applied.

For our second order scheme, we take $N_x = 100$ and $\triangle t = \triangle x/15 = 0.00125$. In comparison, we use the second order Runge-Kutta scheme with MUSCL scheme in space, and take $N_x = 2000$ and $\triangle t = \min\{\frac{\triangle x}{\max|v|}, \epsilon_0 \triangle x, \epsilon_0 \triangle v^2\}/5 = 7.0313\ e-6$. We compute the macroscopic variable ρ, ϕ and flux $j(t,x) = \int_{\mathcal{R}} v f(t,x,v) dv$. Figure 5 shows that the results obtained by our second order scheme agrees very well with the "reference" solution obtained by the explicit scheme with refined mesh.

3.5 The Gravitational case

As already proved in [21], the limit equation of the VPFP system with gravitational force only has a unique weak solution locally in time. In this section, we give an example of this case, and one will see a different behavior from the electrostatic case. Consider the same initial data as in (3.3), but here ϕ^0 is the solution to

$$\triangle_x \phi^0 = \rho^0(x) - h(x).$$

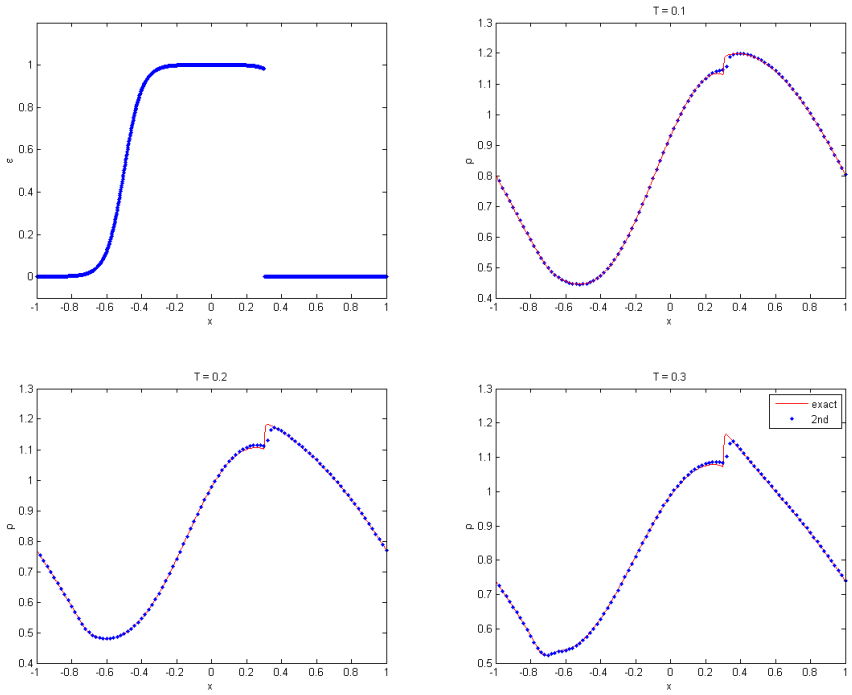

Figure 4: The mixing regime problem. The solid line is computed by an explicit method with refined mesh and serves as the "reference" solution. The dots are obtained by the new second order scheme.

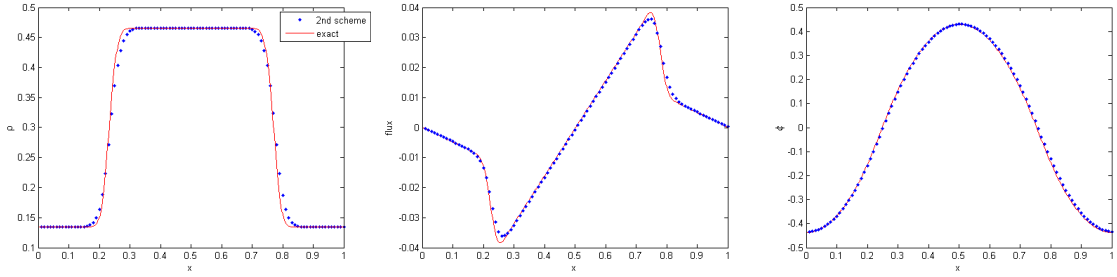

Figure 5: The comparison of density, flux and potential for a Riemann problem at time t = 0.2 between the under-resolved solution by the second order scheme (dots) and resolved solution by the explicit second order Runge-Kutta scheme (solid line).

Then we use the first order scheme (2.8) to check the limit behavior of f when pushing ϵ to 0. The distance between f and the local Maxwellian $\| f - M^{eq} \|_1$ is given in Figure 6. As it can be seen, at the first few steps when solutions remain smooth, $f^n - (M^{eq})^n = O(\epsilon)$, however, things become worse later on, this is because solutions are tempting to blow up.

4 Conclusion

An asymptotic-preserving scheme for the Vlasov-Poisson-Fokker-Planck system in the high field regime has been introduced in this paper. The main idea is to combine the two stiff terms, $\frac{1}{\epsilon}\nabla_x \phi_\epsilon \cdot \nabla_v f_\epsilon$ and $\frac{1}{\epsilon}\mathcal{P}_{non}$ together into a modified form of the Fokker-Planck operator, which contains

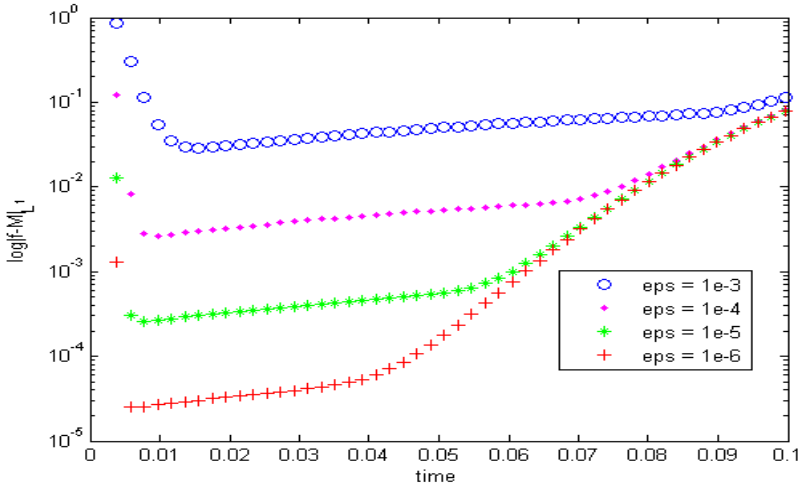

Figure 6: The gravitational case. The time evolution of $\| f - M \|_1$ for different ϵ with equilibrium initial data using the first order scheme. The mesh size are $N_v = 64$, $N_x = 64$, $\triangle t = \triangle x/8$.

the information of the potential. Then we can use the method developed in [18] to discretize this modified collision operator, and resulting in an implicit scheme that only needs to invert a symmetric system, which can be solved by the conjugate gradient method. This scheme shares some good properties: it conserves mass, preserves positivity, is stable and asymptotic preserving. A uniformly second order scheme is also available here. Some numerical experiments are carried out to test the performance of the scheme.

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