ON A NONLOCAL SELECTION-MUTATION MODEL WITH A GRADIENT FLOW STRUCTURE

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Abstract. In this paper, we are interested in an integro-differential model with a nonlinear competition term that describes the evolution of a population structured with respect to a continuous trait. Under some assumption, the steady solution is shown unique and strictly positive, and also globally stable. The exponential convergence rate to the steady state is also established.

1. Introduction

1.1. **The model and its basic properties.** We are interested in the dynamics of a population of individuals with a quantitative trait. The reproduction rate of each individual is determined by its trait and the environment, leading therefore to selection. On the other hand, the influx of mutations into a population over time can counteract the underlying selection forces.

There are different models featuring balance between two evolutionary forces. In this work, we are concerned with the problem governed by

$$\partial_t f(t,x) = \Delta f(t,x) + \frac{1}{2} f(t,x) \left(a(x) - \int_X b(x,y) f^2(t,y)\,dy \right), \text{ for } t>0,\ x \in X, \tag{1.1a}$$

$$f(0,x) = f_0(x) \geq 0, \quad x \in X, \tag{1.1b}$$

$$\frac{\partial f}{\partial \nu} = 0, \quad x \in \partial X, \tag{1.1c}$$

where $f(t,x)$ denotes the density of individuals with trait x, X is a subdomain of $\mathbb{R}^d$, ν is the unit outward normal at a point x on the boundary ∂X.

With some effort, the theory presented here could even be generalized to a complete metric set X endowed with a measure satisfying adequate regularity properties.

In the model, coefficient $a(x)$ is the intrinsic growth rate of individuals with trait x, and $b(x,y) > 0$ represents the competitive interaction between individuals, while the diffusion term plays certain role of mutations in the population dynamics. The trait dependent competition as such appears in many population balance models of Lotka-Volterra type, see e.g., [3, 11, 13, 14]. In particular, the nonlinear competition effect does appear in the model for fish species introduced in [25] in the study of the effect of exploitation on these species. Their model when the number of fish species tend to infinity formally leads to a continuous model of the form

$$\partial_t f(t,x) = \frac{1}{2} f(t,x) \left(a(x) - \int_X b(x,y)(f(t,y) - d(x,y))^2\,dy \right).$$

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This equation with $d=0$ when augmented with a mutation term Δf is exactly (1.1a).

The main difference between (1.1) and more classical models is its non-linear competition term

$$-\int b(x,y)\, f^2(t,y)\, dy,$$

compared to the more usual $\int b(x,y)\, f(t,y)\, dy$. Such a non-linear term significantly changes the effect of competition in particular by increasing the effect of large populations. This may reflect a stricter constraint on resources for instance as in the fish species model above.

Non-linear competitions have seldom been studied from a mathematical point of view and the main goal of the present article is to introduce the required tools and explain how the classical approach for linear competition should be modified. While the analysis is performed for this particular model, we believe it could easily be carried over to other non-linear terms.

Note that from a mathematical point of view, an attractive feature of model (1.1a) is its gradient flow structure in the sense that (1.1a) can be written as

$$\partial_t f = -\frac{1}{2}\frac{\delta F}{\delta f}, \tag{1.2}$$

where the corresponding energy functional is

$$F[f] = \frac{1}{4}\int\int b(x,y) f^2(t,x) f^2(t,y) dx dy - \frac{1}{2}\int a(x) f^2(t,x) dx + \int |\nabla_x f(t,x)|^2 dx, \tag{1.3}$$

so that the energy dissipation law $\frac{d}{dt}F[f] = -2\int |\partial_t f|^2 dx \le 0$ holds for all $t>0$, at least for classical solutions.

Such a gradient flow structure is preserved by the finite volume scheme recently proposed in [7], in which the authors show that both semi-discrete and fully discrete schemes satisfy the two desired properties: positivity of numerical solutions and energy dissipation. These ensure that the positive steady state is asymptotically stable. In addition, the numerical solutions of the model with small mutation are shown to be close to those of the corresponding model with linear competition (1.5). Numerical schemes with similar methodology have been proposed and analyzed for the linear selection dynamics governed by (1.5) in [17, 18]. The objective of this paper is to provide a rigorous analysis on global existence of (1.1), and time-asymptotic convergence to the positive steady state, complementing with the numerical results in [7].

Let us remark as well that under the transformation $u = f^2$, the resulting equation from model (1.1a) becomes

$$\partial_t u(t,x) = \Delta u - \frac{|\nabla u|^2}{2u} + u(t,x)\left(a(x) - \int_X b(x,y) u(t,y) dy\right). \tag{1.4}$$

While the competition term is now linear, the original non-linearity was transfer-ed to the diffusion with a new Hamilton-Jacobi like term. Therefore there does not seem to be any simple way to reduce (1.1a) to an already studied case.

Of course for rare mutation, the diffusion term may be dropped from model (1.1a). The resulting equation then becomes the well-known

$$\partial_t u(t,x) = u(t,x)\left(a(x) - \int_X b(x,y) u(t,y) dy\right). \tag{1.5}$$

Such a simplified model with the usual mutation has been derived from random stochastic models of finite populations (see [8, 9]). This competition model or its variation arises not only in evolution theory but also in ecology for non-local resources (and x denotes the location there, see e.g. [4, 12, 15]). The model without mutation is interesting from the point of view of asymptotic behavior; one expects that the population density concentrates at large times, see, e.g., [1, 6, 11, 16, 24]. The singular steady-state solutions of the competition model correspond to highly concentrated population densities of the form of well separated Dirac masses, which have been shown to happen only asymptotically in models with mutation [2, 10, 19, 20, 21, 22, 23].

The equation (1.5) admits many generalizations, the most popular of which is

$$\partial_t u(t,x) = \int K(x,y)u(t,y)dy - k(x)u(t,x) + u(t,x)\left(a(x) - \int_X b(x,y)u(t,y)dy\right), \tag{1.6}$$

where $K(x,y) \geq 0$ is a mutation kernel satisfying $\int K(x,y)dy = k(x)$. Such a competition-mutation model has been derived from random stochastic models of finite populations (see [8, 9]). The added term has zero integral, and plays certain role of diffusion, so one also adopts

$$\partial_t u(t,x) = \Delta u + u(t,x)\left(a(x) - \int_X b(x,y)u(t,y)dy\right), \tag{1.7}$$

as a competition-mutation model. However, the large time solution behavior of this model remains a challenging issue. Nevertheless, for $b(x,y) = \eta(y) > 0$, the asymptotic solution behavior when mutation tends to vanish has been well studied, see e.g., [2, 21, 23].

Note that the direct competition model (1.5) is a gradient flow $\partial_t u = -\mathrm{grad} H$ under the metric $\langle g,h\rangle_u = \int \frac{g\cdot h}{u}dx$, where

$$H[u] = \frac{1}{2}\int\int b(x,y)u(x)u(y)dxdy - \int a(x)u(x)dx.$$

It is possible to modify the energy to obtain (1.4) directly as a gradient flow. Consider an augmented energy of the form

$$H[u] = \frac{1}{2}\int\int b(x,y)u(x)u(y)dxdy - \int a(x)u(x)dx + \frac{1}{2}\int \frac{|\nabla_x u|^2}{u}dx.$$

By a direct calculation we have

$$\frac{\delta H}{\delta u} = \int b(x,y)u(y)dy - a - \frac{\Delta u}{u} + \frac{|\nabla_x u|^2}{2u^2}.$$

By the metric $\langle g,h\rangle_u = \int \frac{g\cdot h}{u}$ and $\langle \frac{\delta H}{\delta u}, g\rangle = \langle \mathrm{grad} H, g\rangle_u$, we have

$$-\mathrm{grad} H = -u\frac{\delta H}{\delta u} = u\left(a - \int b(x,y)u(y)dy\right) + \Delta u - \frac{|\nabla_x u|^2}{2u}.$$

Hence the gradient flow of H is governed by (1.4) which when transformed with $f = \sqrt{u}$ leads to (1.1a).

In Sect. 1.2, we present the main results of this article, namely positivity and uniqueness of steady solutions in Theorem 1.2, convergence to the steady solution in Theorem 1.2, as well as the exponential convergence rate in Theorem 1.3. In Sect. 2, we prove the results presented in Sect. 1.2.

1.2. **Main results.** The purpose of this paper is to analyze the solution behavior of selection-mutation dynamics (1.1) with the gradient flow structure (1.2) with (1.3). To this end, we make the following assumptions:

$$a \in L^\infty(X), \quad |\{x; \quad a(x) > 0\}| \neq 0; \tag{1.8a}$$

$$b \in L^\infty(X \times X), \quad b_m = \inf_{x,x' \in X} b(x,x') > 0. \tag{1.8b}$$

$$b(x,y) = b(y,x), \forall g \in L^1(X)\backslash\{0\}, \quad \int\int b(x,y)g(x)g(y)dxdy > 0. \tag{1.8c}$$

One can check that b defines then a scalar product over $L^1(X)$,

$$\langle g, h\rangle_b = \int\int b(x,y)g(x)h(y)dxdy$$

with corresponding norm

$$\|g\|_b = \left(\int\int b(x,y)g(x)g(y)dxdy\right)^{1/2}.$$

In what follows we also use the notation

$$R[h] = \frac{1}{2}h\left(a - \int b(x,y)h^2(y)dy\right). \tag{1.9}$$

We work with solutions which are continuous functions of time having values in $L^2(X)$; denoted by $C([0,T]; L^2(X))$, and normed by

$$\|w\| = \sup_{0\leq t\leq T} \|w(t,\cdot)\|_{L^2(X)}.$$

Existence of such solutions can be obtained through a standard fixed point and extension argument leading to

Theorem 1.1. *Let $f_0 \in L^2(X)$, and both a and b satisfy the first two assumptions of (1.8). Then (1.1) admits a global weak solution*

$$f \in L^\infty(\mathbb{R}^+; L^2(X)).$$

Moreover, we have
(a) $\sup_{t>0} \|f(t,\cdot)\|_{L^2(X)} \leq M, \quad (t,x) \in \mathbb{R}^+ \times X.$
(b) f is stable and depends continuously on f_0 in the following sense: if $\tilde{f}$ is another solution with initial data $\tilde{f}_0$, then for every $t > 0$,

$$\int |f - \tilde{f}|^2 dx \leq e^{\lambda t} \int |f_0 - \tilde{f}_0|^2 dx,$$

where λ depends only on a, b and $\|f_0\|$.

Given that the proof of Theorem 1.1 is rather classical, we only give it in appendix B. The strong competition assumption (1.8c) is directly connected to the stability of the steady solution. In fact, under assumption (1.8), the steady solution g is unique and upper-bounded.

If $\int a dx \geq 0$, the steady state is strictly positive. The case $\int a dx < 0$ is less obvious. Recalling [5, Theorem 3.13(b)] which claims that there exists a unique positive λ_1 and the positive function $\psi \in D(L_1)$ such that $\int a\psi^2 dx > 0$ and

$$\lambda_1 = \frac{\int |\nabla_x \psi|^2 dx}{\int a\psi^2 dx} = \inf\left\{ \frac{\int |\nabla_x v|^2 dx}{\int a v^2 dx} \;:\; v \in D(L_1) \text{ and } \int a v^2 dx > 0 \right\}, \tag{1.10}$$

where $D(L_1) = \{u \in H^2(X) : \partial_n u|_{\partial X} = 0\}$ is the domain of the Laplace operator $L_1 u = -\Delta u$, we can show the steady state is still strictly positive if $\lambda_1 < 1/2$. More precisely, we have the following result.

Theorem 1.2. *There exists $g \geq 0$ solution in the sense of distribution to*

$$\Delta g + R[g] = 0, \quad x \in X \quad \partial_\nu g = 0, \quad \text{on } \partial X. \tag{1.11}$$

Moreover,
(i) If $\int a dx \geq 0$ or $\int a dx < 0$ with $\lambda_1 < 1/2$, then there exists a unique positive solution such that $0 < g_{\min} \leq g \leq g_{\max} < \infty$ in X.
(ii) If $\int a dx < 0$ with $\lambda_1 \geq 1/2$, there is no positive steady solution.

Thanks to this result we can show the convergence of $f(t,\cdot)$ towards g:

Theorem 1.3. *Assume both a and b satisfy (1.8). Consider any non-negative $f^0 \in L^1(X) \cap L^\infty(X)$. Then the corresponding solution $f(t,\cdot)$ of (1.1) is such that*

$$\frac{d}{dt} F[f(t,\cdot)] < 0 \quad \text{as long as } f \ \text{ is not a steady solution.} \tag{1.12}$$

As a consequence

$$\lim_{t\to\infty} \|f(t,\cdot) - g(\cdot)\|_{L^2(X)} = 0. \tag{1.13}$$

And moreover, there exists C depending on initial data f_0 and $g \geq 0$ such that

$$\int |f(t,x) - g(x)|^2 dx \leq C e^{-rt} \quad \forall t > 0,$$

for $\int a dx \geq 0$ or $\int a dx < 0$ with $\lambda_1 \neq \frac{1}{2}$, where of course $g = 0$ if $\lambda_1 > 1/2$.
For $\int a dx < 0$ and $\lambda_1 = \frac{1}{2}$,

$$\int |f(t,x)|^2 dx \leq \frac{C}{1+t} \quad \forall t > 0.$$

The proofs of theorems 1.2 and 1.3 rely on a careful use of the competition assumption and is given in the next section.

2. Proofs of the results

2.1. **Proof of Theorem 1.2.** The existence of a non-negative steady state follows a rather classical variational argument which, for the sake of completeness, is given in appendix A.

Consider a non-negative and non 0 steady state g. Multiplying (1.11) by g and then integrating over the domain X, we have

$$\int g^2(a - b * g^2) dx = -2\int g\Delta g dx = 2\int |\nabla g|^2 dx \geq 0.$$

This implies that

$$\|a\|_\infty \|g\|_{L^2}^2 - b_m \|g\|_{L^2}^4 \geq 0,$$

leading to the upper bound

$$\|g\|_{L^2}^2 \leq \frac{\|a\|_\infty}{b_m} < \infty.$$

This directly implies that $\Delta g \in L^2$. By Sobolev embedding, $g \in L^p$ for some $p > 2$ and we can iterate to find $\Delta g \in L^p$. Iterating a finite number of times, this finally gives $\Delta g \in L^\infty$ since $a \in L^\infty$ which is the optimal smoothness on g.

We hence have the elliptic problem $\Delta g + c(x) g = 0$ with bounded $c(x)$ and g. By the standard Harnack inequality we have

$$\sup g \leq C \inf g$$

in any small ball within X. Hence $g > 0$ and bounded from above unless it is identical zero.

We next prove the uniqueness. Let g_1 and g_2 be two positive solutions of (1.11), then using the positivity of b, third assumption in (1.8)

$$\begin{aligned}
0 &\leq \int \int (g_1^2 - g_2^2)(x) b(x,y) (g_1^2 - g_2^2)(y) dy dx \\
&= \int (g_1 - g_2^2/g_1) g_1(x) \int b(x,y) g_1^2(y) dy dx - \int (g_1^2/g_2 - g_2) g_2(x) \int b(x,y) g_2^2(y) dy dx \\
&= \int (g_1 - g_2^2/g_1)(2\Delta g_1(x) + a(x) g_1(x)) dx + \int (g_2 - g_1^2/g_2)(2\Delta g_2(x) + a(x) g_2(x)) dx \\
&= 2 \int (g_1 - g_2^2/g_1)\, \Delta g_1(x) + 2 \int (g_2 - g_1^2/g_2)\, \Delta g_2(x),
\end{aligned}$$

by using the equation (1.11). Hence by integrating by part

$$\begin{aligned}
0 \leq &-2 \int \left(\nabla_x g_1 - \frac{2 g_1 g_2 \nabla_x g_2 - g_2^2 \nabla_x g_1}{g_1^2} \right) \cdot \nabla_x g_1 dx \\
&- 2 \int \left(\nabla_x g_2 - \frac{2 g_1 g_2 \nabla_x g_1 - g_1^2 \nabla_x g_2}{g_2^2} \right) \cdot \nabla_x g_2 dx \\
= &-2 \int \left(\left| \nabla_x g_1 - \frac{g_1}{g_2} \nabla_x g_2 \right|^2 + \left| \nabla_x g_2 - \frac{g_2}{g_1} \nabla_x g_1 \right|^2 \right) dx \leq 0.
\end{aligned}$$

As a conclusion $g_1^2 = g_2^2$, leading to $g_1 = g_2$.

2.2. **Proof of Theorem 1.3: Convergence.** Let us first prove (1.12). A direct calculation shows that

$$\frac{d}{dt} F = -2 \int |\partial_t f|^2 dx \leq 0.$$

Thus it only remains to show that for some $t_0 \geq 0$, if $\partial_t f(t_0, x) \equiv 0$ for all $x \in X$, then $\partial_t f(t,x) \equiv 0$ for all $x \in X$ and $t \geq 0$.

i) For $t > t_0$. Since $\partial_t f(t_0, x) \equiv 0$ for all $x \in X$, then $f(t_0, x)$ is a steady solution. By the uniqueness implied by (b) in Theorem 1.1, we have

$$f(t,x) = f(t_0, x), \; x \in X$$

for all $t > t_0$.

ii) For $0 \le t \le t_0$, we denote $w(t,x) = f(t,x) - f(t_0,x)$ for $x \in X$ and $0 < t < t_0$, so that w is a solution to

$$\begin{aligned}
&\partial_t w = \Delta w + S(t,x), \quad in \quad (t,x) \in (0,t_0) \times X,\\
&\partial_\nu w = 0, \quad x \in \partial X,\\
&w(t_0,x) = 0, \quad x \in X,
\end{aligned}$$

where

$$S(t,x) = R[f(t,\cdot)] - R[f(t_0,\cdot)].$$

A direct estimate using (B.1) gives

$$\|S(t,\cdot)\| \le \lambda \|w(t,\cdot)\| \tag{2.1}$$

for $\lambda = \frac{1}{2}(\|a\|_\infty + 3\|b\|_\infty M^2)$.

Suppose for some $t_1 \in (0,t_0)$, $w(t_1,x) \ne 0$ in X, then

$$\Lambda(t) = \frac{\int_X |\nabla_x w|^2 dx}{\int_X w^2 dx}$$

is well defined in a neighborhood of t_1. By a direct calculation we have

$$\begin{aligned}
\frac{d}{dt}\Lambda(t) &= \frac{2\int \nabla_x w \cdot \partial_t \nabla_x w dx}{\int w^2 dx} - \frac{2\int w \partial_t w dx}{\left(\int w^2 dx\right)^2} \int |\nabla_x w|^2 dx\\
&= -\frac{2}{\int w^2 dx}\left[\int \Delta w \partial_t w dx + \frac{\int |\nabla_x w|^2}{\int w^2 dx} \int w \partial_t w dx\right]\\
&= -\frac{2}{\int w^2 dx}\left[\int (\Delta w + \Lambda w)(\Delta w + S) dx\right],
\end{aligned}$$

since $\partial_t w = \Delta w + S$. This gives

$$\begin{aligned}
\frac{d}{dt}\Lambda(t) &= -\frac{2}{\int w^2 dx}\left[\int (\Delta w + \Lambda w)^2 + \int S(\Delta w + \Lambda w) dx\right]\\
&\le -\frac{2}{\int w^2 dx}\int \left(-\frac{1}{4}S^2\right) dx\\
&\le \frac{\int S^2 dx}{2\int w^2 dx} \le \frac{1}{2}\lambda^2.
\end{aligned}$$

Hence, $\Lambda(t)$ must be bounded in $[t_1, \tilde{t}_0)$, where $\tilde{t}_0 \le t_0$ is the first instant in $(t_1, t_0]$ at which $w(t,\cdot) \equiv 0$ in Ω.

On the other hand,

$$\begin{aligned}\frac{d}{dt}\left(\log\frac{1}{\int w^2dx}\right) &= -\frac{2}{\int w^2dx}\int w\partial_t w dx\\ &= -\frac{2}{\int w^2dx}\int w(\Delta w+S)dx\\ &= 2\Lambda(t)-2\frac{\int Swdx}{\int w^2dx}\\ &\le 2\Lambda(t)+2\lambda.\end{aligned}$$

Thus $\int w^2dx \neq 0$ as long as $\Lambda(t)$ is bounded, contradicting to the assumption that $w(\tilde{t}_0,\cdot)\equiv 0$ in Ω. Hence $w\equiv 0$ in $\Omega\times[0,t_0]$, and our proof of (1.12) is now complete.

Define, for any initial data f^0, the usual ω-limit set as

$$\omega(f^0)=\cap_{s>0}\overline{\{f(t,\cdot),\quad t\ge s\}}.$$

The previous analysis shows that if $f^*\in\omega(f^0)$, then f^* must be a steady solution. Results in Theorem 1.2 and (1.12) then imply that the ω-limit set contains only g as the unique point, hence (1.13).

2.3. **Proof of Theorem 1.3: Exponential convergence.** In the case $g>0$, we introduce the auxiliary functional

$$G=\int\left[\frac{f^2-g^2}{2}-g^2\log\left(\frac{f}{g}\right)\right]dx,$$

which is bounded from below

$$G\ge\int\left[\frac{f^2-g^2}{2}-g^2\left(\frac{f}{g}-1\right)\right]dx=\frac{1}{2}\int(f-g)^2dx.$$

A direct calculation gives

$$\begin{aligned}\frac{d}{dt}G &= \int(f^2-g^2)\frac{f_t}{f}dx\\ &= \int(f^2-g^2)\left[\frac{\Delta f}{f}-\frac{\Delta g}{g}-\frac{1}{2}\int b(x,y)(f^2(y)-g^2(y))dy\right]dx\\ &= -\int\left(\left|\nabla_x f-\frac{f}{g}\nabla_x g\right|^2+\left|\nabla_x g-\frac{g}{f}\nabla_x f\right|^2\right)dx\\ &\quad -\frac{1}{2}\int\int(f^2-g^2)(x)b(x,y)(f^2-g^2)(y)dydx\\ &\le -D(f,g),\end{aligned}$$

where

$$D(f,g)=\int g^2\left|\nabla_x\left(\frac{f}{g}\right)\right|^2dx+\frac{1}{2}\int\int(f^2-g^2)(x)b(x,y)(f^2-g^2)(y)dydx.$$

We claim that there exists $\mu>0$ such that

$$D(f,g)\ge\mu\|f/g-1\|_{L^2}^2. \tag{2.2}$$

Assuming that this is correct for the time being, this gives

$$\frac{d}{dt}G \leq -\mu \int \left(\frac{f}{g}-1\right)^2 dx \leq -\frac{2\mu}{g_{max}^2}G.$$

By Gronwall lemma, we have

$$G(t) \leq G(0)\exp\left(-\frac{2\mu}{g_{max}^2}t\right).$$

Hence

$$\|f(t,\cdot)-g(\cdot)\|_{L^2} \leq \sqrt{2G(t)} \leq \sqrt{2G(0)}\left(-\frac{\mu}{g_{max}^2}t\right).$$

We are thus left to prove (2.2). Using the lower bound and the usual Poincaré's inequality we have

$$\int g^2 \left|\nabla_x\left(\frac{f}{g}\right)\right|^2 dx \geq g_{\min}^2 C_X \inf_c \int \left|\frac{f}{g}-c\right|^2 dx,$$

where C_X is a constant depending only on X. As usual the minimum is achieved at $c^* = \frac{1}{|X|}\int_X \left(\frac{f}{g}\right) dx$.

As a consequence it suffices to find μ independent of $c \geq 0$ such that

$$C_X g_{\min}^2 \int \left|\frac{f}{g}-c\right|^2 dx + \frac{1}{2}\|f^2-g^2\|_b^2 \geq \mu\|\frac{f}{g}-1\|_{L^2}^2.$$

If $c=1$, the inequality is obvious for $\mu \leq \mu_1 = C_X g_{\min}^2$.

For $c \neq 1$, we first estimate

$$\begin{aligned}\|f^2-g^2\|_b^2 &= \|f^2-c^2g^2+(c^2-1)g^2\|_b^2\\ &= (c^2-1)^2\|g^2\|_b^2 + 2(c^2-1)\langle f^2-c^2g^2, g^2\rangle_b + \|f^2-c^2g^2\|_b^2\\ &\geq \epsilon(c^2-1)^2\|g^2\|_b^2 - \frac{\epsilon}{1-\epsilon}\|f^2-c^2g^2\|_b^2,\end{aligned}$$

for any $0<\epsilon<1$ by using Young's inequality.

Note that we have

$$\begin{aligned}\|f^2-c^2g^2\|_b^2 &\leq b_\infty\|f-cg\|_{L^2}^2\|f+cg\|_{L^2}^2\\ &\leq b_\infty g_{\max}^2(\|f\|+c\|g\|)^2\|f/g-c\|_{L^2}^2\\ &\leq b_\infty g_{\max}^2\tilde{M}^2(c+1)^2\|f/g-c\|_{L^2}^2; \quad \tilde{M}=\max\{M,\|g\|\}.\end{aligned}$$

Hence

$$\|f^2-g^2\|_b^2 \geq \epsilon(c^2-1)^2\|g^2\|_b^2 - \frac{\epsilon}{1-\epsilon}b_\infty g_{\max}^2\tilde{M}^2(c+1)^2\|f/g-c\|_{L^2}^2.$$

Therefore

$$\begin{aligned}&C_X g_{\min}^2 \int \left|\frac{f}{g}-c\right|^2 dx + \frac{1}{2}\|f^2-g^2\|_b^2\\ &\quad\geq \left(C_X\, g_{\min}^2 - \frac{\epsilon}{1-\epsilon}\, b_\infty\, g_{\max}^2\, \tilde{M}^2(c+1)^2\right)\|f/g-c\|^2 + \epsilon\,(c^2-1)^2\,\|g\|_b^2\\ &\quad\geq \frac{1}{2}\, C_X\, g_{\min}^2\,\|f/g-c\|^2 + \epsilon\,(c^2-1)^2\,\|g\|_b^2,\end{aligned}$$

by taking $\epsilon = \frac{C_X\, g_{\min}^2}{C_X\, g_{\min}^2 + 2\, b_\infty\, g_{\max}^2\, \tilde{M}^2\, (c+1)^2}$.

On the other hand still using the Young inequality that for any $0 < \eta < 1$

$$|a+b|^2 \geq \eta\,|a|^2 - \frac{\eta}{1-\eta}\,|b|^2,$$

one has that

$$\int \left|\frac{f}{g} - c\right|^2 dx = \int \left|\frac{f}{g} - 1 + 1 - c\right|^2 dx \geq \eta \int \left|\frac{f}{g} - 1\right|^2 dx - \frac{\eta|X|}{1-\eta}\,|c-1|^2.$$

This finally gives

$$\begin{aligned} &C_X g_{\min}^2 \int \left|\frac{f}{g} - c\right|^2 dx + \frac{1}{2}\|f^2 - g^2\|_b^2 \\ &\quad \geq \frac{\eta}{2}\, C_X\; g_{\min}^2 \int \left|\frac{f}{g} - 1\right|^2 dx + |c-1|^2 \left(\epsilon\,(c+1)^2\,\|g\|_b^2 - \frac{\eta|X|}{1-\eta}\right). \end{aligned}$$

Therefore it is enough to take η such that

$$\epsilon\,(c+1)^2\,\|g\|_b^2 - \frac{\eta|X|}{1-\eta} \geq 0,$$

giving us

$$C_X g_{\min}^2 \int \left|\frac{f}{g} - c\right|^2 dx + \frac{1}{2}\|f^2 - g^2\|_b^2 \geq \mu \int \left|\frac{f}{g} - 1\right|^2 dx,$$

for $\mu = \frac{\eta}{2}\, C_X\; g_{\min}^2$.

Finally we investigate the case when 0 is the only non-negative steady solution, which is the case when $\lambda_1 \geq \frac{1}{2}$ and $\int a dx < 0$. In such case, the convergence rate can also be established, by introducing instead

$$G = \frac{1}{2}\int f^2 dx.$$

A direct calculation by integration by parts gives

$$\frac{d}{dt}G = \int f f_t dx = -\int |\nabla_x f|^2 dx + \frac{1}{2}\int a f^2 dx - \frac{1}{2}\|f^2\|_b^2.$$

If $\lambda_1 > \frac{1}{2}$, then from [5, Theorem 3.11] it follows that there exists $\nu > 0$ such that

$$\int |\nabla_x v|^2 dx - \frac{1}{2}\int a v^2 dx \geq \nu\|v\|^2 \tag{2.3}$$

for any $v \in D(L_1)$. Hence we have

$$\frac{d}{dt}G \leq -\nu\|f\|^2 - \frac{1}{2}\|f^2\|_b^2 \leq -2\nu G.$$

This leads to $G \leq G(0)e^{-2\nu t}$, hence $\|f(t,\cdot)\|^2 \leq \|f_0\|^2 e^{-rt}$ with $r = 2\nu$.

If $\lambda_1 = \frac{1}{2}$, we have

$$\frac{d}{dt}G = -\int |\nabla_x f|^2 dx + \lambda_1 \int a f^2 dx - \frac{1}{2}\|f^2\|_b^2 \leq -\frac{1}{2}\|f^2\|_b^2,$$

where we have used the definition for λ_1 when $\int af^2 > 0$, and the inequality remains valid when $\int af^2 \le 0$. Hence

$$\frac{d}{dt}G \le -\frac{b_m}{2}\|f\|^4 = -2b_m G^2.$$

This upon integration over $[0,t]$ using $\|f\|^2 = 2G$ gives

$$\|f(t,\cdot)\|^2 \le \|f_0\|^2(1 + b_m\|f_0\|^2 t)^{-1}, \quad \forall t > 0.$$

This completes the proof of Theorem 1.3.

Appendix A. Existence of a stationary solution

There are several ways to construct a non-negative solution of (1.11). We give here an example of a variational construction. Let $g \in H^1(X)$ be a weak solution to (1.11) in the sense that

$$-\int \nabla_x g \nabla_x w dx + \int R[g] w dx = 0 \ \ \forall w \in H^1(X) \tag{A.1}$$

and $g \ge 0$, $g \not\equiv 0$. We claim that this weak solution is equivalent to the nonzero critical point of the functional

$$F[w] = \int \left[\frac{1}{4}(b * w^2)w^2 - \frac{1}{2}aw_+^2 + |\nabla_x w|^2\right] dx,$$

where $w_+ = \max(w,0)$. Indeed, a weak solution of (A.1) is obviously a critical point of $F[w]$. Conversely, if $g \in H^1(X)$ is a critical point of $F[w]$, then

$$0 = \langle F'[g], g_- \rangle = 2\int [|\partial_x g_-|^2 + \frac{1}{2}g_-^2(b * g^2))dx = 0,$$

where $g_- = \min(g,0)$. We see that $g_- = 0$, which implies $g \ge 0$. Hence g is a weak solution of (A.1).

We next prove the existence of a minimizer for the variational problem F.

Proposition A.1. *There exists $g \in A := \{g \in H^1(X),\ g \ge 0\}$, such that*

$$F(g) = \inf_{w \in H^1(X)} F[w].$$

Moreover,
(i) If $\int adx \ge 0$ or $\int adx < 0$ with $\lambda_1 < 1/2$, then g is not identically 0;
(ii) If $\int adx < 0$ with $\lambda_1 \ge 1/2$, $g \equiv 0$.

Proof. By Young's inequality, we have

$$\begin{aligned} F[w] &= \frac{1}{4}\int (b * w^2)w^2 dx - \frac{1}{2}\int a\, w_+^2\, dx + \int |\nabla_x w|^2 dx \\ &\ge \frac{1}{4}b_m\|w\|_{L^2}^4 - \frac{1}{2}a_\infty\|w\|_{L^2}^2 + \int |\nabla_x w|^2 dx \\ &\ge \int |\nabla_x w|^2 dx - \frac{a_\infty^2}{4b_m}, \end{aligned}$$

so that F is bounded from below. In fact set $m = \inf_{w\in H^1} F[w]$, then $-\frac{a_\infty^2}{4b_m} \leq m < \infty$. Select a minimizing sequence $\{g_k\}_{k=1}^\infty$ so that

$$\lim_{k\to\infty} F[g_k] = m.$$

Then we have $\sup_k \|\partial_x g_k\|_{L^2} < \infty$, and denoting $C = \sup_k F(g_k)$

$$\frac{1}{4} b_m \|g_k\|_{L^2}^4 \leq \frac{1}{2}\, a_\infty\, \|g_k\|_{L^2}^2 + C,$$

which implies that $\|g_k\|_{L^2} \leq \frac{|a|_\infty}{b_m} + \frac{1}{b_m}\sqrt{|a|_\infty^2 + 2b_m C} < \infty$.

Hence $\{g_k\}$ is a bounded sequence in $H^1(X)$. There exists $g \in H^1(X)$ and a subsequence of $\{g_k\}$ (still denoted g_k) such that

$$g_k \rightharpoonup g \text{ weakly in } \ H^1(X), \qquad g_k \longrightarrow g \text{ strongly in } \ L^2(X).$$

On the other hand, note that

$$|\nabla w_+| \leq |\nabla w|.$$

Therefore

$$F(g_+) \leq F(g),$$

clearly one may always replace g_k by its positive part and as a consequence we may assume that $g_k \geq 0$. Hence $g \geq 0$ as well by strong convergence in L^2.

A direct calculation shows that

$$\int |\partial_x^i g_k|^2 dx - \int |\partial_x^i g|^2 dx - \int |\partial_x^i g - \partial_x^i g_k|^2 dx = 2\int 2\partial_x^i g \cdot (\partial_x^i g_k - \partial_x^i g) dx \to 0, \quad i = 0, 1$$

as $k \to \infty$. Note also that

$$\begin{aligned}
&\int b * |g_k|^2 |g_k|^2 dx - \int b * |g_k - g|^2 |g_k - g|^2 dx \\
&= \int b * (|g_k|^2 - |g_k - g|^2)(|g_k|^2 - |g_k - g|^2) dx \\
&\quad + 2\int b * (|g_k|^2 - |g_k - g|^2)|g_k - g|^2 dx \\
&\to \int b * |g|^2 |g|^2 dx \ as \ k \to \infty.
\end{aligned}$$

These together with $\lim_{k\to\infty} \int a|g_k|^2 dx = \int a|g|^2 dx$, ensure that

$$m = \lim_{k\to\infty} F[g_k] \geq F[g].$$

But by $g \in A$, it follows that

$$F[g] = m = \min_{w\in A} F[w].$$

This proves the existence of a non negative minimizer.

To prove that g is not identically 0, when $\int a dx \geq 0$ or $\int a dx < 0$ yet $\lambda_1 < 1/2$, we discuss case by case, keeping in mind that $F(0) = 0$.

(i) If $\int a dx > 0$, one considers the function identically equal to ϵ with ϵ small

$$F(\epsilon) = \frac{\epsilon^4}{4}\, |X| \int b - \frac{\epsilon^2}{2} \int a < 0,$$

for ϵ small enough;

(ii) If $\int adx = 0$, one takes instead $w = \epsilon(1 + \delta v)$ with v satisfying $\int avdx > 0$ and $\partial_n v = 0$ on ∂X, so that

$$F(w) = \frac{\epsilon^2}{2}\left[\frac{\epsilon^2}{2}\int (b * (1+\delta v)^2)(1+\delta v)^2 dx \right.$$
$$\left. +\delta^2 (2\int |\nabla_x v|^2 dx - \int av^2 dx) - 2\delta \int avdx \right] < 0$$

for ϵ, δ suitable small.

(iii) If $\int adx < 0$ and $\lambda_1 < \frac{1}{2}$, we take $w = \tau\psi$ with $\tau > 0$ so that

$$F[\tau\psi] = \frac{\tau^4}{4}\int (b * \psi^2)\psi^2 dx + \tau^2\left(\lambda_1 - \frac{1}{2}\right)\int a\psi^2 dx$$
$$\leq \frac{\tau^2}{4}\|\psi^2\|_b^2\|\left(\tau^2 - 4\left(\frac{1}{2} - \lambda_1\right)\frac{\int a\psi^2 dx}{\|\psi^2\|_b^2}\right) < 0$$

for τ sufficient small. Hence, in all these three cases the minimizer cannot be 0.

Finally we show 0 is the only minimizer if $\int adx < 0$ and $\lambda_1 \geq \frac{1}{2}$. Note that for any v satisfying $\int av^2 dx \leq 0$ we have

$$F[v] = \frac{1}{4}\int (b * v^2)v^2 dx - \frac{1}{2}\int av^2 dx + \int |\nabla_x v|^2 dx$$
$$\geq \frac{1}{4}\|v^2\|_b^2 + \|\nabla_x v\|^2;$$

otherwise if $\int av^2 dx > 0$, then

$$F[v] \geq \frac{1}{4}\|v^2\|_b^2 + \left(\lambda_1 - \frac{1}{2}\right)\int av^2 dx \geq \frac{1}{4}\|v^2\|_b^2.$$

That is we have $F[v] \geq 0 = F[0]$, in such case 0 is the only minimizer.

Finally note that if a and b are not constant then the Euler-Lagrange equation for the variational problem, which is (1.11), does not admit constants as solution. Thus in that case the minimizer cannot be constant either. $\square$

Appendix B. Proof of Theorem 1.1

We begin with a formal a priori estimate, then we indicate how to make the argument rigorous.

(a) The existence theory is based on the following a priori estimate for solutions to (1.1). It is obtained after multiplying the equation by $2f$.

$$\frac{d}{dt}\int f^2 dx + 2\int |\nabla f|^2 dx = \int f^2\left(a - \int b(x,y)f^2(y)dy\right)$$
$$\leq \|a\|_\infty \int f^2 dx - b_m\left(\int f^2 dx\right)^2$$
$$\leq b_m \int f^2 dx\left(\gamma - \int f^2 dx\right).$$

Here $b_m = \inf_{x,y\in X} b(x,y) > 0$ and $\gamma := \frac{\|a\|_\infty}{b_m}$. A direct Gronwall lemma then gives

$$\int f^2(t,x)dx \le \frac{\gamma}{1+(\gamma\|f_0\|_{L^2}^{-2}-1)e^{-\|a\|_\infty t}} \le \max\left\{\int f_0^2 dx,\ \gamma\right\} =: M^2.$$

In addition,

$$\int_0^t \int |\nabla f|^2 dx d\tau \le \frac{1}{2}\int f_0^2 dx + \frac{\|a\|_\infty^2}{8b_m} t, \quad t>0.$$

(b) In order to prove the existence of a solution with the above properties, we consider a standard fixed point and extension argument. For $T>0$, let

$$\Gamma := \{f \in C([0,T];L^2(X)), \|f\| \le M, \quad 0\le t\le T\},$$

where we recall that

$$\|f\| = \sup_{t\in[0,\ T]} \|f(t,.)\|_{L^2(X)}.$$

The set Γ is not empty (since $0\in\Gamma$), and Γ is closed. If $f\in\Gamma$, then $\|f\|\le M$. If $f_1, f_2 \in \Gamma$, then

$$\begin{aligned} R[f_1]-R[f_2] &= \frac{1}{2}\,a\,(f_1-f_2) - \frac{1}{2}\,f_1\,b*f_1^2 + \frac{1}{2}\,f_2\,b*f_2^2 \\ &= \frac{1}{2}\,a\,(f_1-f_2) - \frac{1}{2}\,(f_1-f_2)\,b*f_1^2 + \frac{1}{2}\,f_2\,b*(f_2^2-f_1^2). \end{aligned}$$

It follows that
(B.1)

$$\begin{aligned} \|R[f_1]-R[f_2]\| &\le \frac{1}{2}\|a\|_\infty\|f_1-f_2\| + \frac{1}{2}\|f_1-f_2\|\|b\|_\infty\|f_1\|^2 + \frac{1}{2}\|b\|_\infty\|f_2\|\|f_1-f_2\|\|f_1+f_2\| \\ &\le \lambda\|f_1-f_2\|, \end{aligned}$$

for $\lambda = \frac{1}{2}(\|a\|_\infty + 3\|b\|_\infty M^2)$.

Let $T=\frac{1}{2\lambda}$ and observe that T depends only on $\|f_0\|$, a and b. Define a mapping Φ on $C([0,T];L^2(X))$ onto itself by

$$w = \Phi[f],$$

where w is the unique solution in $C([0,T];L^2(X))$ to

$$\partial_t w = \Delta w + R[f], \quad \partial_\nu w|_{\partial X} = 0,$$

where obviously $R[f]\in C([0,T];L^2(X))$. Observe that a fixed point of Φ is a solution of (1.1).

We claim that Φ maps Γ onto itself. To see this, let $f\in\Gamma$, then

$$\|R[f]\| \le \lambda\|f\|.$$

Therefore

$$\frac{d}{dt}\int w^2 dx + 2\int |\nabla w|^2 dx = 2\int wR[f]dx \le 2\lambda\|w\|\|f\|.$$

Hence $\|w\| \le T\lambda\|f\| \le T\lambda M = \frac{1}{2}M$ which proves our claim.

Next, we show that Φ is a contraction on Γ. If $f_1, f_2\in\Gamma$, we have $w_i = \Phi[f_i]$, and $v = w_1 - w_2$ solves

$$\partial_t v = \Delta v + R[f_1] - R[f_2].$$

Hence

$$\frac{d}{dt}\int v^2 dx + 2\int |\nabla_x v|^2 = 2\int v(R[f_1] - R[f_2])dx$$
$$\leq 2\|v\|\|R[f_1] - R[f_2]\|,$$

which thanks to (B.1) leads to

$$\frac{d}{dt}\|v\| \leq \lambda\|f_1 - f_2\|. \tag{B.2}$$

which upon integration gives

$$\|v\| \leq \lambda T\|f_1 - f_2\| \leq \frac{1}{2}\|f_1 - f_2\|,$$

and so Φ is a contraction on Γ. We may apply Banach's fixed point theorem to conclude that Φ has a unique fixed point in Γ.

Note that we have also proved that if $f_0 \in L^2(X)$, then for any solution f we have $\|f(t,\cdot)\|_{L^2(X)} \leq M$. This allows us to conclude that the solution of the problem exists for all time. To see this, we apply the above local existence result repeatedly on $T + n\sigma \leq t \leq T + (n+1)\sigma$, where $\sigma = \sigma(\|f(t,\cdot)\|_{L^2(X)})$, and eventually we obtain a solution for all time.

Moreover, we have from (B.2) that if f_1 and f_2 are two solutions

$$\|(f_1 - f_2)(t,\cdot)\| \leq e^{\lambda t}\|(f_1 - f_2)(0,\cdot)\|, \quad \forall t > 0.$$

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