Uniform spectral convergence of the stochastic Galerkin method for the linear transport equations with random inputs in diffusive regime and a micro-macro decomposition based asymptotic preserving method*

Shi Jin[†], Jian-Guo Liu [‡]and Zheng Ma [§]

August 8, 2016

Abstract

In this paper we study the stochastic Galerkin approximation for the linear transport equation with random inputs and diffusive scaling. We first establish uniform (in the Knudsen number) stability results in the random space for the transport equation with uncertain scattering coefficients, and then prove the uniform spectral convergence (and consequently the sharp stochastic Asymptotic-Preserving property) of the stochastic Galerkin method. A micro-macro decomposition based fully discrete scheme is adopted for the problem and

*Research was supported by NSF grants DMS-1522184, DMS-1514826, and DMS-1107291 RNMS KI-Net. Shi Jin was also supported by NSFC grant No. 91330203, and by the Office of the Vice Chancellor for Research and Graduate Education at the University of Wisconsin-Madison with funding from the Wisconsin Alumni Research Foundation.

[†]Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706, USA (jin@math.wisc.edu) and Department of Mathematics, Institute of Natural Sciences, MOE-LSEC and SHL-MAC, Shanghai Jiao Tong University, Shanghai 200240, China.

[‡]Department of Physics and Department of Mathematics, Duke University, Durham, NC 27708, USA (jian-guo.liu@duke.edu)

[§]Department of Mathematics, Shanghai Jiao Tong Unviersity, Shanghai 200240, China.

proved to have a uniform stability. Numerical experiments are conducted to demonstrate the stability and asymptotic properties of the method.

Key words. linear transport equation, random inputs, diffusion limit, uncertainty quantification, stochastic Galerkin method, polynomial chaos, asymptotic-preserving scheme.

AMS subject classifications.

1 Introduction

We consider the linear transport equation in one dimensional slab geometry:

$$\varepsilon\partial_t f + v\partial_x f = \frac{\sigma}{\varepsilon}\mathcal{L}f - \varepsilon\sigma^a f + \varepsilon S, \qquad \sigma(x,z) \geq \sigma_{\min} > 0, \tag{1}$$

$$\mathcal{L}f(t,x,v,z) = \frac{1}{2}\int_{-1}^{1} f(t,x,v',z)\,\mathrm{d}v' - f(t,x,v,z). \tag{2}$$

This equation arises in neutron transport, radiative transfer etc. that describes particles (for example neutrons) transport in a background media (for example nuclei), in which $f(t,x,t,z)$ is the density distribution of particles at time $t \geq 0$, position $x \in (0,1)$. $v = \Omega \cdot e_x = \cos\theta \in [-1,1]$ where θ is the angle between the moving direction and x-axis. $\sigma(x,z)$, $\sigma^a(x,z)$ are total and absorption cross-section respectively. $S(x,z)$ is the source term. ε is the dimensionless Knudsen number, the ratio between particle mean free path and the characteristic length (such as the length of the domain). The Dirichlet boundary conditions are given in the incoming direction by

$$\begin{aligned} f(0,v,t) &= f_{\mathrm{L}}(v,t), \qquad \text{for } v \geq 0\,, \\ f(1,v,t) &= f_{\mathrm{R}}(v,t), \qquad \text{for } v \leq 0\,, \end{aligned} \tag{3}$$

while the initial condition is given by

$$f|_{t=0}(x,v,z) = f_0(x,v,z). \tag{4}$$

We are interested in the problem that contains *uncertainty* in the collision cross-section, source, initial or boundary data. The uncertainty is characterized by the random variable $z \in \mathbb{R}^d$ with probability density function $\omega(z)$. Thus in our problem f, σ, σ^a and S all depend on z.

In recent years, there have been extensive activities to study partial differential equations or engineering problems with uncertainties. Many numerical methods have been introduced. In this article, we are interested in the polynomial chaos (originally introduced in Wiener's work [24]) based stochastic Galerkin method which has been shown to be competitive in many applications, see [5, 26, 25]. The stochastic Galerkin method has been used for linear transport equation with uncertain coefficients [4]. Here we are interested in the problem that contains both uncertainty and multiscale. The latter is characterized by the Kundsen number ε, which, in the so-called optically thin region ($\varepsilon \ll 1$), due to high scattering rate of particles, leads the linear transport equation to a diffusion equation, known as the diffusion limit [18, 3, 1]. For the past decades, developing asymptotic-preserving (AP) schemes for (deterministic) linear transport equation with diffusive scaling has seen many activities, see for examples [19, 20, 6, 10, 9, 17, 21, 8]. Only recently AP scheme for linear transport equation with both uncertainty and diffusive scaling was introduced in [15] (in the framework of stochastic Galerkin method, coined as s-AP method). See more related recent works along this line in [7, 12, 13]. A scheme is s-AP if the stochastic Galerkin method for the linear transport equation, as $\varepsilon \to 0$, becomes a stochastic Galerkin method for the limiting diffusion equation. It was realized in [15] that the deterministic AP framework can be easily adopted to study linear transport equations with uncertain coefficients. Moreover, as shown in [7, 12], kinetic equations, linear or nonlinear, *could preserve the regularity in random space of the initial data at later time*, which naturally leads to spectral accuracy of the stochastic Galerkin method.

When $\varepsilon \ll 1$, however, the energy estimates and consequently the convergence rates given in [7, 12] depend on the reciprocal of ε, which implies that one needs the degree of the polynomials used in the stochastic Galerkin method to grow as ε decreases. In fact, this is typical of a standard numerical method for problems that contain small or multiple scales. While AP schemes can be used with numerical parameters independent of ε, to prove this rigorously is not so easy and has been done only in a few occasions [6, 14]. A standard approach to prove a uniform convergence is to use the diffusion limit, as was done first in [6] in the deterministic case and then in [11] for the uncertain transport equation. See also the review article [8]. However such approaches might not give the sharp convergence rate.

In this paper, we provide a *sharp* error estimate for the stochastic Galerkin method for problem (1). This requires a sharp (ε-independent) energy estimate on high order derivatives in the random space for f, as well as $[f] - f$ where $[f]$ is the velocity average of f defined in (5)) which is shown to be bounded even if $\varepsilon \to 0$. Then the uniform in ε spectral convergence naturally follows, without using the diffusion limit.

The s-AP scheme in [15] uses the AP framework of [9] that relies on the even and odd-parity formulation of the transport equation. In this paper, we use the micro-macro decomposition based approach (see [21]) to develop a fully discrete s-AP method. The advantage of this approach is that it allows us to prove *a uniform (in ε) stability* condition, as was done in the deterministic counterpart in [22]. In fact, we will show that one can easily adopt the proof of [22] for the s-AP scheme.

The paper is organized as follows. In Section 2 we summarize the diffusion limit of the linear transport equation. The generalized polynomial chaos based stochastic Galerkin method for the problem is introduced in Section 3 and shown formally to be s-AP. The uniform in ε regularity of the stochastic Galerkin scheme is proven in Section 4, which leads to a uniform spectral convergence proof. The micro-macro decomposition based fully discrete scheme is given in Section 5 and its uniform stability is established in 6. Numerical experiments are carried out in Section 7. The paper is concluded in Section 8.

2 The diffusion limit

Denote

$$[\phi] = \frac{1}{2}\int_{-1}^{1} \phi(v)\,\mathrm{d}v, \tag{5}$$

as the average of velocity dependent function ϕ. For each random realization z, there exists a positive function $\phi(v) > 0$, the so-called absolute equilibrium state that satisfies $[\phi] = 1$, $[v\phi(v)] = 0$, (from Perron-Frobenius theorem, cf. [1]).

Define in the Hilbert space $L^2\big((-1,1);\ \phi^{-1}\,\mathrm{d}v\big)$ the inner product and norm

$$\langle f, g\rangle_\phi = \int_{-1}^{1} f(v)g(v)\phi^{-1}\,\mathrm{d}v, \qquad \|f\|_\phi^2 = \langle f, f\rangle_\phi. \tag{6}$$

The the linear operator $\mathcal{L}$ satisfies the following properties [1]:

- $[\mathcal{L}f] = 0$, for every $f \in L^2([-1,1])$;
- The null space of f is $\mathcal{N}(\mathcal{L}) = \mathrm{Span}\ \{\ \phi \mid \phi = [\phi]\ \}$;

- The range of f is $\mathcal{R}(\mathcal{L}) = \mathcal{N}(\mathcal{L})^\perp = \{ f \mid [f] = 0 \}$;
- $\mathcal{L}$ is non-positive self-adjoint in $L^2((-1,1);\phi^{-1}\,\mathrm{d}v)$, i.e. , there is a positive constant s_m such that

$$\langle f, \mathcal{L} f\rangle_\phi \le -2 s_m \|f\|_\phi^2, \qquad \forall\, f \in \mathcal{N}(\mathcal{L})^\perp; \tag{7}$$

- $\mathcal{L}$ admits a pseudo-inverse, denoted by $\mathcal{L}^{-1}$, from $\mathcal{R}(\mathcal{L})$ to $\mathcal{R}(\mathcal{L})$.

Let $\rho = [f]$. For each fixed z, the classical diffusion limit theory of linear transport equation [18, 3, 1] gives that, as $\varepsilon \to 0$, ρ converges to the following random diffusion equation:

$$\partial_t \rho = \partial_x(\kappa(z)\partial_x \rho) - \sigma^a \rho + S, \tag{8}$$

where the diffusion coefficient

$$\kappa(z) = \frac{1}{3}\sigma(z)^{-1}\,. \tag{9}$$

The micro-macro decomposition, a useful tool for the study of the Boltzmann equation and its fluid dynamics limit [23], and for the design of asymptotic preserving numerical schemes for kinetic equations [16, 2, 21], takes the form

$$f(x,v,z,t) = \rho(x,z,t) + \varepsilon g(x,v,z,t) \tag{10}$$

where $[g] = 0$. Introduce (10) into (1), one gets its micro-macro form:

$$\partial_t \rho + \partial_x\,[vg] = -\sigma^a \rho + S, \tag{11a}$$

$$\partial_t g + \frac{1}{\varepsilon}(I - [.])(v\partial_x g) = -\frac{\sigma(z)}{\varepsilon^2} g - \frac{1}{\varepsilon^2} v \partial_x \rho. \tag{11b}$$

The diffusion limit (8) can be easily seen now. When $\varepsilon \to 0$, (11b) gives

$$g = -\frac{v}{\sigma(z)}\partial_x \rho$$

which, when plugged into (11a), gives the diffusion equation (8)–(9).

3 The gPC-stochastic Galerkin approximation

We assume the complete orthogonal polynomial basis in the Hilbert space $H(\mathbb{R}^d; \omega(z)\,\mathrm{d}z)$ corresponding to the weight $\omega(z)$ is $\{\phi_m(z), m = 0, 1, \cdots, \}$, where $\phi_m(z)$ is a polynomial of degree m and satisfies

$$\langle \phi_i, \phi_j \rangle_\omega = \int \phi_i(z)\phi_j(z)\omega(z)\,\mathrm{d}z = \delta_{ij}.$$

Here $\phi_0(z) = 1$, and δ_{ij} is the Knocker delta function. The inner product and norm in this space are, respectively,

$$\langle f, g \rangle_\omega = \int_{\mathbb{R}^d} f g\, \omega(z)\,\mathrm{d}z, \quad \|f\|_\omega^2 = \langle f, f \rangle_\omega\,. \tag{12}$$

Since the solution $f(x, v, z, t)$ is defined in $L^2\big((0,1)\times(-1,1)\times\mathbb{R}^d; \omega(z)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}z\big)$, one has the generalized Polynomial Chaos (gPC) expansion

$$f(x, v, z, t) = \sum_{i=0}^{\infty} f_i(x, v, t)\,\phi_i(z), \quad \hat{f} = \big(\, f_i \,\big)_{i=1}^{\infty} := \big(\bar{f}, \hat{f}_1\big).$$

The mean and variance of f can be obtained from the expansion coefficients as

$$\bar{f} = E(f) = \int_{\mathbb{R}} f \omega(z)\,\mathrm{d}z = f_0, \qquad \text{var}\,(f) = |\hat{f}_1|^2\,.$$

The idea of the stochastic Galerkin (SG) approximation [5, 26] is to truncate the above infinite series by

$$f_M = \sum_{i=0}^{M} f_i\,\phi_i, \quad \hat{f}^M = \big(\, f_i \,\big)_{i=0}^{M} := \big(\bar{f}, \hat{f}_1^M\big), \tag{13}$$

from which one can extract the mean and variance of f^k from the expansion coefficients as

$$E(f_M) = \bar{f}, \qquad \text{var}\,(f_M) = |\hat{f}_1^M|^2 \le \ \text{var}\,(f)\,.$$

Furthermore, we define

$$\sigma_{ij} = \big\langle \phi_i,\, \sigma\phi_j \big\rangle_\omega, \quad \Sigma = \big(\, \sigma_{ij} \,\big)_{M+1,M+1},$$
$$\sigma_{ij}^a = \big\langle \phi_i,\, \sigma^a\phi_j \big\rangle_\omega, \quad \Sigma^a = \big(\, \sigma_{ij}^a \,\big)_{M+1,M+1},$$

for $0 \le i, j \le M$. Let Id be the $(M+1)\times(M+1)$ identity matrix. Σ, Σ^a are symmetric positive-definite matrices satisfying [25]

$$\Sigma \ge \sigma_{\min} \text{ Id} .$$

If one applies the gPC ansatz (13) into the transport equation (1), and conduct the Galerkin projection, one obtains [4, 15]:

$$\varepsilon \partial_t \hat{f} + v \partial_x \hat{f} = -\frac{1}{\varepsilon}(I - [\cdot])\Sigma \hat{f} - \varepsilon \Sigma^a \hat{f} - \hat{S} \tag{14}$$

where $\hat{S}$ is defined similarly as (13).

We now use the the micro-macro decomposition

$$\hat{f}(x, v, t) = \hat{\rho}(x, t) + \varepsilon \hat{g}(x, v, t) \tag{15}$$

where $\hat{\rho} = [\hat{f}]$ and $[g] = 0$, in (14) to get

$$\partial_t \hat{\rho} + \partial_x [v \hat{g}] = -\Sigma^a \hat{\rho} + \hat{S}, \tag{16a}$$

$$\partial_t \hat{g} + \frac{1}{\varepsilon}(I - [.])(v \partial_x \hat{g}) = -\frac{1}{\varepsilon^2}\Sigma \hat{g} - \frac{1}{\varepsilon^2} v \partial_x \hat{\rho}, \tag{16b}$$

with initial data

$$\hat{\rho}(x, 0) = \hat{\rho}_0(x), \quad \hat{g}(x, v, 0) = \hat{g}_0(x, v) ,$$

that satisfy

$$\frac{1}{2}\int_{-1}^{1} (\hat{\rho}(x, 0) + \varepsilon \hat{g}(x, v, 0))^2 \, \mathrm{d}v = \hat{\rho}(x, 0)^2 + \frac{\varepsilon^2}{2}\int_{-1}^{1} \hat{g}(x, v, 0))^2 \, \mathrm{d}v \le C .$$

It is easy to see that system (16) formally has the diffusion limit as $\varepsilon \to 0$:

$$\partial_t \hat{\rho} = \partial_x (K \partial_x \hat{\rho}) - \Sigma^a \hat{\rho} + \hat{S} , \tag{17}$$

where

$$K = \frac{1}{3}\Sigma^{-1} .$$

Thus the gPC approximation is s-AP in the sence of in [15].

One can easily derive the following energy estimate for system (16)

$$\int_0^1 \hat{\rho}(x, t)^2 \, \mathrm{d}x + \frac{\varepsilon^2}{2}\int_0^1 \int_{-1}^{1} \hat{g}(x, v, t)^2 \, \mathrm{d}v \, \mathrm{d}x$$

$$\leq \int_0^1 \hat{\rho}(x,0)^2 \, \mathrm{d}x + \frac{\varepsilon^2}{2} \int_0^1 \int_{-1}^1 \hat{g}(x,v,0))^2 \, \mathrm{d}v \, \mathrm{d}x \, .$$

On the other hand, the direct gPC approximation of the random diffusion equation (8)–(9) is:

$$\partial_t \hat{\rho} = \partial_x (K_d \partial_x \hat{\rho}) - \Sigma^a \hat{\rho} + \hat{S}, \tag{18}$$

where $K_d = (\kappa_{ij})$, $\kappa_{i,j} = \langle \phi_i, \kappa \phi_j \rangle_\omega$.

4 The regularity in the random space and a uniform spectral convergence analysis of gPC-SG method

In this section, we assume $\sigma^a = S = 0$ for clarity. We prove that, under some suitable assumptions on $\sigma(z)$, the solution to the linear transport equation with random inputs preserves the regularity in the random space of the initial data *uniformly in* ε. Then based on the regularity result, we conduct the spectral convergence analysis and error estimates for the gPC-SG method, and will also obtain error bounds *uniformly in* ε.

4.1 Notations

We first recall the Hilbert space of the random variable introduced in Section 3,

$$H(\mathbb{R}^d;\, \omega \, \mathrm{d}z) = \Big\{ f \mid \mathbb{R}^d \to \mathbb{R}, \, \int_{\mathbb{R}^d} f^2(z) \omega(z) \, \mathrm{d}z < +\infty \Big\}, \tag{19}$$

equipped with the inner product and norm defined in (12). We also define the kth order differential operator with respect to z as

$$D^k f(t,x,v,z) := \partial_z^k f(t,x,v,z), \tag{20}$$

and the Sobolev norm in H as

$$\|f(t,x,v,\cdot)\|_{H^k}^2 := \sum_{\alpha \leq k} \|D^\alpha f(t,x,v,\cdot)\|_\omega^2. \tag{21}$$

Finally, we introduce norms in space and velocity as follows,

$$\|f(t,\cdot,\cdot,\cdot)\|_{\Gamma}^2 := \int_Q \|f(t,x,v,\cdot)\|_\omega^2 \,\mathrm{d}x\,\mathrm{d}v, \qquad t \geq 0, \tag{22}$$

$$\|f(t,\cdot,\cdot,\cdot)\|_{\Gamma^k}^2 := \int_Q \|f(t,x,v,\cdot)\|_{H^k}^2 \,\mathrm{d}x\,\mathrm{d}v, \qquad t \geq 0, \tag{23}$$

where $Q = [0,1] \times [-1,1]$ denotes the domain in the phase space.

4.2 Regularity in the random space

We will study the regularity of f with respect to the random variable z. To this aim, we first prove the following Lemma.

Lemma 4.1. *Assume $\sigma(z) \geq \sigma_{\min} > 0$, then for any integer k and $g \in H$ we have*

$$-\langle D^k(\sigma g), D^k g\rangle_\omega \leq -\frac{\sigma_{\min}}{2}\|D^k g\|_\omega^2 + \frac{4^k}{2\sigma_{\min}}\big(\max_{0\leq\alpha\leq k}\|D^\alpha\sigma\|_{L^\infty}^2\big)\|g\|_{H^{k-1}}^2\,. \tag{24}$$

Proof. Since

$$D^k(\sigma g) = \sum_{\alpha=0}^{k}\binom{k}{\alpha}(D^{k-\alpha}\sigma)(D^\alpha g) = \sigma D^k g + \sum_{\alpha=0}^{k-1}\binom{k}{\alpha}(D^{k-\alpha}\sigma)(D^\alpha g)\,, \tag{25}$$

we have

$$\begin{aligned}
-\langle D^k\big(\sigma g\big), D^k g\rangle_\omega &= -\langle \sigma D^k g, D^k g\rangle_\omega - \Big\langle \sum_{\alpha=0}^{k-1}\binom{k}{\alpha}(D^{k-\alpha}\sigma)(D^\alpha g), D^k g\Big\rangle_\omega \\
&\leq -\sigma_{\min}\|D^k g\|_\omega^2 - \Big\langle \sum_{\alpha=0}^{k-1}\binom{k}{\alpha}(D^{k-\alpha}\sigma)(D^\alpha g), D^k g\Big\rangle_\omega\,.
\end{aligned} \tag{26}$$

By Young's inequality

$$\begin{aligned}
-\Big\langle \sum_{\alpha=0}^{k-1}\binom{k}{\alpha}(D^{k-\alpha}\sigma)(D^\alpha g), D^k g\Big\rangle_\omega &\leq \frac{\sigma_{\min}}{2}\|D^k g\|_\omega^2 \\
&\quad + \frac{1}{2\sigma_{\min}}\Big\|\sum_{\alpha=0}^{k-1}\binom{k}{\alpha}(D^{k-\alpha}\sigma)(D^\alpha g)\Big\|_\omega^2,
\end{aligned} \tag{27}$$

and Cauchy-Schwarz inequality

$$
\begin{aligned}
\Big\| \sum_{\alpha=0}^{k-1} \binom{k}{\alpha} (D^{k-\alpha}\sigma)(D^\alpha g) \Big\|_\omega^2 &\le \Big(\sum_{\alpha=0}^{k-1} \binom{k}{\alpha}^2 \|D^{k-\alpha}\sigma\|_{L^\infty}^2 \Big) \Big(\sum_{\alpha=0}^{k-1} \|D^\alpha g\|_\omega^2 \Big) \\
&\le \Big\{ \sum_{\alpha=0}^{k} \binom{k}{\alpha}^2 \Big\} \max_{0\le\alpha\le k} \|D^\alpha \sigma\|_{L^\infty}^2 \|g\|_{H^{k-1}}^2, \\
&\le 4^k \big(\max_{0\le\alpha\le k} \|D^\alpha \sigma\|_{L^\infty}^2 \big) \|g\|_{H^{k-1}}^2 \,.
\end{aligned}
\tag{28}
$$

Combining (26), (27) and (28), one obtains

$$
-\langle D^k\big(\sigma\cdot g\big), D^k g\rangle \le -\frac{\sigma_{\min}}{2}\|D^k g\|_\omega^2 + \frac{4^k}{2\sigma_{\min}}\big(\max_{0\le\alpha\le k} \|D^\alpha\sigma\|_{L^\infty}^2 \big)\|g\|_{H^{k-1}}^2 \,. \tag{29}
$$

This completes the proof of **Lemma** 4.1. □

Now we are ready to prove the following regularity result,

Theorem 4.1 (**Uniform regularity).** *Assume*

$$
\sigma(z) \ge \sigma_{\min} > 0 \,.
$$

If for some integer $m \ge 0$,

$$
\|D^k \sigma(z)\|_{L^\infty} \le C_\sigma, \qquad \|D^k f_0\|_{\Gamma(0)} \le C_0, \qquad k = 0, \dots, m, \tag{30}
$$

then the solution f to the linear transport equation (1)–(2), with $\sigma^a = S = 0$, satisfies,

$$
\|D^k f\|_{\Gamma(t)} \le C, \qquad k = 0, \cdots, m, \qquad \forall t > 0, \tag{31}
$$

where C_σ, C_0 and C are constants independent of ε.

Proof. For $\sigma^a = S = 0$, the kth ($0 \le k \le m$) order formal differentiation of (1) with respect to z is,

$$
\varepsilon^2 \partial_t (D^k f) + \varepsilon v \partial_x (D^k f) = D^k\big(\sigma(z)([f] - f)\big), \tag{32}
$$

where $[\cdot]$ is the average operator defined in (5). Multiplying $D^k f$ to both sides of (32) and integrating on $Q = [0,1] \times [-1,1]$, one gets

$$
\begin{aligned}
\frac{\varepsilon^2}{2}\partial_t \|D^k f\|_{\Gamma(t)}^2 &+ \varepsilon \int_Q v \langle D^k f, \partial_x(D^k f)\rangle_\omega \,\mathrm{d}x\,\mathrm{d}v \\
&= \int_Q \langle D^k\big(\sigma(z)([f] - f)\big), D^k f\rangle_\omega \,\mathrm{d}x\,\mathrm{d}v \,.
\end{aligned}
\tag{33}
$$

Integration by parts yields

$$\varepsilon \int_Q v\langle D^k f, \partial_x(D^k f)\rangle_\omega \,\mathrm{d}x\,\mathrm{d}v = \frac{\varepsilon}{2}\int_{Q\times\mathbb{R}^d} v\partial_x(D^k f)^2\omega \,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}v = 0. \tag{34}$$

Notice that

$$\int_Q \langle D^k\big(\sigma(z)([f]-f)\big), \big[D^k f\big]\rangle_\omega \,\mathrm{d}x\,\mathrm{d}v = 0\,, \tag{35}$$

combining with (33) one obtains

$$\frac{\varepsilon^2}{2}\partial_t\|D^k f\|^2_{\Gamma(t)} = -\int_Q \langle D^k\big(\sigma(z)([f]-f)\big), D^k([f]-f)\rangle_\omega \,\mathrm{d}x\,\mathrm{d}v \tag{36}$$

Energy estimate: We will establish the following energy estimate by using Mathematical Induction with respect to k: for any $k \geq 0$, there exist k constants $c_{kj} > 0$, $j = 0, \ldots, k-1$ such that

$$\varepsilon^2\partial_t\Big(\|D^k f\|^2_{\Gamma(t)} + \sum_{j=0}^{k-1} c_{kj}\|D^j f\|^2_{\Gamma(t)}\Big) \leq \begin{cases} -2\sigma_{\min}\big\|\,[f]-f\big\|^2_{\Gamma(t)}, & k=0, \\ \\ -\sigma_{\min}\big\|D^k([f]-f)\big\|^2_{\Gamma(t)}, & k\geq 1. \end{cases} \tag{37}$$

When $k = 0$, (36) becomes

$$\begin{aligned} \varepsilon^2\partial_t\|f\|^2_{\Gamma(t)} &= -\int_Q \langle \sigma(z)([f]-f), ([f]-f)\rangle_\omega \,\mathrm{d}x\,\mathrm{d}v \\ &\leq -2\sigma_{\min}\|[f]-f\|^2_{\Gamma(t)}\,, \end{aligned} \tag{38}$$

which satisfies (37).

Assume that for any $k \leq p$ where $p \in \mathbb{N}$, (37) holds. Adding all these inequalities together we get

$$\varepsilon^2\partial_t\Big(\frac{1}{2}\|f\|^2_{\Gamma(t)} + \sum_{i=1}^{p}\|D^i f\|^2_{\Gamma(t)} + \sum_{i=1}^{p}\sum_{j=0}^{i-1} c_{ij}\|D^j f\|^2_{\Gamma(t)}\Big) \leq -\sigma_{\min}\big\|[f]-f\big\|^2_{\Gamma^p(t)}\,, \tag{39}$$

which is equivalent to

$$\varepsilon^2\partial_t\Big(\sum_{j=0}^{p} c'_{p+1,j}\|D^j f\|^2_{\Gamma(t)}\Big) \leq -\sigma_{\min}\big\|[f]-f\big\|^2_{\Gamma^p(t)}\,, \tag{40}$$

where

$$c'_{p+1,j} = \begin{cases} \frac{1}{2} + \sum_{i=1}^{p} c_{i0}, & j = 0, \\ 1 + \sum_{i=1}^{p} c_{ij}, & 1 \le j \le p-1, \\ 1, & j = p. \end{cases} \tag{41}$$

When $k = p + 1$, (36) reads

$$\varepsilon^2 \partial_t \|D^{p+1} f\|^2_{\Gamma(t)} = -2 \int_Q \langle D^{p+1}\big(\sigma(z)([f] - f)\big), D^{p+1}([f] - f)\rangle_\omega \, \mathrm{d}x \, \mathrm{d}v \,. \tag{42}$$

According to **Lemma** 4.1 with $g = D^{p+1}([f] - f)$ and the assumption $\|D^k \sigma(z)\|_{L^\infty} \le C_\sigma$, the right-hand side satisfies the estimate

$$\begin{aligned} \text{RHS} \le & -\sigma_{\min} \int_Q \big\|D^{p+1}([f] - f)\big\|^2_\omega \, \mathrm{d}x \, \mathrm{d}v \\ & + \frac{4^{p+1}}{\sigma_{\min}} \big(\max_{0 \le \alpha \le p+1} \|D^\alpha \sigma\|^2_{L^\infty} \big) \int_Q \big\|[f] - f\big\|^2_{H^p} \, \mathrm{d}x \, \mathrm{d}v \\ \le & -\sigma_{\min} \big\|D^{p+1}([f] - f)\big\|^2_{\Gamma(t)} + \frac{C_\sigma^2 C'_{p+1}}{\sigma_{\min}} \big\|[f] - f\big\|^2_{\Gamma^p(t)} \,. \end{aligned} \tag{43}$$

where $C'_{p+1} = (p + 1)4^{p+1}$. Now we have the estimate

$$\varepsilon^2 \partial_t \|D^{p+1} f\|^2_{\Gamma(t)} \le -\sigma_{\min} \big\|D^{p+1}([f] - f)\big\|^2_{\Gamma(t)} + \frac{C_\sigma^2 C'_{p+1}}{\sigma_{\min}} \big\|[f] - f\big\|^2_{\Gamma^p(t)}. \tag{44}$$

Adding this equation (44) with (40) multiplied by $C_\sigma^2 C'_{p+1}/\sigma^2_{\min}$ gives,

$$\varepsilon^2 \partial_t \Big(\|D^{p+1} f\|^2_{\Gamma(t)} + \sum_{j=0}^{p} c_{p+1,j} \|D^j f\|^2_{\Gamma(t)} \Big) \le -\sigma_{\min} \big\|D^{p+1}([f] - f)\big\|^2_{\Gamma(t)} \,, \tag{45}$$

where

$$c_{p+1,j} = \frac{C_\sigma^2 C'_{p+1}}{\sigma_{\min}} c'_{p+1,j} \,. \tag{46}$$

This shows that (37) still holds for $k = p + 1$. By Mathematical Induction, (37) holds for all integer $k \in \mathbb{N}$.

Finally, according to (37), we have

$$\partial_t \Big(\|D^k f\|^2_{\Gamma(t)} + \sum_{j=0}^{k-1} c_{kj} \|D^j f\|^2_{\Gamma(t)} \Big) \le 0, \qquad c_{kj} > 0, \ k \in \mathbb{N}. \tag{47}$$

which yields

$$
\begin{aligned}
\|D^k f\|^2_{\Gamma(t)} &\le \|D^k f\|^2_{\Gamma(t)} + \sum_{j=0}^{k-1} c_{kj} \|D^j f\|^2_{\Gamma(t)} \\
&\le \|D^k f_0\|^2_{\Gamma(0)} + \sum_{j=0}^{k-1} c_{kj} \|D^j f_0\|^2_{\Gamma(0)} \\
&\le C_0^2 \Big(1 + \sum_{j=0}^{k-1} c_{kj}\Big) := C^2,
\end{aligned} \tag{48}
$$

where C is clearly independent of ε. This completes the proof of the theorem. □

Theorem 4.1 shows the derivatives of the solution with respect to z can be bounded by the derivatives of initial data. In particular, the $\|D^k f\|_{\Gamma(t)}$ bound is *independent* of ε! This is crucial for our later proof that our scheme is s-AP. However, this estimate alon is not sufficient to guarantee the whole gPC-SG method has a spectral convergence uniform in ε (since the projection error is of $O(1/\varepsilon^2)$), which needs a better estimation of $[f] - f$. To this aim, we first provide the following lemma.

Lemma 4.2. *Assume for some integer* $m \ge 0$*,*

$$
\|D^k(\partial_x f_0)\|_{\Gamma(0)} \le C_x, \qquad k = 0, \dots, m, \qquad t > 0. \tag{49}
$$

Then holds:

$$
\int_Q \varepsilon \langle v D^k(\partial_x f), D^k([f] - f)\rangle_\omega \, \mathrm{d}x \, \mathrm{d}v \le \frac{\sigma_{\min}}{4} \|D^k([f] - f)\|^2_{\Gamma(t)} + \frac{C_1 \varepsilon^2}{\sigma_{\min}}. \tag{50}
$$

Proof. First note that $\partial_x f$ satisfies the same equation as f itself,

$$
\varepsilon^2 \partial_t(\partial_x f) + \varepsilon v \partial_x(\partial_x f) = \sigma(z)([\partial_x f] - \partial_x f). \tag{51}
$$

Thus according to the **Theorem 4.1** and our assumption (49),

$$
\|D^k(\partial_x f)\|_{\Gamma(t)} \le C, \qquad t > 0, \tag{52}
$$

with C independent of ε. Then by Young's inequality,

$$
\begin{aligned}
&\int_Q \varepsilon\langle vD^k(\partial_x f), D^k([f]-f)\rangle_\omega \,\mathrm{d}x\,\mathrm{d}v \\
&\le \frac{\sigma_{\min}}{4}\|D^k([f]-f)\|^2_{\Gamma(t)} + \frac{\varepsilon^2}{\sigma_{\min}}\|vD^k(\partial_x f)\|^2_{\Gamma(t)} \\
&\le \frac{\sigma_{\min}}{4}\|D^k([f]-f)\|^2_{\Gamma(t)} + \frac{\varepsilon^2}{\sigma_{\min}}\|D^k(\partial_x f)\|^2_{\Gamma(t)} \\
&\le \frac{\sigma_{\min}}{4}\|D^k([f]-f)\|^2_{\Gamma(t)} + \frac{C_1\varepsilon^2}{\sigma_{\min}},
\end{aligned}
\tag{53}
$$

where $C_1 = C^2$ is a constant. This completes the proof. □

Now we are ready to prove the following theorem.

Theorem 4.2 (**Estimate on** $[\boldsymbol{f}]-\boldsymbol{f}$**).** *With all the assumptions in Theorem 4.1 and Lemma 4.2, for a given time $T>0$, the following regularity result of $[f]-f$ holds:*

$$
\begin{aligned}
&\|D^k([f]-f)\|^2_{\Gamma(t)} \\
&\le \mathrm{e}^{-\sigma_{\min}t/2\varepsilon^2}\|D^k([f_0]-f_0)\|^2_{\Gamma(0)} + C'(T)\varepsilon^2 \\
&\le C(T)\varepsilon^2,
\end{aligned}
\tag{54}
$$

for any $t\in(0,T]$ and $0\le k\le m$,, where $C'(T)$ and $C(T)$ are constants depending on T.

Proof. First notice that $[f]$ satisfies

$$\varepsilon^2\partial_t\,[f] + \varepsilon\partial_x\,[vf] = 0, \tag{55}$$

so $[f]-f$ satisfies the following equation

$$\varepsilon^2\partial_t([f]-f) + \varepsilon\partial_x([vf]-vf) = -\sigma(z)([f]-f). \tag{56}$$

As the proof in **Theorem 4.1**, differentiating this equation k times with respect to z, multiplying by $D^k([f]-f)$ and integrating on Q one obtains,

$$
\begin{aligned}
\varepsilon^2\partial_t\big\|D^k([f]-f)\big\|^2_{\Gamma(t)} =\; &-2\int_Q \varepsilon\langle D^k(\partial_x\,[vf]-v\partial_x f), D^k([f]-f)\rangle_\omega\,\mathrm{d}x\,\mathrm{d}v \\
&-2\int_Q \langle D^k\big(\sigma(z)([f]-f)\big), D^k([f]-f)\rangle_\omega\,\mathrm{d}x\,\mathrm{d}v \\
:=\; & I + II.
\end{aligned}
\tag{57}
$$

Notice that

$$\int_Q \varepsilon\langle D^k(\partial_x [vf]), D^k([f]-f)\rangle_\omega \,\mathrm{d}x\,\mathrm{d}v = 0, \tag{58}$$

and using **Lemma 4.2**, we have

$$I \le \frac{\sigma_{\min}}{2}\|D^k([f]-f)\|^2_{\Gamma(t)} + \frac{2C_1\varepsilon^2}{\sigma_{\min}}. \tag{59}$$

For the second part by **Lemma 4.1**,

$$II \le -\sigma_{\min}\big\|D^k([f]-f)\big\|^2_{\Gamma(t)} + \frac{C_\sigma^2 4^k}{\sigma_{\min}}\big\|[f]-f\big\|^2_{\Gamma^{k-1}(t)}. \tag{60}$$

So we get the following estimate,

$$\begin{aligned}\varepsilon^2\partial_t\big\|D^k([f]-f)\big\|^2_{\Gamma(t)} \le\ & -\frac{\sigma_{\min}}{2}\big\|D^k([f]-f)\big\|^2_{\Gamma(t)} + \frac{2C_1\varepsilon^2}{\sigma_{\min}} \\ & + \frac{C_\sigma^2 4^k}{\sigma_{\min}}\big\|[f]-f\big\|^2_{\Gamma^{k-1}(t)}.\end{aligned} \tag{61}$$

To prove the Thoerem we use Mathematical Induction. When $k=0$ (61) turns to

$$\partial_t\big\|\,[f]-f\big\|^2_{\Gamma(t)} \le -\frac{\sigma_{\min}}{2\varepsilon^2}\big\|[f]-f\big\|^2_{\Gamma(t)} + \frac{2C_1}{\sigma_{\min}}. \tag{62}$$

By Grönwall's inequality,

$$\begin{aligned}\big\|\,[f]-f\big\|^2_{\Gamma(t)} &\le \mathrm{e}^{-\sigma_{\min}t/2\varepsilon^2}\big\|[f_0]-f_0\big\|^2_{\Gamma(0)} + \frac{2C_1 t}{\sigma_{\min}}\varepsilon^2 \\ &\le C_0(T)\varepsilon^2, \qquad \text{for } t>0,\end{aligned} \tag{63}$$

which satisfies the (54).

Assume for any $k \le p$ where $p \in \mathbb{N}$, (54) holds. This implies

$$\big\|[f]-f\big\|^2_{\Gamma^p(t)} \le C_p(T)\varepsilon^2. \tag{64}$$

So when $k = p+1$ by (61),

$$\begin{aligned}\varepsilon^2\partial_t\big\|D^{p+1}([f]-f)\big\|^2_{\Gamma(t)} \le\ & -\frac{\sigma_{\min}}{2}\big\|D^{p+1}([f]-f)\big\|^2_{\Gamma(t)} + \frac{2C_1\varepsilon^2}{\sigma_{\min}} \\ & + \frac{C_\sigma^2 C'_{p+1}}{\sigma_{\min}}C_p(T)\varepsilon^2,\end{aligned} \tag{65}$$

which means

$$\partial_t \big\| D^{p+1}([f]-f) \big\|^2_{\Gamma(t)} \le -\frac{\sigma_{\min}}{2\varepsilon^2} \big\| D^{p+1}([f]-f) \big\|^2_{\Gamma(t)} + C''_{p+1}(T). \tag{66}$$

Again, the Grönwall's inequality yields

$$\begin{aligned} \big\| D^{p+1}([f]-f) \big\|^2_{\Gamma(t)} &\le \mathrm{e}^{-\sigma_{\min} t/2\varepsilon^2} \big\| D^{p+1}([f_0]-f_0) \big\|^2_{\Gamma(0)} + C''_{p+1}(T) T \varepsilon^2 \\ &\le C_{p+1}(T)\varepsilon^2, \qquad \text{for } t>0, \end{aligned} \tag{67}$$

where $C_{p+1}(T)$ is a constant independent of ε. So by Mathematical induction, we complete the proof of the theorem. □

Remark 4.1. We remark that all the above lemma and theorems are proved for $z \in \mathbb{R}$ and σ depending only on z. However, our conclusions and techniques are not limited to these cases. For $z \in \mathbb{R}^d$, it is straightfoward to prove and for $\sigma(x,z)$ also a function of x, we only need to modify the proof of **Lemma 4.2** by using the same technique as in the proof of **Theorem 4.1**.

4.3 A spectral convergence uniformly in ε

Let f be the solution to the linear transport equation (1)–(2). We define the Mth order projection operator

$$\mathcal{P}_M f = \sum_{i=0}^{M} \langle f, \phi_i \rangle_\omega \phi_i.$$

The error arisen from the gPC-SG can be split into two parts r_N and e_N,

$$f - f_M = f - \mathcal{P}_M f + \mathcal{P}_M f - f_M := r_M + e_M, \tag{68}$$

where $r_M = f - \mathcal{P}_M f$ is the truncation error, and $e_M = \mathcal{P}_M f - f_M$ is the projection error.

For the truncation error r_M, we have the following lemma

Lemma 4.3 (**Truncation error**)**.** *Under all the assumption in Theorem 4.1 and Theorem 4.2, we have for $t \in (0,T]$ and any given integer $k = 0, \dots, m$,*

$$\|r_M\|_{\Gamma(t)} \le \frac{C}{M^k}. \tag{69}$$

Moreover,

$$\big\| \, [r_M] - r_M \big\|_{\Gamma(t)} \le \frac{C(T)}{M^k}\varepsilon, \tag{70}$$

where C and $C(T)$ are independent of ε.

Proof. By the standard error estimate for orthogonal polynomial approximations and **Theorem 4.1**, for $0 \le t \le T$,

$$\|r_N\|_{\Gamma(t)} \le CM^{-k}\|D^k f\|_{\Gamma(t)} \le \frac{C}{M^k}, \tag{71}$$

with C independent of M.

In the same way, according to **Theorem 4.2**,

$$\begin{aligned}\big\| [r_M] - r_M\|_{\Gamma(t)} &= \big\|([f]-f) - ([\mathcal{P}_M f] - \mathcal{P}_M f)\big\|_{\Gamma(t)} \\ &\le CM^{-k}\|D^k([f]-f)\|_{\Gamma(t)} \\ &\le \frac{C(T)}{M^k}\varepsilon,\end{aligned} \tag{72}$$

which completes the proof $\square$

It remains to estimate e_N. To this aim, we first notice f_N satisfying

$$\varepsilon^2\partial_t f_M + \varepsilon v\partial_x f_M = \mathcal{P}_M\big\{\sigma(z)([f_M]-f_M)\big\}. \tag{73}$$

On the other hand, by doing the Nth order projection directly on original linear transport equation we get

$$\varepsilon^2\partial_t(\mathcal{P}_M f) + \varepsilon v\partial_x(\mathcal{P}_M f) = \mathcal{P}_M\big\{\sigma(z)([f]-f)\big\}. \tag{74}$$

(74) subtracted by (73) gives

$$\begin{aligned}\varepsilon^2\partial_t e_M + \varepsilon v\partial_x e_M &= \mathcal{P}_M\Big\{\sigma(z)\big\{\,[f]-f-([f_M]-f_M)\big\}\Big\} \\ &= \mathcal{P}_M\Big\{\sigma(z)\big\{\,[f]-f-([\mathcal{P}_M f]-\mathcal{P}_M f) \\ &\qquad + ([\mathcal{P}_M f]-\mathcal{P}_M f) - ([f_M]-f_M)\big\}\Big\} \\ &= \mathcal{P}_M\Big\{\sigma(z)\big(\,[r_M]-r_M\big)\Big\} + \mathcal{P}_M\Big\{\sigma(z)\big(\,[e_M]-e_M\big)\Big\}.\end{aligned} \tag{75}$$

Now we can give the following estimate of the projection error e_N,

Lemma 4.4. *Under all the assumption in Theorem 4.1 and Theorem 4.2, we have for $t \in (0,T]$ and any given integer $k = 0,\dots,m$,*

$$\|e_M\|_{\Gamma(t)} \le \frac{C(T)}{M^k}, \tag{76}$$

where $C(T)$ is a constant independent of ε.

Proof. We use basically the same energy estimate as before: multiply (75) by e_M and integrate on Q, notice that

$$\int_Q \langle \mathcal{P}_M\big\{\sigma(z)\big(\,[r_M]-r_M\big)\big\},[e_M]\rangle_\omega\,\mathrm{d}x\,\mathrm{d}v=0, \tag{77}$$

$$\int_Q \langle \mathcal{P}_M\big\{\sigma(z)\big(\,[e_M]-e_M\big)\big\},[e_M]\rangle_\omega\,\mathrm{d}x\,\mathrm{d}v=0, \tag{78}$$

then one gets

$$\begin{aligned}\varepsilon^2\partial_t\|e_M\|^2_{\Gamma(t)}=&-\int_Q \langle \mathcal{P}_M\big\{\sigma(z)\big(\,[e_M]-e_M\big)\big\},[e_M]-e_M\rangle_\omega\,\mathrm{d}x\,\mathrm{d}v\\&-\int_Q \langle \mathcal{P}_M\big\{\sigma(z)\big(\,[r_M]-r_M\big)\big\},[e_M]-e_M\rangle_\omega\,\mathrm{d}x\,\mathrm{d}v.\end{aligned} \tag{79}$$

Notice the the projection operator $\mathcal{P}_M$ is a self-joint operator

$$\langle \mathcal{P}_M f,g\rangle_\omega=\langle f,\mathcal{P}_M g\rangle_\omega,$$

and

$$\mathcal{P}_M e_M=e_M,$$

thus

$$\begin{aligned}\varepsilon^2\partial_t\|e_M\|^2_{\Gamma(t)}=&-\int_Q \langle \sigma(z)\big(\,[e_M]-e_M\big),[e_M]-e_M\rangle_\omega\,\mathrm{d}x\,\mathrm{d}v\\&-\int_Q \langle \sigma(z)\big(\,[r_M]-r_M\big),[e_M]-e_M\rangle_\omega\,\mathrm{d}x\,\mathrm{d}v\\\le&-\sigma_{\min}\big\|\,[e_M]-e_M\|^2_{\Gamma(t)}+\frac{\sigma_{\min}}{2}\big\|\,[e_M]-e_M\|^2_{\Gamma(t)}\\&+\frac{C_\sigma}{2\sigma_{\min}}\big\|\,[r_M]-r_M\|^2_{\Gamma(t)}\\\le&-\frac{\sigma_{\min}}{2}\big\|\,[e_M]-e_M\|^2_{\Gamma(t)}+\frac{C_\sigma}{2\sigma_{\min}}\Big(\frac{C'(T)}{M^k}\Big)^2\varepsilon^2\\\le&\Big(\frac{C(T)}{M^k}\Big)^2\varepsilon^2,\end{aligned} \tag{80}$$

where for the last two inequalities we have used Young's inequality and **Lemma 4.3**. Then by a integral over t we get

$$\|e_M\|^2_{\Gamma(t)}\le\|e_M^0\|^2_{\Gamma(0)}+\Big(\frac{C(T)}{M^k}\Big)^2, \tag{81}$$

since $e_M^0=\mathcal{P}_M f_0-f_M^0=0$ we complete the proof of this lemma. □

Finally, we are now ready to state the main convergence theorem:

Theorem 4.3 (**Uniform convergence in** ε). *Assume*

$$\sigma(z) \geq \sigma_{\min} > 0\,.$$

If for some integer $m \geq 0$*,*

$$\|\sigma(z)\|_{H^k} \leq C_\sigma, \quad \|D^k f_0\|_{\Gamma(0)} \leq C_0, \quad \|D^k(\partial_x f_0)\|_{\Gamma(0)} \leq C_x, \quad k = 0, \dots, m, \tag{82}$$

Then the error of the whole gPC-SG method is

$$\|f - f_M\|_{\Gamma(t)} \leq \frac{C(T)}{M^k}, \tag{83}$$

where $C(T)$ *is a constant independent of* ε*.*

Proof. From **Lemma 4.3** and **Lemma 4.4**, one has

$$\|f - f_M\|_{\Gamma(t)} \leq \|r_M\|_{\Gamma(t)} + \|e_M\|_{\Gamma(t)} \leq \frac{C(T)}{M^k},$$

which completes the proof. □

Remark 4.2. Theorem 4.3 gives a uniformly in ε spectral convergence rate, thus one can choose M independent of ε, a very strong s-AP property. If anisotropic scattering, namely σ depends on v, then one usually obtains a convergence rate that requires $M \gg \varepsilon$ (see for example [12]). In such cases the proof of s-AP property is much harder, and one usually needs to use the diffusion limit, see [6] in the case of deterministic case and [11] in the random case.

5 The Full discretization

As pointed out in [15], by using the gPC-SG formulation, one obtains a vector version of the original deterministic transport equation. This enables one to use the deterministic AP scheme. In this paper, we adopt the AP scheme developed in [21] for the gPC-SG system (16).

We take a uniform grid $x_i = ih, i = 0, 1, \cdots N$, where $h = 1/N$ is the grid size, and time steps $t^n = n\Delta t$. ρ_i^n is the approximation of ρ at the grid point (x_i, t^n) while $g^{n+1}_{i+\frac{1}{2}}$ is defined at a staggered grid $x_{i+1/2} = (i + 1/2)h$, $i = 0, \cdots N - 1$.

The fully discrete scheme for the gPC system (11) is

$$\frac{\hat{\rho}^{n+1}_i - \hat{\rho}^n_i}{\Delta t} + \left[v \frac{\hat{g}^{n+1}_{i+\frac{1}{2}} - \hat{g}^{n+1}_{i-\frac{1}{2}}}{\Delta x} \right] = -\Sigma^a_i \hat{\rho}^{n+1}_i + \hat{S}_i, \tag{84a}$$

$$\frac{\hat{g}^{n+1}_{i+\frac{1}{2}} - \hat{g}^n_{i+\frac{1}{2}}}{\Delta t} + \frac{1}{\varepsilon \Delta x}(I - [.]) \left(v^+(\hat{g}^n_{i+\frac{1}{2}} - \hat{g}^n_{i-\frac{1}{2}}) + v^-(\hat{g}^n_{i+\frac{3}{2}} - \hat{g}^n_{i+\frac{1}{2}}) \right) \tag{84b}$$
$$= -\frac{1}{\varepsilon^2} \Sigma_i \hat{g}^{n+1}_{i+\frac{1}{2}} - \frac{1}{\varepsilon^2} v \frac{\hat{\rho}^n_{i+1} - \hat{\rho}^n_i}{\Delta x}.$$

It has the formal diffusion limit when $\varepsilon \to 0$ as can be easily checked, which is

$$\frac{\hat{\rho}^{n+1}_i - \hat{\rho}^n_i}{\Delta t} - K \frac{\hat{\rho}^n_{i+1} - 2\hat{\rho}^n_i + \hat{\rho}^n_{i-1}}{\Delta x^2} = -\Sigma^a_i \hat{\rho}^{n+1}_i + \hat{S}_i, \tag{85}$$

where $K = \frac{1}{3}\Sigma^{-1}$. This is the fully discrete scheme for (17). Thus the scheme is stochastically AP as defined in [15].

We will also state the following proposition which will be used later.

Proposition 5.1. $\left[\hat{g}^n_{i+\frac{1}{2}}\right] = 0$ *for every* n.

6 The uniform stability

One important property for an AP scheme is to have a stability condition independent of ε, so one can take $\Delta t \gg O(\varepsilon)$ when ε becomes small. In this section we prove such a result. The proof basically follows that of [22] for the deterministic problem.

For clarity in this section we assume $\sigma^a = S = 0$. The main theoretical result about the stability is the following theorem:

Theorem 6.1. *Denote*

$$\sigma_{ij} = \langle \phi_i, \sigma \phi_j \rangle_\omega, \quad \Sigma = (\sigma_{ij}), \quad \Sigma \geq \sigma_{\min}\ Id\ .$$

If Δt *satisfies the following CFL condition*

$$\Delta t \leq \frac{\sigma_{\min}}{3} \Delta x^2 + \frac{2\varepsilon}{3} \Delta x, \tag{86}$$

then the sequences $\hat{\rho}^n$ and $\hat{g}^n$ defined by scheme (84) satisfy the energy estimate

$$\Delta x \sum_{i=0}^{N-1} \left((\hat{\rho}_i^n)^2 + \frac{\varepsilon^2}{2} \int_{-1}^{1} \left(\hat{g}^n_{i+\frac{1}{2}} \right)^2 \mathrm{d}v \right) \leq \Delta x \sum_{i=0}^{N-1} \left(\left(\hat{\rho}_i^0\right)^2 + \frac{\varepsilon^2}{2} \int_{-1}^{1} \left(\hat{g}^0_{i+\frac{1}{2}} \right)^2 \mathrm{d}v \right)$$

for every n, and hence the scheme (84) is stable.

Remark 6.1. Since the right hand side of (86) has a lower bound when $\varepsilon \to 0$ (and the lower bound being that of the stability condition of the discrete diffusion equation (85)), the scheme is asymptotically stable and Δt remains finite even if $\varepsilon \to 0$.

6.1 Notations and useful lemma

We give some useful notations for norms and inner products that are used in our analysis. For every grid function $\mu = (\mu_i)_{i=0}^{N-1}$ define:

$$\|\mu\|^2 = \Delta x \sum_{i=0}^{N-1} \mu_i^2 \, . \tag{87}$$

For every velocity dependent grid function $v \in [-1, 1] \mapsto \phi(v) = (\phi_{i+\frac{1}{2}}(v))_{i=0}^{N-1}$, define:

$$|||\phi||| = \Delta x \sum_{i=0}^{N-1} \left[\phi^2_{i+\frac{1}{2}} \right] . \tag{88}$$

If ϕ and ψ are two velocity dependent grid functions, their inner product is defined as:

$$\langle \phi \, , \, \psi \rangle = \Delta x \sum_{i=0}^{N-1} \left[\phi_{i+\frac{1}{2}} \psi_{i+\frac{1}{2}} \right] . \tag{89}$$

Now we give some notations for the finite difference operators that are used in scheme (84). For every grid function $\phi = (\phi_{i+\frac{1}{2}})_{i\in\mathbb{Z}}$, we define the following one-sided difference operators:

$$D^- \phi_{i+\frac{1}{2}} = \frac{\phi_{i+\frac{1}{2}} - \phi_{i-\frac{1}{2}}}{\Delta x} \quad \text{and} \quad D^+ \phi_{i+\frac{1}{2}} = \frac{\phi_{i+\frac{3}{2}} - \phi_{i+\frac{1}{2}}}{\Delta x} \tag{90}$$

We also define the following centered difference operators:

$$D^c \phi_{i+\frac{1}{2}} = \frac{\phi_{i+\frac{3}{2}} - \phi_{i-\frac{1}{2}}}{2\Delta x} \quad \text{and} \quad D^0 \phi_i = \frac{\phi_{i+\frac{1}{2}} - \phi_{i-\frac{1}{2}}}{\Delta x} (= D^- \phi_{i+\frac{1}{2}}). \tag{91}$$

Finally, for every grid function $\mu = (\mu_i)_{i\in\mathbb{Z}}$, define the following centered operator:

$$\delta^0 \mu_{i+\frac{1}{2}} = \frac{\mu_{i+1} - \mu_i}{\Delta x}. \tag{92}$$

We first recall some basic facts. For every grid functions $\phi = (\phi_{i+\frac{1}{2}})_{i=0}^{N-1}$, $\psi = (\psi_{i+\frac{1}{2}})_{i=0}^{N-1}$, and $\mu = (\mu_i)_{i=0}^{N-1}$, one has (see [22]):

$$\left(v^+ D^- + v^- D^+\right) \phi_{i+\frac{1}{2}} = v D^c \phi_{i+\frac{1}{2}} - \frac{\Delta x}{2} |v| D^- D^+ \phi_{i+\frac{1}{2}}; \tag{93}$$

$$\Delta x \sum_{i\in\mathbb{Z}} \left(D^+ \phi_{i+\frac{1}{2}}\right)^2 \le \frac{4}{\Delta x^2} \Delta x \sum_i \phi^2_{i+\frac{1}{2}}; \tag{94}$$

$$\left| \left\langle \left(v^+ D^+ + v^- D^-\right) \psi \, , \, \phi \right\rangle \right| \le \alpha |||\phi|||^2 + \frac{1}{4\alpha} ||||v| D^+ \psi|||^2, \ \forall \alpha > 0; \tag{95}$$

$$\Delta x \sum_{i\in\mathbb{Z}} \mu_i D^0 \phi_i = -\Delta x \sum_{i\in\mathbb{Z}} \left(\delta^0 \mu_{i+\frac{1}{2}}\right) \phi_{i+\frac{1}{2}}; \tag{96}$$

$$\Delta x \sum_{i\in\mathbb{Z}} \psi_{i+\frac{1}{2}} D^- \phi_{i+\frac{1}{2}} \Delta x = -\Delta x \sum_{i\in\mathbb{Z}} \left(D^+ \psi_{i+\frac{1}{2}}\right) \phi_{i+\frac{1}{2}}; \tag{97}$$

$$\Delta x \sum_{i\in\mathbb{Z}} \phi_{i+\frac{1}{2}} D^c \phi_{i+\frac{1}{2}} = 0; \tag{98}$$

$$\text{If} \quad g \in L^2([-1,1]), \quad \text{then} \quad [vg]^2 \le \frac{1}{2} \left[|v| g^2\right]. \tag{99}$$

6.2 Energy estimates

Now we provide the details of the energy estimate. The proof is similar to that for deterministic problem in [22].

First, multiplying (84a) and (84b) by $\hat{\rho}^{n+1}$ and $\varepsilon^2 \hat{g}^{n+1}$, respectively. With the assumption that $\sigma_i^a = 0$, $\hat{S}_i = 0$, and using the fact that $\Sigma \ge \sigma_{\min} Id$, one has

$$\begin{aligned}
&\frac{1}{2\Delta t} \left(\|\hat{\rho}^{n+1}\|^2 - \|\hat{\rho}^n\|^2 + \|\hat{\rho}^{n+1} - \hat{\rho}^n\|^2\right) + \Delta x \sum_{i=0}^{N-1} \hat{\rho}_i^{n+1} D^0 \left[v \hat{g}_i^{n+1}\right] \\
&\quad + \frac{\varepsilon^2}{2\Delta t} \left(|||\hat{g}^{n+1}|||^2 - |||\hat{g}^n|||^2 + |||\hat{g}^{n+1} - \hat{g}^n|||^2\right) \\
&\quad + \varepsilon \left\langle \hat{g}^{n+1} \, , \, \left(v^+ D^- + v^- D^+\right) \hat{g}^n \right\rangle \\
&\le - \sigma_{\min} |||\hat{g}^{n+1}|||^2 + \Delta x \sum_{i=0}^{N-1} \left[v D^0 \hat{g}_i^{n+1}\right] \hat{\rho}_i^n.
\end{aligned}$$

Combining the second term on the left hand side and the last term on the right hand side, one gets

$$\begin{aligned}\frac{1}{2\Delta t}&\left(\|\hat\rho^{n+1}\|^2-\|\hat\rho^{n}\|^2+\|\hat\rho^{n+1}-\hat\rho^{n}\|^2\right)\\&+\frac{\varepsilon^2}{2\Delta t}\left(|||\hat g^{n+1}|||^2-|||\hat g^{n}|||^2+|||\hat g^{n+1}-\hat g^{n}|||^2\right)\\&+\varepsilon\left\langle \hat g^{n+1}\,,\,\left(v^+D^-+v^-D^+\right)\hat g^n\right\rangle\\ \le&-\sigma_{\min}|||\hat g^{n+1}|||^2+\Delta x\sum_{i=0}^{N-1}\left[vD^0\hat g_i^{n+1}\right](\hat\rho_i^n-\hat\rho_i^{n+1}).\end{aligned}$$

Using the Cauchy-Schwartz inequality,

$$\Delta x\sum_{i=0}^{N-1}\left[vD^0\hat g_i^{n+1}\right](\hat\rho_i^n-\hat\rho_i^{n+1})\le\frac{1}{2\Delta t}\|\hat\rho^{n+1}-\hat\rho^n\|^2+\frac{\Delta t}{2}\Delta x\sum_{i=0}^{N-1}\left[vD^0\hat g_i^{n+1}\right]^2.$$

This gives

$$\begin{aligned}\frac{1}{2\Delta t}&\left(\|\hat\rho^{n+1}\|^2-\|\hat\rho^{n}\|^2\right)+\frac{\varepsilon^2}{2\Delta t}\left(|||\hat g^{n+1}|||^2-|||\hat g^{n}|||^2+|||\hat g^{n+1}-\hat g^{n}|||^2\right)\\&+\varepsilon\left\langle \hat g^{n+1}\,,\,\left(v^+D^-+v^-D^+\right)\hat g^n\right\rangle\\ \le&-\sigma_{\min}|||\hat g^{n+1}|||^2+\frac{\Delta t}{2}\Delta x\sum_{i=0}^{N-1}\left[vD^0\hat g_{i+\frac12}^{n+1}\right]^2.\end{aligned}$$

We take the following decomposition

$$\begin{aligned}&\left\langle \hat g^{n+1}\,,\,\left(v^+D^-+v^-D^+\right)\hat g^n\right\rangle=\left\langle \hat g^{n+1}\,,\,\left(v^+D^-+v^-D^+\right)\hat g^{n+1}\right\rangle\\&+\left\langle \hat g^{n+1}\,,\,\left(v^+D^-+v^-D^+\right)(\hat g^n-\hat g^{n+1})\right\rangle=:A+B,\end{aligned}$$

where

$$A=\frac{\Delta x}{2}\Delta x\sum_{i=0}^{N-1}\left[|v|\left(D^+\hat g_{i+\frac12}^{n+1}\right)^2\right].$$

$$B=-\left\langle\left(v^+D^++v^-D^-\right)\hat g^{n+1}\,,\,\hat g^n-\hat g^{n+1}\right\rangle.$$

Using the Cauchy-Schwartz inequality,

$$|B|\le\frac{\varepsilon}{2\Delta t}|||\hat g^{n+1}-\hat g^n|||^2+\frac{\Delta t}{2\varepsilon}||||v|D^+\hat g^{n+1}|||^2.$$

This leads to

$$\frac{1}{2\Delta t}\left(\|\hat{\rho}^{n+1}\|^2-\|\hat{\rho}^n\|^2\right)+\frac{\varepsilon^2}{2\Delta t}\left(|||\hat{g}^{n+1}|||^2-|||\hat{g}^n|||^2\right)$$
$$+\varepsilon\frac{\Delta x}{2}\sum_{i=0}^{N-1}\left[|v|\left(D^+\hat{g}^{n+1}_{i+\frac{1}{2}}\right)^2\right]\Delta x-\frac{\Delta t}{2}|||\,|v|D^+\hat{g}^{n+1}|||^2$$
$$\leq -\sigma_{\min}|||\hat{g}^{n+1}|||^2+\frac{\Delta t}{2}\Delta x\sum_{i=0}^{N-1}\left[vD^0\hat{g}^{n+1}_{i+\frac{1}{2}}\right]^2.$$

Since $|v|\leq 1$,

$$\frac{\Delta t}{2}|||\,|v|D^+\hat{g}^{n+1}|||^2\leq\frac{\Delta t}{2}\Delta x\sum_{i=0}^{N-1}\left[|v|\left(D^+\hat{g}^{n+1}_{i+\frac{1}{2}}\right)^2\right],$$

$$\frac{\Delta t}{2}\Delta x\sum_{i\in\mathbb{Z}}\left[vD^0\hat{g}^{n+1}_{i+\frac{1}{2}}\right]^2\leq\frac{\Delta t}{4}\Delta x\sum_{i=0}^{N-1}\left[|v|\left(D^+\hat{g}^{n+1}_{i+\frac{1}{2}}\right)^2\right].$$

These imply

$$\frac{1}{2\Delta t}\left(\|\hat{\rho}^{n+1}\|^2-\|\hat{\rho}^n\|^2\right)+\frac{\varepsilon^2}{2\Delta t}\left(|||\hat{g}^{n+1}|||^2-|||\hat{g}^n|||^2\right)$$
$$\leq -\sigma_{\min}|||\hat{g}^{n+1}|||^2+\left(\frac{3\Delta t}{4}-\varepsilon\frac{\Delta x}{2}\right)\Delta x\sum_{i=0}^{N-1}\left[|v|\left(D^+\hat{g}^{n+1}_{i+\frac{1}{2}}\right)^2\right].$$

Note

$$\left(\frac{3\Delta t}{4}-\varepsilon\frac{\Delta x}{2}\right)\Delta x\sum_{i=0}^{N-1}\left[|v|\left(D^+\hat{g}^{n+1}_{i+\frac{1}{2}}\right)^2\right]\leq\left(\frac{3\Delta t}{4}-\varepsilon\frac{\Delta x}{2}\right)_+\Delta x\sum_{i=0}^{N-1}\left[\left(D^+\hat{g}^{n+1}_{i+\frac{1}{2}}\right)^2\right]$$
$$\leq\left(\frac{3\Delta t}{4}-\varepsilon\frac{\Delta x}{2}\right)_+\frac{4}{\Delta x^2}|||\hat{g}^{n+1}|||^2,$$

where $(a)_+=\max(0,a)$ denotes the positive part of a. Applying this in (6.2) then gives

$$\frac{1}{2\Delta t}\left(\|\hat{\rho}^{n+1}\|^2-\|\hat{\rho}^n\|^2\right)+\frac{\varepsilon^2}{2\Delta t}\left(|||\hat{g}^{n+1}|||^2-|||\hat{g}^n|||^2\right)$$
$$\leq\left(\left(\frac{3\Delta t}{4}-\varepsilon\frac{\Delta x}{2}\right)_+\frac{4}{\Delta x^2}-\sigma_m\right)|||\hat{g}^{n+1}|||^2.$$

This means that we have the final energy estimate

$$\|\hat{\rho}^{n+1}\|^2 + \varepsilon^2 |||\hat{g}^{n+1}|||^2 \leq \|\hat{\rho}^n\|^2 + \varepsilon^2 |||\hat{g}^n|||^2$$

if Δt is such that

$$\left(\frac{3\Delta t}{4} - \varepsilon\frac{\Delta x}{2}\right)^+ \frac{4}{\Delta x^2} \leq \sigma_{\min}.$$

Since $\sigma_{\min} > 0$, an equivalent condition is $(\frac{3\Delta t}{4} - \varepsilon\frac{\Delta x}{2})\frac{4}{\Delta x^2} \leq \sigma_{\min}$, which gives the sufficient condition

$$\Delta t \leq \frac{\Delta x^2 \sigma_{\min}}{3} + \frac{2}{3}\varepsilon\Delta x\,.$$

This completes the proof of **Theorem 6.1**.

7 Numerical Examples

In this section, we present several numerical examples to illustrate the effectiveness of our method.

We consider the linear transport equation with random coefficient $\sigma(z)$:

$$\varepsilon\partial_t f + v\partial_x f = \frac{\sigma(z)}{\varepsilon}([f] - f), \quad 0 < x < 1\,, \tag{100}$$

with initial condition:

$$f(x, v, 0, z) = 0\,,$$

and the boundary conditions are:

$$f(0, v, t, z) = 1, \quad v \geq 0; \qquad f(1, v, t, z) = 0, \quad v \leq 0.$$

7.1 Example 1

First we consider a random coefficient with one dimensional random parameter:

$$\sigma(z) = 2 + z, \quad z \text{ uniformly distributed in } (-1, 1).$$

The limiting random diffusion equation for the kinetic equation (100) is

$$\rho_t = \frac{1}{3\sigma(z)}\rho_{xx}\,, \tag{101}$$

with initial condition and boundary conditions:

$$\rho(0,t,z)=1, \quad \rho(1,t,z)=0, \quad \rho(x,t,z)=0.$$

The analytical solution for (101) with the given initial and boundary conditions in the is

$$\rho(x,t,z)=\frac{3}{2}-\frac{1}{2}\,\mathrm{erf}\left(\frac{x}{2\sqrt{\sigma(z)t}}\right). \tag{102}$$

When ε is small, we use this as the reference solution, as it is accurate with an error of $O(\varepsilon^2)$. Hereafter we set $\varepsilon = 10^{-8}$. In addition, the standard 30-points Gauss-Legendre quadrature set is used for the velocity space to compute ρ in the following example.

To examine the accuracy, we use two error norms: the differences in the mean solutions and in the corresponding standard deviation, with ℓ^2 norm in x:

$$e_{mean}(t)=\left\|\mathbb{E}[u^h]-\mathbb{E}[u]\right\|_{\ell^2},$$
$$e_{std}(t)=\left\|\sigma[u^h]-\sigma[u]\right\|_{\ell^2},$$

where u^h, u are the numerical solutions and the reference solutions, respectively.

In Figure 1, we plot the errors in mean and standard deviation of the gPC numerical solutions at $t = 0.01$ with different gPC orders. Three sets of results are included: solutions with $\Delta x = 0.04$ (squares), $\Delta x = 0.02$ (circles), $\Delta x = 0.01$ (stars). We always use $\Delta t = 0.0002/3$. One observes that the errors become smaller with finer mesh. One can see that the solutions decay rapidly in N and then saturate where spatial discretization error dominates. It is then obvious that the errors due to gPC expansion can be neglected at order $N = 4$ even for $\varepsilon = 10^{-8}$. The solution profiles of the mean and standard deviation are shown on the left and right of Figure 2, respectively.

We also plot the profiles of the mean and standard deviation of the flux vf in Figure 3. Here we observe good agreement among the gPC-Galerkin method, stochastic collocation method with 20 Gauss-Legendre quadrature points, and the analytical solution (102).

In Figure 4, we examine the difference between the solution $t = 0.01$ obtained by the 4th-order gPC method with $\Delta x = 0.01$, $\Delta t = \Delta x^2/12$ and the limiting analytical solution (102). As expected, we observe the differences become smaller as ε is smaller in a quadratic fashion, before the numerical errors become dominant.

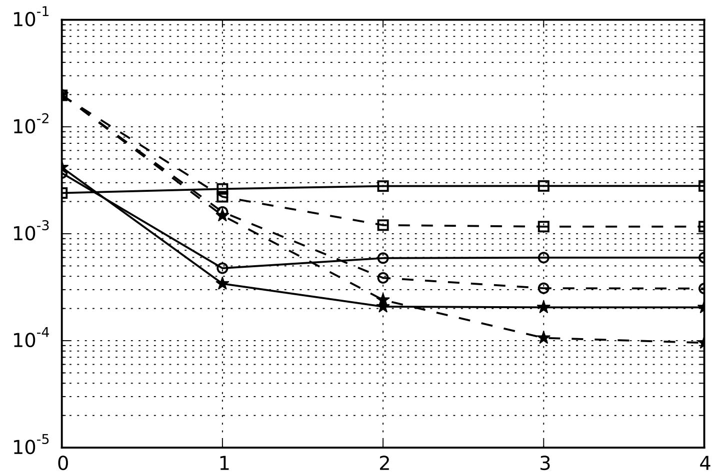

Figure 1: Example 1. Errors of the mean (solid line) and standard deviation (dash line) of ρ with respect to the gPC order at $\varepsilon = 10^{-8}$: $\Delta x = 0.04$ (squares), $\Delta x = 0.02$ (circles), $\Delta x = 0.01$ (stars).

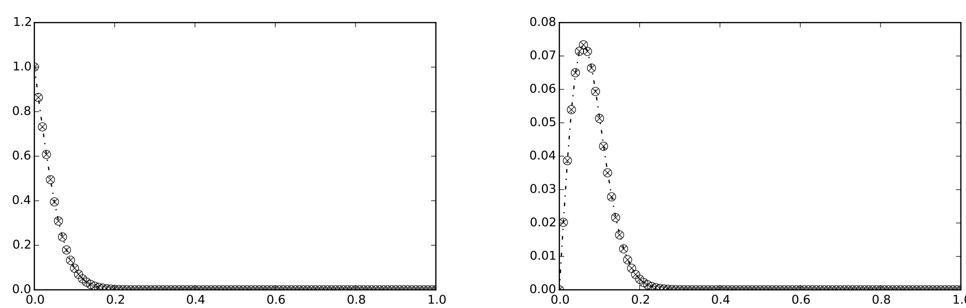

Figure 2: Example 1. The mean (left) and standard deviation (right) of ρ at $\varepsilon = 10^{-8}$, obtained by the gPC Galerkin at order $N = 4$ (circles), the stochastic collocation method (crosses), and the limiting analytical solution (102).

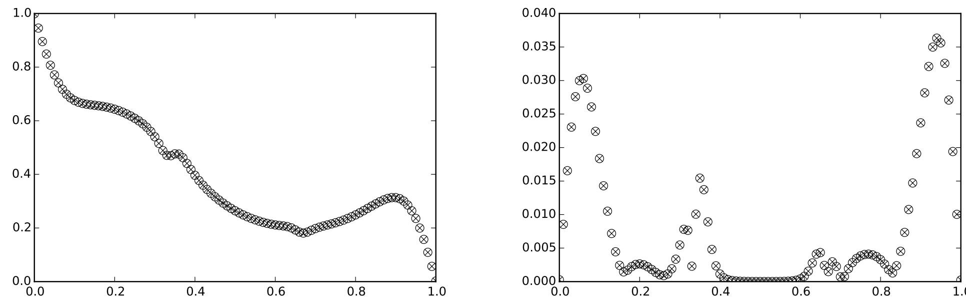

Figure 3: Example 1. The mean (left) and standard deviation (right) obtained by gPC-Galerkin (circle) and collocation method (cross) at time $t = 0.01$

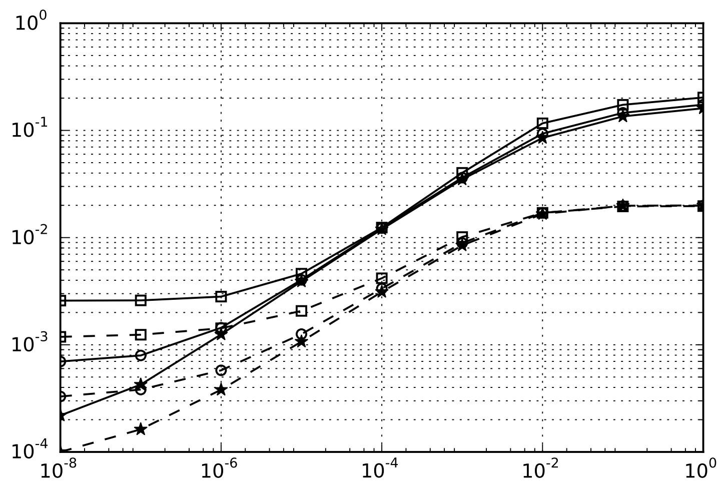

Figure 4: Example 1. Differences in the mean (solid line) and standard deviation (dash line) of ρ with respect to ε^2, between the limiting analytical solution (102) and the 4th-order gPC solution with $\Delta x = 0.04$ (squares), $\Delta x = 0.02$ (circles) and $\Delta x = 0.01$ (stars).

7.2 Example 2: mixing regime

In this test, we still set $\sigma = 2 + z$. We consider $\varepsilon > 0$ depending on the space variable in a wide range of mixing scales:

$$\varepsilon(x) = 10^{-3} + \frac{1}{2}[\tanh(6.5 - 11x) + \tanh(11x - 4.5)] \tag{103}$$

which varies smoothly from 10^{-3} to $O(1)$ as shown in Figure 5. This tests the ability of the scheme for problems with mixing regimes, or its uniform convergence in ε.

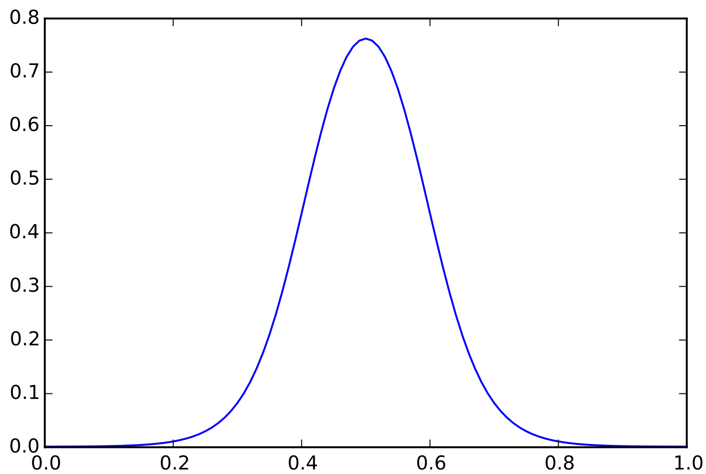

Figure 5: $\varepsilon(x)$

The initial data is

$$f_{\text{in}}(x, v, z) = \frac{\rho_0}{2}\big[\exp\big(-(\frac{v - 0.75}{T_0})^2\big) + \exp\big(-(\frac{v + 0.75}{T_0})^2\big)\big] \tag{104}$$

with

$$\rho_0(x) = \frac{2 + \sin(2\pi x)}{2}, \quad T_0(x) = \frac{5 + 2\cos(2\pi x)}{20}. \tag{105}$$

The reference solution is obtained using collocation method with 30 points. The parameters are set up as the following: the mesh size is $\Delta x = 0.01$, and the corresponding t direction mesh size is $\Delta t = \Delta x^2/3 = 0.00000033$. And we use the 5-th order gPC-Galerkin method to evolve the equation to different time $t = 0.005$, $t = 0.01$, $t = 0.05$, $t = 0.1$. For the v integral, we use Legendre quadrature of 30 points.

Figure 6 shows the ℓ^2 error of the mean and standard deviation with respect to the gPC order. We also see the fast (spectral) convergence of the method.

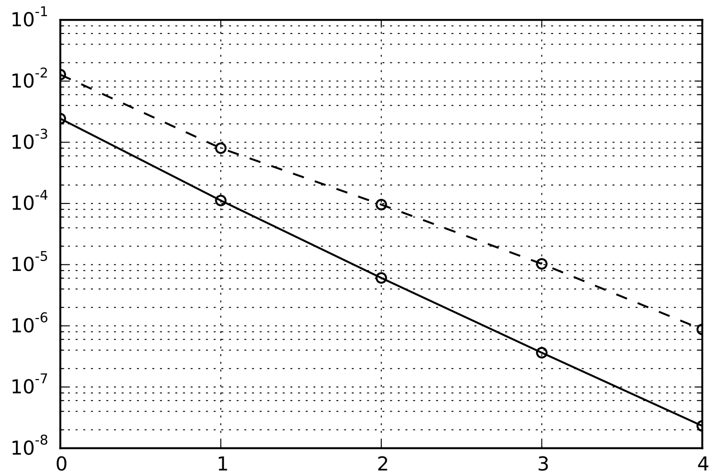

Figure 6: Example 2 with initial data (104)–(105). The l^2 error of mean and standard deviation(dash line) respect to gPC order.

7.3 Example 3: random initial data

We then add randomness on the initial data ($\sigma = 2 + z$ still random).

$$f(x, v, 0, z) = f(x, v, 0) + 0.2z \qquad (106)$$

where $f(x, v, 0)$ is the same as in (104). This time we set $\Delta x = 0.01$ and $\Delta t = \Delta x^2/12$ and final time $T = 0.01$. First we test in the fluid limit regime $\epsilon = 10^{-8}$ as shown in Figure 7. Then we test in $\epsilon = 1$ which is shown in Figure 8. One can see a good agreement between the gPC-SG solutions and the solutions by the collocation method.

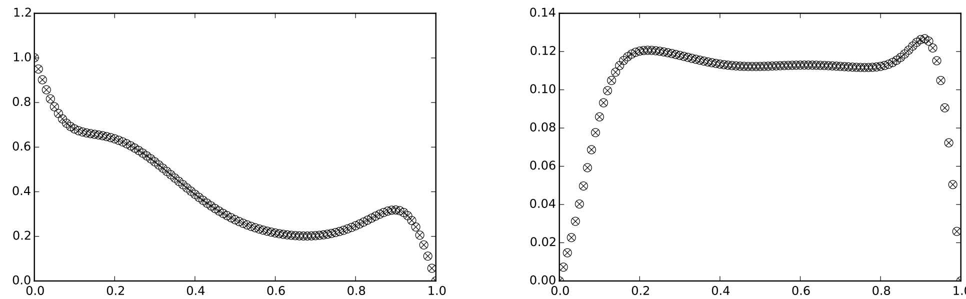

Figure 7: Example 3. The mean (left) and standard deviation (right) obtained by gPC-Galerkin (circle) and collocation method (cross) at time $t = 0.1$, $\epsilon = 10^{-8}$

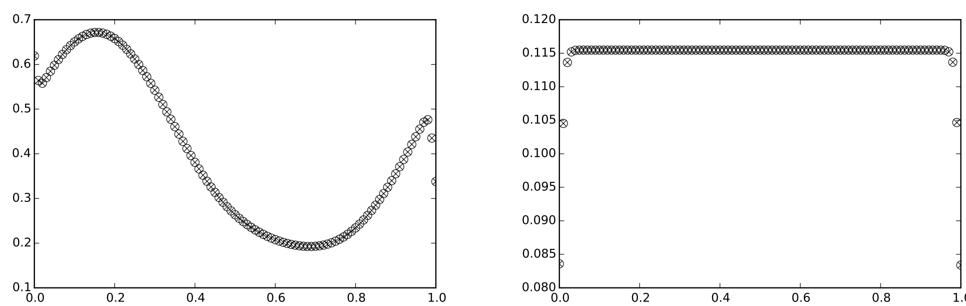

Figure 8: Example 3. The mean (left) and standard deviation (right) obtained by gPC-Galerkin (circle) and collocation method (cross) at time $t = 0.1$, $\epsilon = 1$

7.4 Example 4: random boundary data

For the next example, we then add randomness on the boundary conditions:

$$f_L(v, z) = 2 + z, \quad f_R(v, z) = 1 + z. \tag{107}$$

We also test when $\epsilon = 10^{-8}$ and $\epsilon = 10$ as shown in Figure 9 and Figure 10, again, good agreements are observed between the gPC-SG solutions and the solutions by the collocation

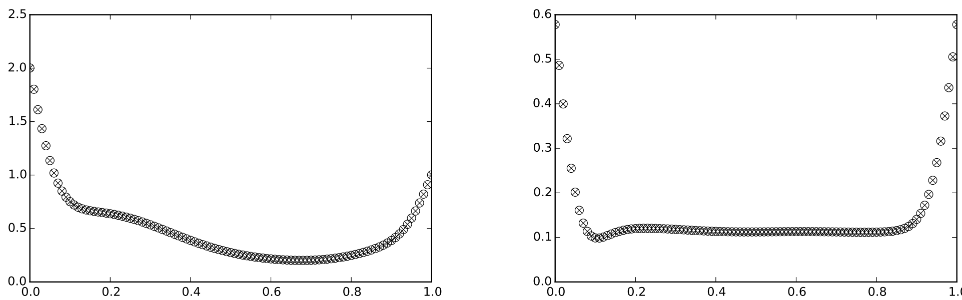

Figure 9: Example 4. The mean (left) and standard deviation (right) obtained by gPC-Galerkin (circle) and collocation method (cross) at time $t = 0.1$, $\epsilon = 10^{-8}$

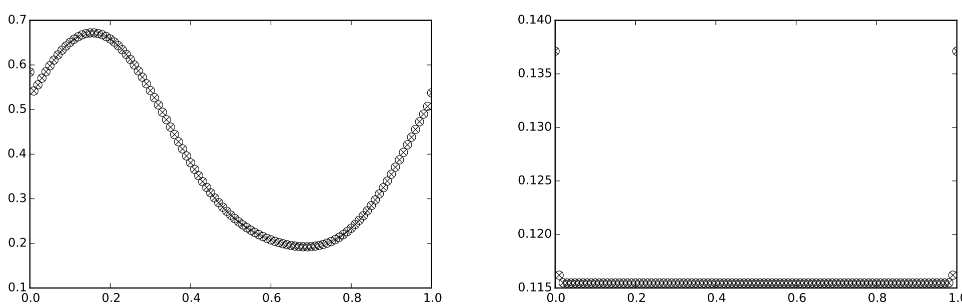

Figure 10: Example 4. The mean (left) and standard deviation (right) obtained by gPC-Galerkin (circle) and collocation method (cross) at time $t = 0.1$, $\epsilon = 10$

7.5 Example 5: 2D random space

Finally, model the random input as a random field, in the following form:

$$\sigma(x, z_1, z_2) = 1 - \frac{\sigma z_1}{\pi^2}\cos(2\pi x) - \frac{\sigma z_2}{4\pi^2}\cos(4\pi x) \tag{108}$$

where we set $\sigma = 4$ and z_1, z_2 are both uniformly distributed in $(-1, 1)$. The mean and standard deviation of the solution ρ at $t = 0.01$ obtained by the 5th-order gPC Galerkin with $\Delta x = 0.025$, $\Delta t = 0.0002/3$ are shown in Figure 11. We then use the high-order stochastic collocation method over 40×40 Gauss-Legendre quadrature points to compute the reference mean and standard deviation of the solutions. In Figure 12, we show the errors of the mean (solid lines) and standard deviation (dash lines) of ρ with respect to the order of gPC expansion. The fast spectral convergence of the errors can be clearly seen.

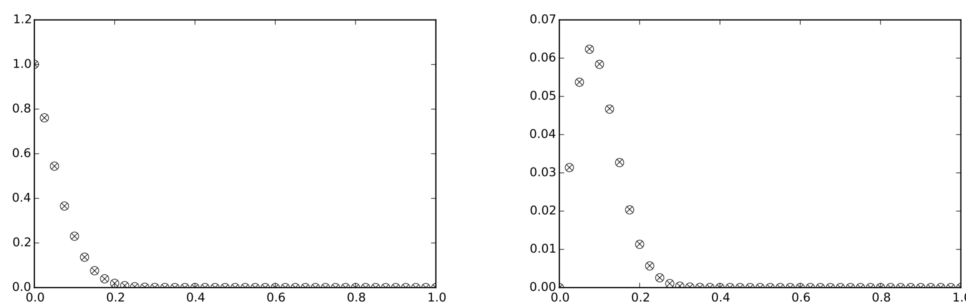

Figure 11: The mean (left) and standard deviation (right) of ρ at $\varepsilon = 10^{-8}$, obtained by 5th-order gPC Galerkin (circles) and the stochastic collocation method (crosses). The random input has dimension $d = 2$.

8 Conclusions

In this paper we establish the uniform spectral accuracy in terms of the Knudsen number, which consequently allows us to justify the stochastic Asymptotic-Preserving property, of the stochastic Galerkin method for the linear transport equation with random scattering coefficients. For the micro-macro decomposition based fully discrete scheme we also prove a uniform

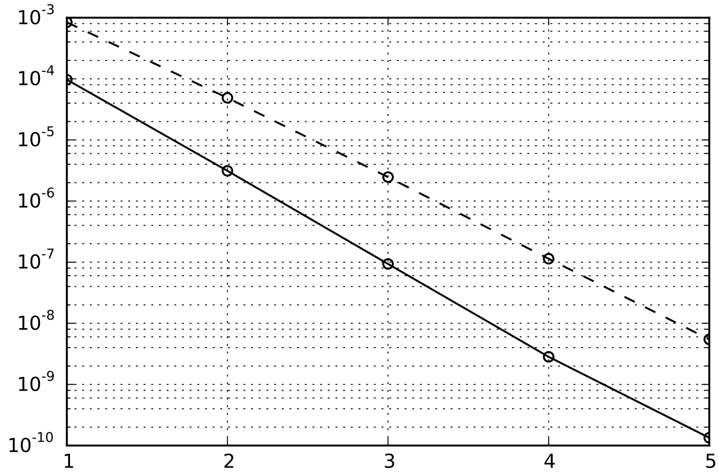

Figure 12: Errors of the mean (solid line) and standard deviation (dash line) of ρ with respect to gPC order, with the $d = 2$ dimensional random input.

stability result. These are the first uniform accuracy and stability results for the underlying problem.

It is expected that our uniform stability proof is useful for more general kinetic or transport equations, which is the subject of our future study.

References

[1] C. Bardos, R. Santos, and R. Sentis. Diffusion approximation and computation of the critical size. *Trans. Amer. Math. Soc.*, 284(2):617–649, 1984.

[2] Mounir Bennoune, Mohammed Lemou, and Luc Mieussens. Uniformly stable numerical schemes for the Boltzmann equation preserving the compressible Navier-Stokes asymptotics. *J. Comput. Phys.*, 227(8):3781–3803, 2008.

[3] Alain Bensoussan, Jacques-Louis Lions, and George Papanicolaou. *Asymptotic analysis for periodic structures*, volume 5 of *Studies in*

Mathematics and its Applications. North-Holland Publishing Co., Amsterdam-New York, 1978.

[4] Anil K. Prinja Erin D. Fichtl and James S. Warsa. Stochastic methods for uncertainty quantification in radiation transport. In *International Conference on Mathematics, Computational Methods & Reactor Physics (M&C 2009), Saratoga Springs, New York, May 3-7, 2009*, 2009.

[5] R.G. Ghanem and P. Spanos. *Stochastic Finite Elements: a Spectral Approach*. Springer-Verlag, 1991.

[6] François Golse, Shi Jin, and C. David Levermore. The convergence of numerical transfer schemes in diffusive regimes. I. Discrete-ordinate method. *SIAM J. Numer. Anal.*, 36(5):1333–1369, 1999.

[7] Jingwei Hu and Shi Jin. A stochastic galerkin method for the boltzmann equation with uncertainty. *J. Comp. Phys.*, to appear.

[8] S. Jin. Asymptotic preserving (ap) schemes for multiscale kinetic and hyperbolic equations: a review. *Lecture Notes for Summer School on Methods and Models of Kinetic Theory(M&MKT), Porto Ercole (Grosseto, Italy)*, 2010.

[9] S. Jin, L. Pareschi, and G. Toscani. Uniformly accurate diffusive relaxation schemes for multiscale transport equations. *SIAM Journal on Numerical Analysis*, 38(3):913–936, 2000.

[10] Shi Jin. Efficient asymptotic-preserving (AP) schemes for some multiscale kinetic equations. *SIAM J. Sci. Comput.*, 21(2):441–454 (electronic), 1999.

[11] Shi Jin, Qin Li, and Li Wang. The uniform convergence of generalized polynomial chaos based methods for multiscale transport equation with random scattering. preprint.

[12] Shi Jin and Liu Liu. An asymptotic-preserving stochastic galerkin method for the semiconductor boltzmann equation with random inputs and diffusive scalings. *Preprint*, 2016.

[13] Shi Jin and Hanqing Lu. An asymptotic-preserving stochastic galerkin method for the radiative heat transfer equations with random inputs and diffusive scalings. preprint.

[14] Shi Jin, Min Tang, and Houde Han. On a uniformly second order numerical method for the one-dimensional discrete-ordinate transport equation and its diffusion limit with interface. *Networks and Heterogeneous Media 4*, pages 35–65, 2009.

[15] Shi Jin, Dongbin Xiu, and Xueyu Zhu. Asymptotic-preserving methods for hyperbolic and transport equations with random inputs and diffusive scalings. *J. Comput. Phys.*, 289:35–52, 2015.

[16] A. Klar and C. Schmeiser. Numerical passage from radiative heat transfer to nonlinear diffusion models. *Math. Models Methods Appl. Sci.*, 11(5):749–767, 2001.

[17] Axel Klar. An asymptotic-induced scheme for nonstationary transport equations in the diffusive limit. *SIAM J. Numer. Anal.*, 35(3):1073–1094 (electronic), 1998.

[18] Edward W. Larsen and Joseph B. Keller. Asymptotic solution of neutron transport problems for small mean free paths. *J. Mathematical Phys.*, 15:75–81, 1974.

[19] Edward W. Larsen and J. E. Morel. Asymptotic solutions of numerical transport problems in optically thick, diffusive regimes. II. *J. Comput. Phys.*, 83(1):212–236, 1989.

[20] Edward W. Larsen, J. E. Morel, and Warren F. Miller, Jr. Asymptotic solutions of numerical transport problems in optically thick, diffusive regimes. *J. Comput. Phys.*, 69(2):283–324, 1987.

[21] Mohammed Lemou and Luc Mieussens. A new asymptotic preserving scheme based on micro-macro formulation for linear kinetic equations in the diffusion limit. *SIAM J. Sci. Comput.*, 31(1):334–368, 2008.

[22] Jian-Guo Liu and Luc Mieussens. Analysis of an asymptotic preserving scheme for linear kinetic equations in the diffusion limit. *SIAM J. Numer. Anal.*, 48(4):1474–1491, 2010.

[23] Tai-Ping Liu and Shih-Hsien Yu. Boltzmann equation: micro-macro decompositions and positivity of shock profiles. *Comm. Math. Phys.*, 246(1):133–179, 2004.

[24] N. Wiener. The homogeneous chaos. *Amer. J. Math.*, 60:897–936, 1938.

[25] D. Xiu. *Numerical methods for stochastic computations.* Princeton Univeristy Press, Princeton, New Jersey, 2010.

[26] D. Xiu and G.E. Karniadakis. The Wiener-Askey polynomial chaos for stochastic differential equations. *SIAM J. Sci. Comput.*, 24(2):619–644, 2002.