LOCAL SENSITIVITY ANALYSIS FOR THE KURAMOTO MODEL WITH RANDOM INPUTS IN A LARGE COUPLING REGIME

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ABSTRACT. Synchronization phenomenon is ubiquitous in strongly correlated oscillatory systems, and the Kuramoto model serves as a prototype synchronization model for phase-coupled oscillators. In this paper, we provide local sensitivity analysis for the Kuramoto model with random inputs in initial data, distributed natural frequencies and coupling strengths, which exhibits the interplay between random effects and synchronization nonlinearity. Our local sensitivity analysis provides some understanding of the robustness of emergent dynamics of the random Kuramoto model in a large coupling regime, including *"propagation and vanishment of uncertainties"* and *"continuous dependence"* of phase and frequency variations in random parameter space with respect to the variations in initial data.

1. Introduction

Complex oscillatory systems often exhibit collective coherent behaviors, e.g., flashing of fireflies, chorusing of crickets, synchronous firing of cardiac pacemaker and metabolic synchrony in yeast cell suspension etc [1, 8, 32, 41]. The rigorous mathematical treatment of such problems began from the pioneering works [29, 41] by Kuramoto and Winfree about half century ago. They introduced simple phase models for weakly coupled limit-cycle oscillators, and showed how collective coherent behavior can emerge from the interplay between intrinsic randomness in natural frequency and nonlinear attractive phase couplings. This coherent motion is often called "*synchronziation*" which means the adjustment of rhythms in an ensemble of weakly coupled oscillators. Recently, the synchronization of oscillators on networks became an emerging research area in different disciplines such as biology, control theory, statistical physics and sociology. After Kuramoto and Winfree's seminal works, several phase models have been used in the phenomenological study of synchronization. Among them, our main interest in this paper lies on the Kuramoto model. We first briefly introduce the Kuramoto model (see Section 2 for its basic mathematical structures).

Let $\theta_i = \theta_i(t)$ be the phase of the i-th limit-cycle oscillator, and we assume that the Kuramoto oscillators are located on a symmetric network whose interaction(connection) topology is denoted by the coupling matrix $K = (\kappa_{ij})$. In this setting, the evolution of

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Date: March 1, 2018.

1991 *Mathematics Subject Classification.* 15B48, 92D25.

Key words and phrases. Cucker-Smale model, flocking, local sensitivity analysis, random communication, uncertainty quantification.

Acknowledgment. The work of S.-Y. Ha was supported by National Research Foundation of Korea(NRF-2017R1A2B2001864), and the work of S. Jin was supported by NSF grants DMS-1522184 and DMS-1107291: RNMS KI-Net, and by the Office of the Vice Chancellor for Research and Graduate Education at the University of Wisconsin.

phases is governed by the first-order system of ordinary differential equations [28, 29]:

$$\dot{\theta}_i = \nu_i + \frac{1}{N}\sum_{j=1}^{N} \kappa_{ij} \sin(\theta_j - \theta_i), \quad t > 0, \tag{1.1}$$

where ν_i is the random natural frequency and κ_{ij} is the symmetric coupling strength between j and i-th oscillators. In previous literature in applied mathematics, control theory, say[1, 15, 23] and references therein, system (1.1) has been studied after the randomness in the natural frequencies is quenched, i.e., ν_i is treated as a time-independent parameter (see [4, 5, 6, 7, 9, 13, 14, 15, 16, 17, 21, 22, 23, 24, 35, 38, 39, 40]) for deterministic data and coupling strengths. However, as one can easily imagine, initial data, natural frequencies and mutual coupling strengths can be uncertain due to incomplete measurement of data and ignorance of exact interaction mechanism between oscillators.

In this paper, in order to address this uncertainty, we employ a UQ (uncertainty quantification) formalism in [25, 26, 30] (and references therein) in the context of synchronization. Recently, UQ analysis has been applied in the collective models in the context of flocking in [2, 10, 18, 20]. In previous studies, most analytical works for (1.1) were restricted to situation where the randomness in natural frequency ν_i is quenched and the coupling strengths are the same constant. Throughout the paper, we consider a more realistic case where the natural frequencies and mutual coupling strengths contain a kind of random component. For this, we introduce random parameters z whose probability density function is given by $g = g(z)$. In this setting, the random phase process $\theta_i(t, z)$ satisfies a random dynamical system:

$$\partial_t \theta_i(t, z) = \nu_i(z) + \frac{1}{N}\sum_{j=1}^{N} \kappa_{ij}(z) \sin(\theta_j(t, z) - \theta_i(t, z)), \quad 1 \le i \le N. \tag{1.2}$$

Note that if randomness in natural frequency and coupling strengths are quenched, then system (1.2) reduces to the deterministic Kuramoto model (1.1) on a symmetric network. Since the R.H.S. of (1.2) is 2π-periodic, the system (1.2) can be regarded as a dynamical system on N-tori $\mathbb{T}^N$. However, if necessary, by lifting the system (1.2) in its covering space $\mathbb{R}^N$, we will regard (1.2) as a dynamical system on $\mathbb{R}^N$. For the proposed random dynamical system (1.2), we are mainly interested in the following questions:

- (Q1): How do the randomness in natural frequencies and coupling strength affect synchronization process?
- (Q2): Are phase-locked states for (1.1) robust in the presence of randomness?

While the mathematical and computational study for self-organization has received tremendous interests in the last decade (see for examples review articles [31, 36]), as far as the authors know, the study of uncertainty quantification for such problems has not been fully addressed in literature until several recent works [2, 10, 18, 20] for the Cucker-Smale model of flocking. The type of analysis conducted here is similar to the ones done in [18], in which the flocking conditions, as well as local sensitivity analysis were studied from the viewpoint of random initial data and communication weights between particles. Such a study not only helps to understand the impact of uncertainty in the dynamic behavior of system under investigation, but it also helps to understand the behavior of numerical approximations for such random systems, since indeed the sensitivity results imply the regularity of the solution in the Sobolev norms which is important to understand the convergence of stochastic algorithms [42]. For notational simplicity, we set

$$z \in \Omega \subset \mathbb{R}, \quad \omega_i := \partial_t \theta_i, \quad \Theta := (\theta_1, \cdots, \theta_N), \quad V := (\omega_1, \cdots, \omega_N), \quad \mathcal{V} := (\nu_1, \cdots, \nu_N).$$

To see the random effect in (1.2), we expand the phase and frequency processes $\theta_i(t, z+dz)$ and via Taylor's expnasion:

$$\begin{aligned} \theta_i(t, z+dz) &= \theta_i(t,z) + \partial_z \theta_i(t,z) dz + \frac{1}{2}\partial_z^2 \theta_i(t,z)(dz)^2 + \cdots, \\ \omega_i(t, z+dz) &= \omega_i(t,z) + \partial_z \omega_i(t,z) dz + \frac{1}{2}\partial_z^2 \omega_i(t,z)(dz)^2 + \cdots \end{aligned} \tag{1.3}$$

Thus, the local sensitivity estimates [33, 34] deal with the dynamic behaviors of the sensitivity vectors $\partial_z^r \Theta$ and $\partial_z^r V$ consisting of coefficients in the R.H.S. of (1.3).

The main results of this papers are two-fold: First, we provide a sufficient framework $(\mathcal{F})$ leading to the uniform bound estimate for the diameter and ℓ^1-stability property of $\partial_z^r \Theta$. Under the framework $(\mathcal{F})$ formulated in terms of initial data, natural frequencies and coupling strengths which are computable from given data and parameters, our results provide the following local sensitivity estimates for $\partial_z^r \Theta$:

- (Uniform bound for the diameter of $\partial_z^r \Theta$): Our first estimate provides the estimate like

$$D(\partial_z^l \Theta(t,z)) \le D(\partial_z^l \Theta^0(z)) e^{-\kappa_m(z)\cos D(\Theta^0(z))t} + C_l(z)(1 - e^{-\kappa_m(z)\cos D(\Theta^0(z))t}), \tag{1.4}$$

  where the random function $C_l(z)$ depends only on given random data and parameters $D(\partial_z^r \mathcal{V}(z))$, $\partial_z^r \kappa_{ij}(z)$ and $D(\partial_z^r \Theta^0(z))$ for $r = 0,\ 1,\ \cdots, l$ (see Theorem 3.1 and (4.1)). In particular, for the ensemble of identical Kuramoto oscillators, we will show a more refined estimate than (1.4):

$$D(\partial_z \Theta(t,z)) \le \mathcal{D}_l(z) e^{-\frac{\kappa_m(z)\cos D(\Theta^0(z))t}{2}}, \quad \text{for every} \quad t \ge 0,$$

  where the random function $\mathcal{D}_l(z)$ depends only on given random data and parameters $\partial_z^r \kappa_{ij}(z)$ and $D(\partial_z^r \Theta^0(z))$ for $r = 0,\ 1,\ \cdots, l$ (see Corollary 3.3 for details).

- (ℓ_1-stability): For two solutions Θ and $\tilde{\Theta}$ to (1.2) with initial data Θ^0 and $\tilde{\Theta}^0$ in random space, respectively: for every $l \in \mathbb{N}$, there exists a nonnegative random variable $\mathcal{E}_l = \mathcal{E}_l(z)$ independent of t and a non-negative functional $\Lambda_l := \Lambda_l(t,z)$ such that

$$\frac{\partial}{\partial t} \|\partial_z^l (\Theta - \tilde{\Theta})(t,z)\|_1 + \Lambda_l(t,z) \le \mathcal{E}_l(z) \sum_{p=0}^{l-1} \|\partial_z^p (\Theta - \tilde{\Theta})(t,z)\|_1,$$

  for every $t \ge 0$. In addition, if we further assume that $\theta_c(0,z) = \tilde{\theta}_c(0,z)$(for the definition of θ_c, see (2.1)), we can find the exponential decay of ℓ^1-difference between two solutions as follows: for every $l \in \mathbb{N}$, there exists a nonnegative random variable

$\tilde{\mathcal{E}}_l := \tilde{\mathcal{E}}_l(z)$ such that

$$\|\partial_z^l(\Theta - \tilde{\Theta})(t,z)\|_1 \le \tilde{\mathcal{E}}_l(z) e^{-\kappa_m(z)\gamma(z)(\cos D(\Theta^0(z)))t} \sum_{p=0}^{l} \|\partial_z^p(\Theta^0 - \tilde{\Theta}^0)(z)\|_1.$$

In general, the aforementioned local sensitivity estimates for $\partial_z^r \Theta$ do not hold in a low coupling regime (see the discussion right after the framework $(\mathcal{F})$ in Section 3). Second, we provide a synchronizing property of the frequency variations $\partial_z^r V$ (see Theorem 5.1) under the same framework $(\mathcal{F})$:

$$D(\partial_z^l V(t,z)) \le \mathcal{F}_l(z) e^{-\frac{\kappa_m(z)\cos D(\Theta^0(z))t}{2}}, \quad \text{for every} \quad t \ge 0,$$

where the random function $\mathcal{D}_l(z)$ depends only on given random data and parameters $\partial_z^r \kappa_{ij}(z)$ and $D(\partial_z^r \Theta^0(z))$ for $r = 0,\ 1,\ \cdots, l$.

The rest of this paper is organized as follows. In Section 2, we provide conservation laws, relative equilibria, gradient flow formulation and pathwise emergent dynamics for the random Kuramoto model (1.2). In Section 3, a uniform bound for the diameter for the phase variation is present, and a uniform ℓ_1-stability of $\partial_z^r \Theta$ is given in Section 4. In Section 5, we present a synchronizing property of $\partial_z^r V$. Finally Section 6 is devoted to a brief summary of our main results and future directions.

Notation: Throughout the paper, we use the following simplified notation: for $Z := (z_1, \cdots, z_N)$ and coupling matrix $K = (\kappa_{ij})$, we set

$$D(Z) := \max_{1 \le i,j \le N} |z_i - z_j|, \quad \|Z\|_p := \Big(\sum_{i=1}^N |z_i|^p \Big)^{\frac{1}{p}}, \quad p \in [1, \infty),$$
$$\|Z\|_\infty := \max_{1 \le i \le N} |z_i|, \quad \kappa_m(z) := \min_{i,j} \kappa_{ij}(z), \quad \|\partial_z^r \kappa(z)\|_\infty := \max_{i,j} |\partial_z^r \kappa_{i,j}(z)|.$$

Let $\pi : \Omega \to \mathbb{R}_+ \cup \{0\}$ be a nonnegative p.d.f. function, and let $y = y(z)$ be a scalar-valued random function defined on Ω. Then, we define the expected value as

$$\mathbb{E}[\varphi] := \int_\Omega \varphi(z)\pi(z)dz,$$

2. Preliminaries

In this section, we study conservation laws and pathwise asymptotic dynamics for the random Kuramoto model (1.2). These estimates are crucial in the local sensitivity analysis in the following three sections.

2.1. **Conservation laws.** First, we consider conservation laws associated with random dynamical system (1.2). In general, for a given dynamical system, it is important to look for conserved quantities which govern overall dynamics of a system. For example, if a Hamiltonian system has enough conserved quantities, then it can be integrable. So far, it is known that the Kuramoto model (1.1) admits two conservation laws, namely the number of oscillators and total sum of phases. Thus, once the complete synchronization happens, where all oscillators rotate with the common frequency, then that constant is given by the average natural frequencies. For given Θ and $\mathcal{V}$, consider a time-dependent random function $\mathcal{C}(\Theta, \mathcal{V}, t)$:

$$\mathcal{C}(\Theta, \mathcal{V}, t) := \sum_{i=1}^N \theta_i - t \sum_{i=1}^N \nu_i.$$

Next, we show that the quantity $\mathcal{C}(\Theta, \mathcal{V}, t)$ is conserved along (1.2).

Lemma 2.1. *Let $\Theta = \Theta(t, z)$ be a random phase vector whose dynamics is governed by the random Kuramoto model* (1.2)*. Then, the quantity $\mathcal{C}(\Theta, \mathcal{V}, t)$ is constant along the path of* (1.2)*: for $z \in \Omega,\ t > 0$,*

$$\partial_t \mathcal{C}(\Theta(t, z), \mathcal{V}(z), t) = 0.$$

Proof. We use the symmetry of $\kappa_{ij} = \kappa_{ji}$ and (1.2) to obtain

$$\partial_t \mathcal{C}(\Theta(t, z), \mathcal{V}(z), t) = \partial_t \Big(\sum_{i=1}^N \theta_i(t, z) - t \sum_{i=1}^N \nu_i(z) \Big) = \sum_{i=1}^N \partial_t \theta_i(t, z) - \sum_{i=1}^N \nu_i(z) = 0,$$

which yields the desired estimate. □

Remark 2.1. *Note that Lemma 2.1 implies*

$$\sum_{i=1}^N \theta_i(t, z) = t \sum_{i=1}^N \nu_i + \sum_{i=1}^N \theta_i^0(z), \quad t \geq 0,\ z \in \Omega.$$

Hence, unless $\sum_{i=1}^N \nu_i$ is zero, the total phase $\sum_{i=1}^N \theta_i$ itself is not a conserved quantity.

Due to the translation invariant property of (1.2), the dynamics for averages and fluctuations around them are completely decoupled in the sense that if one sets

$$\begin{aligned} &\theta_c := \frac{1}{N} \sum_{i=1}^N \theta_i, \quad \nu_c := \frac{1}{N} \sum_{i=1}^N \nu_i, \\ &\tilde{\theta}_i := \theta_i - \theta_c, \quad \tilde{\nu}_i := \nu_i - \nu_c, \quad i = 1, \cdots, N, \end{aligned} \tag{2.1}$$

then θ_c and $\tilde{\theta}_i$ satisfy

$$\begin{aligned} &\partial_t \theta_c(t, z) = \nu_c(z), \\ &\partial_t \tilde{\theta}_i(t, z) = \tilde{\nu}_i(z) + \frac{1}{N} \sum_{j=1}^N \kappa_{ij}(z) \sin(\tilde{\theta}_j(t, z) - \tilde{\theta}_i(t, z)). \end{aligned} \tag{2.2}$$

Note that the fluctuations satisfy the same equation (1.2). Thus, without loss of generality, we will assume zero sum conditions:

$$\sum_{i=1}^N \nu_i = 0, \quad \sum_{i=1}^N \theta_i = 0$$

instead of the system $(2.2)_2$.

2.2. **Relative equilibria.** Note that the equilibrium solution $\Theta = (\theta_1, \cdots, \theta_N)$ to (1.2) is a solution to the following random system: for each $z \in \Omega$,

$$\nu_i(z) + \frac{1}{N}\sum_{j=1}^{N} \kappa_{ij}(z) \sin(\theta_j(t,z) - \theta_i(t,z)) = 0, \quad 1 \le i \le N. \tag{2.3}$$

Due to Remark 2.1, if $\sum_{i=1}^{N} \nu_i \neq 0$, then system (2.3) does not have a solution. This forces us to consider relaxed equilibria.

Definition 2.1. [1, 15, 22] *Let $\Theta(t,z) = (\theta_1(t,z), \cdots, \theta_N(t,z))$ be a time-dependent random phase vector.*

(1) Θ is a random phase-locked state if all relative phase differences are constant over time along the sample path: for $z \in \Omega$,

$$\theta_i(t,z) - \theta_j(t,z) = \theta_i(0,z) - \theta_j(0,z), \quad t \ge 0, \quad 1 \le i,j \le N.$$

(2) Θ exhibits asymptotic phase-locking (complete synchronization) if the relative frequencies tend to zero asymptotically: for $z \in \Omega$,

$$\lim_{t\to\infty} |\partial_t\theta_i(t,z) - \partial_t\theta_j(t,z)| = 0, \quad 1 \le i,j \le N.$$

Note that the random Kuramoto model (1.2) can also be recast as a gradient flow along the sample path. We define a random potential in [22, 24, 37]: for a given random phase vector $\Theta(t,z) = (\theta_1(t,z), \cdots, \theta_N(t,z))$,

$$V(\Theta(t,z)) := -\sum_{k=1}^{N} \nu_k(z)\theta_k(t,z) + \frac{1}{2}\sum_{k,l=1}^{N} \kappa_{ik}(z)\big(1 - \cos(\theta_k(t,z) - \theta_l(t,z))\big). \tag{2.4}$$

Then, it is easy to see that the random Kuramoto model (1.2) can be rewritten as a gradient flow: for each $z \in \Omega$,

$$\partial_t\Theta = -\nabla_\Theta V(\Theta), \quad t > 0. \tag{2.5}$$

For the deterministic case, the gradient flow formulation (2.4) and (2.5) is useful to derive the complete synchronization estimates for generic initial phase configuration in [22] without decay rate. Since the following analysis requires a detailed exponential decay, we will not employ the gradient flow and instead, we will use the framework in [12] where the explicit relaxation rate toward the phase-locked states and uniform ℓ_1-stability have been studied.

2.3. **Pathwise emergent dynamics.** In this subsection, we provide pathwise emergent dynamics of (1.2) which are useful in the following two sections.

Proposition 2.1. *Let $\Theta = \Theta(t,z)$ be a phase vector whose dynamics is governed by the random Kuramoto model (1.2). Then, for a given $z \in \Omega$, the following assertions hold.*

(1) (Identical oscillators) Suppose that the coupling strength, natural frequencies and initial phases satisfy

$$D(\mathcal{V}(z)) = 0, \quad \kappa_m(z) > 0, \quad 0 < D(\Theta^0(z)) < \pi.$$

Then, there exists $\kappa_m > 0$ such that

$$D(\Theta^0(z))e^{-\kappa_m(z)t} \le D(\Theta(t,z)) \le D(\Theta^0(z))e^{-\kappa_m(z)\gamma t}, \quad t \ge 0,$$

where $\gamma(z) := \frac{\sin D(\Theta^0(z))}{D(\Theta^0(z))} \in (0,1)$.

(2) (Nonidentical oscillators) Suppose that the coupling strength, natural frequencies and initial phases satisfy

$$\kappa_m(z) > \frac{D(\mathcal{V}(z))}{\sin D(\Theta^0(z))} > 0, \quad 0 < D(\Theta^0(z)) < \frac{\pi}{2}.$$

Then, we have

$$D(\Theta(t,z)) \le D(\Theta^0(z)), \quad D(V(t,z)) \le D(V^0(z))e^{-\kappa_m(z)\cos D(\Theta^0(z))t}.$$

Proof. The proof is almost the same as in [19] with a slight modification. Thus, we refer to [19] for details. □

Remark 2.2. *Note that the uniform boundedness of $D(V(t,z))$ can be derived directly from* (1.2)*: for $i = 1, \cdots, N$,*

$$|\omega_i(t,z)| = \Big|\nu_i(z) + \frac{1}{N}\sum_{j=1}^N \kappa_{ij}(t,z)\sin(\theta_j^0(z) - \theta_i^0(t,z))\Big| \le \|\nu(z)\|_\infty + \|\kappa(z)\|_\infty,$$

where $\|\nu(z)\|_\infty := \max_i |\nu_i(z)|$, $||\kappa(z)||_\infty := \max_{i,j} |\kappa_{ij}(z)|$. Thus, we have a uniform boundedness for $D(V)$. Similarly, we can also deduce the boundedness of $D(V^l)$ from (3.7)*.*

As a direct application of Proposition 2.1, we obtain statistical estimate for expectation of random phase and frequency configurations.

Corollary 2.1. *Suppose that initial data, natural frequencies and coupling strength satisfy*

$$0 < \sup_{z\in\Omega} D(\Theta^0(z)) \le \frac{\pi}{2} - \varepsilon, \quad \textit{for some} \quad \varepsilon > 0, \quad \sum_{i=1}^N \theta_i^0 = 0, \quad \sum_{i=1}^N \nu_i = 0,$$

$$\sup_{z\in\Omega} D(\mathcal{V}(z)) < \infty, \quad \kappa_m(z) > \frac{D(\mathcal{V}(z))}{\sin D(\Theta^0(z))} \quad \textit{and} \quad \inf_{z\in\Omega}\kappa_m(z) \ge \eta > 0,$$

and let $\Theta = \Theta(t,z)$ be a solution to system (1.2)*. Then, we have*

$$\mathbb{E}[D(\Theta(t))] \le \mathbb{E}[D(\Theta^0)] \quad \textit{and} \quad \mathbb{E}[D(V(t))] \le \mathbb{E}[D(V^0)]e^{-\eta\sin(\varepsilon)t}.$$

Proof. The estimates directly follow from Proposition 2.1. □

2.4. **Elementary key tools.** In this subsection, we study two elementary facts to be crucially used in the later sections. First, for reader's convenience, we quote the formula for the chain rules for higher derivatives of a composition function from [27]. It proof can be made using the mathematical induction. We first introduce an index set: for given positive integer n,

$$\Lambda(n) := \{(k_1, \cdots, k_n) \in (\mathbb{Z}_+ \cup \{0\})^n \ : \ k_1 + 2k_2 + \cdots + nk_n = n\}.$$

Note that $(0, \cdots, 0, 1)$ is an element of $\Lambda(n)$. Then, n-th derivative of $f(g(x))$ is given by the following formula:

$$\begin{aligned} &\frac{d^n}{dx^n} f(g(x)) \\ &\quad = \sum_{(k_1,\cdots,k_n)\in\Lambda(n)} \frac{n!}{k_1!\cdots k_n!} f^{(k)}(g(x))\Big(\frac{g'(x)}{1!}\Big)^{k_1}\Big(\frac{g''(x)}{2!}\Big)^{k_2}\cdots\Big(\frac{g^{(n)}(x)}{n!}\Big)^{k_n}, \end{aligned} \tag{2.6}$$

where $k := k_1 + \cdots + k_n$.

Next, we state a Gronwall type lemma without a proof.

Lemma 2.2. *Let* $y : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\}$ *be a differentiable function satisfying*

$$y' \le -\alpha y + Ce^{-\beta t}, \quad t > 0, \qquad y(0) = y_0, \tag{2.7}$$

where $\alpha > \beta$ *and* C *are non-negative constants. Then* y *satisfies*

$$y(t) \le y_0 e^{-\alpha t} + \frac{C}{\alpha - \beta}(e^{-\beta t} - e^{-\alpha t})$$

Proof. Multiplying (2.7) by $e^{\alpha t}$ and integrating it over $(0, t]$ gives

$$y(t)e^{\alpha t} \le y_0 + \frac{C}{\alpha - \beta}(e^{(\alpha-\beta)t} - 1).$$

Dividing the above equation by $e^{\alpha t}$ yields the desired result.

□

3. A uniform diameter bound for phase process

In this section, we present a uniform bound for sensitivity vectors $\partial_z^l \Theta$ with $l \ge 1$. Note that the diameter $D(\partial_z^l \Theta))$ is given by the relation:

$$D(\partial_z^l \Theta)) = \max_i \partial_z^l \theta_i - \min_i \partial_z^l \theta_i.$$

For $l = 0$, we have already studied the decay and uniform bound estimates of $D(\Theta)$ in Proposition 2.1. For $l \ge 1$, we will use mathematical induction together with modified Gronwall's lemma to derive the bound and decay estimates of $D(\partial_z^l \Theta)$.

Consider the equation for $\partial_z^l \theta_i$ by differentiating (1.2) with respect to z:

$$\partial_t \left(\partial_z^l \theta_i(t, z) \right) = \partial_z^l(\nu_i(z)) + \frac{1}{N} \sum_{\substack{1 \le j \le N \\ 0 \le r \le l}} \binom{l}{r} \partial_z^{l-r} \Big(\kappa_{ij}(z) \Big) \partial_z^r \Big(\sin(\theta_j(t, z) - \theta_i(t, z)) \Big). \tag{3.1}$$

Note that for each $z \in \Omega$ and $l \in \mathbb{N} \cup \{0\}$, we have a real-analytic solution $\partial_z^l \theta_i(\cdot, z)$ to (3.1).

3.0.1. *Nonidentical oscillators.* Now, we state a sufficient framework $(\mathcal{F})$ for the local sensitivity analysis for phase process:

- $(\mathcal{F}1)$: Initial phase processes are confined in a quarter arc and have zero mean:
$$0 < D(\Theta^0(z)) < \frac{\pi}{2}, \quad \sup_{0 \le r \le l} D(\partial_z^r \Theta^0(z)) < \infty, \quad \sum_i \theta_i^0(z) = 0.$$
- $(\mathcal{F}2)$: Natural frequencies satisfy uniform bound and have zero mean:
$$\sup_{0 \le r \le l} D(\partial_z^r \mathcal{V}(z)) < \infty, \quad \sum_i \nu_i(z) = 0.$$
- $(\mathcal{F}3)$: Mutual coupling strengths are sufficiently large such that
$$\kappa_m(z) > \frac{D(\mathcal{V}(z))}{\sin D(\Theta^0(z))}, \quad \max_{0 \le r \le l} \|\partial_z^r \kappa(z)\|_\infty \le \kappa_\infty < \infty.$$

Note that the large coupling condition ($\mathcal{F}3$) is necessary for the uniform boundedness of diameters. For example, in a low coupling regime which is close to zero, the uniform bound for diameter does not hold. Consider the random Kuramoto model (1.2) $\kappa_{ij} = 0$:

$$\partial_t \theta_i(t,z) = \nu_i(z), \quad 1 \le i \le N.$$

Thus, θ_i is completely integrable:

$$\theta_i(t,z) = \theta_i^0(z) + t\nu_i(z), \quad t \ge 0.$$

For a pair of oscillators with $\nu_i \neq \nu_j$,

$$|\theta_i(t,z) - \theta_j(t,z)| \ge t|\nu_i(z) - \nu_j(z)| - |\theta_i^0(z) - \theta_j^0(z)| \to \infty, \quad \text{as } t \to \infty,$$

which means the unboundedness of $D(\Theta)$. The same argument can be applied for $\partial_z^r \Theta$ to derive unboundedness for $\kappa_{ij} = 0$. By the structural stability of (1.2), this unboundedness of diameter also works for low coupling regime $\kappa_{ij} \ll 1$.

We now return to the uniform bound estimate for $D(\partial_z^l \Theta)$. As the first step of the induction, we study the estimate for $D(\partial_z^1 \Theta)$ in the following lemma.

Lemma 3.1. (Uniform bound for $D(\partial_z \Theta)$) *Suppose that the framework* ($\mathcal{F}$) *with* $l = 1$ *holds, and let* $\Theta = \Theta(t,z)$ *be a solution to system* (1.2). *Then, for* $z \in \Omega$, *we have*

$$D(\partial_z \Theta(t,z)) \le D(\partial_z \Theta^0(z)) e^{-\kappa_m(z) \cos D(\Theta^0(z)) t} + C_1(z)(1 - e^{-\kappa_m(z) \cos D(\Theta^0(z)) t}) \quad \forall\, t \ge 0,$$

where the random variable $C_1(z)$ *is given by the following relation:*

$$C_1(z) := \frac{D(\partial_z \mathcal{V}(z)) + 2\|\partial_z \kappa(z)\|_\infty \sin D(\Theta^0(z))}{\kappa_m(z) \cos D(\Theta^0(z))}. \tag{3.2}$$

Proof. Let $\Theta = \Theta(t,z)$ be a solution to system (1.2) with zero sum conditions in ($\mathcal{F}1$) and ($\mathcal{F}2$). Then, it follows from (3.1) that for any $l \in \mathbb{N}$,

$$\sum_i \partial_z \theta_i(t,z) = 0, \quad t \ge 0, \ z \in \Omega,$$

and $\partial_z \theta_i(t,z)$ satisfies

$$\partial_t \partial_z \theta_i = \partial_z \nu_i(z) + \frac{1}{N} \sum_{1 \le j \le N} \Big[(\partial_z \kappa_{ij}(z)) \sin(\theta_j - \theta_i) + \kappa_{ij}(z) \cos(\theta_j - \theta_i)(\partial_z \theta_j - \partial_z \theta_i) \Big]. \tag{3.3}$$

We choose extremal indices $M_1 = M_1(t,z), m_1 = m_1(t,z)$ such that

$$\partial_z \theta_{M_1}(t,z) := \max_i \partial_z \theta_i(t,z), \quad \partial_z \theta_{m_1}(t,z) := \min_i \partial_z \theta_i(t,z).$$

Note that for every $z \in \Omega$, $\partial_z \theta_{M_1}(\cdot, z)$ and $\partial_z \theta_{m_1}(\cdot, z)$ are piecewise differentiable and Lipschitz with respect to t. Then, we have

$$D(\partial_z \Theta(t,z)) := \partial_z \theta_{M_1}(t,z) - \partial_z \theta_{m_1}(t,z), \quad t \ge 0, \ z \in \Omega. \tag{3.4}$$

For the estimate of $D(\partial_z \Theta)$, we estimate time-evolution of $\partial_z \theta_{M_1}$ and $\partial_z \theta_{m_1}$ as follows.

• (Upper bound estimate of $\partial_z\theta_{M_1}$): For a.e. $t>0$, we have

(3.5)

$$
\begin{aligned}
\partial_t\partial_z\theta_{M_1} &\le \partial_z\nu_{M_1}(z) + \|\partial_z\kappa(z)\|_\infty \sin D(\Theta) + \frac{1}{N}\sum_{j=1}^N \kappa_{M_1,j}(z)\cos(\theta_j-\theta_{M_1})(\partial_z\theta_j-\partial_z\theta_{M_1})\\
&\le \partial_z\nu_{M_1}(z) + \|\partial_z\kappa(z)\|_\infty \sin D(\Theta) + \frac{1}{N}\sum_{j=1}^N \kappa_m(z)\cos D(\Theta^0(z))(\partial_z\theta_j-\partial_z\theta_{M_1})\\
&= \partial_z\nu_{M_1}(z) + \|\partial_z\kappa(z)\|_\infty \sin D(\Theta) - \kappa_m(z)\cos D(\Theta^0(z))\partial_z\theta_{M_1},
\end{aligned}
$$

where we used $D(\Theta(t,z)) \le D(\Theta^0(z))$ from Proposition 2.1.

• (Lower bound estimate of $\partial_z\theta_{m_1}$): Similarly for a.e. $t>0$, we have

(3.6)

$$
\begin{aligned}
\partial_t\partial_z\theta_{m_1} &\ge \partial_z\nu_{m_1}(z) - \|\partial_z\kappa(z)\|_\infty \sin D(\Theta) + \frac{1}{N}\sum_{j=1}^N \kappa_{m_1,j}(z)\cos(\theta_j-\theta_{m_1})(\partial_z\theta_j-\partial_z\theta_{m_1})\\
&\ge \partial_z\nu_{m_1}(z) - \|\partial_z\kappa(z)\|_\infty \sin D(\Theta) + \frac{1}{N}\sum_{j=1}^N \kappa_m(z)\cos D(\Theta^0(z))(\partial_z\theta_j-\partial_z\theta_{m_1})\\
&= \partial_z\nu_{m_1}(z) - \|\partial_z\kappa(z)\|_\infty \sin D(\Theta) - \kappa_m(z)\cos D(\Theta^0(z))\partial_z\theta_{m_1}, \quad \text{for a.e.} \quad t>0.
\end{aligned}
$$

We use the relations (3.4), (3.5) and (3.6) to yield the following Gronwall type inequality: for a.e. $t>0$,

$$
\begin{aligned}
\partial_t D(\partial_z\Theta) &\le -\kappa_m(z)\cos D(\Theta^0)D(\partial_z\Theta) + D(\partial_z\mathcal{V}(z)) + 2\|\partial_z\kappa(z)\|_\infty \sin D(\Theta)\\
&\le -\kappa_m(z)\cos D(\Theta^0(z))D(\partial_z\Theta) + D(\partial_z\mathcal{V}) + 2\|\partial_z\kappa(z)\|_\infty \sin D(\Theta^0(z)).
\end{aligned} \tag{3.7}
$$

where we used (2) of Proposition 2.1. Then, Grönwall's lemma in Lemma 2.2 and continuity of $D(\partial_z\Theta(\cdot,z))$ with respect to t, can be used to obtain the following desired estimate: for $z\in\Omega$,

$$
\begin{aligned}
D(\partial_z\Theta(t,z)) &\le D(\partial_z\Theta^0(z))e^{-\kappa_m(z)\cos D(\Theta^0(z))t}\\
&\quad + \frac{D(\partial_z\mathcal{V}(z)) + 2\|\partial_z\kappa(z)\|_\infty \sin D(\Theta^0(z))}{\kappa_m(z)\cos D(\Theta^0(z))}\left(1-e^{-\kappa_m(z)\cos D(\Theta^0(z))t}\right).
\end{aligned}
$$

□

Next, we use induction and provide a local sensitivity analysis for the diameter of higher-order z-derivatives of phases.

Theorem 3.1. *Suppose that the framework* ($\mathcal{F}$) *holds, and let* $\Theta = \Theta(t,z)$ *be a solution to the system* (1.2)*. Then, for* $z\in\Omega$*, we have*

(3.8)

$$
D(\partial_z^l\Theta(t,z)) \le D(\partial_z^l\Theta^0(z))e^{-\kappa_m(z)\cos D(\Theta^0(z))t} + C_l(z)(1-e^{-\kappa_m(z)\cos D(\Theta^0(z))t}) \quad \textit{for all} \quad t\ge 0,
$$

where the random variable $C_l(z)$ ($l\ge 2$) *is inductively given by the following relation:*

$$
C_l(z) := \frac{2}{\kappa_m(z)\cos D(\Theta^0(z))}
$$

$$\times \Bigg\{ \frac{D(\partial_z^l \mathcal{V}(z))}{2} + \|\partial_z^l \kappa(z)\|_\infty D(\Theta^0(z))$$

$$+ \|\kappa(z)\|_\infty \sum_{\substack{(k_1,\ \cdots,\ k_l)\in\Lambda(l)\\ k_l=0}} \frac{l!}{k_1!\cdots k_{l-1}!} \prod_{p=1}^{l-1} \left(\frac{\tilde{C}_p(z)}{p!} \right)^{k_p}$$

$$+ \sum_{\substack{1\le r\le l-1\\ (k_1,\ \cdots,\ k_r)\in\Lambda(r)}} \binom{l}{r} \frac{r!}{k_1!\cdots k_r!} \|\partial_z^{l-r}\kappa(z)\|_\infty \prod_{p=1}^{r} \left(\frac{\tilde{C}_p(z)}{p!} \right)^{k_p} \Bigg\},$$

where $\tilde{C}_p(z) := \max\{D(\partial_z^p \Theta^0(z)),\ C_p(z)\}$ *and* $\Lambda(r)$ *follows from Section 2.4.*

Proof. We use the mathematical induction together with initial step in Lemma 3.1.

• (Initial step): For $l = 1$, the estimate (3.8) is already established in Lemma 3.1.

• (Inductive step): For $l \ge 2$, assume that the estimate (3.8) for $D(\partial_z^k \Theta(t,z))$ with $k \le l-1$ hold, and we will show that the estimate (3.8) holds for $D(\partial_z^l \Theta(t,z))$ below. For $(k_1,\ \cdots,\ k_r) \in (\mathbb{N}\cup\{0\})^r$, $1 \le i,\ j \le N$ and $r \in \mathbb{N}$, we set $\mathcal{M}(r, k_1,\ \cdots,\ k_r, \Theta, i, j)$ as follows:

$$\mathcal{M}(r, k_1,\ \cdots,\ k_r, \Theta, i, j) := \sin^{(k)}(\theta_j - \theta_i) \prod_{p=1}^{r} \left(\frac{\partial_z^p \theta_j - \partial_z^p \theta_i}{p!} \right)^{k_p},$$

where $\sin^{(k)}(\theta_j - \theta_i) := \frac{d^k}{dx^k}(\sin(x))\Big|_{x=\theta_j-\theta_i}$ and $k = k_1 + \cdots k_r$.

We use the chain rule for higher order derivatives at the end of Section 2, and deduce that (3.1) becomes as follows:

$$\begin{aligned}
&\partial_t(\partial_z^l \theta_i)(t,z)\\
&= \partial_z^l \nu_i(z) + \frac{1}{N}\sum_{j=1}^N \kappa_{ij}(z)\cos(\theta_j - \theta_i)(\partial_z^l\theta_j - \partial_z^l\theta_i)\\
&\quad + \frac{1}{N} \sum_{\substack{1\le j\le N\\ (k_1,\ \cdots,\ k_l)\in\Lambda(l)\\ k_l=0}} \kappa_{ij}(z)\frac{l!}{k_1!\cdots k_{l-1}!}\mathcal{M}(l-1, k_1,\ \cdots,\ k_{l-1}, \Theta, i, j)\\
&\quad + \frac{1}{N} \sum_{\substack{1\le j\le N\\ 1\le r\le l-1\\ (k_1,\ \cdots,\ k_r)\in\Lambda(r)}} \binom{l}{r}\partial_z^{l-r}\kappa_{ij}(z)\frac{r!}{k_1!\cdots k_r!}\mathcal{M}(r, k_1,\ \cdots,\ k_r, \Theta, i, j)\\
&\quad + \frac{1}{N}\sum_{1\le j\le N} \partial_z^l \kappa_{ij}(z)\sin(\theta_j - \theta_i).
\end{aligned}$$

For the l-th phase variations $\{\partial_z^l \theta_i(t,z)\}$, we choose extremal indices $M_l = M_l(t,z),\ m_l = m_l(t,z)$ such that

$$\begin{aligned}
&\partial_z \theta_{M_l}(t,z) := \max_i \partial_z^l \theta_i(t,z), \quad \partial_z \theta_{m_l}(t,z) := \min_i \partial_z^l \theta_i(t,z),\\
&D(\partial_z^l \Theta(t,z)) := \partial_z^l \theta_{M_l}(t,z) - \partial_z^l \theta_{m_l}(t,z), \quad t \ge 0,\ z \in \Omega.
\end{aligned}$$

We note that for $z \in \Omega$, $\partial_z^l \theta_{M_l}(\cdot, z)$ and $\partial_z^l \theta_{m_l}(\cdot, z)$ are piecewise differentiable and Lipschitz with respect to t. Now, we estimate $\partial_z^l \theta_{M_l}$ and $\partial_z^l \theta_{m_l}$ separately as follows.

• Case A (Estimate for $\partial_t \partial_z^l \theta_{M_l}$): For a.e. $t \geq 0$, we have

$$
\begin{aligned}
&\partial_t(\partial_z^l \theta_{M_l})(t,z) \\
&\quad \leq \left(\partial_z^l \nu_{M_l}(z) + \frac{1}{N} \sum_{j=1}^N \kappa_{M_l,j}(z) \cos(\theta_j - \theta_{M_l})(\partial_z^l \theta_j - \partial_z^l \theta_{M_l}) \right) \\
&\qquad + \frac{1}{N} \sum_{\substack{1 \leq j \leq N \\ (k_1, \cdots, k_l) \in \Lambda(l) \\ k_l = 0}} \kappa_{M_l,j}(z) \frac{l!}{k_1! \cdots k_{l-1}!} \mathcal{M}(l-1, k_1, \ \cdots, \ k_{l-1}, \Theta, M_l, j) \\
&\qquad + \frac{1}{N} \sum_{\substack{1 \leq j \leq N \\ 1 \leq r \leq l-1 \\ (k_1, \cdots, k_r) \in \Lambda(r)}} \binom{l}{r} \partial_z^{l-r} \kappa_{M_l,j}(z) \frac{r!}{k_1! \cdots k_r!} \mathcal{M}(r, k_1, \ \cdots, \ k_r, \Theta, M_l, j) \\
&\qquad + \frac{1}{N} \sum_{1 \leq j \leq N} \partial_z^l \kappa_{M_l,j}(z) \sin(\theta_j - \theta_{M_l}) \\
&\quad =: \sum_{k=1}^4 \mathcal{I}_{1k}.
\end{aligned} \tag{3.9}
$$

◇ Case A.1 (Estimate of $\mathcal{I}_{11}$): By direct estimate, we have

$$
\begin{aligned}
\mathcal{I}_{11} &= \partial_z^l \nu_{M_l} + \frac{1}{N} \sum_{j=1}^N \kappa_{M_l,j} \cos(\theta_j - \theta_{M_l})(\partial_z^l \theta_j - \partial_z^l \theta_{M_l}) \\
&\leq \partial_z^l \nu_{M_l} - \kappa_m \cos D(\Theta^0) \partial_z^l \theta_{M_l}.
\end{aligned}
$$

◇ Case A.2 (Estimate of $\mathcal{I}_{12}$ and $\mathcal{I}_{13}$): for $r \leq l-1$, use the induction hypothesis to get

$$
D(\partial_z^r \Theta(t,z)) \leq D(\partial_z^r \Theta^0(z)) e^{-\kappa_m \cos D(\Theta^0) t} + C_r(z)(1 - e^{-\kappa_m \cos D(\Theta^0) t}), \quad \text{for each} \quad z \in \Omega,
$$

which can be changed to

$$
D(\partial_z^r \Theta(t,z)) \leq \max \left\{ D(\partial_z^r \Theta^0(z)), \ C_r(z) \right\} =: \tilde{C}_r(z), \quad \text{for each} \quad z \in \Omega,
$$

Hence we obtain

$$
|\mathcal{M}(r, k_1, \ \cdots, \ k_r, \Theta, M_l, j)| \leq \prod_{p=1}^r \left(\frac{D(\partial_z^p \Theta(t,z))}{p!} \right)^{k_p} \leq \prod_{p=1}^r \left(\frac{\tilde{C}_p(z)}{p!} \right)^{k_p}
$$

for $r = 1, \cdots, l-1$. Thus,

$$\mathcal{I}_{12} \le \|\kappa(z)\|_\infty \sum_{\substack{(k_1,\ \cdots,\ k_l)\in\Lambda(l)\\ k_l=0}} \frac{l!}{k_1!\cdots k_{l-1}!} \prod_{p=1}^{l-1} \left(\frac{\tilde{C}_p(z)}{p!}\right)^{k_p},$$

$$\mathcal{I}_{13} \le \sum_{\substack{1\le r\le l-1\\ (k_1,\ \cdots,\ k_r)\in\Lambda(r)}} \binom{l}{r} \frac{r!}{k_1!\cdots k_r!} \|\partial_z^{l-r}\kappa(z)\|_\infty \prod_{p=1}^{r} \left(\frac{\tilde{C}_p(z)}{p!}\right)^{k_p}$$

◇ Case A.3 (Estimate of $\mathcal{I}_{14}$): We deduce from Proposition 2.1 that

$$\mathcal{I}_{14} \le \frac{\|\partial_z^l \kappa(z)\|_\infty}{N} \sum_{1\le j\le N} D(\Theta(t,z)) \le \|\partial_z^l\kappa(z)\|_\infty D(\Theta^0(z)).$$

In (3.9), we combine all results in Case A.1 - Case A.3 to get the following: for a.e. $t \ge 0$,

$$\begin{aligned}
\partial_t\partial_z^l\theta_{M_l}(t,z) &\le \partial_z^l\nu_{M_l} - \kappa_m \cos D(\Theta^0)\partial_z^l\theta_{M_l} + \|\partial_z^l\kappa(z)\|_\infty D(\Theta^0(z)) \\
&\quad + \|\kappa(z)\|_\infty \sum_{\substack{(k_1,\ \cdots,\ k_l)\in\Lambda(l)\\ k_l=0}} \frac{l!}{k_1!\cdots k_{l-1}!} \prod_{p=1}^{l-1} \left(\frac{\tilde{C}_p(z)}{p!}\right)^{k_p} \\
&\quad + \sum_{\substack{1\le r\le l-1\\ (k_1,\ \cdots,\ k_r)\in\Lambda(r)}} \binom{l}{r} \frac{r!}{k_1!\cdots k_r!} \|\partial_z^{l-r}\kappa(z)\|_\infty \prod_{p=1}^{r} \left(\frac{\tilde{C}_p(z)}{p!}\right)^{k_p}.
\end{aligned} \tag{3.10}$$

• Case B (Estimate for $\partial_t\partial_z^l\hat{\theta}_{m_l}$): Next, we estimate $\partial_t\partial_z^l\theta_{m_l}$ as follows: for a.e. $t \ge 0$,

$$\begin{aligned}
&\partial_t(\partial_z^l\theta_{m_l})(t,z) \\
&\quad \ge \left(\partial_z^l\nu_{m_l}(z) + \frac{1}{N}\sum_{j=1}^N \kappa_{m_l,j}(z)\cos(\theta_j - \theta_{m_l})(\partial_z^l\theta_j - \partial_z^l\theta_{m_l})\right) \\
&\quad\quad - \frac{1}{N} \sum_{\substack{1\le j\le N\\ (k_1,\ \cdots,\ k_l)\in\Lambda(l)\\ k_l=0}} \kappa_{m_l,j}(z)\frac{l!}{k_1!\cdots k_{l-1}!} |\mathcal{M}(l-1, k_1,\ \cdots,\ k_{l-1}, \Theta, k_l, j)| \\
&\quad\quad - \frac{1}{N} \sum_{\substack{1\le j\le N\\ 1\le r\le l-1\\ (k_1,\ \cdots,\ k_r)\in\Lambda(r)}} \binom{l}{r}\frac{r!}{k_1!\cdots k_r!} |\partial_z^{l-r}\kappa_{m_l,j}(z)\mathcal{M}(r, k_1,\ \cdots,\ k_r, \Theta, m_l, j)| \\
&\quad\quad - \frac{1}{N}\sum_{1\le j\le N} |\partial_z^l\kappa_{m_l,j}(z)\sin(\theta_j - \theta_{m_l})| \\
&\quad =: \sum_{k=1}^4 \mathcal{I}_{2k}.
\end{aligned} \tag{3.11}$$

$\diamond$ Case B.1 (Estimate of $\mathcal{I}_{21}$): In this case, we have

$$\mathcal{I}_{21} \geq \partial_z^l \nu_{m_l} - \kappa_m \cos D(\Theta^0) \partial_z^l \theta_{m_l}.$$

$\diamond$ Case B.2 (Estimate of $\mathcal{I}_{22}$ and $\mathcal{I}_{23}$): As in Case A.2, we have

$$\mathcal{I}_{22} \geq -\|\kappa(z)\|_\infty \sum_{\substack{(k_1,\ \cdots,\ k_l) \in \Lambda(l) \\ k_l = 0}} \frac{l!}{k_1! \cdots k_{l-1}!} \prod_{p=1}^{l-1} \left(\frac{\tilde{C}_p(z)}{p!} \right)^{k_p},$$

$$\mathcal{I}_{23} \geq - \sum_{\substack{1 \leq r \leq l-1 \\ (k_1,\ \cdots,\ k_r) \in \Lambda(r)}} \binom{l}{r} \frac{r!}{k_1! \cdots k_r!} \|\partial_z^{l-r} \kappa(z)\|_\infty \prod_{p=1}^{r} \left(\frac{\tilde{C}_p(z)}{p!} \right)^{k_p}.$$

$\diamond$ Case B.3 (Estimate of $\mathcal{I}_{24}$): By direct estimate, we have

$$\mathcal{I}_{24} \geq -\frac{\|\partial_z^l \kappa(z)\|_\infty}{N} \sum_{1 \leq j \leq N} D(\Theta(t,z)) \geq -\|\partial_z^l \kappa(z)\|_\infty D(\Theta^0(z)).$$

In (3.11), we combine all results in Case B.1 - Case B.3 to get the following: for a.e. $t \geq 0$,

$$\begin{aligned}
\partial_t \partial_z^l \theta_{m_l}(t,z) &\geq \partial_z^l \nu_{m_l} - \kappa_m \cos D(\Theta^0) \partial_z^l \theta_{m_l} - \|\partial_z^l \kappa(z)\|_\infty D(\Theta^0(z)) \\
&\quad - \|\kappa(z)\|_\infty \sum_{\substack{(k_1,\ \cdots,\ k_l) \in \Lambda(l) \\ k_l = 0}} \frac{l!}{k_1! \cdots k_{l-1}!} \prod_{p=1}^{l-1} \left(\frac{\tilde{C}_p(z)}{p!} \right)^{k_p} \\
&\quad - \sum_{\substack{1 \leq r \leq l-1 \\ (k_1,\ \cdots,\ k_r) \in \Lambda(r)}} \binom{l}{r} \frac{r!}{k_1! \cdots k_r!} \|\partial_z^{l-r} \kappa(z)\|_\infty \prod_{p=1}^{r} \left(\frac{\tilde{C}_p(z)}{p!} \right)^{k_p}.
\end{aligned} \tag{3.12}$$

Next, we combine (3.10) and (3.12) to obtain

$$\partial_t D(\partial_z^l \Theta(t,z)) \leq -\kappa_m(z) \cos D(\Theta^0(z)) \Big(D(\partial_z^l \Theta(t,z)) - C_l(z) \Big),$$

and we set

$$y := D(\partial_z^l \Theta), \quad \alpha = \kappa_m \cos D(\Theta^0), \quad \beta = 0, \quad C = \kappa_m \cos D(\Theta^0) C_l(z).$$

Finally, we use Lemma 2.2 to derive the desired estimate. $\square$

As a direct application of Theorem 3.1, we have the local sensitivity estimate for the phase diameter of $\partial_z^l \theta_i$.

Corollary 3.1. *Suppose that the framework* ($\mathcal{F}$) *holds, and let* $\Theta = \Theta(t,z)$ *be a solution to system* (1.2). *Then, for* $z \in \Omega$, *we have*

$$\mathbb{E}[D(\partial_z^l \Theta(t,\cdot))] \leq \mathbb{E} \max\{D(\partial_z^l \Theta^0), C_l(z)\} \quad t \geq 0. \tag{3.13}$$

Proof. The result of Theorem 3.1 implies

$$D(\partial_z^l \Theta(t,z)) \leq \max\{D(\partial_z^l \Theta^0(z)), C_l(z)\}.$$

Note that the R.H.S. depend only on given data and parameters, i.e., $\Theta^0, \mathcal{V}, \kappa_{ij}(z)$. $\square$

3.1. **Identical oscillators.** In this subsection, we consider an ensemble of identical oscillators, and provide more refined local sensitivity estimates than those in Theorem 3.1. For this, we improve estimates in Lemma 3.1 and Theorem 3.1.

Corollary 3.2. (First-order estimate) *Suppose that the framework* $(\mathcal{F})$ *with* $l = 1$ *and*

$$\nu_i = 0, \quad i = 1, \cdots, N$$

hold, and let $\Theta = \Theta(t,z)$ *be a solution to system* (1.2). *Then, for* $z \in \Omega$,

$$D(\partial_z \Theta(t,z)) \le \mathcal{D}_1(z) e^{-\frac{\kappa_m(z)\cos D(\Theta^0(z)) t}{2}}. \tag{3.14}$$

where the random variable $\mathcal{D}_1(z)$ *is given by*

$$\mathcal{D}_1(z) := \frac{4\|\partial_z \kappa(z)\|_\infty D(\Theta^0(z))}{\kappa_m(z)\cos D(\Theta_0(z))} + D(\partial_z \Theta^0(z)).$$

Proof. We use (3.7) with $D(\partial_z \mathcal{V}) = 0$ and Proposition 2.1 that for every $t \ge 0$ and $z \in \Omega$,

$$\begin{aligned} &\partial_t D(\partial_z \Theta(t,z)) \\ &\le -\kappa_m(z)\cos D(\Theta^0(z)) D(\partial_z \Theta(t,z)) + 2\|\partial_z \kappa(z)\|_\infty D(\Theta(t,z)) \\ &\le -\kappa_m(z)\cos D(\Theta^0(z)) D(\partial_z \Theta(t,z)) + 2\|\partial_z \kappa(z)\|_\infty D(\Theta^0(z)) e^{-\frac{\kappa_m(z)\cos D(\Theta^0(z)) t}{2}}. \end{aligned} \tag{3.15}$$

We set

$$y = D(\partial_z \Theta), \quad \alpha = \kappa_m \cos D(\Theta^0), \quad \beta = \frac{\alpha}{2} \quad \text{and} \quad C = 2\|\partial_z \kappa\|_\infty D(\Theta^0),$$

and apply Lemma 2.2 in (3.15) to derive the exponential decay estimate:

$$D(\partial_z \Theta(t,z)) \le \frac{4\|\partial_z \kappa(z)\|_\infty D(\Theta^0(z))}{\kappa_m(z)\cos D(\Theta^0(z))} e^{-\frac{\kappa_m(z)\cos D(\Theta^0(z)) t}{2}} + D(\partial_z \Theta^0(z)) e^{-\kappa_m(z)\cos D(\Theta_0(z)) t}.$$

This implies our desired result. □

Remark 3.1. *For the constant couplings and initial data that are strictly confined in a quarter arc, there exists a small positive constant* $\varepsilon \in (0, \frac{\pi}{2})$ *such that*

$$\kappa_m = constant, \quad \partial_z \kappa_{ij} = 0, \quad \theta_i^0(z) < \frac{\pi}{2} - \varepsilon,$$

the estimate in (3.14) *implies*

$$D(\partial_z \Theta(t,z)) \le D(\partial_z \Theta^0(z)) e^{-\frac{\kappa_m \cos(\frac{\pi}{2} - \varepsilon) t}{2}}.$$

Hence, we have

$$\mathbb{E} D(\partial_z \Theta(t,\cdot)) \le \mathbb{E} D(\partial_z \Theta^0(\cdot)) e^{-\frac{\kappa_m \cos(\frac{\pi}{2} - \varepsilon) t}{2}}.$$

Next, we provide local sensitivity analysis for the identical case with higher order z-derivatives.

Corollary 3.3. (Higher-order estimates) *Suppose that the framework* $(\mathcal{F})$ *with*

$$\nu_i = 0, \quad i = 1, \cdots, N$$

hold, and let $\Theta = \Theta(t,z)$ *be a solution to system* (1.2). *Then, for* $z \in \Omega$,

$$D(\partial_z^l \Theta(t,z)) \le \mathcal{D}_l(z) e^{-\frac{\kappa_m(z)\cos D(\Theta^0(z)) t}{2}},$$

where the random variable $\mathcal{D}_l(z)$ $(l \ge 2)$ is inductively defined by

$$\begin{aligned}
\mathcal{D}_l(z) &:= D(\partial_z^l \Theta^0(z)) \\
&\quad + \frac{4}{\kappa_m(z)\cos D(\Theta^0(z))} \\
&\quad \times \Bigg\{ D(\Theta^0(z))\|\partial_z^l \kappa(z)\|_\infty + \|\kappa(z)\|_\infty \sum_{\substack{(k_1,\ \cdots,\ k_l)\in\Lambda(l)\\ k_l=0}} \frac{l!}{k_1!\cdots k_{l-1}!} \prod_{p=1}^{l-1} \left(\frac{\mathcal{D}_p(z)}{p!}\right)^{k_p} \\
&\quad + \sum_{\substack{1\le r\le l-1\\ (k_1,\ \cdots,\ k_r)\in\Lambda(r)}} \binom{l}{r} \|\partial_z^{l-r}\kappa(z)\|_\infty \frac{r!}{k_1!\cdots k_r!}\left(\frac{\mathcal{D}_p(z)}{p!}\right)^{k_p} \Bigg\}.
\end{aligned}$$

Proof. We will proceed by induction. Note that we have already proved $l = 1$ case in Corollary 3.2. Then for the induction step, we estimate $\partial_t\partial_z^l\theta_{M_1}$ and $\partial_t\partial_z^l\theta_{m_1}$ as we did in Theorem 3.1.

• Case C (Estimate for $\partial_t\partial_z^l\theta_{M_l}$): For $\partial_t\partial_z^l\theta_{M_l}$,

$$\begin{aligned}
&\partial_t(\partial_z^l\theta_{M_l})(t,z) \\
&\le \frac{1}{N}\sum_{j=1}^N \kappa_{M_l,j}(z)\cos(\theta_j-\theta_{M_l})(\partial_z^l\theta_j - \partial_z^l\theta_{M_l}) \\
&\quad + \frac{1}{N}\sum_{\substack{1\le j\le N\\ (k_1,\ \cdots,\ k_l)\in\Lambda(l)\\ k_l=0}} \kappa_{M_l,j}(z)\frac{l!}{k_1!\cdots k_{l-1}!}\mathcal{M}(l-1,k_1,\ \cdots,\ k_{l-1},\Theta,M_l,j) \\
&\quad + \frac{1}{N}\sum_{\substack{1\le j\le N\\ 1\le r\le l-1\\ (k_1,\ \cdots,\ k_r)\in\Lambda(r)}} \binom{l}{r}\partial_z^{l-r}\kappa_{M_l,j}(z)\frac{r!}{k_1!\cdots k_r!}\mathcal{M}(r,k_1,\ \cdots,\ k_r,\Theta,M_l,j) \qquad (3.16)\\
&\quad + \frac{1}{N}\sum_{1\le j\le N}\partial_z^l\kappa_{M_l,j}(z)\sin(\theta_j-\theta_{M_l}) \\
&=: \sum_{k=1}^4 \mathcal{I}_{3k}.
\end{aligned}$$

⋄ Case C.1 (Estimate of $\mathcal{I}_{31}$): for $\mathcal{I}_{31}$,

$$\mathcal{I}_{31} = \frac{1}{N}\sum_{j=1}^N \kappa_{M_l,j}\cos(\theta_j-\theta_{M_l})(\partial_z^l\theta_j-\partial_z^l\theta_{M_l}) \le -\kappa_m\cos D(\Theta^0)\partial_z^l\theta_{M_l}.$$

⋄ Case C.2 (Estimate of $\mathcal{I}_{32}$ and $\mathcal{I}_{33}$): By induction hypothesis, for each $z\in\Omega$,

$$D(\partial_z^r\Theta(t,z)) \le \mathcal{D}_r(z)e^{-\frac{\kappa_m\cos D(\Theta^0(z))t}{2}}.$$

Thus,

$$\mathcal{M}(r,k_1,\ \cdots,\ k_r,\Theta,M_l,j) \le \prod_{p=1}^{r}\left(\frac{D(\partial_z^p\Theta(t,z))}{p!}\right)^{k_p} \le e^{-\frac{\kappa_m\cos D(\Theta^0(z))t}{2}}\prod_{p=1}^{r}\left(\frac{\mathcal{D}_p(z)}{p!}\right)^{k_p}, \tag{3.17}$$

where we used $k_1+\cdots+k_r\ge 1$. Thus, (3.17) yields

$$\mathcal{I}_{32} \le \left\{\|\kappa(z)\|_\infty \sum_{\substack{(k_1,\ \cdots,\ k_l)\in\Lambda(l)\\ k_l=0}} \frac{l!}{k_1!\cdots k_{l-1}!}\prod_{p=1}^{l-1}\left(\frac{\mathcal{D}_p(z)}{p!}\right)^{k_p}\right\} e^{-\frac{\kappa_m\cos D(\Theta^0(z))t}{2}},$$

$$\mathcal{I}_{33} \le \left\{\sum_{\substack{1\le r\le l-1\\ (k_1,\ \cdots,\ k_r)\in\Lambda(r)}} \binom{l}{r}\|\partial_z^{l-r}\kappa(z)\|_\infty \frac{r!}{k_1!\cdots k_r!}\left(\frac{\mathcal{D}_p(z)}{p!}\right)^{k_p}\right\} e^{-\frac{\kappa_m\cos D(\Theta^0(z))t}{2}}.$$

⋄ Case C.3 (Estimate of $\mathcal{I}_{34}$): In this case,

$$\mathcal{I}_{34} \le \frac{\|\partial_z^l\kappa\|_\infty}{N}\sum_{1\le j\le N} D(\Theta(t,z)) \le D(\Theta^0(z))\|\partial_z^l\kappa(z)\|_\infty e^{-\kappa_m\cos D(\Theta^0(z))t}.$$

In (3.16), we combine all estimates in Case C.1 - Case C.3 to get

$$\begin{aligned}
&\partial_t\partial_z^l\theta_{M_l}(t,z)\\
&\quad\le -\kappa_m\cos D(\Theta^0)\partial_z^l\theta_{M_l} + D(\Theta^0(z))\|\partial_z^l\kappa(z)\|_\infty e^{-\kappa_m\cos D(\Theta^0(z))t}\\
&\qquad + \left\{\|\kappa(z)\|_\infty \sum_{\substack{(k_1,\ \cdots,\ k_l)\in\Lambda(l)\\ k_l=0}} \frac{l!}{k_1!\cdots k_{l-1}!}\prod_{p=1}^{l-1}\left(\frac{\mathcal{D}_p(z)}{p!}\right)^{k_p}\right\} e^{-\frac{\kappa_m\cos D(\Theta^0(z))t}{2}}\\
&\qquad + \left\{\sum_{\substack{1\le r\le l-1\\ (k_1,\ \cdots,\ k_r)\in\Lambda(r)}} \binom{l}{r}\|\partial_z^{l-r}\kappa(z)\|_\infty \frac{r!}{k_1!\cdots k_r!}\left(\frac{\mathcal{D}_p(z)}{p!}\right)^{k_p}\right\} e^{-\frac{\kappa_m\cos D(\Theta^0(z))t}{2}}
\end{aligned} \tag{3.18}$$

• Case D (Estimate for $\partial_t \partial_z^l \theta_{m_l}$): Now we evaluate $\partial_t \partial_z^l \theta_{m_l}$ as follows.

$$
\begin{aligned}
&\partial_t(\partial_z^l \theta_{m_l})(t,z) \\
&\quad \geq \frac{1}{N}\sum_{j=1}^N \kappa_{m_l,j}\cos(\theta_j-\theta_{m_l})(\partial_z^l\theta_j - \partial_z^l\theta_{m_l}) \\
&\quad - \frac{1}{N}\sum_{\substack{1\leq j\leq N \\ (k_1,\ \cdots,\ k_l)\in\Lambda(l) \\ k_l=0}} \kappa_{m_l,j}\frac{l!}{k_1!\cdots k_{l-1}!}|\mathcal{M}(l-1,k_1,\ \cdots,\ k_{l-1},\Theta,k_l,j)| \\
&\quad - \frac{1}{N}\sum_{\substack{1\leq j\leq N \\ 1\leq r\leq l-1 \\ (k_1,\ \cdots,\ k_r)\in\Lambda(r)}} \binom{l}{r}\frac{r!}{k_1!\cdots k_r!}|\partial_z^{l-r}\kappa_{m_l,j}|\ |\mathcal{M}(r,k_1,\ \cdots,\ k_r,\Theta,m_l,j)| \qquad (3.19)\\
&\quad - \frac{1}{N}\sum_{1\leq j\leq N}|\partial_z^l\kappa_{m_l,j}\sin(\theta_j-\theta_{m_l})| \\
&\quad =: \sum_{k=1}^4 \mathcal{I}_{4k}.
\end{aligned}
$$

◇ Case D.1 (Estimate of $\mathcal{I}_{41}$): In this case,

$$
\mathcal{I}_{41} \geq -\kappa_m \cos D(\Theta^0)\partial_z^l\theta_{m_l}.
$$

◇ Case D.2 (Estimate of $\mathcal{I}_{42}$ and $\mathcal{I}_{43}$): Similar to estimates $\mathcal{I}_{32}$ and $\mathcal{I}_{33}$,

$$
\mathcal{I}_{42} \geq -\left\{\|\kappa(z)\|_\infty \sum_{\substack{(k_1,\ \cdots,\ k_l)\in\Lambda(l) \\ k_l=0}} \frac{l!}{k_1!\cdots k_{l-1}!}\prod_{p=1}^{l-1}\left(\frac{\mathcal{D}_p(z)}{p!}\right)^{k_p}\right\} e^{-\frac{\kappa_m\cos D(\Theta^0(z))t}{2}},
$$

$$
\mathcal{I}_{43} \geq -\left\{\sum_{\substack{1\leq r\leq l-1 \\ (k_1,\ \cdots,\ k_r)\in\Lambda(r)}} \binom{l}{r}\|\partial_z^{l-r}\kappa(z)\|_\infty \frac{r!}{k_1!\cdots k_r!}\left(\frac{\mathcal{D}_p(z)}{p!}\right)^{k_p}\right\} e^{-\frac{\kappa_m\cos D(\Theta^0(z))t}{2}}.
$$

◇ Case D.3 (Estimate of $\mathcal{I}_{44}$): We have

$$
\mathcal{I}_{44} \geq -\frac{\|\partial_z^l\kappa\|_\infty}{N}\sum_{1\leq j\leq N} D(\Theta(t,z)) \geq -D(\Theta^0(z))\|\partial_z^l\kappa(z)\|_\infty e^{-\kappa_m\cos D(\Theta^0(z))t}.
$$

In (3.19), we combine all esitmates in Case D.1 - Case D.3 to get

$$
\begin{aligned}
&\partial_t \partial_z^l \theta_{m_l}(t,z) \\
&\quad \geq -\kappa_m \cos D(\Theta^0) \partial_z^l \theta_{m_l} - D(\Theta^0(z)) \|\partial_z^l \kappa(z)\|_\infty e^{-\kappa_m \cos D(\Theta^0(z))t} \\
&\qquad - \left\{ \|\kappa(z)\|_\infty \sum_{\substack{(k_1,\, \cdots,\, k_l) \in \Lambda(l) \\ k_l = 0}} \frac{l!}{k_1! \cdots k_{l-1}!} \prod_{p=1}^{l-1} \left(\frac{\mathcal{D}_p(z)}{p!} \right)^{k_p} \right\} e^{-\frac{\kappa_m \cos D(\Theta^0(z))t}{2}} \\
&\qquad - \left\{ \sum_{\substack{1 \leq r \leq l-1 \\ (k_1,\, \cdots,\, k_r) \in \Lambda(r)}} \binom{l}{r} \|\partial_z^{l-r} \kappa(z)\|_\infty \frac{r!}{k_1! \cdots k_r!} \left(\frac{\mathcal{D}_p(z)}{p!} \right)^{k_p} \right\} e^{-\frac{\kappa_m \cos D(\Theta^0(z))t}{2}}
\end{aligned} \tag{3.20}
$$

Finally, we combine (3.18) and (3.20) to obtain

$$
\partial_t D(\partial_z^l \Theta(t,z)) \leq -\kappa_m \cos D(\Theta^0) D(\partial_z^l \Theta(t,z)) + E(z) e^{-\frac{\kappa_m \cos D(\Theta^0(z))t}{2}}. \tag{3.21}
$$

where the random variable $E(z)$ is given by

$$
\begin{aligned}
E(z) := 2 \Bigg\{ & D(\Theta^0(z)) \|\partial_z^l \kappa(z)\|_\infty + \|\kappa(z)\|_\infty \sum_{\substack{(k_1,\, \cdots,\, k_l) \in \Lambda(l) \\ k_l = 0}} \frac{l!}{k_1! \cdots k_{l-1}!} \prod_{p=1}^{l-1} \left(\frac{\mathcal{D}_p(z)}{p!} \right)^{k_p} \\
& + \sum_{\substack{1 \leq r \leq l-1 \\ (k_1,\, \cdots,\, k_r) \in \Lambda(r)}} \binom{l}{r} \|\partial_z^{l-r} \kappa(z)\|_\infty \frac{r!}{k_1! \cdots k_r!} \left(\frac{\mathcal{D}_p(z)}{p!} \right)^{k_p} \Bigg\},
\end{aligned}
$$

Then, we apply Grönwall's lemma in Lemma 2.2 to (3.21), to yield

$$
D(\partial_z^l \Theta(t,z)) \leq D(\partial_z^l \Theta^0(z)) e^{-\kappa_m \cos D(\Theta^0)t} + \frac{2}{\kappa_m \cos D(\Theta^0)} E(z) e^{-\frac{\kappa_m \cos D(\Theta^0)t}{2}},
$$

which implies the desired result. □

4. Uniform ℓ^1-stability estimate for the phase process

In this subsection, we provide ℓ^1-stability estimates for phase variations $\partial_z^l \Theta$ with respect to initial phase variations in random space. Let Θ and $\tilde{\Theta}$ be solutions to (1.2) with sufficiently regular initial data Θ^0 and $\tilde{\Theta}^0$ in random space, respectively. In the sequel, we will derive an estimate like

$$
\frac{\partial}{\partial t} \|\partial_z^l (\Theta - \tilde{\Theta})(t,z)\|_1 + \Lambda_l(t,z) \leq \mathcal{E}_l(z) \sum_{p=0}^{l-1} \|\partial_z^p (\Theta - \tilde{\Theta})(t,z)\|_1,
$$

where $l \in \mathbb{N}$, the positive random variable $\mathcal{E}_l(z)$ is independent of t and $\Lambda_l(t,z)$ is a non-negative functional.

Let Θ and $\tilde{\Theta}$ be two random Kuramoto flows whose dynamics are governed by (1.2). Then, for $r \in \mathbb{N} \cup \{0\}$ and $z \in \Omega$, we set

$$\begin{aligned}
&I_r^0(t,z) := \{1 \le i \le N \mid \partial_z^r \theta_i(t,z) - \partial_z^r \tilde{\theta}_i(t,z) = 0\},\\
&I_r^+(t,z) := \{1 \le i \le N \mid \partial_z^r \theta_i(t,z) - \partial_z^r \tilde{\theta}_i(t,z) > 0\},\\
&I_r^-(t,z) := \{1 \le i \le N \mid \partial_z^r \theta_i(t,z) - \partial_z^r \tilde{\theta}_i(t,z) < 0\},\\
&I^0(t,z) := I_0^0(t,z), \quad I^+(t,z) := I_0^+(t,z), \quad I^-(t,z) := I_0^-(t,z).
\end{aligned} \tag{4.22}$$

First, we provide pathwise ℓ^1- stability for (1.2) following the arguments line by line in [12]

Proposition 4.1. (Pathwise ℓ_1-stability) *Suppose that the framework $(\mathcal{F})$ with $l = 0$ hold, and let $\Theta = \Theta(t,z)$ and $\tilde{\Theta} = \tilde{\Theta}(t,z)$ be two solutions to system (1.2) with initial data Θ^0 and $\tilde{\Theta}^0$, respectively. Then, for $z \in \Omega$, the following assertions hold:*

(1) If $\theta_c(0) \neq \tilde{\theta}_c(0)$, then

$$\frac{\partial}{\partial t}\|(\Theta - \tilde{\Theta})(t,z)\|_1 + \Lambda(t,z) \le 0,$$

for all $t \ge 0$ and each $z \in \Omega$, where the non-negative functional $\Lambda(s,z)$ is defined by

$$\begin{aligned}
\Lambda(t,z) := &\frac{\kappa_m(z)\sin D(\Theta^0(z))}{ND(\Theta^0(z))}\cos D(\Theta^0(z))\\
&\times \Bigg[\big(|I^0(t)| + 2|I^-(t)|\big) \sum_{i \in I^+(t)} |(\theta_i - \tilde{\theta}_i)(t)| + \big(|I^0(t)| + 2|I^+(t)|\big) \sum_{i \in I^-(t)} |(\theta_i - \tilde{\theta}_i)(t)| \Bigg].
\end{aligned}$$

(2) If $\theta_c(0,z) = \tilde{\theta}_c(0,z)$ for every $z \in \Omega$, then

$$\|(\Theta^0 - \tilde{\Theta}^0)(z)\|_1 e^{-\|\kappa(z)\|_\infty t} \le \|(\Theta - \tilde{\Theta})(t,z)\|_1 \le \|(\Theta^0 - \tilde{\Theta}^0)(z)\|_1 e^{-\kappa_m(z)\cos D(\Theta^0(z))\gamma(z)t},$$

for all $t \ge 0$ and each $z \in \Omega$, where $\gamma(z) := \frac{\sin D(\Theta^0(z))}{D(\Theta^0(z))}$.

Proof. The proof is almost the same as in the deterministic case in [12]. So we omit the proof. □

For $\mathcal{A}(t,z) = \{\alpha(t,z)\}_{i=1}^N \in \mathbb{R}^N$, and define J^0, $J^\pm$ and Δ_{ij} as in (4.22): for $t \ge 0$, $z \in \Omega$,

$$\begin{aligned}
&J^0(t,z) := \{1 \le i \le N \mid \alpha_i(t,z) = 0\},\\
&J^+(t,z) := \{1 \le i \le N \mid \alpha_i(t,z) > 0\},\\
&J^-(t,z) := \{1 \le i \le N \mid \alpha_i(t,z) < 0\},\\
&\Delta_{ij}(t,z) := (sgn(\alpha_i(t,z)) - sgn(\alpha_j(t,z)))(\alpha_j(t,z) - \alpha_i(t,z)).
\end{aligned}$$

Lemma 4.1. *The following assertions hold.*

(1) We have

$$\sum_{i,j=1}^N \Delta_{ij} = -2\left\{ \big(|J^0| + 2|J^-|\big) \sum_{i \in J^+} |\alpha_i| + \big(|J^0| + 2|J^+|\big) \sum_{i \in J^-} |\alpha_i| \right\}.$$

(2) Moreover, if $\displaystyle\sum_{i=1}^N \alpha_i = 0$*, then*

$$\sum_{i,j=1}^N \Delta_{ij} = -2N \sum_{i=1}^N |\alpha_i|.$$

Proof. (i) Note that if $\alpha_i\alpha_j > 0$, then $\Delta_{ij} = 0$. So we consider the other cases for evaluating Δ_{ij} as follows:

$$\begin{array}{llll}
\alpha_i > 0, \quad \alpha_j = 0 & \Longrightarrow & \Delta_{ij} = -|\alpha_i|; \quad \alpha_i < 0, \quad \alpha_j = 0 & \Longrightarrow \quad \Delta_{ij} = -|\alpha_i|, \\
\alpha_i = 0, \quad \alpha_j > 0 & \Longrightarrow & \Delta_{ij} = -|\alpha_j|; \quad \alpha_i = 0, \quad \alpha_j < 0 & \Longrightarrow \quad \Delta_{ij} = -|\alpha_j|, \\
\alpha_i > 0, \quad \alpha_j < 0 & \Longrightarrow & \Delta_{ij} = -2(|\alpha_i| + |\alpha_j|), & \\
\alpha_i < 0, \quad \alpha_j > 0 & \Longrightarrow & \Delta_{ij} = -2(|\alpha_i| + |\alpha_j|). &
\end{array}$$

These imply

$$\begin{aligned}
&\sum_{(i,j)\in J^+\times J^0} \Delta_{ij} = -|J^0| \sum_{i\in J^+} |\alpha_i|, \qquad \sum_{(i,j)\in J^-\times J^0} \Delta_{ij} = -|J^0| \sum_{i\in J^-} |\alpha_i|, \\
&\sum_{(i,j)\in J^0\times J^+} \Delta_{ij} = -|J^0| \sum_{i\in J^+} |\alpha_i|, \qquad \sum_{(i,j)\in J^0\times J^-} \Delta_{ij} = -|J^0| \sum_{i\in J^-} |\alpha_i|, \\
&\sum_{(i,j)\in J^+\times J^-} \Delta_{ij} = -2|J^-| \sum_{i\in J^+} |\alpha_i| - 2|J^+| \sum_{i\in J^-} |\alpha_i|, \\
&\sum_{(i,j)\in J^+\times J^0} \Delta_{ij} = -2|J^-| \sum_{i\in J^+} |\alpha_i| - 2|J^+| \sum_{i\in J^-} |\alpha_i|.
\end{aligned} \tag{4.23}$$

We combine all estimates in (4.23) to derive the estimate (i).

(ii) Now, assume that $\sum_{i=1}^N \alpha_i = 0$. Then

$$\sum_{i\in J^-} |\alpha_i| = -\sum_{i\in J^-} \alpha_i = \sum_{i\in J^+} \alpha_i = \sum_{i\in J^+} |\alpha_i|.$$

Thus,

$$\sum_{i\in J^-} |\alpha_i| = \sum_{i\in J^+} |\alpha_i| = \frac{1}{2} \sum_{i=1}^N |\alpha_i|.$$

This yields

$$\begin{aligned}
\sum_{i,j=1}^N \Delta_{ij} &= -2 \left\{ \big(|J^0| + 2|J^-|\big) \sum_{i\in J^+} |\alpha_i| + \big(|J^0| + 2|J^+|\big) \sum_{i\in J^-} |\alpha_i| \right\} \\
&= -2 \left\{ \big(|J^0| + |J^-| + |J^+|\big) \sum_{i=1}^N |\alpha_i| \right\} = -2N \sum_{i=1}^N |\alpha_i|.
\end{aligned}$$

□

Now we ready to prove the ℓ^1-stability result. First we provide $l = 1$ case.

Proposition 4.2. *Suppose that the framework* $(\mathcal{F})$ *with* $l = 1$ *for initial data* Θ^0 *and* $\tilde{\Theta}^0$ *hold, and let* $\Theta := \Theta(t,z)$ *and* $\tilde{\Theta} := \tilde{\Theta}(t,z)$ *be two solutions* (1.2) *with initial data* Θ^0 *and* $\tilde{\Theta}^0$*, respectively satisfying*

$$D(\partial_z^r \tilde{\Theta}^0(z)) \le D(\partial_z^r \Theta^0(z)), \quad \text{for} \quad r = 0,\ 1, \quad \text{and each} \quad z \in \Omega.$$

Then for all $t \ge 0$ *and each* $z \in \Omega$*,*

$$\frac{\partial}{\partial t}\|\partial_z(\Theta - \tilde{\Theta})(t,z)\|_1 + \Lambda_1(t,z) \le \mathcal{E}_1(z)\|(\Theta - \tilde{\Theta})(t,z)\|_1,$$

where the non-negative functional $\Lambda_1(s,z)$ *is defined by*

$$\begin{aligned}\Lambda_1(s,z) := \frac{\kappa_m(z)\cos D(\Theta^0(z))}{N}\bigg[&\big(|I_1^0(s)| + 2|I_1^-(s)|\big) \sum_{i \in I_1^+(s)} |(\partial_z\theta_i - \partial_z\tilde{\theta}_i)(s)| \\ &+ \big(|I_1^0(s)| + 2|I_1^+(s)|\big) \sum_{i \in I_1^-(s)} |(\partial_z\theta_i - \partial_z\tilde{\theta}_i)(s)|\bigg],\end{aligned}$$

and the random variable $\mathcal{E}_1(z)$ *is given by*

$$\mathcal{E}_1(z) := 2\|\partial_z\kappa(z)\|_\infty \cos D(\Theta^0(z)) + 2\|\kappa(z)\|_\infty \sin D(\Theta^0(z))\tilde{C}_1(z).$$

Proof. For each $z \in \Omega$, it follows from (3.11) that

$$\begin{aligned}&\frac{\partial}{\partial t}\sum_{i=1}^N |(\partial_z\theta_i - \partial_z\tilde{\theta}_i)(t)| \\ &= \frac{1}{N}\sum_{i,j=1}^N \mathrm{sgn}(\partial_z\theta_i - \partial_z\tilde{\theta}_i)\partial_z\kappa_{ij}(z)(\sin(\theta_j - \theta_i) - \sin(\tilde{\theta}_j - \tilde{\theta}_i)) \\ &\quad + \frac{1}{N}\sum_{i,j=1}^N \mathrm{sgn}(\partial_z\theta_i - \partial_z\tilde{\theta}_i)\kappa_{ij}(z)(\cos(\theta_j - \theta_i) - \cos(\tilde{\theta}_j - \tilde{\theta}_i))(\partial_z\theta_j - \partial_z\theta_i) \\ &\quad + \frac{1}{N}\sum_{i,j=1}^N \mathrm{sgn}(\partial_z\theta_i - \partial_z\tilde{\theta}_i)\kappa_{ij}(z)\cos(\tilde{\theta}_j - \tilde{\theta}_i)(\partial_z\theta_j - \partial_z\theta_i - \partial_z\tilde{\theta}_j + \partial_z\tilde{\theta}_i) \\ &=: \sum_{k=1}^3 \mathcal{I}_{5k}.\end{aligned}$$

Now we estimate $\mathcal{I}_{5k}$ (k =1, 2, 3) separately.

• Case E.1 (Estimate for $\mathcal{I}_{51}$) : In this case,

$$\begin{aligned}\mathcal{I}_{51} &= \frac{2}{N}\sum_{i,j=1}^N \mathrm{sgn}(\partial_z\theta_i - \partial_z\tilde{\theta}_i)\partial_z\kappa_{ij}\cos\left(\frac{\theta_j - \theta_i}{2} + \frac{\tilde{\theta}_j - \tilde{\theta}_i}{2}\right)\sin\left(\frac{\theta_j - \theta_i}{2} - \frac{\tilde{\theta}_j - \tilde{\theta}_i}{2}\right) \\ &\le \frac{2}{N}\sum_{i,j=1}^N |\partial_z\kappa_{ij}|\cos D(\Theta^0)\left(\left|\frac{\theta_j - \tilde{\theta}_j}{2}\right| + \left|\frac{\theta_i - \tilde{\theta}_i}{2}\right|\right) \\ &\le 2\|\partial_z\kappa\|_\infty \cos D(\Theta^0)\|(\Theta - \tilde{\Theta})(t,z)\|_1.\end{aligned}$$

• Case E.2 (Estimate for $\mathcal{I}_{52}$): Similarly, we have

$$\begin{aligned}\mathcal{I}_{52} &\le \frac{2}{N}\sum_{i,j=1}^{N}\kappa_{ij}\sin\left|\frac{\theta_j-\theta_i}{2}+\frac{\tilde\theta_j-\tilde\theta_i}{2}\right|\sin\left|\frac{\theta_j-\theta_i}{2}-\frac{\tilde\theta_j-\tilde\theta_i}{2}\right||\partial_z\theta_j-\partial_z\theta_i|\\ &\le \frac{2\|\kappa\|_\infty\sin D(\Theta^0)D(\partial_z\Theta(t,z))}{N}\sum_{i,j=1}^{N}\sin\left|\frac{\theta_j-\tilde\theta_j}{2}-\frac{\theta_i-\tilde\theta_i}{2}\right|\\ &\le \frac{2\|\kappa\|_\infty\sin D(\Theta^0)D(\partial_z\Theta(t,z))}{N}\sum_{i,j=1}^{N}\left(\left|\frac{\theta_j-\tilde\theta_j}{2}\right|+\left|\frac{\theta_i-\tilde\theta_i}{2}\right|\right)\\ &\le 2\|\kappa\|_\infty\sin D(\Theta^0)\tilde C_1(z)\|(\Theta-\tilde\Theta)(t,z)\|_1.\end{aligned}$$

• Case E.3 (Estimate for $\mathcal{I}_{53}$) : We get

$$\begin{aligned}\mathcal{I}_{53} &= \frac{1}{N}\sum_{i,j=1}^{N}\mathrm{sgn}(\partial_z\theta_i-\partial_z\tilde\theta_i)\kappa_{ij}\cos(\tilde\theta_j-\tilde\theta_i)(\partial_z\theta_j-\partial_z\theta_i-\partial_z\tilde\theta_j+\partial_z\tilde\theta_i)\\ &= \frac{1}{2N}\sum_{i,j=1}^{N}(\mathrm{sgn}(\partial_z\theta_i-\partial_z\tilde\theta_i)-\mathrm{sgn}(\partial_z\theta_j-\partial_z\tilde\theta_j))\kappa_{ij}\cos(\tilde\theta_j-\tilde\theta_i)(\partial_z\theta_j-\partial_z\theta_i-\partial_z\tilde\theta_j+\partial_z\tilde\theta_i)\\ &\le \frac{\kappa_m\cos D(\Theta^0)}{2N}\sum_{i,j=1}^{N}(\mathrm{sgn}(\partial_z\theta_i-\partial_z\tilde\theta_i)-\mathrm{sgn}(\partial_z\theta_j-\partial_z\tilde\theta_j))(\partial_z\theta_j-\partial_z\theta_i-\partial_z\tilde\theta_j+\partial_z\tilde\theta_i),\end{aligned}$$

where we used

$$(\mathrm{sgn}(\alpha)-\mathrm{sgn}(\beta))(\beta-\alpha)\le 0,\quad \alpha,\beta\in\mathbb{R}. \tag{4.24}$$

Now we let $\alpha_i=\partial_z\theta_i-\partial_z\tilde\theta_i$. Then applying Lemma 4.1 completes the proof. Note that J^0, J^+ and J^- become I_1^0, I_1^+ and I_1^- respectively. □

For the case $\theta_c(0,z)=\tilde\theta_c(0,z)$, we have the exponential decay estimate.

Corollary 4.1. *Suppose that the framework* ($\mathcal{F}$) *with* $l=1$ *for initial data* Θ^0 *and* $\tilde\Theta^0$ *hold, and further assume that* $\theta_c(0,z):=\tilde\theta_c(0,z)$. *Let* $\Theta:=\Theta(t,z)$ *and* $\tilde\Theta:=\tilde\Theta(t,z)$ *be two solutions* (1.2) *with initial data* Θ^0 *and* $\tilde\Theta^0$, *respectively satisfying*

$$D(\partial_z^r\tilde\Theta^0(z))\le D(\partial_z^r\Theta^0(z)),\quad \textit{for}\quad r=0,\ 1,\quad \textit{and each}\quad z\in\Omega.$$

Then for all $t\ge 0$ *and each* $z\in\Omega$,

$$\|\partial_z(\Theta-\tilde\Theta)(t,z)\|_1\le\tilde{\mathcal{E}}_1(z)(z)e^{-\kappa_m(z)\cos D(\Theta^0(z))\gamma(z)t}\Big(\|(\Theta^0-\tilde\Theta^0)(z)\|_1+\|\partial_z(\Theta^0-\tilde\Theta^0)(z)\|_1\Big),$$

where $\gamma(z):=\frac{\sin D(\Theta^0(z))}{D(\Theta^0(z))}\in(0,1)$ *and the random variable* $\tilde{\mathcal{E}}_1(z)$ *is given by*

$$\tilde{\mathcal{E}}_1(z):=\max\left\{1,\ \frac{\mathcal{E}_1(z)}{\kappa_m(z)\cos D(\Theta^0(z))(1-\gamma(z))}\right\}.$$

Proof. We set

$$\alpha_i=\partial_z\theta_i-\partial_z\tilde\theta_i.$$

Since $\theta_c(0,z) = \tilde{\theta}_c(0,z)$,

$$\sum_{i=1}^N \alpha_i = 0.$$

On the other hand, in the course of proof of Proposition 4.1,

$$\mathcal{I}_{53} \le -\kappa_m \cos D(\Theta^0)\|\partial_z(\Theta - \tilde{\Theta})(t,z)\|_1.$$

Hence

$$\begin{aligned}
&\frac{\partial}{\partial t}\|\partial_z(\Theta - \tilde{\Theta})(t,z)\|_1 \\
&\quad \le -\kappa_m \cos D(\Theta^0)\|\partial_z(\Theta - \tilde{\Theta})(t,z)\|_1 + \mathcal{E}_1(z)\|(\Theta - \tilde{\Theta})(t,z)\|_1 \\
&\quad \le -\kappa_m \cos D(\Theta^0)\|\partial_z(\Theta - \tilde{\Theta})(t,z)\|_1 + \mathcal{E}_1(z)|(\Theta^0 - \tilde{\Theta}^0)(z)\|_1 e^{-\kappa_m \gamma \cos D(\Theta^0)t},
\end{aligned} \tag{4.25}$$

where we used the estimate (ii) of Proposition 4.1:

$$\|(\Theta - \tilde{\Theta})(t,z)\|_1 \le \|(\Theta^0 - \tilde{\Theta}^0)(z)\|_1 e^{-\kappa_m \gamma \cos D(\Theta^0)t}.$$

We now apply Grönwall's lemma in Lemma 2.2 to (4.25) to obtain

$$\begin{aligned}
\|\partial_z(\Theta - \tilde{\Theta})(t,z)\|_1 &\le \|\partial_z(\Theta^0 - \tilde{\Theta}^0)(z)\|_1 e^{-\kappa_m \cos D(\Theta^0)t} \\
&\quad + \frac{\mathcal{E}_1(z)e^{-\kappa_m \cos D(\Theta^0)\gamma t}}{\kappa_m \cos D(\Theta^0)(1-\gamma)}\|(\Theta^0 - \tilde{\Theta}^0)(z)\|_1(1 - e^{-\kappa_m \cos D(\Theta^0)(1-\gamma)t}),
\end{aligned}$$

which yields our desired result. □

Finally, we provide the stability result for higher-order derivatives.

Theorem 4.1. (Higher-order estimates) *Suppose that the framework $(\mathcal{F})$ for initial data Θ^0 and $\tilde{\Theta}^0$ hold, and let $\Theta := \Theta(t,z)$ and $\tilde{\Theta} := \tilde{\Theta}(t,z)$ be two solutions* (1.2) *with initial data Θ^0 and $\tilde{\Theta}^0$, respectively satisfying*

$$D(\partial_z^r \tilde{\Theta}^0(z)) \le D(\partial_z^r \Theta^0(z)), \quad \text{for any} \quad r = 0,\ 1,\ \cdots,\ l \quad \text{and each} \quad z \in \Omega.$$

Then for all $t \ge 0$ and each $z \in \Omega$,

$$\frac{\partial}{\partial t}\|\partial_z^l(\Theta - \tilde{\Theta})(t,z)\|_1 + \Lambda_l(t,z) \le \mathcal{E}_l(z)\sum_{p=0}^{l-1}\|\partial_z^p(\Theta - \tilde{\Theta})(t,z)\|_1,$$

where the non-negative functional $\Lambda_l(s,z)$ is defined by

$$\begin{aligned}
\Lambda_l(s,z) := &\frac{\kappa_m(z)\cos D(\Theta^0(z))}{N} \\
&\times \Big[\big(|I_l^0(s)| + 2|I_l^-(s)|\big) \sum_{i \in I_l^+(s)} |(\partial_z^l \theta_i - \partial_z^l \tilde{\theta}_i)(s)| \\
&\quad + \big(|I_l^0(s)| + 2|I_l^+(s)|\big) \sum_{i \in I_l^-(s)} |(\partial_z^l \theta_i - \partial_z^l \tilde{\theta}_i)(s)| \Big],
\end{aligned}$$

and the random variable $\mathcal{E}_l(z)$ is given by

$$\mathcal{E}_l(z) := 2\|\kappa(z)\|_\infty \sin D(\Theta^0(z))\tilde{C}_l(z)$$

$$
\begin{aligned}
&+ 2\|\kappa\|_\infty \left(\sum_{\substack{(k_1,\ \cdots,\ k_l)\in\Lambda(l)\\ k_l=0}} \frac{l!}{k_1!\cdots k_{l-1}!} P(l-1, k_1, \cdots, k_{l-1}) \right) \\
&+ \left(2(l-1) \sum_{\substack{1\le r\le l-1\\ (k_1,\ \cdots,\ k_r)\in\Lambda(r)}} \binom{l}{r} \|\partial_z^{l-r}\kappa(z)\|_\infty \frac{r!}{k_1!\cdots k_r!} P(r, k_1, \cdots, k_r, z) \right) \\
&+ 2\|\partial_z^l \kappa(z)\|_\infty \cos D(\Theta^0(z)),
\end{aligned}
$$

where the random variable $P(r, k_1, \cdots, k_r, z)$ *is given by*

$$
P(r, k_1, \cdots, k_r, z) := \prod_{p=1}^{r} \left| \frac{\tilde{C}_p(z)}{p!} \right|^{k_p} + \left[\sum_{\substack{1\le p\le r\\ k_p\neq 0}} k_p \left(\tilde{C}_p(z) \right)^{k_p-1} \left\{ \prod_{\substack{0\le \mu\le r+1\\ \mu\neq p}} \left(\frac{\tilde{C}_\mu(z)}{\mu!} \right)^{k_\mu} \right\} \right].
$$

Proof. It follows from (2.6) that

$$
\begin{aligned}
&\partial_t(\partial_z^l \theta_i)(t,z) \\
&= \partial_z^l \nu_i(z) + \frac{1}{N}\sum_{j=1}^{N} \kappa_{ij}(z)\cos(\theta_j-\theta_i)(\partial_z^l\theta_j - \partial_z^l\theta_i) \\
&+ \frac{1}{N} \sum_{\substack{1\le j\le N\\ (k_1,\ \cdots,\ k_l)\in\Lambda(l)\\ k_l=0}} \kappa_{ij}(z) \frac{l!}{k_1!\cdots k_{l-1}!} \mathcal{M}(l-1, k_1, \cdots, k_{l-1}, \Theta, i, j) \\
&+ \frac{1}{N} \sum_{\substack{1\le j\le N\\ 1\le r\le l-1\\ (k_1,\ \cdots,\ k_r)\in\Lambda(r)}} \binom{l}{r} \partial_z^{l-r}\kappa_{ij}(z) \frac{r!}{k_1!\cdots k_r!} \mathcal{M}(r, k_1, \cdots, k_r, \Theta, i, j) \\
&+ \frac{1}{N} \sum_{1\le j\le N} \partial_z^l \kappa_{ij}(z) \sin(\theta_j - \theta_i),
\end{aligned}
$$

where $\mathcal{M}(r, k_1, \cdots, k_r, \Theta, i, j)$ is defined in the proof of Theorem 3.1. Recall that the functional $\mathcal{M}(r, k_1, \cdots, k_r, \Theta, i, j)$ has the following form:

$$
\mathcal{M}(r, k_1, \cdots, k_r, \Theta, i, j) := \sin^{(k)}(\theta_j - \theta_i) \prod_{p=1}^{r} \left(\frac{\partial_z^p \theta_j - \partial_z^p \theta_i}{p!} \right)^{k_p},
$$

where $(k_1,\ \cdots,\ k_r) \in (\mathbb{N}\cup\{0\})^r$, $1 \le i,\ j \le N$, $r \in \mathbb{N}$ and $k = k_1 + \cdots + k_r$. And also for more simplicity, we let

$$
\mathcal{M}_r := \mathcal{M}(r, k_1, \cdots, k_r, \Theta, i, j), \quad \tilde{\mathcal{M}}_r := \mathcal{M}(r, k_1, \cdots, k_r, \tilde{\Theta}, i, j).
$$

Now we proceed by induction on l. By using above notations, we can get the following relation:

$$
\begin{aligned}
&\frac{\partial}{\partial t}\|\partial_z^l(\Theta-\tilde{\Theta})(t,z)\|_1 \\
&= \frac{1}{N}\sum_{i,j=1}^{N} \mathrm{sgn}(\partial_z^l\theta_i-\partial_z^l\tilde{\theta}_i)\kappa_{ij}(z)(\cos(\theta_j-\theta_i)-\cos(\tilde{\theta}_j-\tilde{\theta}_i))(\partial_z^l\theta_j-\partial_z^l\theta_i) \\
&\quad + \frac{1}{N}\sum_{i,j=1}^{N} \mathrm{sgn}(\partial_z^l\theta_i-\partial_z^l\tilde{\theta}_i)\kappa_{ij}(z)\cos(\tilde{\theta}_j-\tilde{\theta}_i)(\partial_z^l\theta_j-\partial_z^l\theta_i-\partial_z^l\tilde{\theta}_j+\partial_z^l\tilde{\theta}_i) \\
&\quad + \frac{1}{N}\sum_{\substack{1\le i,j\le N\\ (k_1,\,\cdots,\,k_l)\in\Lambda(l)\\ k_l=0}} \mathrm{sgn}(\partial_z^l\theta_i-\partial_z^l\tilde{\theta}_i)\kappa_{ij}(z)\frac{l!}{k_1!\cdots k_{l-1}!}(\mathcal{M}_{l-1}-\tilde{\mathcal{M}}_{l-1}) \\
&\quad + \frac{1}{N}\sum_{\substack{1\le i,j\le N\\ 1\le r\le l-1\\ (k_1,\,\cdots,\,k_r)\in\Lambda(r)}} \mathrm{sgn}(\partial_z^l\theta_i-\partial_z^l\tilde{\theta}_i)\binom{l}{r}\partial_z^{l-r}\kappa_{ij}(z)\frac{r!}{k_1!\cdots k_r!}(\mathcal{M}_r-\tilde{\mathcal{M}}_r) \\
&\quad + \frac{1}{N}\sum_{i,j=1}^{N} \mathrm{sgn}(\partial_z^l\theta_i-\partial_z^l\tilde{\theta}_i)\partial_z^l\kappa_{ij}(z)(\sin(\theta_j-\theta_i)-\sin(\tilde{\theta}_j-\tilde{\theta}_i)) \\
&=: \sum_{k=1}^{5}\mathcal{I}_{6k}.
\end{aligned}
\tag{4.26}
$$

As we did in Proposition 4.2, we estimate each $\mathcal{I}_{6k}$ (k =1, 2, 3, 4, 5) separately:

- Case F.1 (Estimate for $\mathcal{I}_{61}$):

$$
\begin{aligned}
\mathcal{I}_{61} &\le \frac{2}{N}\sum_{i,j=1}^{N}\kappa_{ij}\sin\left|\frac{\theta_j-\theta_i}{2}+\frac{\tilde{\theta}_j-\tilde{\theta}_i}{2}\right|\sin\left|\frac{\theta_j-\theta_i}{2}-\frac{\tilde{\theta}_j-\tilde{\theta}_i}{2}\right||\partial_z^l\theta_j-\partial_z^l\theta_i| \\
&\le \frac{2\|\kappa\|_\infty\sin D(\Theta^0)D(\partial_z^l\Theta(t,z))}{N}\sum_{i,j=1}^{N}\sin\left|\frac{\theta_j-\tilde{\theta}_j}{2}-\frac{\theta_i-\tilde{\theta}_i}{2}\right| \\
&\le \frac{2\|\kappa\|_\infty\sin D(\Theta^0)D(\partial_z^l\Theta(t,z))}{N}\sum_{i,j=1}^{N}\left(\left|\frac{\theta_j-\tilde{\theta}_j}{2}\right|+\left|\frac{\theta_i-\tilde{\theta}_i}{2}\right|\right) \\
&\le 2\|\kappa\|_\infty\sin D(\Theta^0)\tilde{C}_l(z)\|(\Theta-\tilde{\Theta})(t,z)\|_1.
\end{aligned}
$$

• Case F.2 (Estimate for $\mathcal{I}_{62}$) :

$$
\begin{aligned}
\mathcal{I}_{62} &= \frac{1}{N}\sum_{i,j=1}^{N} \mathrm{sgn}(\partial_z^l\theta_i - \partial_z^l\tilde\theta_i)\kappa_{ij}\cos(\tilde\theta_j - \tilde\theta_i)(\partial_z^l\theta_j - \partial_z^l\theta_i - \partial_z^l\tilde\theta_j + \partial_z^l\tilde\theta_i)\\
&= \frac{1}{2N}\sum_{i,j=1}^{N} (\mathrm{sgn}(\partial_z^l\theta_i - \partial_z^l\tilde\theta_i) - \mathrm{sgn}(\partial_z^l\theta_j - \partial_z^l\tilde\theta_j))\kappa_{ij}\cos(\tilde\theta_j - \tilde\theta_i)(\partial_z^l\theta_j - \partial_z^l\theta_i - \partial_z^l\tilde\theta_j + \partial_z^l\tilde\theta_i)\\
&\le \frac{\kappa_m\cos D(\Theta^0)}{2N}\sum_{i,j=1}^{N} (\mathrm{sgn}(\partial_z^l\theta_i - \partial_z^l\tilde\theta_i) - \mathrm{sgn}(\partial_z^l\theta_j - \partial_z^l\tilde\theta_j))(\partial_z^l\theta_j - \partial_z^l\theta_i - \partial_z^l\tilde\theta_j + \partial_z^l\tilde\theta_i),
\end{aligned}
$$

where we used (4.24). Now we set $\alpha_i := \partial_z^l\theta_i - \partial_z^l\tilde\theta_i$, and apply Lemma 4.1 to get

$$
\mathcal{I}_{82} \le -\Lambda_l(t,z),
$$

and note that J^0, J^+ and J^- become I_l^0, I_l^+ and I_l^- respectively.

• Case F.3 (Estimates for $\mathcal{I}_{63}$ and $\mathcal{I}_{64}$) : Here, it suffices to estimate $\mathcal{M}_r - \tilde{\mathcal{M}}_r$:

$$
\begin{aligned}
&\mathcal{M}_r - \tilde{\mathcal{M}}_r\\
&= \Big(\sin^{(k)}(\theta_j - \theta_i) - \sin^{(k)}(\tilde\theta_j - \tilde\theta_i)\Big)\prod_{p=1}^{r}\left(\frac{\partial_z^p\theta_j - \partial_z^p\theta_i}{p!}\right)^{k_p}\\
&\quad + \Big(\sin^{(k)}(\tilde\theta_j - \tilde\theta_i)\Big)\sum_{p=1}^{r}\left[\left\{\prod_{\mu_1=0}^{p-1}\left(\frac{\partial_z^{\mu_1}\tilde\theta_j - \partial_z^{\mu_1}\tilde\theta_i}{\mu_1!}\right)^{k_{\mu_1}}\right\}\left\{\prod_{\mu_2=p+1}^{r+1}\left(\frac{\partial_z^{\mu_2}\theta_j - \partial_z^{\mu_2}\theta_i}{\mu_2!}\right)^{k_{\mu_2}}\right\}\right.\\
&\quad \left.\times\left\{\left(\frac{\partial_z^p\theta_j - \partial_z^p\theta_i}{p!}\right)^{k_p} - \left(\frac{\partial_z^p\tilde\theta_j - \partial_z^p\tilde\theta_i}{p!}\right)^{k_p}\right\}\right]\\
&=: \mathcal{K}_1 + \mathcal{K}_2,
\end{aligned}
$$

where we set $k_0 = k_{r+1} = 0$. For $\mathcal{K}_1$, note that

$$
\left|\sin^{(k)}(\theta_j - \theta_i) - \sin^{(k)}(\tilde\theta_j - \tilde\theta_i)\right| \le 2\sin\left|\frac{\theta_j - \theta_i}{2} - \frac{\tilde\theta_j - \tilde\theta_i}{2}\right|.
$$

Hence, for $\mathcal{K}_1$,

$$
\begin{aligned}
\mathcal{K}_1 &\le 2\sin\left|\frac{\theta_j - \theta_i}{2} - \frac{\tilde\theta_j - \tilde\theta_i}{2}\right|\prod_{p=1}^{r}\left|\frac{\partial_z^p\theta_j - \partial_z^p\theta_i}{p!}\right|^{k_p}\\
&\le 2\left(\left|\frac{\theta_j - \tilde\theta_j}{2}\right| + \left|\frac{\theta_i - \tilde\theta_i}{2}\right|\right)\prod_{p=1}^{r}\left|\frac{D(\partial_z^p\Theta)}{p!}\right|^{k_p} \le \left(\left|\theta_j - \tilde\theta_j\right| + \left|\theta_i - \tilde\theta_i\right|\right)\prod_{p=1}^{r}\left|\frac{\tilde C_p(z)}{p!}\right|^{k_p}.
\end{aligned}
$$

For the term $\mathcal{K}_2$,

$$
\left|\left(\frac{\partial_z^p\theta_j - \partial_z^p\theta_i}{p!}\right)^{k_p} - \left(\frac{\partial_z^p\tilde\theta_j - \partial_z^p\tilde\theta_i}{p!}\right)^{k_p}\right|
$$

$$\leq \left\{ \begin{array}{ll} 0 & \text{if} \quad k_p = 0, \\ k_p \left(\tilde{C}_p(z)\right)^{k_p - 1} |\partial_z^p \theta_j - \partial_z^p \theta_i - \partial_z^p \tilde{\theta}_j + \partial_z^p \tilde{\theta}_i| & \text{if} \quad k_p > 0, \end{array} \right.$$

where we used the relation: for $n \in \mathbb{N}$ and $0 \leq p \leq l$

$$a^n - b^n = (a-b) \sum_{k=0}^{n-1} a^k b^{n-1-k}, \quad D(\partial_z^p \Theta(t,z)),\ D(\partial_z^p \tilde{\Theta}(t,z)) \leq \tilde{C}_p(z).$$

Note that the second relation follows from the assumption $D(\partial_z^r \tilde{\Theta}^0) \leq D(\partial_z^r \Theta^0)$ for $r = 0,\ 1,\ \cdots, l$. Now this yields

$$\begin{aligned}
\mathcal{K}_2 &\leq \sum_{p=1}^{r} \left[\left\{ \prod_{\substack{0 \leq \mu \leq r+1 \\ \mu \neq p}} \left(\frac{\tilde{C}_\mu(z)}{\mu!} \right)^{k_\mu} \right\} \left\{ \left(\frac{\partial_z^p \theta_j - \partial_z^p \theta_i}{p!} \right)^{k_p} - \left(\frac{\partial_z^p \tilde{\theta}_j - \partial_z^p \tilde{\theta}_i}{p!} \right)^{k_p} \right\} \right] \\
&\leq \sum_{\substack{1 \leq p \leq r \\ k_p \neq 0}} \left[\left\{ \prod_{\substack{0 \leq \mu \leq r+1 \\ \mu \neq p}} \left(\frac{\tilde{C}_\mu(z)}{\mu!} \right)^{k_\mu} \right\} k_p \left(\tilde{C}_p(z)\right)^{k_p - 1} \left(|\partial_z^p \theta_j - \partial_z^p \tilde{\theta}_j| + |\partial_z^p \theta_i - \partial_z^p \tilde{\theta}_i| \right) \right] \\
&\leq \left[\sum_{\substack{1 \leq p \leq r \\ k_p \neq 0}} k_p \left(\tilde{C}_p(z)\right)^{k_p - 1} \left\{ \prod_{\substack{0 \leq \mu \leq r+1 \\ \mu \neq p}} \left(\frac{\tilde{C}_\mu(z)}{\mu!} \right)^{k_\mu} \right\} \right] \\
&\quad \times \sum_{p=1}^{r} \left(|\partial_z^p \theta_j - \partial_z^p \tilde{\theta}_j| + |\partial_z^p \theta_i - \partial_z^p \tilde{\theta}_i| \right).
\end{aligned}$$

We combine estimates for $\mathcal{K}_1$ and $\mathcal{K}_2$ to get

$$|\mathcal{M}_r - \tilde{\mathcal{M}}_r| \leq P(r, k_1, \cdots, k_r, z) \sum_{p=0}^{r} \left(|\partial_z^p \theta_j - \partial_z^p \tilde{\theta}_j| + |\partial_z^p \theta_i - \partial_z^p \tilde{\theta}_i| \right),$$

Now, the terms $\mathcal{I}_{83}$ and $\mathcal{I}_{84}$ can be estimated as follows.

$$\begin{aligned}
\mathcal{I}_{63} &\leq \frac{1}{N} \sum_{\substack{1 \leq i,j \leq N \\ (k_1,\ \cdots,\ k_l) \in \Lambda(l) \\ k_l = 0}} \kappa_{ij} \frac{l!}{k_1! \cdots k_{l-1}!} |\mathcal{M}_{l-1} - \tilde{\mathcal{M}}_{l-1}| \\
&\leq \frac{\|\kappa\|_\infty}{N} \left(\sum_{\substack{(k_1,\ \cdots,\ k_l) \in \Lambda(l) \\ k_l = 0}} \frac{l!}{k_1! \cdots k_{l-1}!} P(l-1, k_1, \cdots, k_{l-1}, z) \right) \\
&\quad \times \sum_{\substack{1 \leq i,j \leq N \\ 0 \leq p \leq l-1}} \left(|\partial_z^p \theta_j - \partial_z^p \tilde{\theta}_j| + |\partial_z^p \theta_i - \partial_z^p \tilde{\theta}_i| \right) \\
&\leq 2\|\kappa\|_\infty \left(\sum_{\substack{(k_1,\ \cdots,\ k_l) \in \Lambda(l) \\ k_l = 0}} \frac{l!}{k_1! \cdots k_{l-1}!} P(l-1, k_1, \cdots, k_{l-1}, z) \right) \sum_{0 \leq p \leq l-1} \|\partial_z^p \Theta - \partial_z^p \tilde{\Theta}\|_1
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{I}_{64} &\le \frac{1}{N} \sum_{\substack{1\le i,j\le N\\ 1\le r\le l-1\\ (k_1,\ \cdots,\ k_r)\in\Lambda(r)}} \binom{l}{r} \|\partial_z^{l-r}\kappa\|_\infty \frac{r!}{k_1!\cdots k_r!} |\mathcal{M}_r - \tilde{\mathcal{M}}_r| \\
&\le \frac{1}{N} \sum_{\substack{1\le r\le l-1\\ (k_1,\ \cdots,\ k_r)\in\Lambda(r)}} \binom{l}{r} \|\partial_z^{l-r}\kappa\|_\infty \frac{r!}{k_1!\cdots k_r!} P(r,k_1,\cdots,k_r,z) \\
&\quad \times \sum_{i,j=1}^N \sum_{r=1}^{l-1} \sum_{p=0}^{r} \Big(|\partial_z^p\theta_j - \partial_z^p\tilde{\theta}_j| + |\partial_z^p\theta_i - \partial_z^p\tilde{\theta}_i| \Big) \\
&\le \left(2(l-1) \sum_{\substack{1\le r\le l-1\\ (k_1,\ \cdots,\ k_r)\in\Lambda(r)}} \binom{l}{r} \|\partial_z^{l-r}\kappa\|_\infty \frac{r!}{k_1!\cdots k_r!} P(r,k_1,\cdots,k_r,z) \right) \\
&\quad \times \sum_{p=0}^{l-1} \|\partial_z^p(\Theta - \tilde{\Theta})(t,z)\|_1.
\end{aligned}$$

- Case F.4 (Estimate for $\mathcal{I}_{65}$) : Finally, we can estimate

$$\begin{aligned}
\mathcal{I}_{65} &= \frac{2}{N} \sum_{i,j=1}^N \mathrm{sgn}(\partial_z^l\theta_i - \partial_z^l\tilde{\theta}_i)\partial_z^l\kappa_{ij} \cos\left(\frac{\theta_j-\theta_i}{2} + \frac{\tilde{\theta}_j-\tilde{\theta}_i}{2}\right) \sin\left(\frac{\theta_j-\theta_i}{2} - \frac{\tilde{\theta}_j-\tilde{\theta}_i}{2}\right) \\
&\le \frac{2}{N} \sum_{i,j=1}^N |\partial_z^l\kappa_{ij}| \cos D(\Theta^0) \left(\left|\frac{\theta_j-\tilde{\theta}_j}{2}\right| + \left|\frac{\theta_i-\tilde{\theta}_i}{2}\right| \right) \\
&\le 2\|\partial_z^l\kappa\|_\infty \cos D(\Theta^0) \|(\Theta-\tilde{\Theta})(t,z)\|_1.
\end{aligned}$$

In (4.26), we combine all estimates for $\mathcal{I}_{6k}$ to obtain

$$\frac{\partial}{\partial t}\|\partial_z^l(\Theta-\tilde{\Theta})(t,z)\|_1 \le -\Lambda_l(t,z) + \mathcal{E}_l(z)\sum_{p=0}^{l-1}\|\partial_z^p(\Theta-\tilde{\Theta})(t,z)\|_1.$$

Then, Gronwall's lemma yields the desired estimate. □

Corollary 4.2. *Suppose that the framework* $(\mathcal{F})$ *with* $r = l \in \mathbb{N}$ *for initial data* Θ^0 *and* $\tilde{\Theta}^0$ *hold, and further assume* $\theta_c(0,z) = \tilde{\theta}_c(0,z)$ *for every* $z \in \Omega$. *Let* $\Theta := \Theta(t,z)$ *and* $\tilde{\Theta} := \tilde{\Theta}(t,z)$ *be two solutions* (1.2) *with initial data* Θ^0 *and* $\tilde{\Theta}^0$, *respectively satisfying*

$$D(\partial_z^r\tilde{\Theta}^0(z)) \le D(\partial_z^r\Theta^0(z)), \quad \textit{for any} \quad r = 0,\ 1,\ \cdots,\ l \quad \textit{and each} \quad z\in\Omega.$$

Then for all $t \ge 0$ *and each* $z \in \Omega$,

$$\|\partial_z^l(\Theta-\tilde{\Theta})(t,z)\|_1 \le \tilde{\mathcal{E}}_l(z) e^{-\kappa_m(z)\gamma(\cos D(\Theta^0(z)))t} \sum_{p=0}^{l} \|\partial_z^p(\Theta^0-\tilde{\Theta}^0)(z)\|_1,$$

where the random variable $\tilde{\mathcal{E}}_l(z)$ $(l \geq 2)$ is inductively given by

$$\tilde{\mathcal{E}}_l(z) := \max\left\{1,\ \frac{\left(l\mathcal{E}_l(z)\sum_{p=0}^{l-1}\tilde{\mathcal{E}}_p(z)\right)}{\kappa_m \cos D(\Theta^0)(1-\gamma)}\right\}.$$

Proof. The proof will be done inductively on l.

- (Initial step): In this case, we have already proved this case in Corollary 4.1.

- (Inductive step): Set

$$\alpha_i = \partial_z^l \theta_i - \partial_z^l \tilde{\theta}_i.$$

Then, we have $\sum_{i=1}^N \alpha_i = 0$, since we assumed $\theta_c(0,z) = \tilde{\theta}_c(0,z)$. Thus, it follows from Proposition 4.2 that

$$\Lambda_l(t,z) = -\kappa_m \cos D(\Theta^0)\|\partial_z^l(\Theta - \tilde{\Theta})(t,z)\|_1.$$

On the other hand, it follows from Theorem 4.1 that

$$\begin{aligned}
&\frac{\partial}{\partial t}\|\partial_z^l(\Theta - \tilde{\Theta})(t,z)\|_1 \\
&\quad \leq -\kappa_m \cos D(\Theta^0)\|\partial_z^l(\Theta - \tilde{\Theta})(t,z)\|_1 + \mathcal{E}_l(z)\sum_{p=0}^{l-1}\|\partial_z^p(\Theta - \tilde{\Theta})(t,z)\|_1.
\end{aligned} \tag{4.27}$$

Note that from (ii) of Proposition 4.2 and induction hypothesis,

$$\|\partial_z^p(\Theta - \tilde{\Theta})(t,z)\|_1 \leq \tilde{\mathcal{E}}_p(z) e^{-\kappa_m \cos D(\Theta^0)\gamma t}\sum_{s=0}^{p}\|\partial_z^s(\Theta^0 - \tilde{\Theta}^0)(z)\|_1$$

for each $p \in \mathbb{N} \cup \{0\}$. We apply this to (4.27) and for every $l \geq 1$, to get

$$\begin{aligned}
&\frac{\partial}{\partial t}\|\partial_z^l(\Theta - \tilde{\Theta})(t,z)\|_1 \\
&\quad \leq -\kappa_m \cos D(\Theta^0)\|\partial_z^l(\Theta - \tilde{\Theta})(t,z)\|_1 \\
&\qquad + \mathcal{E}_l(z) e^{-\kappa_m \cos D(\Theta^0)\gamma t}\sum_{p=0}^{l-1}\tilde{\mathcal{E}}_p(z)\sum_{s=0}^{p}\|\partial_z^s(\Theta^0 - \tilde{\Theta}^0)(z)\|_1 \\
&\quad \leq -\kappa_m \cos D(\Theta^0)\|\partial_z^l(\Theta - \tilde{\Theta})(t,z)\|_1 \\
&\qquad + \left(l\mathcal{E}_l(z)\sum_{p=0}^{l-1}\tilde{\mathcal{E}}_p(z)\right)\sum_{p=0}^{l-1}\|\partial_z^p(\Theta^0 - \tilde{\Theta}^0)(z)\|_1 e^{-\kappa_m \cos D(\Theta^0)\gamma t}.
\end{aligned} \tag{4.28}$$

For (4.28), we use Grönwall's inquality in Lemma 2.2 to yield

$$\begin{aligned}
&\|\partial_z^l(\Theta - \tilde{\Theta})(t,z)\|_1 \\
&\quad \le \|\partial_z^l(\Theta^0 - \tilde{\Theta}^0)(z)\|_1 e^{-\kappa_m \cos D(\Theta^0) t} \\
&\quad + \frac{\left(l\mathcal{E}_l(z)\sum_{p=0}^{l-1}\tilde{\mathcal{E}}_p(z)\right)e^{-\kappa_m \cos D(\Theta^0)\gamma t}}{\kappa_m \cos D(\Theta^0)(1-\gamma)} \sum_{p=0}^{l-1} \|\partial_z^p(\Theta^0 - \tilde{\Theta}^0)(z)\|_1 (1 - e^{-\kappa_m \cos D(\Theta^0) t}), \\
&\quad \le \tilde{\mathcal{E}}_l(z) e^{-\kappa_m(z)\gamma \cos D(\Theta^0) t} \sum_{p=0}^{l} \|\partial_z^p(\Theta^0 - \tilde{\Theta}^0)(z)\|_1.
\end{aligned}$$

This yields our desired result. □

5. Local sensitivity analysis for frequency process

In this section, we present a synchronizing property of frequency variations in a random space in a large coupling regime. As noticed in Proposition 2.1, under some conditions on natural frequencies, coupling strengths and initial data, we can find a positively invariant set and "vanishing of uncertainty" for frequency processes which are consistent with emergent dynamics of the deterministic Kuramoto model. Below, we will show that the variations $\partial_z^l V = (\partial_z^l \omega_1, \cdots, \partial_z^l \omega_N)$ will also enjoy a synchronizing property in a large coupling regime, which clearly exhibits vanishing of uncertainty.

Lemma 5.1. *Suppose that the framework $(\mathcal{F})$ in Section 3 with $l = 1$ hold, and let $\Theta = \Theta(t,z)$ be a solution to system (1.2). Then, for $z \in \Omega$,*

$$D(\partial_z V(t,z)) \le \mathcal{F}_1(z) e^{-\frac{\kappa_m(z)(\cos D(\Theta^0))t}{2}},$$

where $C(z)$ is a nonnegative random variable given by

$$\mathcal{F}_1(z) := D(\partial_z V^0(z)) + \frac{4\left(\|\partial_z \kappa(z)\|_\infty D(V^0(z)) + \|\kappa(z)\|_\infty \tilde{C}_1(z) D(V^0(z))\right)}{\kappa_m(z)\cos D(\Theta^0(z))}.$$

Proof. First, we differentiate (1.2) with respect to t to obtain

$$\partial_t \omega_i(t,z) = \frac{1}{N}\sum_{j=1}^N \kappa_{ij}(z)\cos(\theta_j(t,z) - \theta_i(t,z))(\omega_j(t,z) - \omega_i(t,z)). \tag{5.1}$$

We again differentiate the above relation with respect to z to get

$$\begin{aligned}
&\partial_t \partial_z \omega_i(t,z) \\
&\quad = \frac{1}{N}\sum_{j=1}^N \partial_z \kappa_{ij}(z)\cos(\theta_j(t,z) - \theta_i(t,z))(\omega_j(t,z) - \omega_i(t,z)) \\
&\quad - \frac{1}{N}\sum_{j=1}^N \kappa_{ij}(z)\sin(\theta_j(t,z) - \theta_i(t,z))(\partial_z\theta_j(t,z) - \partial_z\theta_i(t,z))(\omega_j(t,z) - \omega_i(t,z)) \\
&\quad + \frac{1}{N}\sum_{j=1}^N \kappa_{ij}(z)\cos(\theta_j(t,z) - \theta_i(t,z))(\partial_z\omega_j(t,z) - \partial_z\omega_i(t,z)).
\end{aligned} \tag{5.2}$$

We choose indices M_1' and m_1' such that for $t > 0$ and $z \in \Omega$,

$$\partial_z \omega_{M_1'}(t,z) := \max_i \partial_z \omega_i(t,z), \quad \partial_z \omega_{m_1'}(t,z) := \min_i \partial_z \omega_i(t,z).$$

Note that for every $z \in \Omega$, $\partial_z \omega_{M_1'}(\cdot, z)$ and $\partial_z \omega_{m_1'}(\cdot, z)$ are piecewise differentiable and Lipschitz continuous with respect to t.

• Case F (Estimate for $\partial_z \omega_{M_1'}$): We use (5.2) to obtain, for a.e. $t \geq 0$,

$$\begin{aligned}
&\partial_t \partial_z \omega_{M_1'}(t) \\
&\quad = \frac{1}{N}\sum_{j=1}^N \partial_z \kappa_{M_1',j}(z) \cos(\theta_j(t) - \theta_{M_1'}(t))(\omega_j(t) - \omega_{M_1'}(t)) \\
&\qquad - \frac{1}{N}\sum_{j=1}^N \kappa_{M_1',j}(z) \sin(\theta_j(t) - \theta_{M_1'}(t))(\partial_z\theta_j(t) - \partial_z\theta_{M_1'}(t))(\omega_j(t) - \omega_{M_1'}(t)) \\
&\qquad + \frac{1}{N}\sum_{j=1}^N \kappa_{M_1',j}(z) \cos(\theta_j(t) - \theta_{M_1'}(t))(\partial_z\omega_j(t) - \partial_z\omega_{M_1'}(t)) \\
&\quad =: \sum_{k=1}^3 \mathcal{I}_{7k}.
\end{aligned} \tag{5.3}$$

We use the result and same arguments in Proposition 2.1 and Lemma 3.1 to obtain

$$\begin{aligned}
\mathcal{I}_{71} &\leq \|\partial_z \kappa\|_\infty D(V(t,z)) \leq \|\partial_z\kappa\|_\infty D(V^0(z)) e^{-\kappa_m \cos D(\Theta^0) t}, \\
\mathcal{I}_{72} &\leq \frac{\|\kappa\|_\infty}{N} \sum_{j=1}^N |\partial_z\theta_j(t,z) - \partial_z\theta_{M_1'}(t,z)||\omega_j(t,z) - \omega_{M_1'}(t,z)| \\
&\leq \|\kappa\|_\infty D(\partial_z\Theta(t,z)) D(V(t,z)) \\
&\leq \|\kappa\|_\infty \tilde{C}_1(z) D(V^0(z)) e^{-\kappa_m \cos D(\Theta^0) t}, \\
\mathcal{I}_{73} &\leq \frac{1}{N}\sum_{j=1}^N \kappa_m \cos D(\Theta^0)(\partial_z\omega_j(t,z) - \partial_z\omega_{M_1'}(t,z)) \\
&\leq -\kappa_m \cos D(\Theta^0) \partial_z \omega_{M_1'}(t,z).
\end{aligned} \tag{5.4}$$

In (5.3), we combine all estimates in (5.4) to obtain

$$\begin{aligned}
\partial_t \partial_z \omega_{M_1'}(t,z) &\leq -\kappa_m \cos D(\Theta^0) \partial_z \omega_{M_1'} \\
&\qquad + \Big(\|\partial_z\kappa\|_\infty D(V^0(z)) + \|\kappa\|_\infty \tilde{C}_1(z) D(V^0(z)) \Big) e^{-\kappa_m \cos D(\Theta^0) t}.
\end{aligned} \tag{5.5}$$

Similarly,

$$\begin{aligned}
\partial_t \partial_z \omega_{m_1'}(t,z) &\geq -\kappa_m \cos D(\Theta^0) \partial_z \omega_{m_1'} \\
&\qquad - \Big(\|\partial_z\kappa\|_\infty D(V^0(z)) + \|\kappa\|_\infty \tilde{C}_1(z) D(V^0(z)) \Big) e^{-\kappa_m \cos D(\Theta^0) t}.
\end{aligned} \tag{5.6}$$

Now, we combine (5.5) and (5.6) to get

$$\partial_t D(\partial_z V(t,z)) \leq -\kappa_m \cos D(\Theta^0(z)) D(\partial_z V(t,z))$$

$$+ 2 \left(\|\partial_z \kappa\|_\infty D(V^0(z)) + \|\kappa\|_\infty \tilde{C}_1(z) D(V^0(z)) \right) e^{-\kappa_m \cos D(\Theta^0(z))t}.$$

Then, set

$$y := D(\partial_z V(t,z)), \quad \alpha := \kappa_m \cos D(\Theta^0(z)) \quad \beta := \frac{\alpha}{2} \quad \text{and}$$
$$C := 2 \left(\|\partial_z \kappa\|_\infty D(V^0(z)) + \|\kappa\|_\infty \tilde{C}_1(z) D(V^0(z)) \right),$$

and apply Lemma 2.2, one gets the desired estimate. □

Next, we provide synchronizing property of $\partial_z^l V$ as follows.

Theorem 5.1. *Suppose that the framework $(\mathcal{F})$ in Section 3 hold, and let $\Theta = \Theta(t,z)$ be a solution to system (1.2). Then, for all $t \geq 0$ and each $z \in \Omega$,*

$$D(\partial_z^l V(t,z)) \leq \mathcal{F}_l(z) e^{-\frac{\kappa_m \cos D(\Theta^0)t}{2}},$$

where the random variable $\mathcal{F}_l(z)$ is inductively given by

$$\begin{aligned} \mathcal{F}_l(z) &:= D(\partial_z^l V^0(z)) + \frac{4}{\kappa_m \cos D(\Theta^0(z))} \\ &\times \Bigg[\|\kappa(z)\|_\infty \tilde{C}_l(z) D(V^0(z)) + \|\partial_z^l \kappa(z)\|_\infty D(V^0(z)) \\ &+ \frac{l}{N} \sum_{\substack{1 \leq j \leq N \\ (k_1, \cdots, k_l) \in \Lambda(l) \\ k_l = 0}} \|\kappa(z)\|_\infty \frac{(l-1)!}{k_1! \cdots k_{l-1}!} Q(l-1, k_1, \cdots, k_{l-1}, z) \\ &+ \frac{1}{N} \sum_{\substack{1 \leq j \leq N \\ 1 \leq r \leq l-1 \\ (k_1, \cdots, k_r) \in \Lambda(r)}} \binom{l}{r} \|\partial_z^{l-r} \kappa(z)\|_\infty \frac{r!}{k_1! \cdots k_r!} Q(r, k_1, \cdots, k_r, z) \Bigg], \end{aligned}$$

where the random variable $Q(r, k_1, \cdots, k_r, z)$ is given by

$$Q(r, k_1, \cdots, k_r, z) := D(V^0(z)) \prod_{p=1}^r \left(\frac{\tilde{C}_p(z)}{p!} \right)^{k_p} + \sum_{\substack{1 \leq p \leq r \\ k_p \neq 0}} k_p \left\{ \prod_{s \neq p} \left(\frac{\tilde{C}_s(z)}{s!} \right)^{k_s} \right\} \mathcal{F}_p(z) \left(\frac{\tilde{C}_p(z)}{p!} \right)^{k_p - 1}.$$

Proof. We apply ∂_z^l to (5.1) to obtain

$$\begin{aligned} &\partial_t (\partial_z^l \omega_i)(t,z) \\ &= \frac{1}{N} \sum_{j=1}^N \kappa_{ij}(z) \cos(\theta_j - \theta_i)(\partial_z^l \omega_j - \partial_z^l \omega_i) \\ &- \frac{1}{N} \sum_{j=1}^N \kappa_{ij}(z) \sin(\theta_j - \theta_i)(\partial_z^l \theta_j - \partial_z^l \theta_i)(\omega_j - \omega_i) \\ &+ \frac{l}{N} \sum_{\substack{1 \leq j \leq N \\ (k_1, \cdots, k_l) \in \Lambda(l) \\ k_l = 0}} \kappa_{ij}(z) \frac{(l-1)!}{k_1! \cdots k_{l-1}!} \frac{\partial}{\partial t} \mathcal{M}(l-1, k_1, \cdots, k_{l-1}, \Theta, i, j) \end{aligned}$$

$$+\frac{1}{N}\sum_{\substack{1\le j\le N\\ 1\le r\le l-1\\ (k_1,\ \cdots,\ k_r)\in\Lambda(r)}}\binom{l}{r}\partial_z^{l-r}\kappa_{ij}(z)\frac{r!}{k_1!\cdots k_r!}\frac{\partial}{\partial t}\mathcal{M}(r,k_1,\ \cdots,\ k_r,\Theta,i,j)$$
$$+\frac{1}{N}\sum_{1\le j\le N}\partial_z^l\kappa_{ij}(z)\cos(\theta_j-\theta_i)(\omega_j-\omega_i).$$

As in the proof of Theorem 3.1, we will proceed by induction. First we choose indices M_l' and m_l' such that for $t>0$ and $z\in\Omega$,

$$\partial_z^l\omega_{M_l'}(t,z):=\max_i\partial_z^l\omega_i(t,z),\quad \partial_z^l\omega_{m_l'}(t,z):=\min_i\partial_z^l\omega_i(t,z).$$

Note that for every $z\in\Omega$, $\partial_z^l\omega_{M_l'}(\cdot,z)$ and $\partial_z^l\omega_{m_l'}(\cdot,z)$ are piecewise differentiable and Lipschitz continuous with respect to t. Now we consider the estimate $\partial_z^l\omega_{M_l'}$ as follows: for a.e. $t\ge 0$,

$$\begin{aligned}
&\partial_t(\partial_z^l\omega_{M_l'})(t,z)\\
&=\frac{1}{N}\sum_{j=1}^N\kappa_{M_l',j}(z)\cos(\theta_j-\theta_{M_l'})(\partial_z^l\omega_j-\partial_z^l\omega_{M_l'})\\
&-\frac{1}{N}\sum_{j=1}^N\kappa_{M_l',j}(z)\sin(\theta_j-\theta_{M_l'})(\partial_z^l\theta_j-\partial_z^l\theta_{M_l'})(\omega_j-\omega_{M_l'})\\
&+\frac{l}{N}\sum_{\substack{1\le j\le N\\ (k_1,\ \cdots,\ k_l)\in\Lambda(l)\\ k_l=0}}\kappa_{M_l',j}(z)\frac{(l-1)!}{k_1!\cdots k_{l-1}!}\frac{\partial}{\partial t}\mathcal{M}(l-1,k_1,\ \cdots,\ k_{l-1},\Theta,M_l^{'},j)\\
&+\frac{1}{N}\sum_{\substack{1\le j\le N\\ 1\le r\le l-1\\ (k_1,\ \cdots,\ k_r)\in\Lambda(r)}}\binom{l}{r}\partial_z^{l-r}\kappa_{M_l',j}(z)\frac{r!}{k_1!\cdots k_r!}\frac{\partial}{\partial t}\mathcal{M}(r,k_1,\ \cdots,\ k_r,\Theta,M_l^{'},j)\\
&+\frac{1}{N}\sum_{1\le j\le N}\partial_z^l\kappa_{M_l'j}(z)\cos(\theta_j-\theta_{M_l'})(\omega_j-\omega_{M_l'})\\
&=:\sum_{k=1}^5\mathcal{I}_{8k}.
\end{aligned}$$

Next, we estimate the terms $\mathcal{I}_{8k}$ as follows.

- Case G.1 (Estimate of $\mathcal{I}_{81}$ and $\mathcal{I}_{82}$): In this case,

$$\begin{aligned}
\mathcal{I}_{81}&\le -\kappa_m\cos D(\Theta^0)\partial_z^l\omega_{M_l'},\\
\mathcal{I}_{82}&\le \|\kappa\|_\infty D(\partial_z^l\Theta(t,z))D(V(t,z))\le\|\kappa\|_\infty\tilde{C}_l(z)D(V^0(z))e^{-\kappa_m\cos D(\Theta^0)t}.
\end{aligned}$$

- Case G.2 (Estimate of $\mathcal{I}_{83}$ and $\mathcal{I}_{84}$): We first estimate $\frac{\partial}{\partial t}\Big(\mathcal{M}(r,k_1,\cdots,k_r,\Theta,M_l^{'},j)\Big)$ as follows:

$$
\begin{aligned}
&\frac{\partial}{\partial t}\Big(\mathcal{M}(r,k_1,\cdots,k_r,\Theta,M_l',j)\Big)\\
&\quad=\frac{\partial}{\partial t}\Big(\sin^{(k)}(\theta_j-\theta_{M_l'})\Big)\prod_{p=1}^{r}\left(\frac{\partial_z^p\theta_j-\partial_z^p\theta_{M_l'}}{p!}\right)^{k_p}\\
&\qquad+\sum_{\substack{1\le p\le r\\ k_p\neq 0}}k_p\left\{\prod_{s\neq p}\left(\frac{\partial_z^s\theta_j-\partial_z^s\theta_{M_l'}}{s!}\right)^{k_s}\right\}(\partial_z^p\omega_j-\partial_z^p\omega_{M_l'})\left(\frac{\partial_z^p\theta_j-\partial_z^p\theta_{M_l'}}{p!}\right)^{k_p-1}\\
&\quad\le D(V(t,z))\prod_{p=1}^{r}\left(\frac{\tilde{C}_p(z)}{p!}\right)^{k_p}\\
&\qquad+\sum_{\substack{1\le p\le r\\ k_p\neq 0}}k_p\left\{\prod_{s\neq p}\left(\frac{\tilde{C}_s(z)}{s!}\right)^{k_s}\right\}D(\partial_z^pV(t,z))\left(\frac{\tilde{C}_p(z)}{p!}\right)^{k_p-1}\\
&\quad\le D(V^0(z))\prod_{p=1}^{r}\left(\frac{\tilde{C}_p(z)}{p!}\right)^{k_p}e^{-\kappa_m\cos D(\Theta^0)t}\\
&\qquad+\left[\sum_{\substack{1\le p\le r\\ k_p\neq 0}}k_p\left\{\prod_{s\neq p}\left(\frac{\tilde{C}_s(z)}{s!}\right)^{k_s}\right\}\mathcal{F}_p(z)\left(\frac{\tilde{C}_p(z)}{p!}\right)^{k_p-1}\right]e^{-\frac{\kappa_m\cos D(\Theta^0)t}{2}}\\
&\quad\le Q(r,k_1,\cdots,k_r)e^{-\frac{\kappa_m\cos D(\Theta^0)t}{2}}.
\end{aligned}
$$

This yields

$$
\mathcal{I}_{83}\le\left[\frac{l}{N}\sum_{\substack{1\le j\le N\\ (k_1,\cdots,k_l)\in\Lambda(l)\\ k_l=0}}\|\kappa\|_\infty\frac{(l-1)!}{k_1!\cdots k_{l-1}!}Q(l-1,k_1,\cdots,k_{l-1})\right]e^{-\frac{\kappa_m\cos D(\Theta^0)t}{2}},
$$

$$
\mathcal{I}_{84}\le\left[\frac{1}{N}\sum_{\substack{1\le j\le N\\ 1\le r\le l-1\\ (k_1,\cdots,k_r)\in\Lambda(r)}}\binom{l}{r}\|\partial_z^{l-r}\kappa\|_\infty\frac{r!}{k_1!\cdots k_r!}Q(r,k_1,\cdots,k_r)\right]e^{-\frac{\kappa_m\cos D(\Theta^0)t}{2}}.
$$

• Case G.3 (Estimate of $\mathcal{I}_{85}$): In this case,

$$
\mathcal{I}_{85}\le\|\partial_z^l\kappa\|_\infty D(V(t,z))\le\|\partial_z^l\kappa\|_\infty D(V^0(z))e^{-\kappa_m\cos D(\Theta^0)t}.
$$

Now we combine all results in Case G.1 - Case G.3 to obtain

$$
\partial_t(\partial_z^l\omega_{M_l'})(t,z)\le-\kappa_m\cos D(\Theta^0)\partial_z^l\omega_{M_l'}(t,z)+G(z)e^{-\frac{\kappa_m\cos D(\Theta^0)t}{2}}, \tag{5.7}
$$

where the random variable $G(z)$ is given by

$$
\begin{aligned}
G(z) := {} & \|\kappa\|_\infty \tilde{C}_l(z) D(V^0(z)) + \|\partial_z^l \kappa\|_\infty D(V^0(z)) \\
& + \frac{l}{N} \sum_{\substack{1 \le j \le N \\ (k_1, \cdots, k_l) \in \Lambda(l) \\ k_l = 0}} \|\kappa\|_\infty \frac{(l-1)!}{k_1! \cdots k_{l-1}!} Q(l-1, k_1, \cdots, k_{l-1}) \\
& + \frac{1}{N} \sum_{\substack{1 \le j \le N \\ 1 \le r \le l-1 \\ (k_1, \cdots, k_r) \in \Lambda(r)}} \binom{l}{r} \|\partial_z^{l-r} \kappa\|_\infty \frac{r!}{k_1! \cdots k_r!} Q(r, k_1, \cdots, k_r).
\end{aligned}
$$

Similarly, we have

$$
\partial_t (\partial_z^l \omega_{m_l'})(t, z) \ge -\kappa_m \cos D(\Theta^0) \partial_z^l \omega_{m_l'}(t, z) - G(z) e^{-\frac{\kappa_m \cos D(\Theta^0) t}{2}}. \tag{5.8}
$$

Finally, it follows from (5.7) and (5.8) that we have

$$
\partial_t D(\partial_z^l V(t, z)) \le -\kappa_m \cos D(\Theta^0) D(\partial_z^l V(t, z)) + 2G(z) e^{-\frac{\kappa_m \cos D(\Theta^0) t}{2}}.
$$

Then, Grönwall's lemma yields the desired result. □

6. Conclusion

In this paper, we studied local sensitivity analysis for the random Kuramoto model with pairwise symmetric coupling strengths. More precisely, we provided a sufficient framework leading to the uniform bound for diameter and uniform stability estimate for phase variations and synchronization property of frequency variations. Our framework is explicitly expressed in terms of initial data, distributed natural frequencies and coupling strengths. Our results reveal the stochastic robustness of synchronizing property of the Kuramoto ensemble in a large coupling regime. Of course, there are several unresolved problems to be explored. For example, in a small coupling regime and intermediate coupling regime, the dynamics of the Kuramoto model in a deterministic setting is itself not clearly understood at present, not to mention of uncertainty quantification. More precisely, the phase transition like phenomena from the disordered state to ordered state occurs at a critical coupling strength in a mean-field setting. Thus, how does the uncertainty affects in this phase-transition like process? Another interesting project is to understand the interplay between the mean-field limit and uncertainty, which will be be pursued in the near future.

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