Mean-field sparse optimal control

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We introduce the rigorous limit process connecting finite dimensional sparse optimal control problems with ODE constraints, modelling parsimonious interventions on the dynamics of a moving population divided into leaders and followers, to an infinite dimensional optimal control problem with a constraint given by a system of ODE for the leaders coupled with a PDE of Vlasov-type, governing the dynamics of the probability distribution of the followers. In the classical mean-field theory, one studies the behaviour of a large number of small individuals *freely interacting* with each other, by simplifying the effect of all the other individuals on any given individual by a single averaged effect. In this paper, we address instead the situation where the leaders are actually influenced also by an external *policy maker*, and we propagate its effect for the number N of followers going to infinity. The technical derivation of the sparse mean-field optimal control is realized by the simultaneous development of the mean-field limit of the equations governing the followers dynamics together with the Γ-limit of the finite dimensional sparse optimal control problems.

1. Introduction

In several individual-based models for multi-agent motion the finite-dimensional dynamics in $2d \times N$ variables, where N is the number of individuals and d is the dimension of the space in which the motion of such individuals evolves, is given by

$$\dot{x}_i = v_i$$
$$\text{and} \quad \dot{v}_i = H \star \mu_N(x_i, v_i), \quad i = 1, \ldots N, \ t \in [0, T], \tag{1.1}$$
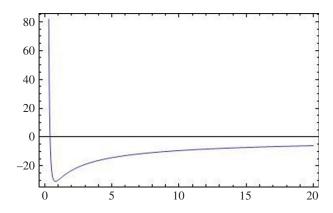
Figure 1. Model function $f(r) = \sigma_r/r^4 - \sigma_a/r^{0.4}$, $\sigma_r > 0$, $\sigma_a > 0$, of small range repulsion and medium-long range attraction. (Online version in colour.)

where $H : \mathbb{R}^{2d} \to \mathbb{R}^d$ is a locally Lipschitz interaction kernel with sublinear growth whose action on the group is modelled by convolution, where the atomic measure

$$\mu_N(t) = \frac{1}{N} \sum_{i=1}^{N} \delta_{(x_i(t), v_i(t))} \tag{1.2}$$

differently represents the group of agents. As a relevant example of this setting, we mention the interaction kernel $H(x, v) := a(|x|)v$, for a bounded non-increasing function $a : \mathbb{R}_+ \to \mathbb{R}_+$, which gives the well-known alignment model of Cucker & Smale [1,2], see also the generalizations in [3], as well as interaction kernels of the type $H(x, v) := f(|x|)x$, where the function $f : \mathbb{R}_+ \to \mathbb{R}$ can encode small range repulsion and medium–long range attraction, as considered in [4] (figure 1).

As discussed in details in the aforementioned papers, such systems can exhibit convergence to certain interesting attractors, representing a higher level of global organization, although such spontaneous coordination may be conditional, depending on the initial configuration. In recent works [5,6], the external control of such systems has been considered in order to promote the collective organization of the group of agents also in those situations where the initial conditions are out of the basin of attraction of the interesting configurations. The emphasis given in this context was on *sparse* controls, meaning that we consider systems

$$\left. \begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= H \star \mu_N(x_i, v_i) + u_i, \quad i = 1, \ldots N, \ t \in [0, T], \end{aligned} \right\} \tag{1.3}$$

and

where $u_i : [0, T] \to \mathbb{R}^d$ are measurable control functions which we wish being vanishing for most of the $i = 1, \ldots, N$ and possibly for most of the $t \in [0, T]$. This choice of controls models the *parsimonious* and *moderate* external intervention of a government of the group, for instance the role of a mediator in an assembly, where the group needs to reach unanimous consensus on a common conduct, as it is the case for the voting system in the Council of the European Union, where unanimous decision are usually targeted.

When the number of the involved agents N is very large, the solution of an optimal control problem for a system of the type (1.3) unfortunately becomes an impossible task because of the *curse of dimensionality*. Already dealing with systems of a few hundreds agents is computationally extremely demanding and often numerically inaccurate. Therefore, we may wonder whether we can describe an appropriate limit dynamics and an optimal control problem for the limit case $N \to +\infty$, which can be re-conducted to computationally manageable dimensionalities. When no control is involved, this procedure is well known as in the classical mean-field theory one studies the evolution of a large number of small individuals *freely interacting* with each other, by simplifying the effect of all the other individuals on any given individual by a single averaged

effect. This results in considering the evolution of the particle density distribution in the state variables, leading to so-called mean-field partial differential equations of Vlasov- or Boltzmann-type [7]. In particular, for our system (1.1), the corresponding mean-field equations are

$$\partial_t \mu + v \cdot \nabla_x \mu = \nabla_v \cdot [(H \star \mu)\mu].$$

We refer to [8] and the references therein for a recent survey on some of the most relevant mathematical aspects on this approach to swarm models. Nevertheless, the proper definition of a limit dynamics when an external control is added to the system, and it is supposed to have some *sparsity* surprisingly remains a difficult task. In fact, the most immediate and perhaps natural approach would be to assign as well to the finite-dimensional control u an atomic vector-valued time-dependent measure

$$\nu_N(t) = \sum_{i=1}^{N} u_i \delta_{(x_i(t), v_i(t))}$$

and consider a proper limit ν for $N \to +\infty$, leading to the controlled PDE

$$\partial_t \mu + v \cdot \nabla_x \mu = \nabla_v \cdot [(H \star \mu)\mu + \nu], \tag{1.4}$$

where now ν represents an external source field. Unfortunately, despite the fact that ν_N are supposed to be the minimizers of certain cost functionals which may allow for the necessary compactness to derive the limit $\nu_N \to \nu$, it seems eventually hard to design a cost functional with a proper meaning in the finite-dimensional model and at the same time promoting a good behaviour of the measure ν. In fact, for the optimal control problems considered for instance in [6, section 5], such a limit procedure does not prevent ν being singular with respect to μ. This means that in the weak formulation of the equation (1.4) the role of ν is essentially mute, it does not interact at all with μ, hence it loses completely its steering purpose. Imaginatively, it is like trying to steer a river by means of toothpicks! Even if we considered in (1.4) the absolutely continuous part $\mu_a = f\mu$ of ν only with respect to μ, if there was any, we would end up with an equation of the type

$$\partial_t \mu + v \cdot \nabla_x \mu = \nabla_v \cdot [(H \star \mu + f)\mu], \tag{1.5}$$

where now f is a force field which is just an L^1-function with respect to the measure μ. Unfortunately, existence and stability of solutions for equations of the type (1.5) is established only for fields f with at least *some* regularity [9]. At this point, it seems that our quest for a proper definition of a *mean-field optimal control* reaches to a dead end, unless we allow for some modelling compromise. The first successful approach actually starts from equation (1.5), by assuming the vector-valued function $f(t, x, v)$ being in a proper compact set of a function space of Carathéodory functions in t and locally Lipschitz continuous functions in (x, v), and proceeding back to reformulate the finite-dimensional modelling, leading to systems of the type

$$
\begin{aligned}
\dot{x}_i &= v_i \\
\dot{v}_i &= H \star \mu_N(x_i, v_i) + f(t, x_i, v_i), \quad i = 1, \dots N, \ t \in [0, T],
\end{aligned}
\Bigg\} \tag{1.6}
$$

and

where now f is a feedback control. This approach has been recently explored in [10], where a proof of a simultaneous Γ-limit and mean-field limit of the finite-dimensional optimal controls for (1.6) to a corresponding infinite dimensional optimal control for (1.5) has been established. We also mention the related work Bensoussan *et al.* [11], where first-order conditions are derived for optimal control problems of equations of the type (1.5) for Lipschitz feedback controls $f(t, x, v)$ in a stochastic setting. Such conditions result in a coupled system of a forward Vlasov-type equation and a backward Hamilton–Jacobi equation, similar to situations encountered in the context of *mean-field games* [12] or the *Nash certainty equivalence* [13]. Certainly, this calls for a renewed enthusiasm and hope, until one realizes that actually the problem of characterizing the optimal controls $f(t, x, v)$ with the purpose of an efficient and manageable numerical computation may not have simplified significantly, as it is not a trivial task to obtain a rigorous derivation and the well-posedness of the corresponding first-order conditions as in [11] in a fully deterministic setting.

This introduces us to the main scope of this paper. Inspired by the successful construction of the coupled $\Gamma-$ and mean-field limits in [10] and the multiscale approach in [14,15], to describe a mixed granular-diffuse dynamics of a crowd, we modify here our modelling: not starting from (1.5), but actually from the initial system (1.1).

The idea is to add to (1.1), or, better, to elect m particular individuals, which interact freely with the N individuals given above. We denote by (y, w) the space-velocity variables of these new individuals. We can consider these m individuals as 'leaders' of the crowd, whereas the other N individuals may be called 'followers'. However, the interpretation given in this paper to the leaders is considering them as few 'discrete representatives' of the entire crowd. In particular, we shall assume that we have a small amount m of leaders/representatives who have a great influence on the population, and a large amount N of followers who have a small influence on the population.

Then, the dynamics, we shall study, is

$$
\left.
\begin{aligned}
\dot{y}_k &= w_k, \\
\dot{w}_k &= H \star \mu_N(y_k, w_k) + H \star \mu_m(y_k, w_k) \quad k = 1, \ldots m, \ t \in [0, T], \\
\dot{x}_i &= v_i \\
\dot{v}_i &= H \star \mu_N(x_i, v_i) + H \star \mu_m(x_i, v_i) \quad i = 1, \ldots N, \ t \in [0, T],
\end{aligned}
\right\}
\tag{1.7}
$$

and

where we considered the additional atomic measure

$$
\mu_m(t) = \frac{1}{m} \sum_{k=1}^{m} \delta_{(y_k(t), w_k(t))},
\tag{1.8}
$$

supported on the trajectories $t \mapsto (y_k(t), w_k(t))$, $k = 1, \ldots, m$. (One can generalize this model to the one where different kernels for the interaction between a leader and a follower, two leaders, etc., are considered. All the results of this paper easily generalize to this setting.) From now on, the notations μ_N and μ_m for the atomic measures representing followers and leaders, respectively, are considered fixed and we shall use them extensively in the rest of the paper. Up to now, the dynamics of the system is similar to a standard multi-agent dynamics for $N + m$ individuals, with the only difference that the actions of leaders and followers have different weights on a single individual, $1/m$ and $1/N$, respectively. Let us now add controls on the m leaders. We obtain the system

$$
\left.
\begin{aligned}
\dot{y}_k &= w_k, \\
\dot{w}_k &= H \star \mu_N(y_k, w_k) + H \star \mu_m(y_k, w_k) + u_k \quad k = 1, \ldots m, \ t \in [0, T], \\
\dot{x}_i &= v_i \\
\dot{v}_i &= H \star \mu_N(x_i, v_i) + H \star \mu_m(x_i, v_i) \quad i = 1, \ldots N, \ t \in [0, T],
\end{aligned}
\right\}
\tag{1.9}
$$

and

where $u_k : [0, T] \to \mathbb{R}^d$ are measurable controls for $k = 1, \ldots, m$, and we define the control map $u : [0, T] \to \mathbb{R}^{md}$ by $u(t) = (u_1(t), \ldots, u_m(t))$ for $t \in [0, T]$. The main difficulty arising in this context is that one usually deals with control functions $u(\cdot)$ that are discontinuous in time. In fact, one needs to consider solutions of the finite-dimensional problem (1.9) in the Carathéodory sense, i.e. functions $t \mapsto (y(t), w(t), x(t), v(t))$ that are absolutely continuous with respect to time and satisfy the integral formulation of (1.9). For the sake of completeness and readability of our results, we report some well-known facts on such solutions in appendix A. In this setting, it makes sense to choose $u \in L^1([0, T], \mathcal{U})$, where $\mathcal{U}$ is a fixed non-empty compact subset of $\mathbb{R}^{d \times m}$ and $\mathcal{U} \subset B(0, U)$ for $U > 0$. Finite-dimensional control problems in this setting are of interest, and we focus on a specific class of control problems, namely optimal control problems in a finite-time horizon with fixed final time. We design a *sparse* control u to drive the whole population of $m + N$ individuals to a given configuration. We model this situation by solving the following optimization problem

$$
\min_{u \in L^1([0, T], \mathcal{U})} \int_0^T \left\{ L(y(t), w(t), \mu_N(t)) + \frac{1}{m} \sum_{k=1}^{m} |u_k(t)| \right\} dt,
\tag{1.10}
$$

where $L(\cdot)$ is a suitable continuous map in its arguments. (For example, one can use L to model the distance between the state variables and the basin of attraction to the interesting configurations. Then, the optimization leads the system to goal-driven dynamics.) The use of (scalar) ℓ^1-norms to penalize controls as in (1.10) dates back to the 1960s with the models of linear fuel consumption [16]. More recent work in dynamical systems [17] resumes again ℓ^1-minimization emphasizing its sparsifying power, i.e. the optimal control $u(t) = (u_1(t), \ldots, u_m(t))$ has mostly vanishing components, in contrast with more classical ℓ^2-norm penalization terms, corresponding to controls with simultaneously many active components. Also in optimal control with partial differential equation constraints, it became rather popular to use L^1-minimization to enforce sparsity of controls [18–24], for instance in the modelling of optimal placing of actuators or sensors.

In order to give precise meaning to the limit of the optimal control problems (1.9)–(1.10) for the number N of followers tending to infinity, we need to address a few technical challenges. As already observed above, owing to the presence of the control $u(\cdot)$, the classical results for the mean-field limit of (1.9) cannot be directly applied, because here the right-hand side is discontinuous in time, see for instance [25–28], where continuity of the right-hand side is assumed. Moreover, only a part of the $m + N$ variables increases in number, whereas the number m of leaders is kept constant. Finally, even a description of the whole population of leaders and followers by a unique measure would not catch the possibility of acting on the leaders only.

As one of our main results, we shall show in theorem 3.3 that, given a control strategy $u \in L^1([0, 1], \mathcal{U})$, it is possible to formally define a mean-field limit of (1.9) when $N \to \infty$ in the following sense: the population is represented by the vector of positions–velocities (y, w) of the leaders coupled with the compactly supported probability measure $\mu \in \mathcal{P}_1(\mathbb{R}^{2d})$ of the followers in the position–velocity space. Here, $\mathcal{P}_1(\mathbb{R}^n)$ denotes the set consisting of all probability measures on $\mathbb{R}^n$ of finite first moment. Then, the mean-field limit will result in a coupled system of an ODE with control for (y, w) and a PDE without control for μ. More precisely, the limit dynamics is described by

$$
\left.
\begin{aligned}
&\dot{y}_k = w_k, \\
&\dot{w}_k = H \star (\mu + \mu_m)(y_k, w_k) + u_k, \quad k = 1, \ldots m, \ t \in [0, T]
\end{aligned}
\right\}
\tag{1.11}
$$

and
$$
\partial_t \mu + v \cdot \nabla_x \mu = \nabla_v \cdot [(H \star (\mu + \mu_m))\mu],
$$

where the weak solutions of the equations have to be interpreted in the Carathéodory sense.

Let us emphasize that in this paper we shall interpret μ_m as a discrete measure 'immersed' in the diffused one μ (i.e. the support of μ_m shall be contained in the support of μ), for instance we shall choose $\mu_m(t = 0)$ to be at the time $t = 0$ an empirical m dimensional realization of $\mu(t = 0)$. The atoms (y_k, w_k) constituting the support of μ_m are interpreted as representatives of the entire distribution μ, which we are indirectly controlling, by acting directly on its representatives. See figure 2 from [15] for an example of a dynamics similar to the one of (1.11) for a multiscale pedestrian crowd mixing a granular discrete part and a diffuse part, where a first-order model was considered.

Besides the mean-field limit of (1.9) to (1.11) for $N \to +\infty$, we shall simultaneously prove in theorem 5.3 a Γ-convergence result, implying that the optimal controls u_N^* of the finite-dimensional optimal control problems (1.9)–(1.10) converge weakly in $L^1([0, T], \mathcal{U})$ for $N \to +\infty$ to optimal controls u^*, which are minimal solutions of

$$
\min_{u \in L^1([0,T],\mathcal{U})} \int_0^T \left\{ L(y(t), w(t), \mu(t)) + \frac{1}{m} \sum_{k=1}^m |u_k(t)| \right\} \, dt.
\tag{1.12}
$$

This is actually an existence result of solutions for the infinite-dimensional optimal control problem (1.11)–(1.12). Different from the one proposed in [10] though, this model retains the controls only on a finite and small group of agents, despite the fact that the entire population can be very large (here modelled by the limit $N \to +\infty$). Hence, by the stratagem of dividing the populations in two groups and allowing only one of them to have growing size, we do not need

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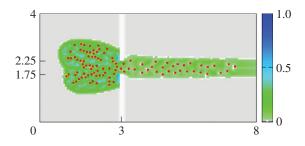rsta.royalsocietypublishing.org *Phil. Trans. R. Soc. A* **372**: 20130400

Figure 2. A mixed granular-diffuse crowd leaving a room through a door. This figure was kindly provided by Cristiani *et al.* [15]. Copyright © 2011 Society for Industrial and Applied Mathematics. Reprinted with permission. All rights reserved. (Online version in colour.)

anymore to be necessarily exposed to the curse of dimensionality when it comes to numerically solving the corresponding optimal control problem. We shall address the concrete analysis of the first-order optimality conditions for (1.9)–(1.10) and their relationship to (1.11)–(1.12) in a follow-up paper. This will be the basis for the numerical implementations.

The paper is organized as follows. In §2, we apply basic results recalled from appendix A to ensure the well-posedness of the finite-dimensional system (1.9). Section 3 is devoted to the mean-field limit of (1.9) to the coupled system (1.11) and the well-posedness of the latter. For the sake of self-containedness, we sketch in §4 known existence results for the finite dimensional problems (1.9)–(1.10). In §5, we develop our main result of Γ-convergence of the finite-dimensional optimal control problems (1.9)–(1.10) to the corresponding infinite-dimensional ones (1.11)–(1.12). The appendix A recalls classical well-posedness results of Carathéodory differential equations and certain stability results of transport flows specifically formulated for the systems of equations (1.9) and (1.11).

2. The finite-dimensional dynamics

We state the following assumptions

(H) Let $H : \mathbb{R}^{2d} \to \mathbb{R}^d$ be a locally Lipschitz function such that, for a constant $C > 0$

$$|H(\xi)| \leq C(1 + |\xi|), \quad \text{for all } \xi \in \mathbb{R}^{2d}. \tag{2.1}$$

We consider now the system (1.9) with N followers and the control u. We shall prove results of existence and uniqueness of the solution of (1.9), where time-dependent support estimates will be given independently of the number N of followers. With this goal, we endow each space of configurations $\mathbb{R}^{2d(m+N)}$ with the following norm and the corresponding distance

$$\|(y, w, x, v)\| := \frac{1}{m} \sum_{k=1}^{m} (|y_k| + |w_k|) + \frac{1}{N} \sum_{i=1}^{N} (|x_i| + |v_i|), \tag{2.2}$$

where the norm $|\cdot|$ on $\mathbb{R}^d$ is the Euclidean. The choice of this norm (2.2) is eventually related to the use of the 1-Wasserstein distance on the space $\mathcal{P}_1(\mathbb{R}^{2d})$ of probability measures of bounded first moment. For the sake of compact writing, we shall denote the trajectories of (1.9) by $\zeta(t) = (y(t), w(t), x(t), v(t))$ and trajectories of leaders or followers by ξ, i.e. $\xi(t) = (y(t), w(t))$ or $\xi(t) = (x(t), v(t))$ depending on the context. We can write (1.9) in the following compact form

$$\dot{\zeta}(t) = g(t, \zeta(t)), \tag{2.3}$$

where the right-hand side is

$$g(t, \zeta) = (w, [H \star (\mu_N + \mu_m)(y_k, w_k) + u_k]_{k=1}^{m}, v, [H \star (\mu_N + \mu_m)(x_i, v_i)]_{i=1}^{N}). \tag{2.4}$$

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Lemma 2.1. *Given H satisfying condition (H) and $\mu_n = (1/n) \sum_{\ell=1}^{n} \delta_{\xi_\ell}$ for $\xi_\ell \in \mathbb{R}^{2d}$ for all $\ell = 1, \ldots, n$, an arbitrary atomic measure, we have*

$$|H \star \mu_n(\xi)| \leq C \left(1 + |\xi| + \frac{1}{n} \sum_{\ell=1}^{n} |\xi_n| \right). \tag{2.5}$$

Proof. By sublinear growth of H, we have immediately the estimate

$$|H \star \mu_n(\xi)| \leq \frac{1}{n} \sum_{\ell=1}^{n} |H(\xi - \xi_\ell)| \leq C \left(1 + |\xi| + \frac{1}{n} \sum_{\ell=1}^{n} |\xi_n| \right).$$

∎

Proposition 2.2. *Let H be a map satisfying (H). Then, given a control $u \in L^1([0, T], \mathcal{U})$ and an initial datum $\zeta^0 = (y^0, w^0, x^0, v^0)$, there exists a unique Carathéodory solution $\zeta(t) = (y(t), w(t), x(t), v(t))$ of (1.9) such that*

$$\|\zeta(t)\| \leq (\|\zeta^0\| + \bar{C}T)e^{\bar{C}T}, \tag{2.6}$$

for all $t \in [0, T]$, where $\bar{C} > 0$ is a constant depending on $C > 0$, $U > 0$ but not depending on N. Moreover, the trajectory is Lipschitz continuous in time, i.e.

$$\|\zeta(t_1) - \zeta(t_2)\| \leq \mathcal{L}|t_1 - t_2|, \quad \text{for all } t_1, t_2 \in [0, T], \tag{2.7}$$

for the Lipschitz constant $\mathcal{L} = \bar{C}(1 + (\|\zeta^0\| + \bar{C}T)e^{\bar{C}T})$.

Proof. Given the explicit form of (2.4) and, thanks to condition (H) and lemma 2.1, the right-hand side $g(t, \zeta)$ of the system (2.3) fulfills the linear growth condition

$$\|g(s, \zeta)\| \leq \bar{C}(1 + \|\zeta\|), \quad \text{for all } \zeta \in \mathbb{R}^{2d},$$

allowing us to apply theorem A.2 in appendix A, which ensures the well-posedness of (1.9). Moreover,

$$\|\zeta(t_1) - \zeta(t_2)\| = \left\| \int_{t_2}^{t_1} g(s, \zeta(s)) \, ds \right\| \leq \int_{t_2}^{t_1} \bar{C}(1 + \|\zeta(s)\|) \, ds$$

$$\leq \bar{C}(1 + (\|\zeta^0\| + \bar{C}T)e^{\bar{C}T})|t_1 - t_2|.$$

∎

3. The coupled ODE and PDE system

In the following, we consider the space $\mathcal{P}_1(\mathbb{R}^n)$, consisting of all probability measures on $\mathbb{R}^n$ of finite first moment. On this set, we shall consider the following distance, called the *Monge–Kantorovich–Rubinstein distance*,

$$\mathcal{W}_1(\mu, \nu) = \sup \left\{ \left| \int_{\mathbb{R}^n} \varphi(x) d(\mu - \nu)(x) \right| : \varphi \in \text{Lip}(\mathbb{R}^n), \ \text{Lip}(\varphi) \leq 1 \right\}, \tag{3.1}$$

where $\text{Lip}(\mathbb{R}^n)$ is the space of Lipschitz continuous functions on $\mathbb{R}^n$ and $\text{Lip}(\varphi)$ is the Lipschitz constant of a function φ. Such a distance can also be represented in terms of optimal transport plans by Kantorovich duality in the following manner: if we denote by $\Pi(\mu, \nu)$, the set of transference plans between the probability measures μ and ν, i.e. the set of probability measures on $\mathbb{R}^n \times \mathbb{R}^n$ with first and second marginals equal to μ and ν, respectively, then we have

$$\mathcal{W}_1(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \left\{ \int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y| \, d\pi(x, y) \right\}. \tag{3.2}$$

In the form (3.2), the distance $\mathcal{W}_1$ is also known as the 1-Wasserstein distance. We refer to [29,30] for more details. Note that if $\mu_m = (1/m) \sum_{k=1}^{m} \delta_{\xi_k}$ and $\mu'_m = (1/m) \sum_{k=1}^{m} \delta_{\xi'_k}$ are two atomic measures, then (3.1) immediately yields

$$\mathcal{W}_1(\mu_m, \mu'_m) \leq \frac{1}{m} \sum_{k=1}^{m} |\xi_k - \xi'_k|. \tag{3.3}$$

This is the reason for having fixed the norm notation $\| \cdot \|$ as in (2.2).

We formally define now a proper concept of solutions for the system (1.11).

Definition 3.1. Let $u \in L^1([0, T], \mathcal{U})$ be given. We say that a map $(y, w, \mu) : [0, T] \to \mathcal{X} := \mathbb{R}^{2d \times m} \times \mathcal{P}_1(\mathbb{R}^{2d})$ is a solution of the controlled system with interaction kernel H

$$\left. \begin{aligned} \dot{y}_k &= w_k, \\ \dot{w}_k &= H \star (\mu + \mu_m)(y_k, w_k) + u_k, \quad k = 1, \ldots m, \ t \in [0, T] \\ \text{and} \qquad \partial_t \mu + v \cdot \nabla_x \mu &= \nabla_v \cdot [(H \star (\mu + \mu_m))\mu], \end{aligned} \right\} \tag{3.4}$$

with control u, where μ_m is the time-dependent atomic measure as in (1.8), if

(i) the measure μ is equi-compactly supported in time, i.e. there exists $R > 0$ such that $\mathrm{supp}(\mu(t)) \subset B(0, R)$ for all $t \in [0, T]$;

(ii) the solution is continuous in time with respect to the following metric in $\mathcal{X}$

$$\|(y, w, \mu) - (y', w', \mu')\|_{\mathcal{X}} := \frac{1}{m} \sum_{k=1}^{m} (|y_k - y'_k| + |w_k - w'_k|) + \mathcal{W}_1(\mu, \mu'), \tag{3.5}$$

where $\mathcal{W}_1(\mu, \mu')$ is the 1-Wasserstein distance in $\mathcal{P}_1(\mathbb{R}^{2d})$;

(iv) the (y, w) coordinates define a Carathéodory solution of the following controlled problem with interaction kernel H, control $u(\cdot)$, and the external field $H \star \mu$:

$$\left. \begin{aligned} \dot{y}_k &= w_k \\ \text{and} \qquad \dot{w}_k &= H \star (\mu + \mu_m)(y_k, w_k) + u_k, \quad k = 1, \ldots m, \ t \in [0, T] \end{aligned} \right\} \tag{3.6}$$

(v) the μ component satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^{2d}} \phi(x, v) \, \mathrm{d}\mu(t)(x, v) = \int_{\mathbb{R}^{2d}} \nabla \phi(x, v) \cdot \omega_{H, \mu, y, w}(t, x, v) \, \mathrm{d}\mu(t)(x, v) \tag{3.7}$$

for every $\phi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, in the sense of distributions, where $\omega_{H, \mu, y, w}(t, x, v) : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ is the time-varying vector field defined as follows

$$\omega_{H, \mu, y, w}(t, x, v) := (v, H \star \mu(t)(x, v) + H \star \mu_m(t)(x, v)). \tag{3.8}$$

Let, moreover, $(y^0, w^0, \mu^0) \in \mathcal{X}$ be given, with $\mu^0 \in \mathcal{P}_1(\mathbb{R}^{2d})$ of compact support. We say that $(y, w, \mu) : [0, T] \to \mathcal{X}$ is a solution of (3.4) with initial data (y^0, w^0, μ^0) and control u if it is a solution of (3.4) with control u and it satisfies $(y(0), w(0), \mu(0)) = (y^0, w^0, \mu^0)$.

Let us emphasize that the PDE appearing in (3.4) is simply the strong formulation of (3.7), which allows us to write it in a more compact form.

Following the well-known arguments in [29, section 8.1], once $\mu_m(t)$ is a fixed time-dependent atomic measure of the type (1.8), a measure $\mu(t)$ is a weak equi-compactly supported solution of

$$\partial_t \mu + v \cdot \nabla_x \mu = \nabla_v \cdot [(H \star (\mu + \mu_m))\mu] \tag{3.9}$$

in the sense of (v) in the above definition if and only if it satisfies (i) and the measure–theoretical fixed point equation

$$\mu(t) = (T_t^{\mu,\mu_m})_\sharp \mu^0, \tag{3.10}$$

with $\mu_0 := \mu(0)$ and T_t^{μ,μ_m} is the flow function defined by (A 11) in appendix A. Here, $(T_t^{\mu,\mu_m})_\sharp$ denotes the push-forward of μ_0 through T_t^{μ,μ_m}.

Before actually proving the existence of solutions of (3.4) as in definition 3.1, it will be convenient to address the stability of the system (3.4) first.

Proposition 3.2. *Let $u \in L^1([0,T],\mathcal{U})$ be a given fixed control for (3.4) and two solutions (y^1,w^1,μ^1) and (y^2,w^2,μ^2) of (3.4) relative to the control u and given respective initial data $(y^{0,i}, w^{0,i}, \mu^{0,i}) \in \mathcal{X}$, with $\mu^{0,i}$ compactly supported, $i = 1, 2$. Then, there exists a constant $C_T > 0$ such that*

$$\|(y^1(t), w^1(t), \mu^1(t)) - (y^2(t), w^2(t), \mu^2(t))\|_{\mathcal{X}}$$
$$\leq C_T \|(y^{0,1}, w^{0,1}, \mu^{0,1}) - (y^{0,2}, w^{0,2}, \mu^{0,2})\|_{\mathcal{X}}, \quad \text{for all } t \in [0,T] \tag{3.11}$$

Proof. We show the stability estimate by chaining the stability of (3.6) with the one of (3.9). Let us first address the stability of (3.6) given μ^1, μ^2. By integration, we have

$$|y_k^1(t) - y_k^2(t)| \leq |y_k^{0,1} - y_k^{0,2}| + \int_0^t |w_k^1(s) - w_k^2(s)| \, ds, \tag{3.12}$$

and, by lemma A.7, there exists a constant $\Lambda_R > 0$, such that

$$|w_k^1(t) - w_k^2(t)| \leq |w_k^{0,1} - w_k^{0,2}| + \int_0^t (|(H \star \mu^1(y_k^1(s), w_k^1(s)) - (H \star \mu^2(y_k^2(s), w_k^2(s))|$$
$$+ (H \star \mu_m^1(y_k^1(s), w_k^1(s)) - (H \star \mu_m^2)(y_k^2(s), w_k^2(s))| \, ds),$$
$$\leq |w_k^{0,1} - w_k^{0,2}| + \Lambda_R \int_0^t \left[\left(\frac{1}{m} \sum_{k=1}^m |y_k^1(s) - y_k^2(s)| + |w_k^1(s) - w_k^2(s)| \right) \right.$$
$$\left. + \mathcal{W}_1(\mu^1(s), \mu^2(s)) \right] ds \tag{3.13}$$

Now, we consider the stability of (3.9) given μ_m^1, μ_m^2. In view of the representation (3.10) of solutions by means of mass transportation, there exist constants $\mathcal{L} > 0$, $\Lambda_R > 0$, and $\rho > 0$ such that

$$\mathcal{W}_1(\mu^1(t), \mu^2(t)) = \mathcal{W}_1((T_t^{\mu^1,\mu_m^1})_\sharp \mu^{0,1}, (T_t^{\mu^2,\mu_m^2})_\sharp \mu^{0,2})$$
$$\leq W_1((T_t^{\mu^1,\mu_m^1})_\sharp \mu^{0,1}, (T_t^{\mu^1,\mu_m^1})_\sharp \mu^{0,2}) + W_1((T_t^{\mu^1,\mu_m^1})_\sharp \mu^{0,2}, (T_t^{\mu^2,\mu_m^2})_\sharp \mu^{0,2})$$
$$\leq \mathcal{L} \mathcal{W}_1(\mu^{0,1}, \mu^{0,2}) + \mathcal{W}_1((T_t^{\mu^1,\mu_m^1})_\sharp \mu^{0,2}, (T_t^{\mu^2,\mu_m^2})_\sharp \mu^{0,2})$$
$$\leq \mathcal{L} \mathcal{W}_1(\mu^{0,1}, \mu^{0,2}) + \|T_t^{\mu^1,\mu_m^1} - T_t^{\mu^2,\mu_m^2}\|_{L^\infty(B(0,R))}$$
$$\leq \mathcal{L} \mathcal{W}_1(\mu^{0,1}, \mu^{0,2}) + \int_0^t e^{\mathcal{L}(s-t)} \|(H \star \mu^1(s) - H \star \mu^2(s))$$
$$+ (H \star \mu_m^1(s) - H \star \mu_m^2(s))\|_{L^\infty(B(0,\rho))} \, ds$$
$$\leq \mathcal{L} \mathcal{W}_1(\mu^{0,1}, \mu^{0,2}) + \Lambda_R \int_0^t e^{\mathcal{L}(s-t)} \left[\left(\frac{1}{m} \sum_{k=1}^m |y_k^1(s) - y_k^2(s)| + |w_k^1(s) - w_k^2(s)| \right) \right.$$
$$\left. + \mathcal{W}_1(\mu^1(s), \mu^2(s)) \right] ds, \tag{3.14}$$

where we first applied the triangle inequality, in the second inequality, we used the Lipschitz continuity of the flow map $\mathcal{T}_t^{\mu^1,\mu_m^1}$ given by (A 14) for $\mu^1 = \mu_2$ and $\mu_m^1 = \mu_m^2$, and lemma A.6 also for the third inequality, the fourth inequality is again a consequence of (A 14), and the last one again owing to an application of lemma A.7. By combining (3.12), (3.13) and (3.14), and recalling the definition of the norm $\|\cdot\|_{\mathcal{X}}$, we easily recognize and conclude the estimate

$$
\|(y^1(t), w^1(t), \mu^1(t)) - (y^2(t), w^2(t), \mu^2(t))\|_{\mathcal{X}}
$$
$$
\leq C_0 \left(\|(y^{0,1}, w^{0,1}, \mu^{0,1}) - (y^{0,2}, w^{0,2}, \mu^{0,2})\|_{\mathcal{X}} + \int_0^t \|(y^1(s), w^1(s), \mu^1(s)) - (y^2(s), w^2(s), \mu^2(s))\|_{\mathcal{X}} \, ds \right),
$$

for a suitable constant $C_0 > 0$ depending on $\mathcal{L}, \Lambda_R, T$. An application of Gronwall's inequality concludes the stability estimate. ∎

This latter result also implies that, once a control $u \in L^1([0,T], \mathcal{U})$ is fixed, the solution of (3.4), if it exists, is uniquely determined by the initial conditions. We shall derive now the existence of solutions of (3.4) in the sense of definition 3.1 by a limit process for $N \to \infty$, where we allow for a variable control u_N depending on N.

Theorem 3.3. *Let $(y^0, w^0, \mu^0) \in \mathcal{X}$ be given, with μ^0 of bounded support in $B(0, R)$, for $R > 0$. Define a sequence $(\mu_N^0)_{N \in \mathbb{N}}$ of atomic probability measures equi-compactly supported in $B(0, R)$ such that each μ_N^0 is given by $\mu_N^0 := \sum_{i=1}^N \delta_{x_{i,N}^0, v_{i,N}^0}$ and $\lim_{N \to \infty} \mathcal{W}_1(\mu_N^0, \mu^0) = 0$. Fix now a weakly convergent sequence $(u_N)_{N \in \mathbb{N}} \subset L^1([0,T], \mathcal{U})$ of control functions, i.e. $u_N \rightharpoonup u_*$ in $L^1([0,T], \mathcal{U})$. For each initial datum $\zeta_N^0 = (y^0, w^0, x_N^0, v_N^0)$ depending on N, denote with $\zeta_N(t) = (y_N(t), w_N(t), \mu_N(t)) := (y_N(t), w_N(t), x_N(t), v_N(t))$, the unique solution of the finite-dimensional control problem (1.9) with control u_N. (Here, we apply the identification of the trajectories $(x(t), v(t))$ and the measure $\mu_N(t)$ by means of (1.2).) Then, the sequence (y_N, w_N, μ_N) converges in $C^0([0,T], \mathcal{X})$ to some (y_*, w_*, μ_*), which is a solution of (3.4) with initial data (y^0, w^0, μ^0) and control u_*, in the sense of definition 3.1.*

Proof. Because the initial data ζ_N^0 are equi-compactly supported, the trajectories

$$
\zeta_N(t) = (y_N(t), w_N(t), \mu_N(t))
$$

are equibounded and equi-Lipschitz continuous in $C^0([0,T], \mathcal{X})$, because of (3.3), combined with (2.6) and (2.7). By an application of the Ascoli–Arzelà theorem for functions on $[0,T]$ and values in the complete metric space $\mathcal{X}$, there exists a subsequence, again denoted by $\zeta_N(\cdot) = (y_N(\cdot), w_N(\cdot), \mu_N(\cdot))$ converging uniformly to a limit $\zeta_* = (y_*(\cdot), w_*(\cdot), \mu_*(\cdot))$, which is also equi-compactly supported in a ball $B(0, R_T)$ for a suitable $R_T > 0$. Owing to equi-Lipschitz continuity in time of the trajectories and the continuity of the Wasserstein distance, we also obtain

$$
\|\zeta_*(t_2) - \zeta_*(t_1)\|_{\mathcal{X}} = \lim_N \|\zeta_N(t_2) - \zeta_N(t_1)\|_{\mathcal{X}} \leq \Lambda_T |t_2 - t_1|,
$$

for all $t_1, t_2 \in [0,T]$, where $\Lambda_T > 0$ is a uniform Lipschitz constant. Hence, the limit trajectory ζ_* belongs as well to $C^0([0,T], \mathcal{X})$.

We need now to show that ζ_* is a solution of (3.4) in the sense of definition 3.1. We first verify that (y_*, w_*) is a solution of the ODEs part of (3.4) for $\mu = \mu_*$. (The symbol '$\rightrightarrows$' used below actually means uniform convergence.) To this end, we observe that the limit $\zeta_N \to \zeta_*$ in particular specifies into

$$
\xi_N \rightrightarrows \xi_*, \quad \text{in } [0,T]
$$

and

$$
\dot{\xi}_N \rightharpoonup \dot{\xi}_*, \quad \text{in } L^1([0,T], \mathbb{R}^{2d}),
$$

(3.15)

where $\xi_N(t) = (y_N(t), w_N(t))$ and $\xi_* = (y_*, w_*)$ and

$$\lim_N \mathcal{W}_1(\mu_N(t), \mu_*(t)) = 0, \tag{3.16}$$

uniformly with respect to $t \in [0, T]$. In particular, the limits (3.15) imply $\dot{y}_{k,*}(t) = w_{k,*}(t)$ in $[0, T]$ for all $k = 1, \ldots, m$. We shall now show that $(y_*(t), w_*(t))$ is actually the Carathéodory solution of (3.6) by verifying also its second equation.

Let us denote now

$$\mu_{m,N}(t) = \frac{1}{m} \sum_{k=1}^{m} \delta_{(y_{k,N}(t), w_{k,N}(t))} \quad \text{and} \quad \mu_{m,*}(t) = \frac{1}{m} \sum_{k=1}^{m} \delta_{(y_{k,*}(t), y_{k,*}(t))}.$$

As a consequence of (3.3), lemma A.7 in appendix A, and the uniform convergence of the trajectories, we have that

$$\mathcal{W}_1(\mu_{m,N}(t), \mu_{m,*}(t)) \to 0 \tag{3.17}$$

as $N \to +\infty$, uniformly in $t \in [0, T]$. By (3.16) and (3.17), and the linear growth of H, we deduce

$$H \star (\mu_N + \mu_{m,N})(y_{k,N}, v_{k,N}) \rightrightarrows H \star (\mu_* + \mu_{m,*})(y_{k,*}, w_{k,*}), \text{ in } [0, T], \quad \text{for } N \to \infty, \tag{3.18}$$

again by applying lemma A.7 in appendix A.

To prove that $(y_*(t), w_*(t))$ is actually the Carathéodory solution of (3.6), we have only to show that for all $k = 1, \ldots, m$ one has

$$\dot{w}_{k,*} = (H \star \mu_* + \mu_{m,*})(y_{k,*}, w_{k,*}) + u_{k,*}. \tag{3.19}$$

This is clearly equivalent to the following: for every $\eta \in \mathbb{R}^d$ and every $\hat{t} \in [0, T]$, it holds

$$\eta \cdot \int_0^{\hat{t}} \dot{w}_{k,*}(t)\, dt = \eta \cdot \int_0^{\hat{t}} [(H \star \mu_*(t) + \mu_{m,*}(t))(y_{k,*}(t), w_{k,*}(t)) + u_{k,*}(t)]\, dt, \tag{3.20}$$

which follows from the weak L^1 convergence of $\dot{w}_{k,N}$ to $\dot{w}_{k,*}$ and of u_N to u_* for $N \to +\infty$, and from (3.18).

We are now left with verifying that μ_* is a solution of (3.9) for $\mu_m = \mu_{m,*}$ in the sense of definition 3.1 (v). For all $\hat{t} \in [0, T]$ and for all $\phi \in C_c^1(\mathbb{R}^{2d})$, we infer that

$$\langle \phi, \mu_N(\hat{t}) - \mu_N(0) \rangle = \int_0^{\hat{t}} \left[\int_{\mathbb{R}^{2d}} \nabla \phi(x, v) \cdot w_{H, \mu_N, y_N, w_N}(t, x, v)\, d\mu_N(t, x, v) \right] dt,$$

which is verified by considering the differentiation

$$\frac{d}{dt} \langle \phi, \mu_N(t) \rangle = \frac{1}{N} \frac{d}{dt} \sum_{i=1}^{N} \phi(x_i(t), v_i(t))$$

$$= \frac{1}{N} \left[\sum_{i=1}^{N} \nabla_x \phi(x_i(t), v_i(t)) \cdot \dot{x}_i(t) + \sum_{i=1}^{N} \nabla_v \phi(x_i(t), v_i(t)) \cdot \dot{v}_i(t) \right],$$

and directly applying the substitutions as in (1.9) for the followers variables (x, v). Moreover,

$$\lim_{N \to \infty} \langle \phi, \mu_N(\hat{t}) - \mu_N(0) \rangle = \langle \phi, \mu_*(\hat{t}) - \mu^0 \rangle, \tag{3.21}$$

for all $\phi \in C_c^1(\mathbb{R}^{2d})$. By possibly extracting an additional subsequence, by weak-$*$ convergence, and the dominated convergence theorem, we obtain the limit

$$\lim_{k \to \infty} \int_0^{\hat{t}} \int_{\mathbb{R}^{2d}} (\nabla_v \phi(x, v) \cdot v)\, d\mu_{N_k}(t)(x, v)\, dt = \int_0^{\hat{t}} \int_{\mathbb{R}^{2d}} (\nabla_v \phi(x, v) \cdot v)\, d\mu_*(t)(x, v)\, dt, \tag{3.22}$$

for all $\phi \in C_c^1(\mathbb{R}^{2d})$. By lemma A.7 in appendix A, we also have that for every $\rho > 0$

$$\lim_{k \to \infty} \|H \star (\mu_{N_k}(t) + \mu_{m,N}(t)) - H \star (\mu_*(t) + \mu_{m,*}(t))\|_{L^\infty(B(0,\rho))} = 0,$$

and, as $\phi \in C_c^1(\mathbb{R}^{2d})$ has compact support, it follows that

$$\lim_{k\to\infty} \|\nabla_v \phi \cdot (H \star (\mu_{N_k}(t) + \mu_{m,N_k}(t)) - H \star (\mu_*(t) + \mu_{m,*}(t)))\|_\infty = 0.$$

Denote with $\mathcal{L}^1\llcorner_{[0,\hat{t}]}$ the Lebesgue measure on the time interval $[0,\hat{t}]$. Because the product measures $\mathcal{L}^1\llcorner_{[0,\hat{t}]}\times(1/\hat{t})\mu_{N_k}(t)$ converge in $\mathcal{P}_1([0,\hat{t}]\times\mathbb{R}^{2d})$ to $\mathcal{L}^1\llcorner_{[0,\hat{t}]}\times(1/\hat{t})\mu_*(t)$, we finally have

$$\lim_{k\to\infty} \int_0^{\hat{t}} \int_{\mathbb{R}^{2d}} (\nabla_v \phi(x,v) \cdot H \star (\mu_{N_k}(t) + \mu_{m,N_k}(t))) \, d\mu_{N_k}(t)(x,v) \, dt$$

$$= \int_0^{\hat{t}} \int_{\mathbb{R}^{2d}} (\nabla_v \phi(x,v) \cdot H \star (\mu_*(t) + \mu_{m,*}(t))) \, d\mu_*(t)(x,v) \, dt. \tag{3.23}$$

The statement now follows by combining (3.21)–(3.23). ∎

Remark 3.4. In the proof of the previous theorem, we consider a converging subsequence of ζ_N after application of the Ascoli–Arzelà theorem. Let us stress that in view of the uniqueness of the solution of (3.4), we do not need to restrict ourselves to a subsequence, but we can infer the convergence of the entire sequence ζ_N to the solution of (3.4). This observation will play an important role below, when we shall prove the Γ-convergence of finite-dimensional optimal control problems constrained by the ODE system (1.9) to the infinite-dimensional optimal control problem constrained by the ODE–PDE system (3.4).

4. The finite-dimensional optimal control problem

We state the following assumptions:

(L) Let $L: \mathcal{X} \to \mathbb{R}_+$ be a continuous function with respect to the distance induced on $\mathcal{X}$ by the norm $\|\cdot\|_{\mathcal{X}}$;

Given $N \in \mathbb{N}$ and an initial datum $(y_1(0),\dots,y_m(0),w_1(0),\dots,w_m(0),x_1(0),\dots,x_N(0),v_1(0),\dots,v_N(0)) \in (\mathbb{R}^d)^m \times (\mathbb{R}^d)^m \times (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$, we consider the following optimal control problem:

$$\min_{u=(u_1,\dots,u_k)\in L^1([0,T],\mathcal{U})} \int_0^T \left\{ L(y(t),w(t),\mu_N(t)) + \frac{1}{m}\sum_{k=1}^m |u_k(t)| \right\} dt, \tag{4.1}$$

where

$$\mu_m(t)(x,v) = \frac{1}{m}\sum_{k=1}^m \delta_{(y_k(t),w_k(t))}(x,v) \quad \text{and} \quad \mu_N(t)(x,v) = \frac{1}{N}\sum_{i=1}^N \delta_{(x_i(t),v_i(t))}(x,v) \tag{4.2}$$

are the time-dependent atomic measures supported on the phase space trajectories $(y_k(t),w_k(t)) \in \mathbb{R}^{2d}$, for $k=1,\dots m$ and $(x_i(t),v_i(t)) \in \mathbb{R}^{2d}$, for $i=1,\dots N$, respectively, constrained by being the solution of the system

$$\left. \begin{aligned} \dot{y}_k &= w_k, \\ \dot{w}_k &= H \star \mu_N(y_k,w_k) + H \star \mu_m(y_k,w_k) + u_k \quad k=1,\dots m, \ t \in [0,T], \\ \dot{x}_i &= v_i \\ \dot{v}_i &= H \star \mu_N(x_i,v_i) + H \star \mu_m(x_i,v_i) \qquad\qquad i=1,\dots N, \ t \in [0,T], \end{aligned} \right\} \tag{4.3}$$

and

for a given datum $\zeta^0 = (y^0,w^0,x^0,v^0) \in \mathbb{R}^{2d(m+N)}$ and control $u \in L^1([0,T],\mathcal{U})$.

Let us recall that the existence of Carathéodory solutions of (4.3) for any given $u = (u_1,\dots,u_k) \in L^1([0,T],\mathcal{U})$ is ensured by proposition 2.2.

Theorem 4.1. *The finite horizon optimal control problem (4.1)–(4.3) with initial datum $\zeta^0 = (y^0,w^0,x^0,v^0) \in \mathbb{R}^{4d}$ has solutions.*

Proof. For the sake of self-containedness and broad readability, we just sketch briefly the proof of this statement, which follows from very classical results in optimal control [31, theorem 5.2.1]. Let $(u^h)_{h\in\mathbb{N}}$ be a minimizing sequence realizing at its limit the minimum of the cost functional in (4.1). As this sequence is necessarily bounded in $L^1([0,T],\mathcal{U})$, it admits a subsequence, which we simply rename as $(u^h)_{h\in\mathbb{N}}$, weakly converging to a $u^* \in L^1([0,T],\mathcal{U})$. At the same time, the corresponding solutions $\zeta^h(t) = (y^h(t), w^h(t), x^h(t), v^h(t))$ of (4.3) given the control u^h in $L^1([0,T],\mathcal{U})$ are equi-bounded and equi-Lipschitz continuous in time, thanks to an argument identical to the one given at the beginning of the proof of proposition 3.3. We similarly conclude that ζ^h has a subsequence, again not relabelled, converging uniformly to a trajectory ζ^* which is actually the solution of (4.3) given the control u^* in $L^1([0,T],\mathcal{U})$. The uniform convergence of the trajectories and their compact support also allow us to conclude by the use of condition (L) that

$$\lim_{h\to+\infty} \int_0^T L(y^h(t), w^h(t), \mu_N^h(t))\,dt = \int_0^T L(y^*(t), w^*(t), \mu_N^*(t))\,dt,$$

and the weak convergence of $(u^h)_{h\in\mathbb{N}}$ to $u^* \in L^1([0,T],\mathcal{U})$ implies the lower-semicontinuity of the norm

$$\lim_{h\to+\infty} \inf \frac{1}{m} \int_0^T \sum_{k=1}^m |u_k^h(t)|\,dt \geq \frac{1}{m} \int_0^T \sum_{k=1}^m |u_k^*(t)|\,dt.$$

We conclude by these two limits that u^* is an optimal control for (4.1)–(4.3). ∎

5. The Γ-limit to the infinite-dimensional optimal control problem

We shall now recall the concept of Γ-limit, which, together with the mean-field limit established by theorem 3.3 will allow us to prove that solutions of the optimal control problems (4.1)–(4.3) converges to optimal controls for the system (3.4).

Definition 5.1 (Γ-convergence) [32, definition 4.1, proposition 8.1]. Let X be a metrizable separable space and $F_N : X \to (-\infty, \infty]$, $N \in \mathbb{N}$ be a sequence of functionals. Then, we say that F_N *Γ-converges* to F, written as $F_N \xrightarrow{\Gamma} F$, for an $F : X \to (-\infty, \infty]$, if

(i) lim inf-*condition:* for every $u \in X$ and every sequence $u_N \to u$,

$$F(u) \leq \liminf_{N\to\infty} F_N(u_N);$$

(ii) lim sup-*condition:* for every $u \in X$, there exists a sequence $u_N \to u$, called *recovery sequence*, such that

$$F(u) \geq \limsup_{N\to\infty} F_N(u_N).$$

Furthermore, we call the sequence $(F_N)_N$ *equi-coercive* if for every $c \in \mathbb{R}$, there is a compact set $K \subseteq X$ such that $\{u : F_N(u) \leq c\} \subseteq K$ for all $N \in \mathbb{N}$. As a direct consequence, assuming $u_N^* \in \arg\min F_N \neq \emptyset$, for all $N \in \mathbb{N}$, there is a subsequence $(u_{N_k}^*)_k$ and $u^* \in X$ such that

$$u_{N_k}^* \to u^* \in \arg\min F.$$

In the following, we assume that H is a function satisfying (H) so that (1.9) and (3.4) are well-posed, for a given control u and suitable initial conditions. In view of the definition of Γ-convergence, let us fix as our domain $X = L^1([0,T],\mathcal{U})$ which, endowed with the weak L^1-topology, is actually a metrizable space.

Fix now an initial datum $(y^0, w^0, \mu^0) \in \mathcal{X}$, with μ^0 compactly supported, $\text{supp}(\mu^0) \subset B(0,R)$, $R > 0$, and choose a sequence of equi-compactly supported atomic measures μ_N^0, $\text{supp}(\mu_N^0) \subset B(0,R)$, $\mu_N^0 = (1/N)\sum_{i=1}^N \delta_{(x_i^0, v_i^0)}$ such that $\mathcal{W}_1(\mu_N^0, \mu^0) \to 0$ for $N \to +\infty$.

We define the following functional on X

$$F(u) = \int_0^T \left\{ L(y(t), w(t), \mu(t)) + \frac{1}{m} \sum_{k=1}^m |u_k(t)| \right\} dt, \tag{5.1}$$

where the triplet (y, w, μ) defines the unique solution of (3.4) with initial datum (y^0, w^0, μ^0) and control u, i.e.

$$\left. \begin{aligned} \dot{y}_k &= w_k, \\ \dot{w}_k &= H \star (\mu + \mu_m)(y_k, w_k) + u_k, \quad k = 1, \ldots m, \ t \in [0, T] \\ \partial_t \mu + v \cdot \nabla_x \mu &= \nabla_v \cdot [(H \star (\mu + \mu_m))\mu], \end{aligned} \right\} \tag{5.2}$$

and

in the sense of definition 3.1. Similarly, we define the functionals on X given by

$$F_N(u) = \int_0^T \left\{ L(y_N(t), w_N(t), \mu_N(t)) + \frac{1}{m} \sum_{k=1}^m |u_k(t)| \right\} dt, \tag{5.3}$$

where $\mu_N(t) = (1/N) \sum_{i=1}^N \delta_{(x_{i,N}(t), v_{i,N}(t))}$ is the time-dependent atomic measure supported on the trajectories defining the Carathéodory solution of the system

$$\left. \begin{aligned} \dot{y}_k &= w_k, \\ \dot{w}_k &= H \star \mu_N(y_k, w_k) + H \star \mu_m(y_k, w_k) + u_k \quad k = 1, \ldots m, \ t \in [0, T], \\ \dot{x}_i &= v_i \\ \dot{v}_i &= H \star \mu_N(x_i, v_i) + H \star \mu_m(x_i, v_i) \quad i = 1, \ldots N, \ t \in [0, T], \end{aligned} \right\} \tag{5.4}$$

and

with initial datum (y^0, w^0, x_N^0, v_N^0) and control u.

Remark 5.2. Observe that the choice of the functionals F_N depends on the choice of the sequence μ_N^0 approximating μ^0.

The rest of this section is devoted to the proof of the Γ-convergence of the sequence of functionals $(F_N)_{N \in \mathbb{N}}$ on X to the target functional F. Let us mention that Γ-convergence in optimal control problems has been already considered, see for instance [33], but, to the best of our knowledge, it has been only recently specified in connection to mean-field limits in [10].

Theorem 5.3. *Let H and L be maps satisfying conditions (H) and (L) respectively. Given an initial datum $(y^0, w^0, \mu^0) \in \mathcal{X}$ and an approximating sequence μ_N^0, with μ^0, μ_N^0 equi-compactly supported, i.e. $\mathrm{supp}(\mu^0) \cup \mathrm{supp}(\mu_N^0) \subset B(0, R)$, $R > 0$, for all $N \in \mathbb{N}$, then the sequence of functionals $(F_N)_{N \in \mathbb{N}}$ on $X = L^1([0, T], \mathcal{U})$ defined in (5.3) Γ-converges to the functional F defined in (5.1).*

Proof. Let us start by showing the $\Gamma - \liminf$ condition. Let us fix a weakly convergent sequence of controls $u_N \rightharpoonup u_*$ in $L^1([0, T], \mathcal{U})$. As done in the proof of theorem 3.3, we can associate with each of these controls a sequence of solutions $\zeta_N(t) = (y_N(t), w_N(t), \mu_N(t)) := (y_N(t), w_N(t), x_N(t), v_N(t))$ of (5.4) uniformly convergent to a solution $\zeta_*(t) = (y_*(t), w_*(t), \mu_*(t))$ of (5.2) in the sense of definition 3.1 with control u_* and initial datum (y^0, w^0, μ^0). In view of the fact that solutions $\zeta_N(t)$ and $\zeta_*(t)$ will have supports uniformly bounded with respect to N and $t \in [0, T]$ and by the uniform convergence of trajectories $(y_N(t), w_N(t)) \rightrightarrows (y_*(t), w_*(t))$ as well as the uniform convergence $\mathcal{W}_1(\mu_N(t), \mu_*(t)) \to 0$ for $t \in [0, T]$, it follows from condition (L) that

$$\lim_{N \to +\infty} \int_0^T L(y_N(t), w_N(t), \mu_N(t)) = \int_0^T L(y_*(t), w_*(t), \mu_*(t)) \, dt. \tag{5.5}$$

Note that, thanks to remark 3.4, here we are allowed to consider the convergence of the entire sequence $(\zeta_N)_{N\in\mathbb{N}}$ and we do not need to restrict to a subsequence (and this is a crucial issue in order to properly derive the $\Gamma - \liminf$ condition!). By the assumed weak convergence of $(u_N)_{N\in\mathbb{N}}$ to $u^* \in L^1([0,T],\mathcal{U})$, we obtain the lower-semicontinuity of the norm

$$\lim_{N\to+\infty}\inf \frac{1}{m}\int_0^T \sum_{k=1}^m |u_{k,N}(t)|\,dt \geq \frac{1}{m}\int_0^T \sum_{k=1}^m |u_{k,*}(t)|\,dt. \tag{5.6}$$

By combining (5.5) and (5.6), we immediately obtain the $\Gamma - \liminf$ condition

$$\liminf_{N\to\infty} F_N(u_N) \geq F(u_*).$$

We need now to address the $\Gamma - \limsup$ condition. Let us fix u_* and we consider the trivial recovery sequence $u_N \equiv u_*$ for all $N \in \mathbb{N}$. Similarly as above for the argument of the $\Gamma - \liminf$ condition, we can associate with each of these controls a sequence of solutions $\zeta_N(t) = (y_N(t), w_N(t), \mu_N(t)) := (y_N(t), w_N(t), x_N(t), v_N(t))$ of (5.4) uniformly convergent to a solution $\zeta_*(t) = (y_*(t), w_*(t), \mu_*(t))$ of (5.2) in the sense of definition 3.1 with control u_* and initial datum (y^0, w^0, μ^0) and we can similarly conclude the limit (5.5). Additionally, being the sequence $(u_N)_{N\in\mathbb{N}}$ trivially a constant sequence, we have

$$\lim_{N\to+\infty} \frac{1}{m}\int_0^T \sum_{k=1}^m |u_{k,N}(t)|\,dt = \frac{1}{m}\int_0^T \sum_{k=1}^m |u_{k,*}(t)|\,dt. \tag{5.7}$$

Hence, combining (5.5) and (5.7) we can easily infer

$$\limsup_{N\to\infty} F_N(u_N) = \lim_{N\to\infty} F_N(u_*) = F(u_*).$$

∎

Corollary 5.4. *Let H and L be maps satisfying conditions (H) and (L) respectively. Given an initial datum $(y^0, w^0, \mu^0) \in \mathcal{X}$, with μ^0 compactly supported, $\mathrm{supp}(\mu^0) \subset B(0,R)$, $R > 0$, the optimal control problem*

$$\min_{u\in L^1([0,T],\mathcal{U})} \int_0^T \left\{ L(y(t), w(t), \mu(t)) + \frac{1}{m}\sum_{k=1}^m |u_k(t)| \right\} dt, \tag{5.8}$$

has solutions, where the triplet (y, w, μ) defines the unique solution of (3.4) with initial datum (y^0, w^0, μ^0) and control u of

$$\left. \begin{aligned} \dot{y}_k &= w_k, \\ \dot{w}_k &= H \star (\mu + \mu_m)(y_k, w_k) + u_k, \quad k = 1,\ldots m, \ t \in [0,T] \\ \partial_t\mu + v \cdot \nabla_x\mu &= \nabla_v \cdot [(H \star (\mu + \mu_m))\mu], \end{aligned} \right\} \tag{5.9}$$

and

in the sense of definition 3.1, and

$$\mu_m(t) = \frac{1}{m}\sum_{k=1}^n \delta_{(y_k(t), w_k(t))}. \tag{5.10}$$

Moreover, solutions to (5.8) can be constructed as weak limits u^* of sequences of optimal controls u_N^* of the finite-dimensional problems

$$\min_{u\in L^1([0,T],\mathcal{U})} \int_0^T \left\{ L(y_N(t), w_N(t), \mu_N(t)) + \frac{1}{m}\sum_{k=1}^m |u_k(t)| \right\} dt, \tag{5.11}$$

where $\mu_N(t) = (1/N)\sum_{i=1}^{N} \delta_{(x_{i,N}(t),v_{i,N}(t))}$ and $\mu_{m,N}(t) = (1/m)\sum_{k=1}^{m} \delta_{(y_{k,N}(t),w_{k,N}(t))}$ are the time-dependent atomic measures supported on the trajectories defining the solution of the system

$$\left.\begin{aligned}
\dot{y}_k &= w_k, \\
\dot{w}_k &= H \star \mu_N(y_k, w_k) + H \star \mu_{m,M}(y_k, w_k) + u_k \quad k = 1, \ldots m, \ t \in [0, T], \\
\dot{x}_i &= v_i \\
\dot{v}_i &= H \star \mu_N(x_i, v_i) + H \star \mu_{m,M}(x_i, v_i) \qquad\qquad i = 1, \ldots N, \ t \in [0, T],
\end{aligned}\right\} \tag{5.12}$$

and

with initial datum (y^0, w^0, x_N^0, v_N^0), control u, and $\mu_N^0 = (1/N)\sum_{i=1}^{N} \delta_{(x_i^0, v_i^0)}$ is such that $\mathcal{W}_1(\mu_N^0, \mu^0) \to 0$ for $N \to +\infty$.

Proof. Note that the optimal controls u_N^* of the finite-dimensional optimal control problems (5.11)–(5.12) belongs to $X = L^1([0, T], \mathcal{U})$, which is a compact set with respect to the weak topology of L^1. Hence, $(u_N^*)_{N \in \mathbb{N}}$ admits a subsequence, which we do not relabel, weakly convergent to some $u^* \in L^1([0, T], \mathcal{U})$. Moreover, as done in the proof of theorem 3.3, we can associate with each of these controls a sequence of solutions $\zeta_N(t) = (y_N(t), w_N(t), \mu_N(t)) := (y_N(t), w_N(t), x_N(t), v_N(t))$ of (5.4) uniformly convergent to a solution $\zeta_*(t) = (y_*(t), w_*(t), \mu_*(t))$ of (5.2) in the sense of definition 3.1 with control u^*. In order to conclude that u^* is an optimal control for (5.4), we need to show that it is actually a minimizer of F. For that, we use the fact that F is the Γ-limit of the sequence $(F_N)_{N \in \mathbb{N}}$ as proved in theorem 5.3. Let $u \in X$ be an arbitrary control and let $(u_N)_{N \in \mathbb{N}}$ be a recovery sequence given by the $\Gamma - \limsup$ condition, so that

$$F(u) \geq \limsup_{N \to \infty} F_N(u_N). \tag{5.13}$$

By using now the optimality of $(u_N^*)_{N \in \mathbb{N}}$

$$\limsup_{N \to \infty} F_N(u_N) \geq \limsup_{N \to \infty} F_N(u_N^*) \geq \liminf_{N \to \infty} F_N(u_N^*). \tag{5.14}$$

Applying the $\Gamma - \liminf$ condition yields

$$\liminf_{N \to \infty} F_N(u_N^*) \geq F(u^*). \tag{5.15}$$

By chaining the inequalities (5.13)–(5.15), we eventually obtain that

$$F(u) \geq F(u^*), \quad \text{for all } u \in X,$$

or that u^* is an optimal control. ∎

Remark 5.5. Observe that the previous result does not state uniqueness of the optimal control for the infinite-dimensional problem. Moreover, in general, we do not ensure that such optimal controls are always limits of sequences of optimal controls of (5.11)–(5.12).

Appendix A

For the reader's convenience, we start by briefly recalling some well-known results about solutions to Carathéodory differential equations. We fix an interval $[0, T]$ on the real line, and let $n \geq 1$. Given a domain $\Omega \subset \mathbb{R}^n$, a Carathéodory function $g : [0, T] \times \Omega \to \mathbb{R}^n$, and $0 < \tau \leq T$, a

function $y : [0, \tau] \rightarrow \Omega$ is called a solution of the Carathéodory differential equation

$$\dot{y}(t) = g(t, y(t)) \tag{A 1}$$

on $[0, \tau]$ if and only if y is absolutely continuous and (A 1) is satisfied a.e. in $[0, \tau]$. The following existence and uniqueness result holds.

Theorem A.1. *Consider an interval $[0, T]$ on the real line, a domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$, and a Carathéodory function $g : [0, T] \times \Omega \rightarrow \mathbb{R}^n$. Assume that there exists a constant $C > 0$ such that*

$$|g(t, y)| \leq C$$

for a.e. $t \in [0, T]$ and every $y \in \Omega$. Then, given $y_0 \in \Omega$, there exists $0 < \tau \leq T$ and a solution $y(t)$ of (A 1) on $[0, \tau]$ satisfying $y(0) = y_0$.

If, in addition, there exists another constant $\Lambda > 0$ such that

$$|g(t, y_1) - g(t, y_2)| \leq \Lambda |y_1 - y_2| \tag{A 2}$$

for a.e. $t \in [0, T]$ and every $y_1, y_2 \in \Omega$, the solution is uniquely determined on $[0, \tau]$ by the initial condition y_0.

Proof. See, for instance, [34, ch. 1, theorems 1 and 2]. ∎

In addition, the global existence theorem and a Gronwall estimate on the solutions can be easily generalized to this setting.

Theorem A.2. *Consider an interval $[0, T]$ on the real line and a Carathéodory function $g : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. Assume that there exists a constant $C > 0$ such that*

$$|g(t, y)| \leq C(1 + |y|) \tag{A 3}$$

for a.e. $t \in [0, T]$ and every $y \in \mathbb{R}^n$. Then, given $y_0 \in \mathbb{R}^n$, there exists a solution $y(t)$ of (A 1) defined on the whole interval $[0, T]$ which satisfies $y(0) = y_0$. Any solution satisfies

$$|y(t)| \leq (|y_0| + Ct)e^{Ct} \tag{A 4}$$

for every $t \in [0, T]$.

If, in addition, for every relatively compact open subset of $\mathbb{R}^n$, (A 2) holds, the solution is uniquely determined on $[0, T]$ by the initial condition y_0.

Proof. Let $C_0 := (|y_0| + CT)e^{CT}$. Take a ball $\Omega \subset \mathbb{R}^p$ centred at 0 with radius strictly greater than C_0. Existence of a local solution defined on an interval $[0, \tau]$ and taking values in Ω follows now easily from (A 3) and theorem A.1. Using (A 3), any solution of (A 1) with initial datum y_0 satisfies

$$|y(t)| \leq |y_0| + Ct + \int_0^t C|y(s)| \, ds$$

for every $t \in [0, \tau]$, therefore (A 4) follows from Gronwall's lemma. In particular, the graph of a solution $y(t)$ cannot reach the boundary of $[0, T] \times \Omega$ unless $\tau = T$, therefore existence of a global solution follows for instance from [34, ch. 1, theorem 4]. If (A 2) holds, uniqueness of the global solution follows from theorem A.1. ∎

The usual results on continuous dependence on the data also hold in this setting: in particular, we will use this lemma, following from (A 4) and the Gronwall inequality.

Lemma A.3. *Let g_1 and $g_2 : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ be Carathéodory functions both satisfying (A 3) for a constant $C > 0$. Let $r > 0$ and define*

$$\rho_{r,m,T} := (r + CT)e^{CT}.$$

Assume, in addition, that there exists a constant $\Lambda = \Lambda(\rho_{r,m,T}) > 0$

$$|g_1(t, y_1) - g_1(t, y_2)| \leq \Lambda |y_1 - y_2|$$

for every $t \in [0, T]$ and every y_1, y_2 such that $|y_i| \leq \rho_{r,m,T}$, $i = 1, 2$. (Note that here we refer exclusively to g_1 and that the constant Λ actually depends on $\rho_{r,m,T}$.) Set

$$q(t) := \|g_1(t, \cdot) - g_2(t, \cdot)\|_{L^\infty(B(0,\rho_{r,m,T}))}.$$

Then, if $\dot{y}_1(t) = g(t, y_1(t))$, $\dot{y}_2(t) = g_2(t, y_2(t))$, $|y_1(0)| \leq r$ and $|y_2(0)| \leq r$, one has

$$|y_1(t) - y_2(t)| \leq e^{\Lambda t}|y_1(0) - y_2(0)| + \int_0^t e^{\int_s^t Lt} q(s) \, ds \tag{A 5}$$

for every $t \in [0, T]$.

We mention again that $\mathcal{P}_1(\mathbb{R}^n)$ denotes the space of probability measures on $\mathbb{R}^n$ with finite first moment. This is a metric space when endowed with the Wasserstein distance $\mathcal{W}_1$. We recall below several useful results from [10,26] concerning Lipschitz continuity estimates for transport flows induced by the dynamics (1.9), which may be found in slightly different form and generality in several other papers [25,27,28]. Lemma A.9 is recalled from [10, lemma 6.4].

Lemma A.4. *Let $H : \mathbb{R}^n \to \mathbb{R}^p$, $n \geq p \geq 1$ be a locally Lipschitz function such that*

$$|H(y)| \leq C(1 + |y|), \quad \text{for all } y \in \mathbb{R}^n, \tag{A 6}$$

and $\mu : [0, T] \to \mathcal{P}_1(\mathbb{R}^n)$ be a continuous map with respect to $\mathcal{W}_1$. Then, there exists a constant C' such that

$$|H \star \mu(t)(y)| \leq C'(1 + |y|), \tag{A 7}$$

for every $t \in [0, T]$ and every $y \in \mathbb{R}^n$. Furthermore, if

$$\text{supp } \mu(t) \subset B(0, R), \tag{A 8}$$

for every $t \in [0, T]$, then for every compact subset K of $\mathbb{R}^n$, there exists a constant $\Lambda_{R,K}$ such that

$$|H \star \mu(t)(y_1) - H \star \mu(t)(y_2)| \leq \Lambda_{R,K}|y_1 - y_2|, \tag{A 9}$$

for every $t \in [0, T]$ and every $y_1, y_2 \in K$.

Let us consider $\mu : [0, T] \to \mathcal{P}_1(\mathbb{R}^{2d})$ a continuous map with respect to $\mathcal{W}_1$ such that $\text{supp } \mu(t) \subset B(0, R)$ for all $t \in [0, T]$, and $\mu_m(t)(y, w) = (1/m) \sum_{k=1}^m \delta_{(y_k(t), w_k(t))}$ a time-dependent atomic measure supported on the absolutely continuous trajectories $t \mapsto (y_k(t), w_k(t))$, $k = 1, \ldots, m$. We now consider the system of ODEs on $\mathbb{R}^{2d}$

$$\left. \begin{aligned} \dot{X}(t) &= V(t) \\ \text{and} \qquad \dot{V}(t) &= H \star \mu(t)(X(t), V(t)) + H \star \mu_m(t)(X(t), V(t)) \end{aligned} \right\} \tag{A 10}$$

on an interval $[0, T]$. Here, X, V are both mappings from $[0, T]$ to $\mathbb{R}^d$ and $H : \mathbb{R}^{2d} \to \mathbb{R}^d$ is a locally Lipschitz function satisfying $|H(\xi)| \leq C(1 + |\xi|)$ for all $\xi \in \mathbb{R}^{2d}$ and for a constant $C > 0$. It follows then from these assumptions and lemma A.4 that all the hypothesis of theorem A.2 are satisfied. Therefore, however given $P_0 := (X_0, V_0)$ in $\mathbb{R}^{2d}$, there exists a unique solution $P(t) := (X(t), V(t))$ to (A 10) with initial datum P_0 defined on the whole interval $[0, T]$. We can therefore consider the family of flow maps $\mathcal{T}_t^{\mu,\mu_m} : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ indexed by $t \in [0, T]$ and defined by

$$\mathcal{T}_t^{\mu,\mu_m}(P_0) := P(t), \tag{A 11}$$

where $P(t)$ is the value of the unique solution to (A 10) starting from P_0 at time $t = 0$. The notation aims also at stressing the dependence of these flow maps on the given mappings $\mu(t), \mu_m(t)$. We

can easily recover, as consequence of (A 5), similar estimates as in [26, lemmas 3.7 and 3.8]: we report the statement and a sketch of the proof of this result to allow the reader to keep track of the dependence of these constants on the data of the problem.

Lemma A.5. *Let $H : \mathbb{R}^{2d} \to \mathbb{R}^d$ be a locally Lipschitz function satisfying*

$$|H(\xi)| \le C(1 + |\xi|), \quad \text{for all } \xi \in \mathbb{R}^{2d},$$

for a constant $C > 0$, and let $\mu^1 : [0, T] \to \mathcal{P}_1(\mathbb{R}^{2d})$ and $\mu^2 : [0, T] \to \mathcal{P}_1(\mathbb{R}^{2d})$ be continuous maps with respect to $\mathcal{W}_1$ both satisfying

$$\operatorname{supp} \mu(t) \subset B(0, R) \quad \text{and} \quad \operatorname{supp} \nu(t) \subset B(0, R) \tag{A 12}$$

for every $t \in [0, T]$, and μ^1_m, μ^2_m two time-dependent atomic measures supported on the respective absolutely continuous trajectories $t \mapsto (y^i_k(t), w^i_k(t))$, $i = 1, 2$ and $k = 1, \dots, m$. Consider the flow maps $\mathcal{T}^{\mu^i, \mu^i_m}_t$, $i = 1, 2$, associated with the systems

$$\left. \begin{aligned} \dot{X}(t) &= V(t) \\ \dot{V}(t) &= H \star \mu^i(t)(X(t), V(t)) + H \star \mu^i_m(t)(X(t), V(t)) \end{aligned} \right\} \tag{A 13}$$

and

for $i = 1, 2$ respectively, on $[0, T]$. Fix $r > 0$: then there exist a constant ρ and a constant $\Lambda > 0$, both depending only on r, C, R, and T such that

$$\begin{aligned} |\mathcal{T}^{\mu^1, \mu^1_m}_t(P_1) - \mathcal{T}^{\mu^2, \mu^2_m}_t(P_2)| &\le e^{\Lambda t} |P_1 - P_2| \\ &+ \int_0^t e^{\Lambda(s-t)} \|(H \star \mu^1(s) - H \star \mu^2(s)) + (H \star \mu^1_m(s) - H \star \mu^2_m(s))\|_{L^\infty(B(0,\rho))} \, ds \end{aligned} \tag{A 14}$$

whenever $|P_1| \le r$ and $|P_2| \le r$, for every $t \in [0, T]$.

Proof. Let g_1 and $g_2 : [0, T] \times \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ be the right-hand sides of (A 10), and (A 13), respectively. As in (A 7), we can find a constant C' which depends only on C and R such that

$$|H \star \mu^i(t)(P)| + |H \star \mu^i_m(t)(P)|\} \le C'(1 + |P|) \quad i = 1, 2, \tag{A 15}$$

for every $t \in [0, T]$ and every $P \in \mathbb{R}^{2d}$. Setting now $\hat{C} = 1 + C'$, it follows that g_1 and g_2 both satisfy (A 3) with C replaced by $\hat{C}$. Therefore, for every P_1 and $P_2 \in \mathbb{R}^{2d}$ such that $|P_i| \le r$, $i = 1, 2$ and every $t \in [0, T]$, (A 4) gives

$$|\mathcal{T}^{\mu^i, \mu^i_m}_t(P_1)| \le (r + \hat{C}T)e^{\hat{C}T}, \quad i = 1, 2.$$

Set $\rho := (r + \hat{C}T)e^{\hat{C}T}$. Now, obviously

$$\begin{aligned} \|g_1(t, \cdot) - g_2(t, \cdot)\|_{L^\infty(B(0,\rho))} &= \|(H \star \mu^1(s) - H \star \mu^2(s)) \\ &+ (H \star \mu^1_m(s) - H \star \mu^2_m(s))\|_{L^\infty(B(0,\rho))} \end{aligned}$$

for every $t \in [0, T]$.

Furthermore, by (A 9) and the definition of ρ, the Lipschitz constant of $g_1(t, \cdot)$ on $B(0, \rho)$ can be estimated by a constant $\Lambda > 0$ only depending on R, C, r and T. With this, the conclusion follows at once from (A 5). ∎

We additionally recall the following lemmata (see, e.g. [26, lemma 3.11, lemma 3,13, lemma 3.15, and lemma 4.7] for their proofs).

Lemma A.6. *Let E_1 and $E_2 : \mathbb{R}^n \to \mathbb{R}^n$ be two bounded Borel measurable functions. Then, for every $\mu \in \mathcal{P}_1(\mathbb{R}^n)$, one has*

$$\mathcal{W}_1((E_1)_\sharp \mu, (E_2)_\sharp \mu) \leq \|E_1 - E_2\|_{L^\infty(\text{supp}\,\mu)}.$$

If, in addition, E_1 is locally Lipschitz continuous, and $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^n)$ are both compactly supported on a ball B_r of $\mathbb{R}^n$, then

$$\mathcal{W}_1((E_1)_\sharp \mu, (E_1)_\sharp \nu) \leq \Lambda_r \mathcal{W}_1(\mu, \nu), \tag{A 16}$$

where $\Lambda_r > 0$ is the Lipschitz constant of E_1 on B_r.

Lemma A.7. *Let $H : \mathbb{R}^{2d} \to \mathbb{R}^d$ be a locally Lipschitz function satisfying (2.1), let $\mu : [0,T] \to \mathcal{P}_1(\mathbb{R}^{2d})$ and $\nu : [0,T] \to \mathcal{P}_1(\mathbb{R}^{2d})$ be continuous maps with respect to $\mathcal{W}_1$ both satisfying*

$$\text{supp}\,\mu(t) \subset B(0,R) \quad and \quad \text{supp}\,\nu(t) \subset B(0,R)$$

for every $t \in [0,T]$. Then, for every $\rho > 0$, there exists a constant $\Lambda_{\varrho,R}$ such that

$$\|H \star \mu(t) - H \star \nu(t)\|_{L^\infty(B(0,\rho))} \leq \Lambda_{\varrho,R} \mathcal{W}_1(\mu(t), \nu(t))$$

for every $t \in [0,T]$.

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