Mean-field Optimal Control by Leaders

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Abstract— **In this paper we deal with a social dynamics model, where one controls a small number of leaders in order to influence the behavior of the whole group (leaders and followers). We first provide a general mathematical framework to deal with optimal control of the microscopic problem, where the number of agents is finite, and its mean-field limit with an infinite number of followers.**
Then we focus on a migration-type model and develop optimal control strategies for the microscopic model. Such strategies are tested in their behavior both for the number of agents tending to infinity and for different initial conditions and initial locations of the leaders.

I. INTRODUCTION

The interest in social dynamics, i.e. multi-agent systems with structured interaction patterns, increased in the last decade in various research domains. One of the interests is in defining mathematical models which can capture and predict the main phenomenology. The models were proposed by researchers with very different backgrounds, such as: biologists to model animal groups [7], [21], [24], [25]; physicists to study the dynamics of crowds [8]; engineers interested in robot formations [19], [17]; economists interested in socio-economic networks [1]; general models from mathematicians as the celebrated Cucker-Smale one [9].

Following the model design, many papers were devoted to the analysis of the models. This activity included the understanding of *self-organization* phenomena, which is one of the main features of the social dynamics, see [22], [3], [21]. A mathematical definition of self-organization is that of a stable group configuration toward which the system tends naturally. For instance in the Cucker-Smale model (briefly CS) all agents tend to align their velocity to the average one. Such a phenomenon is called alignment and is encompassed in the more general concept of consensus [23]. The convergence to consensus is not guaranteed in all cases, but depends on the characteristics of the dynamics as well as on the initial conditions. For the CS model it is possible to define a *consensus region*, so that every initial datum in this region converges to consensus.

The natural question then is to design control policies inducing consensus, in particular centralized policies which are *sparse*, i.e. act only on a small number of agents. In [2], [5] a feedback control strategy, defined by the solution to a variational problem, was designed for a generalized version of the CS model. Another direction of investigation is the study of the limiting behavior when the number of agents tends to infinity, see [14], [13], [6]. The recent work [12] combined the two issues of control strategies and mean-field limit, by providing convergence results based on the concept of Γ-limit. Other authors addressed the problem of controlling a whole group by a limited number of leaders mostly at a microscopic level, see for instance [18], [4], [11], [15], [16] and references therein.

In the present paper we further push the investigation along this line by testing how the leaders control strategies behave for increasing numbers of agents. More precisely, we focus on a migration-type model, see also [20], which can be seen as a generalization of the Cucker-Smale one. In rough words, while the CS model leads to alignment of all velocities to the average one, the migration model tends to align all velocities to an ideal preassigned *migration* velocity.
We first adapt the result of [12] to deal with the case of Mayer-type problems (as opposed to the Bolza-type problems analysed in [12]). The main idea is that all ingredients of the Γ-limit are still valid for the Mayer-type problem.
Then we design optimal controls for the migration model. The controls act on the individual choice of weighting more the migration velocity (which in practice can be sensed) or the consensus mechanism, which acts by averaging with the other agents' velocities. The cost function is given by the sum of squared distances from the migration velocity. To make the analysis easier we neglect the dependence on agents' positions in space and consider constant-in-time interaction coefficients. We will choose a control bound so that we can act with full strength only on a single agent. This choice leads to a naturally sparse control and allows a simple strategy which coincides with the instantaneous decrease of the cost function.
We compare different control strategies: 1) control only on the leaders; 2) control on all agents (but only one at a time); 3) no control; 4) leaders staying "close" to the other agents. The achieved results can be summarized as follows. The performances of Strategies 2 and 4 clearly outperform the no control strategy (as expected), but strategy 4 suffers from a lack of optimality. For what concerns sending the number of agents to infinity, we verify that the Strategy 2 still works when the number of agents N increases but the performance deteriorates. This is in line with the convergence results proved in the first part of the paper.

The organization of the paper is as follows. In section II we introduce the general mathematical framework for social dynamics. Then in section III we describe a coupled

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ODE-PDE model and convergence results (Γ-limits) for the optimal controls. In section IV we describe a migration model and study the optimal controls to steer the group towards consensus at migration velocity.

II. General Framework

We consider a multi-agent control system, with N agents, each moving in a $2d$ dimensional space (d coordinates for position and d for velocity). The systems is thus defined in a Euclidean space with $2d \times N$ dimensions. We assume that a consensus dynamics is represented by a convolution operator, thus writes:

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \frac{1}{N}\sum_i a_{ij}\,(v_j - v_i) + u_i, \quad i = 1,\dots N, \quad t \in [0,T], \end{cases} \tag{1}$$

where the coefficients a_{ij} may depend both on $x = (x_1,\dots,x_N)$ and/or $v = (v_1,\dots,v_N)$. For instance in the Cucker-Smale (briefly CS) model one has $a_{ij} = 1/(1+\|x_i - x_j\|^2)^\beta$. Such a system can be written in general form as:

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = (H \star \mu_N)(x_i, v_i) + u_i, \quad i = 1,\dots N, \quad t \in [0,T], \end{cases} \tag{2}$$

where the interaction kernel $H : \mathbb{R}^{2d} \to \mathbb{R}^d$ is locally Lipschitz, $\star$ is the convolution operator and and μ_N is the atomic measure

$$\mu_N(t) = \frac{1}{N}\sum_{i=1}^{N} \delta_{(x_i(t), v_i(t))}. \tag{3}$$

We look for control functions $u_i : [0,T] \to \mathbb{R}^d$ which will be vanishing for most indices i and whose number of switchings in time is limited. We call such controls *sparse*. Results for the CS model were achieved in [5] with control on all agents, but here we are interested in the case of a limited number m of *leaders* and to N tending to infinity. The microscopic dynamics with m leaders can be written as:

$$\begin{cases} \dot{y}_k = w_k, \\ \dot{w}_k = (H \star \mu_N)(y_k, w_k) + (H \star \mu_m)(y_k, w_k) + u_k \\ \dot{x}_i = v_i, \\ \dot{v}_i = (H \star \mu_N)(x_i, v_i) + (H \star \mu_m)(x_i, v_i) \end{cases} \tag{4}$$

with $k = 1,\dots m$ and $i = 1,\dots N$, and where we defined a new atomic measure similar to the one applying to the followers (3), but with a different weight $\frac{1}{m}$:

$$\mu_m(t) = \frac{1}{m}\sum_{k=1}^{m} \delta_{(y_k(t), w_k(t))}; \tag{5}$$

this is because we want to keep the weight of leaders w.r.t. the whole group independent from the total number of agents.

The functions $u_k : [0,T] \to \mathbb{R}^d$ are measurable controls for $k = 1, ..., m$ and only act on the leaders. In this setting, it makes sense to choose $u \in L^1([0,T], \mathcal{U})$ where $\mathcal{U}$ is a fixed nonempty compact subset of $\mathbb{R}^{d\times m}$.

We are interested in minimizing a certain functional V at final time T, i.e. solving the following Mayer optimization problem:

$$\min_{u \in L^1([0,T],\mathcal{U})} V(y(T), w(T), x(T), v(T)) = V(\mu_m, \mu_N). \tag{6}$$

III. The coupled ODE and PDE System

Let us now describe the limiting dynamics for $N \to \infty$. We can define a mean-field limit of (4) in a certain sense specified later.

Let the population be represented by the vector of positions-velocities (y, w) of the leaders, coupled with the compactly supported probability measure $\mu \in \mathcal{P}_1(\mathbb{R}^{2d})$ (the space of probability measures with bounded first moment) of the followers in the position-velocity space. Then, the mean-field limit will result in a coupled system of an ODE with control $u \in L^1([0,T], \mathcal{U})$ for (y, w) and a PDE without control for μ. More precisely the limit dynamics will be described by

$$\begin{cases} \dot{y}_k = w_k, \\ \dot{w}_k = (H \star (\mu + \mu_m))(y_k, w_k) + u_k, \qquad k = 1,\dots m, \\ \partial_t \mu + v \cdot \nabla_x \mu = \nabla_v \cdot \left[(H \star (\mu + \mu_m))\,\mu\right], \end{cases} \tag{7}$$

where the weak solutions of the equations have to be interpreted in the Carathéodory sense.

We first address the question of existence of solution to the coupled ODE-PDE system (7). On the space $\mathcal{P}_1(\mathbb{R}^{2d})$ of probability measures of bounded first moments, we define the Monge-Kantorovich-Rubistein distance (also known as the 1-Wasserstein distance) by:

$$\mathcal{W}_1(\mu, \nu) = \sup\left\{ \left| \int_{\mathbb{R}^n} \varphi(x)\, d(\mu - \nu)(x) \right| \right\} \tag{8}$$

where φ varies among Lipschitz function with Litpschitz constant bounded by 1. This allows us to give a definition of solution which we report in the Appendix.

This allows us to state:

Theorem 3.1: Let $(y^0, w^0, \mu^0) \in \mathcal{X}$ be given, with μ^0 of bounded support, and let $(\mu_N^0)_{N\in\mathbb{N}}$ be the sequence of atomic probability measures such that each μ_N^0 is given by $\mu_N^0 := \sum_{i=1}^N \delta_{x^0_{i,N}, v^0_{i,N}}$ and $\lim_{N\to\infty} \mathcal{W}_1(\mu_N^0, \mu^0) = 0$. Given a weakly convergent sequence $(u_N)_{N\in\mathbb{N}} \subset L^1([0,T], \mathcal{U})$ of controls and an initial datum $\zeta_N^0 = (y^0, w^0, x_N^0, v_N^0)$ depending on N, let us denote with $\zeta_N(t) = (y_N(t), w_N(t), \mu_N(t)) := (y_N(t), w_N(t), x_N(t), v_N(t))$ the unique solution of the finite-dimensional control problem (4) with control u_N. Then, the sequence (y_N, w_N, μ_N) converges in $C^0([0,T], \mathcal{X})$ to some (y_*, w_*, μ_*), which is a solution of (7) with initial data (y^0, w^0, μ^0) and control u_*.

We thus showed that the limit when $N \to \infty$ of the solutions to the finite-dimensional dynamics (4) corresponds to a solution to the infinite-dimensional dynamics (7). We may now ask ourselves whether solutions to finite-dimensional optimal control problems indeed converge to the optimal solution to the infinite-dimensional problem.

The finite-dimensional optimal control problem can be stated as follows: Given $N \in \mathbb{N}$ and an initial datum $(y(0), w(0), x(0), v(0)) \in (\mathbb{R}^d)^m \times (\mathbb{R}^d)^m \times (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$, we consider the following Mayer optimal control problem:

$$\min_{u=(u_1,\dots,u_k)} V(y(T), w(T), \mu_N(T)) = V(\mu_m(T), \mu_N(T)) \tag{9}$$

where μ_m and μ_N are the time dependent atomic measures supported on the phase space trajectories $(y_k(t), w_k(t)) \in \mathbb{R}^{2d}$, for $k = 1, \dots m$ and $(x_i(t), v_i(t)) \in \mathbb{R}^{2d}$, for $i = 1, \dots N$, respectively, constrained by being the solution of the system (4).

Let us then recall the concept of Γ-limit.

Definition 3.2 (Γ-convergence): [10, Definition 4.1, Proposition 8.1] Let X be a metrizable separable space and $F_N \colon X \to (-\infty, \infty]$, $N \in \mathbb{N}$ be a sequence of functionals. Then we say that F_N Γ-*converges* to F, written as $F_N \xrightarrow{\Gamma} F$, for an $F \colon X \to (-\infty, \infty]$, if

1) $\liminf$-*condition:* For every $u \in X$ and every sequence $u_N \to u$,

$$F(u) \le \liminf_{N\to\infty} F_N(u_N);$$

2) $\limsup$-*condition:* For every $u \in X$, there exists a sequence $u_N \to u$, called *recovery sequence*, such that

$$F(u) \ge \limsup_{N\to\infty} F_N(u_N).$$

We define the following functional on X

$$F(u) = V(y(T), w(T), \mu(T)) \tag{10}$$

where the triplet (y, w, μ) defines the unique solution of (7) with initial datum (y^0, w^0, μ^0) and control u.

Similarly, we define the functionals on X given by

$$F_N(u) = V(y(T), w(T), \mu_N(T)) \tag{11}$$

where $\mu_N(t) = \frac{1}{N}\sum_{i=1}^N \delta_{(x_{i,N}(t), v_{i,N}(t))}$ is the time-dependent atomic measure supported on the trajectories defining the Carathéodory solution of the finite-dimensional system (4) with initial datum (y^0, w^0, x_N^0, v_N^0) and control u.

Theorem 3.3: Given an initial datum $(y^0, w^0, \mu^0) \in \mathcal{X}$ and an approximating sequence μ_N^0, with μ^0, μ_N^0, equi-compactly supported, the sequence of functionals $(F_N)_{N\in\mathbb{N}}$ on $X = L^1([0,T], \mathcal{U})$ defined in (11) Γ-converges to the functional F defined in (10).

A proof can be obtained by modifying that of [12], see the Appendix.

IV. Application to a Migration System

We now introduce a migration model, where the agents' dynamics is determined by two forces: the attraction towards a migration velocity V (which we assume can be sensed) and the consensus dynamics as in the CS model. More precisely, each agent has a parameter α_i which provides the balance between the two forces. The system can be written as:

$$\begin{cases} \dot{x_i} = v_i \\ \dot{v_i} = \alpha_i (V - v_i) + (1-\alpha_i)\frac{1}{N}\sum\limits_{j=1}^N a_{ij}(x,v)\,(v_j - v_i) \end{cases} \tag{12}$$

where: V is the desired velocity (given by a magnetic field, an economic trend...), which from now on we will set to zero for simplicity; $\alpha_i \in [0,1]$ is the control allowing to choose whether the agent senses the desired velocity or follows the group, with the constraint $\sum_i \alpha_i \le M$ (M a given constant); we also simplify the consensus dynamics assuming that a_{ij} does not depend on the agents' positions in this simplified case (even choosing $a_{ij} = 1$ for all i, j).

This new system is a special case of the general dynamics (1), obtained by choosing $u_i = \alpha_i(-\frac{1}{N}\sum_i a_{ij}(v_i - v_j) + V - v_i)$. Since we have not bounded the control u, we can decide to take α in $[0,1]^{N+m}$.

As done above, we divide the agents into m leaders and N followers, where only the leaders can sense the migration direction, while the followers are simply guided by the leaders. The system thus becomes:

$$\begin{cases} \dot{x_i} = v_i, \quad i = 1, ..., N+m, \\ \dot{v_i} = -\alpha_i v_i + (1-\alpha_i)\sum\limits_{j=1}^{m+N} q_j (v_j - v_i), \quad i = 1, .., m, \\ \dot{v_i} = \sum\limits_{j=1}^{m+N} q_j (v_j - v_i), \quad i = m+1, .., m+N, \end{cases}$$

where q_i, $i = 1, ..., m+N$, represent the weights given to each agents' influence on the others. We give the leaders a greater weight than the followers: $q_i = \frac{1}{m}$ for $i \in \{1, ..., m\}$ and $q_i = \frac{1}{N}$ for $i \in \{m+1, ..., m+N\}$. In this way the total weight of leaders is the same as the total weight of followers.

For more simplicity, we design a control strategy for a system composed only of leaders (i.e. supposing that all n agents can be controlled). We thus choose to minimize the functional $\mathbb{V}(T) = \frac{1}{n}\sum_i \|v_i(T) - V\|^2$ at final time T. $\mathbb{V}$ measures the distance between each agent's velocity and the desired velocity V.

First of all we define the average velocity by $\bar{v} = \frac{1}{n}\sum_i v_i$ and project the dynamics over $\bar{v}$ by considering the new variables $\xi_i = \langle v_i, \frac{\bar{v}}{\|\bar{v}\|}\rangle$. This gives: $\dot{\xi_i} = -\xi_i + (1-\alpha_i)\bar{\xi}$, where $\bar{\xi} = \frac{1}{n}\sum_i \xi_i$. The orthogonal component $\omega_i = v_i - \xi_i \bar{v}$ simply satisfies $\dot{\omega}_i = -\omega_i$. Thus we consider the optimal control problem for the ξ_i with cost $\mathbb{V} = \frac{1}{n}\sum_i \xi_i^2$. For this problem the Hamiltonian and covector equations read:

$$\begin{cases} H = \sum\limits_{i=1}^N \lambda_i \left(-\xi_i + (1-\alpha_i)\bar{\xi}\right) \\ \dot{\lambda}_i = -\dfrac{\partial H}{\partial \xi_i} \quad \text{ for every } i \in \{1, ..., n\}, \end{cases}$$

which gives:

$$H = -\bar{\xi}\,(\sum_j \alpha_j \lambda_j) + \tilde{H}, \quad \dot{\lambda}_i = \lambda_i - \bar{\lambda} + \frac{\sum_j \alpha_j \lambda_j}{n}, \tag{13}$$

where $\tilde{H}$ does not depend on α_j and $\bar{\lambda} = \frac{1}{n}\sum_i \lambda_i$. The global strategy consists of setting $\alpha_j = 1$ on the biggest covector λ_j and $\alpha_k = 0$ on the others ($k \neq j$), if and only if $\lambda_j \geq 0$. Since $\dot{\lambda}_k - \dot{\lambda}_j = \lambda_k - \lambda_j$, we know that if $\lambda_j = \max_i \lambda_i$ at time t, then it is maximal at all time. Let $\lambda_{\max} := \max_i \lambda_i$. According to equation (13), $\lambda_{\max}$ is increasing. Hence, if $\lambda_{\max}$ is negative, it is necessarily on a time interval $[0, \delta]$, with $\delta < T$. The transversality condition reads $\lambda_i(T) = \frac{2}{n}\xi_i(T)$. If there exists a j such that $\xi_j(T) > \xi_k(T)$ for every $k \neq j$, then we deduce $\lambda_j(t) > \lambda_k(t)$ for every t. Furthermore, we can assume with no loss of generality that the ξ_i stay in the same order. From the expression of H we conclude that if $\lambda_j(t) \geq 0$ for all t, where $\lambda_j = \lambda_{\max}$, then $\alpha_j \equiv 1$ and the other controls vanish. If there are more than one $\lambda_j(T)$ with maximal value, then every control driving the corresponding agents to the same final position would be optimal. Simulations showed that the event $\lambda_{\max} < 0$ on some interval $[0, \delta[$ happens only in a few isolated cases, and that the gain in performance in setting $\alpha_k = 0$ instead of $\alpha_k = 1$ on $[0, \delta[$ is very modest. Therefore, we simply use the ansatz of always choosing the maximal ξ_i and act only on agent i with control $\alpha_i = 1$.
This strategy also coincides with the instantaneous maximal decrease of $\mathbb{V}$. Indeed, computing the derivative of $\mathbb{V}$ w.r.t. time gives: $\frac{d\mathbb{V}}{dt} = -2\mathbb{V} + \frac{2}{n}\bar{\xi}\sum_i(1-\alpha_i)\xi_i$.

We test the potency of control using only a selected group of leaders based on their initial positions. To do this, we design four different strategies:

1) Only m of the $m+N$ agents can be controlled.
2) Any of the $m+N$ agents can be controlled.
3) No control at all is applied. This is the reference case.
4) Only m of the $m+N$ agents can be controlled, with the additional constraint that if the leaders become too far away from the followers, they stop being controlled in order to "wait" for the remaining agents. This creates a better cohesion in the group. We set the threshold such that the leaders stop being controlled if $\max_{j\in\{m+1,\dots,m+N\}}\langle v_j, \bar{v}\rangle > \frac{6}{5}\max_{i\in\{1,\dots,m\}}\langle v_i, \bar{v}\rangle$.

We apply these various control strategies to three initial configurations on a plane. Configuration a) consists of spreading the followers randomly within a disc of radius 1 centered around the point $(1,1)$, while placing the leaders in a semi-circle around the followers opposite from the target velocity $(0,0)$. In configurations b) and c), all agents are given random initial velocities within that same disc. Moreover, in configuration c), the leaders are chosen to be the ones with the biggest projected velocities over the mean velocity.

The various simulations show that consensus is reached in the cases where several or all of the agents are controlled. See Figure 1 for an example of the resulting dynamics. Figure 2 shows the evolution of the functional $\mathbb{V}$ for a group of agents initially in configuration a) (i.e. with the controlled agents spread in a semi-circle around the uncontrolled ones), for a group of 10 leaders and 10 followers ($m = N = 10$).

The simulation is consistent with expectations: without any control on the agents, $\mathbb{V}$ hardly decreases and does

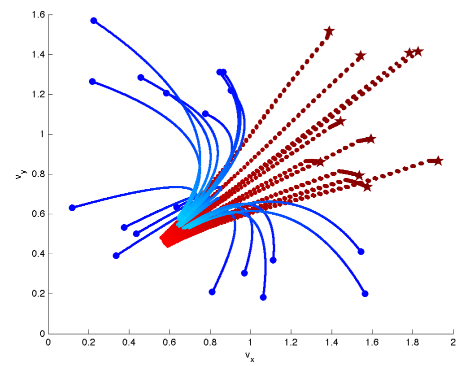

Fig. 1: Evolution of the leaders' (thick red lines) and followers' (thin blue lines) velocities starting from configuration a), with control strategy 4. Initial positions are marked by stars (leaders) and circles (followers).

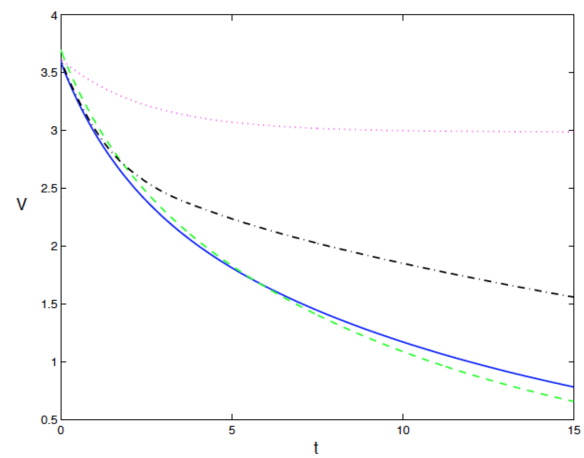

Fig. 2: Evolution of the functional $\mathbb{V}$ for initial configuration a), using control strategies 1 (full line), 2 (dashed), 3 (dotted) and 4 (dot-dashed line)

not tend to zero, whereas there is a clear decrease in the cases where the agents are controlled. It is interesting to notice that the control strategies 1 and 2 do not differ much. As expected, controlling all agents is more efficient than controlling only a group of leaders. However, the difference is small, which proves that controlling only some agents can be effective. When using strategy 4, in which the leaders "wait" for the followers, the functional $\mathbb{V}$ first follows the curves of strategies 1 and 2, but then its decrease rate becomes lower as the leaders become too far from the followers and thus stop being controlled.

It is also informative to compare the evolution of $\mathbb{V}$ when starting from initial configurations b) and c). The first striking difference is the behavior of $\mathbb{V}$ when using the fourth control strategy. When controlled and uncontrolled agents are spread out randomly, the initial decrease of $\mathbb{V}$ is much slower than when the controlled agents are chosen on one side. Indeed, since the control attracts the leaders towards the opposite side of the group, it takes them a longer time to cross the threshold at which they are considered to be too far from the leaders when starting from configurations a) and c). Therefore, they are controlled for a longer period of time and are better able to drive the group to consensus.

As displayed in Table I, a quantitative analysis shows that when using the second control strategy (i.e. controlling only the group of leaders), the final value of $\mathbb{V}$ is lower in configuration c) than in configuration b). This confirms the intuition that the leaders are better able to guide the followers when pushing them from behind rather than being spread out among the group.

Control strategies	1	2	3	4
Initial average	3.277			
Initial stand. dev.	0.249			
Final average	0.620	0.709	3.012	1.458
Final stand. dev.	0.060	0.070	0.256	0.144

Initial configuration a)

Control strategies	1	2	3	4
Initial average	2.521			
Initial stand. dev.	0.308			
Final average	0.447	0.514	2.050	1.2180
Final stand. dev.	0.063	0.073	0.292	0.147

Initial configuration b)

Control strategies	1	2	3	4
Initial average	2.456			
Initial stand. dev.	0.215			
Final average	0.470	0.529	2.367	0.940
Final stand. dev.	0.051	0.059	0.236	0.114

Initial configuration c)

TABLE I: Average and standard deviation of the functional $\mathbb{V}$ at initial and final time where 40 simulations were run for each control strategy ($T = 20$, $m = 10$, $N = 20$)

We now focus on the first control strategy. Starting from the third configuration, we run simulations for a total number of agents respectively equal to 10, 20, 40, 80 and 120, while keeping a constant number of 10 leaders. Since the initial configuration is random, we run a total of 40 simulations in each of these cases and average the resulting functional $\mathbb{V}$. Figure 3 shows the evolution of the average of $\mathbb{V}$ with respect to time. One notices that as the number of agents increases, consensus is slower to reach. In the first case, $m = 10$ and $N = 0$: all agents are controlled, and therefore, consensus is reached very fast. In the second case, $m = N = 10$, so only half of the agents are controlled. In the third case, $N = 30$, so there are thrice as many followers as leaders. Although convergence is slower, the small group of agents is still able to drive the group to consensus, as in the fifth case, in which leaders represent one-twelfth of the agents. Even more, the difference in the decay of $\mathbb{V}$ from the case of 80 agents to that of 120 is modest, inline with the expectation of Γ-convergence of cost functionals.

V. CONCLUSIONS

We considered a general mathematical framework for social dynamics, which allows to deal both with microscopic models and their mean-field limits. Moreover, we provide convergence of cost functional in the sense of Γ-convergence. We then test our framework on a model for migration, where we want to drive the group to consensus at a given migration velocity. After designing a control strategy, we apply that to a small number of leaders. Via simulations, we show the effectiveness of our control strategy both w.r.t. the case of control over all group and for the number of agents tending to infinity. Further investigations will concern the optimal control of the mean-field limit and control strategies for other social dynamic models.

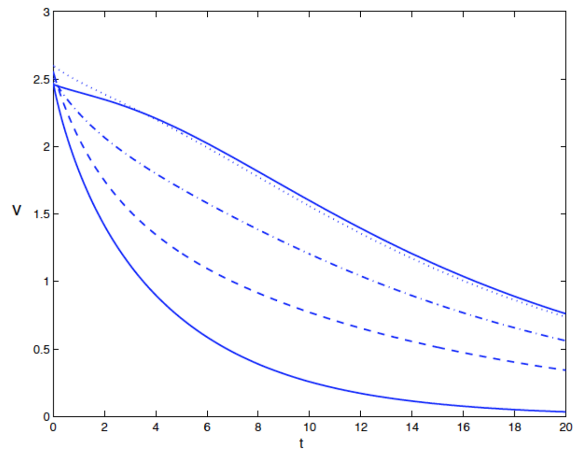

Fig. 3: Evolution of the functional $\mathbb{V}$ from initial configuration c), using control strategy 1, for a total number of 10 (solid line), 20 (dashed), 40 (dot-dash), 80 (dotted) and 120 (solid line) agents

APPENDIX

Definition 5.1: Let $u \in L^1([0,T],\mathcal{U})$ be given. We say that a map $(y,w,\mu) : [0,T] \to \mathcal{X} := \mathbb{R}^{2d\times m} \times \mathcal{P}_1(\mathbb{R}^{2d})$ is a solution of the controlled system with interaction kernel H

$$\begin{cases} \dot{y}_k = w_k, \quad k = 1,\dots m,\ t \in [0,T], \\ \dot{w}_k = H \star (\mu + \mu_m)(y_k, w_k) + u_k, \\ \partial_t \mu + v \cdot \nabla_x \mu = \nabla_v \cdot \left[(H \star (\mu + \mu_m))\, \mu \right], \end{cases} \tag{14}$$

with control u, where μ_m is the time-dependent atomic measure as in (5), if

(i) the measure μ is equi-compactly supported in time, i.e., there exists $R > 0$ such that $\mathrm{supp}(\mu(t)) \subset B(0,R)$ for all $t \in [0,T]$;

(ii) the solution is continuous in time with respect to the following metric in $\mathcal{X}$

$$\|(y,w,\mu) - (y',w',\mu')\|_{\mathcal{X}} := \\ := \frac{1}{m}\sum_{k=1}^{m} (|y_k - y_k'| + |w_k - w_k'|) + \mathcal{W}_1(\mu,\mu'), \tag{15}$$

where $\mathcal{W}_1(\mu,\mu')$ is the 1-Wasserstein distance in $\mathcal{P}_1(\mathbb{R}^{2d})$;

(iv) the (y, w) coordinates define a Carathéodory solution of the following controlled problem with interaction kernel H, control $u(\cdot)$, and the external field $H \star \mu$:

$$\begin{cases} \dot{y}_k = w_k, \\ \dot{w}_k = H \star (\mu + \mu_m)(y_k, w_k) + u_k, \quad k = 1, \dots m; \end{cases} \tag{16}$$

(v) the μ component satisfies

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}^{2d}} \phi(x, v) \, d\mu(t)(x, v) = \\ &= \int_{\mathbb{R}^{2d}} \nabla \phi(x, v) \cdot \omega_{H,\mu,y,w}(t, x, v) \, d\mu(t)(x, v) \end{aligned} \tag{17}$$

for every $\phi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, in the sense of distributions, where $\omega_{H,\mu,y,w}(t, x, v) : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ is the time-varying vector field defined as follows

$$\omega_{H,\mu,y,w}(t, x, v) := (v, H \star \mu(t)(x, v) + H \star \mu_m(t)(x, v)). \tag{18}$$

Let moreover $(y^0, w^0, \mu^0) \in \mathcal{X}$ be given, with $\mu^0 \in \mathcal{P}_1(\mathbb{R}^{2d})$ of bounded support. We say that $(y, w, \mu) : [0, T] \to \mathcal{X}$ is a solution of (14) with initial data (y^0, w^0, μ^0) and control u if it is a solution of (14) with control u and it satisfies $(y(0), w(0), \mu(0)) = (y^0, w^0, \mu^0)$.

Proof of Theorem 3.3: *Proof:* Fix a weakly convergent sequence of controls $u_N \rightharpoonup u_*$ in $L^1([0, T], \mathcal{U})$. Let $\zeta_N(t) = (y_N(t), w_N(t), \mu_N(t)) := (y_N(t), w_N(t), x_N(t), v_N(t))$ be the associated solutions. Then the sequence ζ_N converges to a solution $\zeta_*(t) = (y_*(t), w_*(t), \mu_*(t))$ of (7) in the sense of Definition 5.1 with control u_* and initial datum (y^0, w^0, μ^0). All supports are uniformly bounded w.r.t. N and we do have uniform convergence of the trajectories (y_N, w_N) (the leader trajectories, whose number does not tend to infinity). Moreover $\mathcal{W}_1(\mu_N(t), \mu_*(t)) \to 0$ uniformly in $t \in [0, T]$ thus we get the $\Gamma - \liminf$ condition

$$\liminf_{N \to \infty} F_N(u_N) \geq F(u_*).$$

For the $\Gamma - \limsup$ condition, let us fix u_* and we consider the trivial recovery sequence $u_N \equiv u_*$ for all $N \in \mathbb{N}$. We can proceed as above associating a corresponding sequence of solutions, which have the same convergence properties, thus we deduce:

$$\limsup_{N \to \infty} F_N(u_N) = \lim_{N \to \infty} F_N(u_*) = F(u_*).$$

■

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