Sparse control of second-order cooperative systems and partial differential equations to approximate alignment*

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***Abstract*— Second-order cooperative systems are second-order systems in which the forces among agents are non-repulsive. The celebrated Cucker-Smale model is a well known example. Depending on the strength of the forces, the free evolution of such systems is to converge either to a single group of strongly connected agents, or to clustering.**

In this article, we design a simple and robust control strategy steering any second-order cooperative system to approximate alignment. The computation of the control at any instant of time only requires the knowledge of the size of the support and the Lipschitz constant of the interaction force. Moreover, the control is *sparse* in the sense that the (time-varying) support of the control is a small subset of the configuration space.

Our strategy provides approximate alignment, on the one hand, for second-order cooperative systems with any number N of agents, and on the other, for the mean-field limit of such systems, i.e., when N tends to infinity. Such a limit is a transport partial differential equation involving nonlocal terms, called a *cooperative Partial Differential Equation*.

Keywords: control of transport PDEs, PDEs with nonlocal terms, cooperative systems, collective behavior.

I. INTRODUCTION

In recent years, the study of collective behavior of a crowd of autonomous agents has drawn a great interest from scientific communities, e.g. in civil engineering (for evacuation problems), robotics (coordination of robots), computer science and sociology (social networks), and biology (crowds of animals). In particular, it is well known that some simple rules of interaction between agents can promote the formation of special patterns, like in formations of bird flocks, lines, etc... This phenomenon is often referred to as *self-organization*.

Beside the problem of analyzing the collective behavior of a "closed" system, it is interesting to understand what changes of behavior can be induced by an external agent (e.g. a policy maker). For example, one can try to enforce the creation of patterns when they are not formed naturally, or break the formation of such patterns. This is the problem of **control of crowds**, that we address here in a specific case.

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From the mathematical analysis point of view, passing from a huge set of simple rules for each individual to a model capable of capturing the dynamics of the whole crowd, is performed by mean-field process, which permits to consider the limit of a set of ordinary differential equations (one for each agent) and yields a partial differential equation (PDE) for the whole crowd.

We focus here on a well-known family of models for crowds dynamics, called the *cooperative systems* (or *monotone* systems, see, e.g., [1], [2]). Such models may reproduce the behavior of a human or animal crowd, in which each agent tries to align its velocity with the velocities of its neighbors. The dynamics of the i-th agent is given by

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \frac{1}{N}\sum_{j=1}^N \psi(x_j - x_i, v_j - v_i), \quad i = 1, \dots N, \end{cases} \tag{1}$$

where the function $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ provides an account for the influence between two individuals. From now on, we will require the following condition on ψ, that is the natural definition of a second-order binary cooperative system:

$$\psi(x, v) \,//\, v \qquad \text{and} \qquad \psi(x, v) \cdot v \geq 0. \tag{2}$$

For more details on this assumption, see Section II. We will also assume that

$$\psi \text{ is a } L\text{-Lipschitz function.} \tag{3}$$

Under (2), it is easy to prove that the set of velocities is invariant with respect to the dynamics, i.e.

$$\min_{i=1,\dots,N} v_i^k(t) \leq v_j^k(t+s) \leq \max_{i=1,\dots,N} v_i^k(t),$$

for each agent $j = 1, \dots, N$, each dimension $k = 1, \dots, d$ and all $t \in \mathbb{R}, s \geq 0$. Nevertheless, this does not imply that the system converges to alignment or approximate alignment, in general. In this article, **approximate alignment** means that velocities of all agents are in a small neighborhood of a given value (see Definition 10 further).

Then, it is interesting to understand how an external controller can enforce alignment. In this article, we define a simple strategy providing convergence of the system to approximate alignment. Observe that conditions (2) and (3) are the only ones that we require to ensure that our strategy drives any initial data to alignment. No more precise knowledge of ψ than the Lipschitz constant L is required to define the control strategy.

We choose a specific action of the control, which can be extended to the so-called mean-field limit of the system (1), discussed below. Our control $u = u(t, x, v)$ will be a

Lipschitz function with respect to x, v and it will act as follows on the system (1):

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \frac{1}{N}\sum_{j=1}^N \psi(x_j - x_i, v_j - v_i) \\ \qquad +\chi_{\omega(t)} u(t, x_i, v_i), \qquad\qquad i = 1, \dots N. \end{cases} \tag{4}$$

where ω is defined as the **control set**. In other words, for every t the function $u(t, \cdot, \cdot)$ is a Lipschitz vector field, not depending on the specific agent. Its action on the i-th agent is given by the evaluation of u in (x_i, v_i).

We also impose constraints on the control function. In particular, we assume that the control strength is bounded:

$$|u(t, x, v)| \leq 1 \tag{5}$$

for any $t \geq 0$ and that

$$\omega = \mathrm{supp}(u(t, \cdot, \cdot)) \text{ is compact, with } |\omega| \leq c, \tag{6}$$

where $c > 0$ is a small, fixed parameter, and $|\omega|$ is the Lebesgue measure of ω. This constraint models a *sparsity* property of the control, that is the fact that the control can only act on a small part of the configuration space (see [3], [4], [5]).

We now present the mean-field limit, describing the behavior of the system (1) when the number N of agents tends to infinity (see Section II-B). The crowd at time t is then represented by the density of agents $\mu(t, x, v)$, and its dynamics is given by the transport PDE

$$\partial_t \mu + v \cdot \nabla_x \mu + \nabla_v \cdot (\Psi[\mu] \mu) = 0, \tag{7}$$

with $\Psi[\mu](x, v) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(y - x, w - v) d\mu(y, w)$. This velocity field is *non-local* since its value at some point depends on the value of μ in a whole neighborhood of it (see Section II-A). The controlled version is given by

$$\partial_t \mu + v \cdot \nabla_x \mu + \nabla_v \cdot ((\Psi[\mu] + \chi_\omega u(t, \cdot, \cdot)) f) = 0, \tag{8}$$

The fact that the mean-field limit of (4) is (8) uses the specific form of the control u introduced in (4). Indeed, u is Lipschitz and independent on the agents, hence it passes to the mean-field limit (see Section II-B).

Our objective is to treat the control of both (4) and (8) in a unified way. We will first prove that our strategy drives the finite-dimensional system (4) to approximate alignment. Then, a mean-field limit argument will provide the result for the mean-field limit (8).

Theorem 1: Let $v^* \in \mathbb{R}^d$ and $\varepsilon > 0$ be arbitrary.

Let $(x_1(0), v_1(0), \dots, x_N(0), v_N(0))$ be a given initial data for (4). There exist a time $T > 0$ and a control u satisfying (5)-(6) defined on a time interval $[0, T]$ such that the corresponding solution of (4) satisfies $|v_i(t) - v^*| < \varepsilon$, for every $i = 1, \dots, N$ and every $t \geq T$.

Similarly, let $\mu(0)$ be a given initial data for (8) with compact support. There exist a time T and a control u satisfying (5)-(6) defined on a time interval $[0, T]$ such that the corresponding solution of (8) satisfies $\mathrm{supp}(\mu(t)) \subset \mathbb{R}^d \times B(v^*, \varepsilon)$ for every $t \geq T$.

The interest of controlling cooperative systems, both in finite and infinite dimension, was motivated by some specific examples, such as the Cucker-Smale model for flocking of birds (see [6]), the finite-dimensional version of which was studied in [5], [7] and its mean-field limit in [8], [9]. Here, this is the first time that the control problem is treated with a unified approach. In contrast to [9, Section 3] where our control results for the Cucker-Smale model were deeply rooted in precise estimates of the dynamics, in the present paper only a rough knowledge of the dynamics is required, then providing a more robust control strategy. The drawback is that our alignment result is slightly weaker than consensus for the Cucker-Smale models (see Remark 11).

II. COOPERATIVE SYSTEMS

We consider the system (1), that is a second-order system with binary interactions.

In general, a system $\dot{x} = f(x)$ is said to be cooperative if $\frac{\partial f^k}{\partial x^j} \geq 0$ (see, e.g., [2]). In the particular case of binary interactions, this would give $\frac{\partial \dot{v}_i^k}{\partial x_j^l} \geq 0$ for all $i, j = 1, \dots, N$ and all $k, l = 1, \dots, d$, and $\frac{\partial \dot{v}_i^k}{\partial x_j^l} \geq 0$ for $i \neq j$ or $k \neq l$. Simple computations show that this would imply $\frac{\partial \psi(x,v)}{\partial x} = 0$ and the condition (2).

Here, in our study, we only keep the condition (2), showing that it is sufficient to achieve control to alignment. This is why, in order to underline the difference, we have used the wording "second-order cooperative system".

Given an initial data $(x_1(0), v_1(0), \dots, x_N(0), v_N(0))$ for (1), the Lipschitz property of ψ guarantees existence and uniqueness of the solution of (1) for all positive times. Moreover, (2) implies that the set of velocities is invariant, in the following sense.

Proposition 2: Let

$$\mathcal{V} = [\underline{V}^1, \overline{V}^1] \times [\underline{V}^2, \overline{V}^2] \times \cdots \times [\underline{V}^d, \overline{V}^d] \tag{9}$$

be a box such that $v_i(0) \in \mathcal{V}$ for every $i = 1, \dots, N$. Then, the solution of (1) satisfies $v_i(t) \in \mathcal{V}$ for every $t \geq 0$.

Proof: It suffices to prove that the set $\mathbb{R}^d \times \mathcal{V}$ is invariant for the dynamics (1), which is proved by noticing that (2) implies that:

- for each agent i satisfying $v_i^k = \underline{V}^k$ for some $k = 1, \dots, d$, we have $\dot{v}_i^k \geq 0$;
- for each agent i satisfying $v_i^k = \overline{V}^k$ for some $k = 1, \dots, d$, we have $\dot{v}_i^k \leq 0$.

This means that the dynamics is pointing inwards the box domain. We prove the first condition, the second being completely equivalent. We have $\psi(x_j - x_i, v_j - v_i) = \xi(x_j - x_i, v_j - v_i)(v_j - v_i)$, where ξ is a nonnegative function due to (2). Let now the agent i satisfy $v_i^k = \underline{V}^k$: then $v_j^k - v_i^k \geq 0$ for every $j = 1, \dots, N$, hence $\psi(x_j - x_i, v_j - v_i) \cdot e^k = \xi(x_j - x_i, v_j - v_i)(v_j^k - v_i^k) \geq 0$, and thus $\dot{v}_i^k = \frac{1}{N}\sum_{j=1}^N \psi(x_j - x_i, v_j - v_i) \cdot e^k \geq 0$. The result then follows by Cauchy uniqueness. ■

Corollary 3: Let

$$\mathcal{X}(0) = [\underline{X}^1, \overline{X}^1] \times [\underline{X}^2, \overline{X}^2] \times \ldots \times [\underline{X}^d, \overline{X}^d] \qquad (10)$$

and $\mathcal{V}$ given by (9) be two boxes such that $(x_i(0), v_i(0)) \in \mathcal{X}(0) \times \mathcal{V}$ for every $i = 1, \ldots, N$. Then, the solution of (1) satisfies $x_i(t) \in \mathcal{X}(t)$ with

$$\begin{aligned}\mathcal{X}(t) \quad &= \quad [\underline{X}^1 + t\underline{V}^1, \overline{X}^1 + t\overline{V}^1] \times [\underline{X}^2 + t\underline{V}^2, \overline{X}^2 + t\overline{V}^2] \\ &\quad \times \cdots \times [\underline{X}^d + t\underline{V}^d, \overline{X}^d + t\underline{V}^d]. \qquad (11)\end{aligned}$$

Proof: This follows from Proposition 2 and from the fact that $\dot{x}_i(t) \in \mathcal{V}$ for all times. ■

In the following, the lower index $i = 1, \ldots, N$ refers to the i-th agent, while the upper index $k = 1, \ldots, d$ refers to the k-th coordinate in $\mathbb{R}^d$.

A key property for the class of systems (1) is their invariance under translations. Consider a translation in the position variables at time $t = 0$, such as $y_i(0) = x_i(0) + a$ with $a \in \mathbb{R}^d$. Then, the solution $(x(t), v(t))$ of (1) with initial data $(x_1(0), v_1(0), \ldots, x_N(0), v_N(0))$ and the solution $(y(t), w(t))$ with initial data $(y_1(0), v_1(0), \ldots, y_N(0), v_N(0))$ satisfy $y_i(t) = x_i(t) + a$ and $w_i(t) = v_i(t)$ for all times and all agents. Similarly, consider a translation in the velocity variable at time $t = 0$, such as $w_i(0) = v_i(0) + a$, then the solution $(x(t), v(t))$ of (1) with initial data $(x_1(0), v_1(0), \ldots, x_N(0), v_N(0))$ and the solution $(y(t), w(t))$ with initial data $(x_1(0), w_1(0), \ldots, x_N(0), w_N(0))$ satisfy $y_i(t) = x_i(t) + at$ and $w_i(t) = v_i(t) + a$ for all times and all agents.

A. Transport Equations with non-local velocities

In this section, we present a general theory for transport equations with non-local velocities, of which (7) and (8) are particular cases. Let $\mathcal{P}_c(\mathbb{R}^m)$ be the space of probability measures in $\mathbb{R}^m$ with compact support, endowed with the weak-star topology. In the following, we will take $m = 2d$. We recall that $\mu_N \rightharpoonup \mu$ if $\int f \, d\mu_N \to \int f \, d\mu$ for every $f \in C_c^\infty(\mathbb{R}^m)$. Let us recall the definition of the Wasserstein distance in $\mathcal{P}_c(\mathbb{R}^m)$ (see [10], [11]).

Definition 4: Let $\mu, \nu \in \mathcal{P}_c(\mathbb{R}^m)$ be two measures. A transference plan from μ to ν is a probability density in $\mathcal{P}_c(\mathbb{R}^{2m})$ satisfying $\int_{\mathbb{R}^{2m}} (f(x) + g(y)) \, d\pi(x, y) = \int_{\mathbb{R}^m} f(x) \, d\mu(x) + \int_{\mathbb{R}^m} g(y) \, d\nu(y)$ for all functions $f, g \in C_c^\infty(\mathbb{R}^m)$, i.e., π has marginals μ, ν. The Wasserstein distance is defined by

$$W_p(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \left\{ \left(\int_{\mathbb{R}^{2m}} |x - y|^p \, d\pi(x, y) \right)^{1/p} \right\},$$

where $\Pi(\mu, \nu)$ is the set of transference plans from μ to ν.

A crucial property of the Wasserstein distance is that it metrizes weak convergence in a compact space.

Proposition 5: Let $\mu_n \in \mathcal{P}_c(K)$ with K compact. Then $\mu_n \rightharpoonup \mu$ if and only if $W_p(\mu_n, \mu) \to 0$. Moreover, in this case we have $\mu \in \mathcal{P}_c(K)$.

Proof: The condition above is a particular case of [11, Thm 7.12]. It is also clear that $\mu \in \mathcal{P}_c(K)$: indeed, for any function f with support outside K we have $\int f \, d\mu_n = 0$, hence $\int f \, d\mu = 0$. ■

The fundamental result for existence and uniqueness of solutions of

$$\partial_t \mu + \nabla \cdot (\Phi[\mu, t]\mu) = 0 \qquad (12)$$

is then stated in terms of the Wasserstein distance (see [3], [12], [13], [14]).

Theorem 6: We assume that

$$\Phi\,[\mu, t] : \begin{cases} \mathcal{P}_c(\mathbb{R}^m) \times \mathbb{R} & \to \quad C^1(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m) \\ \quad (\mu, t) & \mapsto \quad \Phi\,[\mu, t] \end{cases}$$

satisfies:

- $\Phi\,[\mu, t]$ is measurable with respect to t;
- $\Phi\,[\mu, t]$ is uniformly Lipschitz and with sublinear growth, i.e., there exist $\mathcal{L}$, $\mathcal{M}$ not depending on μ, t, such that $|\Phi\,[\mu, t]\,(x) - \Phi\,[\mu, t]\,(y)| \leq \mathcal{L}|x - y|$ and $|\Phi\,[\mu, t]\,(x)| \leq \mathcal{M}(1 + x)$, for all $\mu \in \mathcal{P}_c(\mathbb{R}^m), t \in \mathbb{R}, x, y \in \mathbb{R}^m$;
- Φ is a Lipschitz function with respect to μ, i.e., there exists $\mathcal{K}$ not depending on t such that $\|\Phi\,[\mu, t] - \Phi\,[\nu, t]\,\|_{C^0} \leq \mathcal{K} W_p(\mu, \nu)$.

For every $\mu_0 \in \mathcal{P}_c(\mathbb{R}^m)$, there exists an unique solution $\mu(t)$ of (12), i.e., a curve in $\mathcal{P}_c(\mathbb{R}^m)$ continuous with respect to time and satisfying (12) in weak form. Moreover, for all initial data $\mu_0, \nu_0 \in \mathcal{P}_c(\mathbb{R}^m)$, the corresponding solutions $\mu(t), \nu(t)$ satisfy $W_p(\mu(t), \nu(t)) \leq e^{4(\mathcal{L}+\mathcal{K})|t|} W_p(\mu_0, \nu_0)$ (continuous dependence of solutions).

Clearly, (7) is a particular case of (12), with $\Phi\,[\mu] = (v, \Psi\,[\mu])^\top$, independent of t. Moreover, if the control function $\chi_{\omega(t)} u(t, x, v)$ is measurable with respect to time and Lipschitz with respect to (x, v) with Lipschitz constant independent of t, then (8) is as well a particular case of (12), with $\Phi\,[\mu, t] = (v, \Psi\,[\mu] + \chi_\omega u)^\top$. For both cases (7) and (8), the assumptions of Theorem 6 are satisfied with $p = 1$. While first and second conditions can be checked directly, the third condition is a consequence of the following estimate:

$$\left\| \begin{pmatrix} v \\ \Psi\,[\mu] + \chi_\omega u \end{pmatrix} - \begin{pmatrix} v \\ \Psi\,[\nu] + \chi_\omega u \end{pmatrix} \right\| (x, v) =$$

$$= \|\Psi\,[\mu] - \Psi\,[\nu]\,\|(x, v) \leq L W_1(\mu, \nu),$$

which follows from the Kantorovich-Rubinstein duality formula (see, e.g., [11])

$$L W^1(\mu, \nu) = \sup_{f \in \mathrm{Lip}(\mathbb{R}^{2d}), |f|_{\mathrm{Lip}} \leq L} \int f \, d(\mu - \nu).$$

Hence we have existence and uniqueness for (7) and (8).

B. Mean-field limits

Definition 7: Let $(y_1, y_2, \ldots, y_N) \in (\mathbb{R}^m)^N$ be a vector representing the position of N particles in $\mathbb{R}^m$. The empirical measure is defined as $\mu_N = \frac{1}{N} \sum_{i=1}^N \delta_{y_i} \in \mathcal{P}_c(\mathbb{R}^m)$.

Definition 8: Consider a family of finite-dimensional models describing a system of N particles. Given a trajectory

$(y_1^N(t), \ldots, y_N^N(t))$ of the system, consider the corresponding empirical measure $\mu_N(t)$. The family of models indexed by N converges to a mean-field model (to be defined) if for any sequence of empirical measures indexed by N and satisfying $\mu_N(0) \rightharpoonup \mu(0)$, we have $\mu_N(t) \rightharpoonup \mu(t)$, where $\mu(t)$ is the solution of the mean-field model starting at $\mu(0)$.

Let us now prove that the mean-field limit of the system (1) is (7). If the initial data for (7) is $\mu(0) = \mu_N(0) = \frac{1}{N}\sum_{i=1}^N \delta_{(x_i(0),v_i(0))}$, an empirical measure of N agents, then the solution at time t of (7) is the empirical measure $\mu_N(t) = \frac{1}{N}\sum_{i=1}^N \delta_{(x_i(t),v_i(t))}$, where $(x_1(t), v_1(t), \ldots, x_N(t), v_N(t))$ is the unique solution of (1) with initial data $(x_1(0), v_1(0), \ldots, x_N(0), v_N(0))$. By density of the empirical measures in the space $\mathcal{P}_c(\mathbb{R}^{2d})$ with respect to the convergence of measures, and continuous dependence of solutions to (7) with respect to the initial data, we obtain that the PDE (7) is the mean-field limit of (1) (see, e.g., [15], [16] for more details on mean-field limits).

Similarly, the mean-field limit of the family of control systems (4) is the controlled PDE (8). The key observation is that $u(t, \cdot, \cdot)$ is a Lipschitz function, not depending on the specific i-th agent (see [4], [9]).

The same technique allows one to extend several results established for the finite-dimensional system (1) to the mean-field limit (7). In particular, Proposition 2 and Corollary 3 yield the following result.

Proposition 9: Let $\mathcal{X}(0), \mathcal{V}$ given by (9)-(10) be boxes such that $\operatorname{supp}(\mu(0)) \in \mathcal{X}(0) \times \mathcal{V}$. Then the solution of (7) satisfies $\operatorname{supp}(\mu(t)) \subset \mathcal{X}(t) \times \mathcal{V}$ for every $t \geq 0$, with $\mathcal{X}(t)$ given by (11).

The invariance properties described above for the finite-dimensional system (1) are as well valid for the mean-field limit (7). We do not provide any details.

C. Approximate alignment and consensus

Definition 10: A solution of the finite-dimensional system (1) or of the mean-field system (7) is said to be ε-approximately aligned around $v^* \in \mathbb{R}^d$ from time T if, for every $t \geq T$, we have

- for (1): $v_i(t) \in B(v^*, \varepsilon)$ for every $i = 1, \ldots, N$;
- for (7): $\operatorname{supp}(\mu(t)) \in \mathbb{R}^d \times B(v^*, \varepsilon)$.

The concept of approximate alignment is weaker than the concept of consensus. In the finite-dimensional case, we recall that a crowd of agents converges to consensus (or flocking) if there exists X such that $|x_i(t) - x_j(t)| \leq X$ for every $t \geq 0$ and $|v_i(t) - v_j(t)| \to 0$ as $t \to +\infty$, for all $i, j = 1, \ldots, N$.

Remark 11: Approximate alignment does not imply uniform compactness of the position variable for $t \geq 0$. Moreover, approximate alignment does not provide convergence of velocities to v^*, as it is the case for flocking.

We recall that we require only weak conditions on the dynamics for (1); as a consequence, even an ε-approximately aligned system with ε very small can result in non-compactness of the position variables and in non-convergence of the velocity variable. As a degenerate example, choose no interaction among agents, i.e., $\psi = 0$. This is similar to the difference between stability and asymptotic stability for dynamical systems.

III. Control of cooperative systems (4) and (8)

In this section, we define a strategy driving any initial data of (4) or (8) with compact support to approximate alignment. We first perform the analysis for the finite-dimensional system (4), and then for the mean-field limit (8).

Following the arguments of Proposition 2 and Corollary 2, we have the following result on the evolution of the support of solutions of (4), provided that the control vector points inwards along the boundary of the velocity support.

Proposition 12: Let $u(t, x, v)$ be a control for (4), and $\mathcal{X}(0), \mathcal{V}$ be boxes defined in (9)-(10) such that $(x_i(0), v_i(0)) \in \mathcal{X}(0) \times \mathcal{V}$ for every $i = 1, \ldots, N$. We assume that, if v satisfies $v = \underline{V}^k$ (resp., $v = \overline{V}^k$) for some k, then $u(t, x, v) \cdot e^k \geq 0$ (resp., $u(t, x, v) \cdot e^k \leq 0$). Then, the solution of (4) satisfies $(x_i(t), v_i(t)) \in \mathcal{X}(t) \times \mathcal{V}$ for all $i = 1, \ldots, N$ and $t \geq 0$, with $\mathcal{X}(t)$ given by (11).

We are going to design a control satisfying the assumptions of Proposition 12. Using the invariance properties of the system, we focus on alignment around $v^* = 0 \in \mathbb{R}^d$ and we assume that $\underline{X} = 0$. Also, without loss of generality, we assume throughout that $\underline{V}^k = 0$ and $\overline{V}^k > 0$ for any dimension $k = 1, \ldots, d$. Indeed, if $\underline{V}^j < 0 < \overline{V}^j$ for a given dimension j and one wants to achieve approximate alignment around 0, one can use the following strategy:

1) Define the translated variables
$$y_i^j(t) = x_i^j(t) - t\underline{V}^j, \qquad w_i^j(t) = v_i^j(t) - \underline{V}^j,$$
$$y_i^k = x_i^k, \qquad w_i^k = v_i^k \qquad \text{for} \quad k \neq j,$$
and note that they follow the dynamics (4);
2) Control this system to approximate alignment around 0 in dimension j with precision $\frac{\varepsilon}{\sqrt{d}} + \underline{V}^j$, with the strategy described below, then reducing the support of velocities of the original system to $[\underline{V}^j, \frac{\varepsilon}{\sqrt{d}})$ in dimension j;
3) Define the reversed variables
$$\bar{y}_i^j(t) = -x_i^j(t) - t\frac{\varepsilon}{\sqrt{d}}, \; \bar{w}_i^j(t) = -v_i^j(t) + \frac{\varepsilon}{\sqrt{d}},$$
$$\bar{y}_i^k = x_i^k, \qquad w_i^k = v_i^k \qquad \text{for} \quad k \neq j.$$
They follow the dynamics (4) with $\bar{\psi}(x, v) = -\psi(-x, -v)$, which satisfies (2)-(3). The support of velocities in dimension j is contained in $(0, -\underline{V}^j + \frac{\varepsilon}{\sqrt{d}}]$.
4) Control this system to approximate alignment with precision $2\frac{\varepsilon}{\sqrt{d}}$. Then, the support of the original velocities v_i in dimension j is contained in $\left(-\frac{\varepsilon}{\sqrt{d}}; \frac{\varepsilon}{\sqrt{d}}\right)$, i.e., we have achieved approximate alignment around 0 in dimension j with precision $\frac{\varepsilon}{\sqrt{d}}$.

Repeating the same strategy for each dimension, we obtain ε-approximate alignment around 0.

To simplify notation, we denote $X = \overline{X}$ and $V = \overline{V}$ in what follows. Using Proposition 12, the solution satisfies $(x_i(t), v_i(t)) \in [0, X+tV] \times [0, V]$ for all $i = 1, \ldots, N$ and $t \geq 0$.

The control strategy is based on the repetition of a fundamental step, until reaching approximate alignment. In Section III-A, we define the fundamental step of our strategy, and we establish precise estimates of its action on the dynamics (4). In Section III-B, we apply the fundamental step repeatedly on a 1D system, and we prove that it yields approximate alignment. In Section III-C, we prove that the use of the strategy on the sequence of dimensions $k = 1, \ldots, d$ yields approximate alignment for (4). Finally, in Section III-D, we prove that our strategy yields approximate alignment for the mean-field limit (8).

A. The fundamental step

We assume that $d = 1$. In Section III-C we will show how to deal with any dimension d.

To define the fundamental step, we only need to know three parameters[1]: the Lipschitz constant L of ψ, and the values X and V (support of the initial data).

Remark 13: It is interesting to observe that one does not need to know the precise values of the parameters X, V, L, but only upper bounds. Indeed, having the knowledge of upper bounds, the control strategy given below drives the system to approximate alignment, with a possibly larger final time T. For this reason, the strategy presented below is very robust to perturbation of the parameters and/of the dynamics.

We now define the fundamental step of the control strategy, based on three parameters l, η, W to be chosen later. We set $A = [X, X+l] \times [\eta, V]$ and

$$\tilde{u}(x, v) = \max\left(0, 1 - \frac{d((x, v), A)}{\eta}\right), \tag{13}$$

where d is the Euclidean distance. The function $\tilde{u}$ is non-negative, $\frac{1}{\eta}$-Lipschitz, and is equal to 1 in A. Note that $\tilde{u}(x, 0) = 0$, i.e., $\tilde{u}$ is zero for the target velocity. Now, we define the control function by

$$u(t, x, v) = -\tilde{u}(x + tW, v)\frac{v}{|v|}, \tag{14}$$

for $t \in [0, T]$ with

$$T = \frac{X + l}{W}. \tag{15}$$

The control function u is well defined, even for $v = 0$, since $\tilde{u}(x, 0) = 0$. The control strategy is based on the following idea: at time $t = 0$ the control u coincides with $\tilde{u}$. Then, the same function is shifted to lower values of positions with velocity W, until the value of $\tilde{u}$ at $X+l$ reaches the minimal value of the position variable, that is larger or equal than 0. Note that u satisfies the assumptions of Proposition 12.

[1] In general, one also needs to know the minimal values $\underline{X}, \underline{V}$ to perform translations.

The support $\omega(t)$ of u at time t is the rectangle

$$\omega(t) = [X - \eta - tW, X + l + \eta - tW] \times [0, V + \eta].$$

We have $|\omega(t)| = (l+2\eta)(V+\eta)$. Note that the control $\chi_\omega u$ is a Lipschitz vector field at each time, as required to ensure existence and uniqueness of solutions of (4) and (8).

Remark 14: It is possible to perform the same strategy with a more regular control, by replacing $\tilde{u}(x, v)$ by $f(\tilde{u}(x, v))$ with $f(a) = \sin^2\left(\frac{\pi}{2}x\right)$. The control is then a function in $C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ for each time.

Consider an initial data $(x_1(0), v_1(0), \ldots, x_N(0), v_N(0))$ for a N-particle system (4). Assuming that $(x_i(0), v_i(0)) \in [0, X] \times [0, V]$ for each $i = 1, \ldots, N$, we want to estimate the solution to (4) at time T defined by (15) with the control u given by (14). In particular, we want to prove that the interval of velocity variables is reduced to a set $[0, V'] \subset [0, V]$. Considering the i-th particle in (4), we have

$$\begin{aligned} \dot{v}_i &= \frac{1}{N}\sum_{j=1}^N \psi(x_j - x_i, v_j - v_i) + u(t, x_i, v_i) \\ &\leq \frac{1}{N}\sum_{j=1}^N L(v_j - v_i) + u(t, x_i, v_i) \\ &\leq L(V - v_i) + u(t, x_i, v_i), \end{aligned}$$

where we used that $\psi(x, v) \leq \psi(x, 0) + Lv = Lv$. By applying the Gronwall lemma to $v_i - V$, we get that

$$\begin{aligned} v_i(T) \leq\ & e^{-LT}\left(v_i(0) - V\right) + V \\ & + e^{-LT}\int_0^T u(t, x_i(t), v_i(t))\,dt. \end{aligned} \tag{16}$$

We now have two cases. First, assume that $v(t) \geq \eta$ for every $t \in [0, T]$. Then $u(t, x_i(t), v_i(t)) = -1$ in an interval of length l in the space variable, hence in a time interval at least $\frac{l}{W-V}$, since the relative velocity of the particle with respect to the moving control is smaller than W. Then (16) gives

$$v_i(T) \leq e^{-LT}(v_i(0) - V) + V - e^{-LT}\frac{l}{W}. \tag{17}$$

The second case is when there exists $t \in [0, T]$ such that $v_i(t) < \eta$. If such a condition is satisfied for any t, then $v_i(T) < \eta$. Otherwise, consider the maximal time t for which $v_i(t) = \eta$ and note that $v_j(s) - v_i(s) \leq V - \eta$ for $s \in [t, T]$, thus $\dot{v}_i \leq L(V - \eta)$, hence

$$v_i(T) \leq \eta + L(V - \eta)(T - t) \leq \eta(1 - LT) + LVT. \tag{18}$$

We now merge the two cases. For all agents, we have $v_i(0) - V \leq 0$, hence $v_i(T) \in [0, V']$ with

$$V' = \max\left\{V - \frac{e^{-LT}l}{W}, \eta(1 - LT) + LVT\right\}. \tag{19}$$

Note that the support may actually be smaller, but the estimate given here is sufficient to prove approximate alignment.

B. Proof of Theorem 1 in 1D

Let us prove ε-approximate alignment of a solution of (4) around 0 in 1D. The proof is constructive and the strategy follows a sequence of fundamental steps (i.e., a sequence of W, l, η) steering the system to approximate alignment.

We denote by $[0, X^0] \times [0, V^0]$ the box containing $(x_i(0), v_i(0))$ for every $i = 1, \ldots, N$. We apply the fundamental step with parameters W^0, l^0, η^0 for the time T^0 given by (15), with $W^0 > V^0$. The corresponding solution of (4) satisfies $(x_i(T^0), v_i(T^0)) \in [0, X^1] \times [0, V^1]$ for every $i = 1, \ldots, N$. We then reiterate the same strategy, with parameters W^1, l^1, η^1 for the time T^1, and iteratively for K times (to be chosen), until reaching ε-approximate alignment around 0. The objective of this section is to design an appropriate choice of W^k, l^k, η^k, with $k = 0, \ldots, K-1$. The index x^k represents here the k-th step of the algorithm.

The key observation is that, as a consequence of (19) and $\dot{X} \leq V$, it holds

$$
\begin{aligned}
V^{k+1} &= \max\left\{ V^k - \frac{e^{-LT^k} l^k}{W^k}, \right. \\
&\qquad \left. \eta^k(1 - LT^k) + LV^kT^k \right\}, \qquad (20) \\
X^{k+1} &= X^k + T^kV^k \qquad (21)
\end{aligned}
$$

Now, we define

$$
\begin{aligned}
l^k &= \frac{k+1}{k+4}\frac{c}{V^k}, \qquad (22) \\
\eta^k &= \min\left\{ \frac{1}{k+4}\frac{c}{V^k}, \frac{1}{k+4}V^k, \frac{3}{4\alpha^2}V^k \right\}, \\
W^k &= \max\left\{ \alpha L(X^k + l^k), \alpha\frac{k+1}{k+4}\frac{c}{(V^k)^2} \right\},
\end{aligned}
$$

until reaching $V^k < \varepsilon$. Here, $\alpha > 2$ is arbitrary. Note that

$$
|\omega| = (l^k + 2\eta^k)(V^k + \eta^k) \leq \frac{(k+3)(k+5)}{(k+4)^2}c < c. \qquad (23)
$$

We have $LT^k = L\frac{X^k + l^k}{W^k} \leq \frac{1}{\alpha}$. Using $e^{-LT^k} \leq 1 - LT^k + L^2\frac{(T^k)2}{2}$, we infer that

$$
\begin{aligned}
&V^k - \frac{e^{-LT^k} l^k}{W^k} - LV^kT^k - \eta^k \geq V^k - \\
&\left(1 - LT^k + L^2\frac{(T^k)^2}{2}\right)\frac{k+1}{k+4}\frac{c}{V^k}\frac{(k+4)(V^k)^2}{\alpha(k+1)c} + \\
&-LV^kT^k - \frac{3}{4\alpha^2}V^k \geq \\
&V^k\left(1 - \frac{1}{\alpha} + \frac{LT^k}{\alpha} - L^2\frac{(T^k)^2}{4} - LT^k - \frac{3}{4\alpha^2}\right) \geq \\
&V^k\left(\left(1 - \frac{1}{\alpha}\right)^2 - \frac{1}{\alpha^2}\right) = \left(1 - \frac{2}{\alpha}\right)V^k. \qquad (24)
\end{aligned}
$$

Since $\alpha > 2$, the maximum in (20) is always reached by the first term, and

$$
V^{k+1} = V^k - \frac{e^{-LT^k} l^k}{W^k}. \qquad (25)
$$

The sequence V^k is then decreasing and thus has a limit V^*. If $V^* = 0$, then we have approximate alignment for any precision $\varepsilon > 0$.

We study this limit in two cases. First, assume that $W^k = \alpha\frac{k+1}{k+4}\frac{c}{(V^k)^2}$ for an infinite number of indices k_j. For such indices, using $e^{-LT^{k_j}} \geq 1 - LT^{k_j}$, similarly to estimates for (24), from (25) we get

$$
V^{k_j+1} \leq V^{k_j}\left(1 - \frac{1}{\alpha} + \frac{1}{\alpha^2}\right) < V^{k_j}\left(1 - \frac{1}{2\alpha}\right).
$$

Since for the other indexes the sequence is decreasing, we have $V^{k_{j+1}} \leq V^{k_j}\left(1 - \frac{1}{2\alpha}\right)$, hence $\lim_{j\to\infty} V^{k_j} = 0$, and thus $\lim_{k\to\infty} V^k = 0$. Hence we have approximate alignment for any precision $\varepsilon > 0$.

In the second case, we assume to have a finite number of indices for which $W^k = \alpha\frac{k+1}{k+4}\frac{c}{(V^k)^2}$. Then, there exists k_0 such that $W^k = \alpha L(X^k + l^k)$ for every $k \geq k_0$. We also have $T^k = \frac{1}{\alpha L}$.

Let us now establish, by contradiction, that $\lim_{k\to\infty} V^k = 0$. Assume that $\lim_{k\to\infty} V^k = V^* > 0$. As a first consequence, we have $\lim_{k\to\infty} l^k = l^* > 0$. Let $\delta < V^*$ and note that there exists $k_1 \geq k_0$ such that

$$
X^{k+1} = X^k + T^kV^k \in \left(X^k + \frac{V^* - \delta}{\alpha L}, X^k + \frac{V^* + \delta}{\alpha L}\right), \qquad (26)
$$

for every $k \geq k_1$. In particular $\lim_{k\to\infty} X^k = +\infty$. As a consequence, $\lim_{k\to\infty} W^k = +\infty$, hence from (25) there exists $k_2 \geq k_1$ such that

$$
V^{k+1} \leq V^k - \frac{r_1 l^*}{W^k} \leq V^k - \frac{r_2}{X^k}, \qquad (27)
$$

for every $k \geq k_2$, with constants $r_1, r_2 > 0$ computed with limits. Note that $X^{k+k_2} \leq X^{k_2} + (V^* + \delta)k$, due to (26). Then, from (27), we have

$$
V^{k+k_2} \leq V^{k_2} - \sum_{j=0}^{k} \frac{r_2}{X^{k_2} + (V^* + \delta)j}.
$$

But the right-hand side tends to $-\infty$, since

$$
\sum_{j=0}^{\infty} \frac{r_2}{X^{k_2} + (V^* + \delta)j} \geq \sum_{j=J}^{\infty} \frac{r_3}{j} = +\infty,
$$

with $J \geq \frac{X^{k_2}}{V^* + \delta}$ and $r_3 = \frac{r_2}{V^* + \delta}$. As a consequence, we have $\lim_k V^k = -\infty < V^*$, which is a contradiction.

Summing up, the strategy yields $\lim_{k\to\infty} V^k = 0$. Hence, for every $\varepsilon > 0$, there exists K such that $V^K < \varepsilon$, and then the algorithm terminates. Moreover, since $T^k \leq \frac{1}{\alpha L}$, the total time of the algorithm is finite, bounded by $\frac{K}{\alpha L}$, and the space variable satisfies $X^K \leq X^0 + V^0\frac{K}{\alpha L}$.

Finally, note that the control is a Lipschitz function for each time, with Lipschitz constant bounded by $\max\left\{ \frac{1}{\eta^k} \mid k = 0, \ldots, K-1 \right\}$.

C. Proof of Theorem 1 in dimension d

We still focus on the system (4), but now in dimension d. Without loss of generality, we assume that $(x_i(0), v_i(0)) \in \mathcal{X}(0) \times \mathcal{V}$ for every $i = 1, \ldots, N$, with $\mathcal{X}(0), \mathcal{V}$ defined by (9)-(10). In what follows, we use the double index $x^{j,k}$ to denote the j-th dimension at the k-th step of the algorithm.

To reach ε-approximate alignment around 0, we first achieve $\frac{\varepsilon}{\sqrt{d}}$-approximate alignment in the first coordinate. We define the sequence $(l^{1,k}, \eta^{1,k}, W^{1,k})$ by

$$\begin{aligned} l^{1,k} &= \frac{k+1}{k+4}\frac{c^{1,k}}{V^{1,k}}, \qquad (28)\\ \eta^{1,k} &= \min\left\{\frac{1}{k+4}\frac{c}{V^{1,k}}, \frac{1}{k+4}V^{1,k}, \frac{3}{4\alpha^2}V^{1,k}\right\},\\ W^{1,k} &= \max\left\{\alpha L(X^{1,k}+l^{1,k})^d, \alpha\tfrac{k+1}{k+4}\tfrac{c}{(V^{1,k})^2}\right\}, \end{aligned}$$

where $c^{1,k} = \frac{c}{(X^{2,k}+V^{2,k})\ldots(X^{d,k}+V^{d,k})(V^{2,k}+1)\ldots(V^{d,k}+1)}$. We define the sets

$$\begin{aligned} &A^{1,k} = [X^{1,k}, X^{1,k}+l^{1,k}] \times [\eta^{1,k}, V^{1,k}],\\ &B^{1,k} = \mathbb{R} \times \left[0, X^{2,k}+\frac{1}{2}V^{2.k}\right] \times \left[0, X^{3,k}+\frac{1}{2}V^{3,k}\right]\\ &\qquad \times \left[X^{d,k}+\frac{1}{2}V^{d,k}\right] \times \mathbb{R} \times \left[V^{2,k}+1\right] \ldots \left[V^{d,k}+1\right], \end{aligned}$$

and the functions

$$\begin{aligned} \tilde{u}^{1,k}(x,v) &= \max\left(0, 1-\frac{d((x^1,v^1), A^{1,k})}{\eta^{1,k}}\right), \qquad (29)\\ \bar{u}^{1,k}(x,v) &= \max\left(0, 1-\frac{d((x,v), B^{1,k})}{\min\left\{\frac{1}{2}, \frac{1}{V^{2,k}}, \ldots, \frac{1}{V^{d,k}}\right\}}\right), \end{aligned}$$

where (x^1, v^1) are the coordinates in the first dimension of (x, v). We define the control u by

$$u^{1,k}(t,x,v) = -\tilde{u}^{1,k}(x+tW^{1,k}e^1, v)\bar{u}^{1,k}(x,v)\frac{v^1}{|v^1|}, \qquad (30)$$

where e^1 is the unit vector in the first dimension. The control set ω is the support of $u^{1,k}$. Similarly to (23), we have

$$\begin{aligned} |\omega| \quad \leq \quad &(l^{1,k}+2\eta^{1,k})(X^{2,k}+V^{2,k})\ldots(X^{d,k}+V^{d,k})\\ &\times(V^{1,k}+\eta^{1,k})(V^{2,k}+1)\ldots(V^{d,k}+1) < c. \end{aligned}$$

Note that, at each step, the support of any uncontrolled variable passes from $X^{j,k}$ to $X^{j,k+1} = X^{j,k} + \frac{1}{2}V^{j,k}$ as a consequence of the fact that, for each step of the algorithm, the final time $T^{1,k}$ is bounded by $\frac{1}{\alpha} < \frac{1}{2}$. Hence, if $(x_i(0), v_i(0)) \in \mathcal{X}(0)\times\mathcal{V}$, then $(x_i(T^{1,k}), v_i(T^{1,k})) \in B^{1,k}$, hence the value of the control function u depends only on the coordinate (x^1, v^1).

The main difference with respect to the 1D case is that, here, the limit of $l^{1,k}$ is $\frac{c}{V^{1,k}k^{d-1}}$, whereas in the previous section the limit of l^k was $\frac{c}{V^k}$.

We now prove that a finite number of fundamental steps ensures approximate alignment in the first variable. The idea is to repeat the proof of the 1D case, by highlighting the differences due to the change in $l^{1,k}, W^{1,k}$. First, it is easy to prove that estimates equivalent to (20)-(26) hold for $X^{1,k}, V^{1,k}, T^{1,k}$ too, since they hold for any positive value of $c^{1,k}$. This also implies that (24) holds, hence (25) can be rewritten as

$$V^{1,k+1} = V^{1,k} - \frac{e^{-LT^{1,k}}l^{1,k}}{W^{1,k}}. \qquad (31)$$

As in the 1D case, the sequence $V^{1,k}$ is decreasing and has a limit. We study this limit in two cases. First, assume that $W^k = \alpha\frac{k+1}{k+4}\frac{c}{(V^k)^2}$ for an infinite number of indices k_j. This implies that $V^{1,k_{j+1}} \leq V^{1,k_j}\left(1-\frac{1}{2\alpha}\right)$, hence $\lim_{k\to\infty} V^{1,k} = 0$.

In the second case, we assume to have a finite number of indices for which $W^{1,k} = \alpha\frac{k+1}{k+4}\frac{c^{1,k}}{(V^{1,k})^2}$. Then, there exists a k_0 such that $W^{1,k} = \alpha L(X^{1,k}+l^{1,k})^d$ for every $k \geq k_0$. We also have $T^{1,k} = \frac{1}{\alpha L}$.

Let us establish, by contradiction, that $\lim_{k\to\infty} V^k = 0$. Denoting by $V^* > 0$ the limit, we have $X^{1,k} \geq r_1 k$ and $W^{1,k} \geq r_2(X^{1,k})^d$ for a sufficiently large k and constants $r_1, r_2 > 0$. Since $X^{j,k+1} = X^{j,k} + \frac{1}{\alpha}V^{j,k}$ and $V^{j,k+1} = V^{j,k}$ for $j = 2, \ldots, d$, we have

$$\lim_{k\to\infty} k^{d-1}c^{1,k} = c^* > 0, \qquad \lim_{k\to\infty} k^{d-1}l^{1,k} = \frac{c^*}{V^*} > 0.$$

It follows that

$$V^{1,k+1} \leq V^{1,k} - \frac{r_3 l^{1,k}}{W^{1,k}} = V^{1,k} - \frac{r_4}{X^{1,k}},$$

for some constants r_3, r_4. This implies convergence of $V^{1,k}$ to $-\infty$ similarly to the 1D case, raising a contradiction.

As a consequence, after a finite number K^1 of steps, we have $V^{1,K^1} < \frac{\varepsilon}{\sqrt{d}}$, and the total time to reach approximate alignment is $T^1 = \sum_{k=0}^{K-1} T^{1,k} \leq \frac{K^1}{\alpha L}$. Then, the solution of (4) at time T^1 satisfies, for every $i = 1, \ldots, N$,

$$\begin{aligned} x_i(T^1) \quad &\in \quad [0, X^1+V^1T^1] \times [0, X^2+V^2T^1]\\ &\qquad \times \ldots \times [0, X^d+V^dT^1],\\ v_i(T^1) \quad &\in \quad \left[0, \frac{\varepsilon}{\sqrt{d}}\right) \times [0, V^2] \times \ldots \times [0, V^d]. \end{aligned}$$

Now, we achieve $\frac{\varepsilon}{\sqrt{d}}$-approximate alignment in the second coordinate by defining $l^{2,k}, \eta^{2,k}, W^{2,k}, A^{2,k}, u^{2,k}$ similarly to the previous case. This control strategy yields approximate alignment in K^2 steps, with finite total time T^2. The key observation is, due to invariance of the velocities, this strategy does not expand the velocity set in the first variable, i.e., $v_i^1(T^1+T^2) \in \left[0, \frac{\varepsilon}{\sqrt{d}}\right)$ for every $i = 1, \ldots, N$.

Repeating the same process for all other dimensions, we have a total final time $T^* = T^1 + T^2 + \ldots + T^d$, that is finite, and the solution of (4) at time T^* satisfies

$$\begin{aligned} x_i(T^*) \quad &\in \quad [0, X^1+V^1T^*] \times [0, X^2+V^2T^*]\\ &\qquad \times \ldots \times [0, X^d+V^dT^*],\\ v_i(T^*) \quad &\in \quad \left[0, \frac{\varepsilon}{\sqrt{d}}\right)^d \end{aligned}$$

for every $i = 1, \ldots, N$. In particular, $|v_i(T^*)| < \varepsilon$, as required.

Finally, note that the control is a Lipschitz function for each time, with Lipschitz constant bounded by

$$\begin{aligned} L' &= \max\left\{2, V^{i,k} \;\middle|\; \begin{array}{c} i=1,\ldots,d, \\ k=0,\ldots,K^i-1 \end{array}\right\} + \\ &\quad \max\left\{\frac{1}{\eta^{i,k}} \;\middle|\; \begin{array}{c} i=1,\ldots,d, \\ k=0,\ldots,K^i-1 \end{array}\right\}. \end{aligned} \tag{32}$$

Remark 15: The control that we have designed is piecewise smooth with respect to t, and for each t is Lipschitz with respect to (x, v). This is exactly what is required to pass to the mean-field limit, by letting the number N of particles go to infinity. Indeed, for the mean-field limit, we need regular flows of the ODE, and then L^1 regularity in time, and Lipschitz in (x, v), suffices to have Filippov solutions. As said in Remark 14, we could design a control that is smooth in (x, v). We could also have smoothness in time, by designing a control moving fast left and right (W positive and negative), but then this would be at the price of having a larger time to reach approximate alignment.

D. *Proof of Theorem 1 for the mean-field model (8)*

In this section, we prove that the strategy yielding approximate alignment for the finite-dimensional system (4) also yields approximate alignment for the mean-field model (8).

The crucial observation is that the control strategy does not focus on a specific i-th agent, but focuses on agents that are in a given subset of the space. In this sense, the limit of the control strategy "passes to the limit" with respect to the mean-field limit from (4) to (8). Another key property is that the definition of the control strategy is based on the size of the support of the solution $X^{i,k}, V^{i,k}$, that are quantities that are well defined for the finite-dimensional and the mean-field dynamics.

Consider an initial measure $\mu(0)$ and a sequence of empirical measures $\mu_N(0) = \frac{1}{N}\sum_{i=1}^N \delta_{(x_{i,N}(0), v_{i,N}(0))}$ satisfying $\mathrm{supp}(\mu_N(0)) \subset \mathrm{supp}(\mu_*(0))$ for every N and $\lim_{N\to\infty} W_1(\mu_N(0), \mu(0)) = 0$. Such a sequence is built for instance by space discretization. Let X^i, V^i be such that

$$\begin{aligned} \mathrm{supp}(\mu(0)) \quad \subset \quad & [0, X^1] \times [0, X^2] \times \ldots \times [0, X^d] \\ & \times [0, V^1] \times [0, V^2] \times \ldots \times [0, V^d]. \end{aligned}$$

For a fixed ε, define the strategy to achieve ε-approximate alignment of a finite-dimensional system (4) with support in such a set. Note that the strategy does not depend on N, and that all parameters l^k, η^k, W^k are completely determined by the size of the support at step k, that is in turn completely determined by the size of the support at step $k-1$ thanks to (20)-(21). As a consequence, the strategy is completely determined by parameters $X^1, \ldots, X^d, V^1, \ldots, V^d$ for the initial support. Moreover, the final time T^* is given, and the control is a Lipschitz function for all times, with Lipschitz constant L' given by (32).

We now study the solution of (8) on the time interval $[0, T^*]$ with the control $\chi_\omega u$ given by the strategy, hence not depending on μ or on μ_N. For any initial data $\mu(0), \mu_N(0)$, the solution $\mu(t), \mu_N(t)$ exists for $t \in [0, T^*]$ and is unique, and supports of $\mu(t), \mu_N(t)$ are contained in

$$\begin{aligned} & [0, X^1 + V^1 T^*] \times [0, X^2 + V^2 T^*] \times \ldots \\ & \times [0, X^d + V^d T^*] \times [0, V^1] \times [0, V^2] \times \ldots \times [0, V^d]. \end{aligned}$$

These properties follow from Section II-A. Moreover, continuous dependence for solutions to (8) implies that $\lim_{N\to\infty} W_1(\mu(T^*), \mu_N(T^*)) = 0$, and therefore $\mu_N(T^*) \rightharpoonup \mu_N(T^*)$.

Now, note that the solution of (8) with initial data $\mu_N(0) = \frac{1}{N}\sum_{i=1}^N \delta_{(x_{i,N}(0), v_{i,N}(0))}$ is the empirical measure $\mu_N(t) = \frac{1}{N}\sum_{i=1}^N \delta_{(x_{i,N}(t), v_{i,N}(t))}$, where $(x_{i,N}(t), v_{i,N}(t))$ is the solution of the finite-dimensional system (4) with N particles and control $\chi_\omega u$. The results of Section III-C imply that $\mathrm{supp}(\mu_N(T^*)) \subset A^*$ with

$$\begin{aligned} A^* \quad = \quad & [0, X^1 + V^1 T^*] \times [0, X^2 + V^2 T^*] \\ & \times \ldots \times [0, X^d + V^d T^*] \times \left[0, \frac{\varepsilon}{\sqrt{d}}\right)^d. \end{aligned}$$

Since A^* does not depend on N, Proposition 5 implies that $\mathrm{supp}(\mu(T^*)) \subset A^*$, i.e., $\mu(T^*)$ is ε-approximately aligned around 0. This proves Theorem 1.

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