Uniform regularity in the random space and spectral accuracy of the stochastic Galerkin method for a kinetic-fluid two-phase flow model with random initial inputs in the light particle regime*

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Abstract

We consider a kinetic-fluid model with random initial inputs which describes disperse two-phase flows. In the light particle regime, using energy estimates, we prove the uniform regularity in the random space of the model for random initial data near the global equilibrium in some suitable Sobolev spaces, with the randomness in the initial particle distribution and fluid velocity. By hypocoercivity arguments, we prove that the energy decays exponentially in time, which means that the long time behavior of the solution is insensitive to such randomness in the initial data. Then we consider the generalized polynomial chaos stochastic Galerkin method (gPC-sG) for the same model. For initial data near the global equilibrium and smooth enough in the physical and random spaces, we prove that the gPC-sG method has spectral accuracy, uniformly in time and the Knudsen number, and the error decays exponentially in time.

1 Introduction

In this paper we consider a kinetic-fluid model for disperse two-phase flows, known as the Navier-Stokes-Vlasov-Fokker-Planck system, first proposed in [10, 11]. Similar two-phase flow models appear in combustion theory [8, 5, 29], the dynamic of sprays [27, 15, 14] and granular flow [1, 7], to name a few. The model we consider describes a mixture of two types of material, called the primary phase and the secondary phase. They are assumed to satisfy the following physical assumptions:

1. The primary phase is liquid or dilute gas, and therefore modeled by the incompressible Navier-Stokes equations.

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2. The secondary phase is small particles (or droplets, bubbles), scattered in the fluid, and it is modeled by a kinetic equation.

3. The interaction between the two phases is assumed to be the Stokes drag force, i.e., a particle is subject to a force proportional to the relative velocity between it and the fluid.

4. The particles are assumed to be subject to the Brownian motions.

There are two scalings that are physically important: one is the light particle regime [10], which assumes:

1. The velocity of the fluid is small compared to the typical molecular velocity of the particles.

2. The particles are light, and thus its effect on the fluid is small.

3. The relaxation time is much smaller than the typical time scale.

Another one is the fine particle regime [11], which assumes:

1. The particle size is small compared to the typical length scale.

2. The density of the fluid and particles are of the same order.

3. The relaxation time is much smaller than the typical time scale.

In this paper we focus on the light particle regime. For simplicity the space is taken as $\mathbb{T}^3 = [-\pi, \pi]^3$ with periodic boundary condition. The equations for the model are given by

$$\begin{cases} u_t + u \cdot \nabla_x u + \nabla_x p - \Delta_x u = \dfrac{1}{\epsilon} \displaystyle\int (v - \epsilon u) F \,\mathrm{d}v, \\ \nabla_x \cdot u = 0, \\ F_t + \dfrac{1}{\epsilon} v \cdot \nabla_x F = \dfrac{1}{\epsilon^2} \nabla_v \cdot (\nabla_v F + (v - \epsilon u) F), \end{cases} \tag{1.1}$$

with initial data

$$u|_{t=0} = u_0, \quad \nabla_x \cdot u_0 = 0, \quad F|_{t=0} = F_0, \tag{1.2}$$

where $t \in \mathbb{R}^+$ is the time variable, $x \in \mathbb{T}^3$ is the space variable, and $v \in \mathbb{R}^3$ is the velocity variable. $u = u(t,x)$ is the velocity field of the fluid, and $F = F(t,x,v)$ is the distribution function of the particles. ϵ is the Knudsen number, which satisfies $0 < \epsilon \le 1$. $\epsilon = O(1)$ corresponds to the kinetic regime, while $\epsilon \to 0$ corresponds to the fluid regime.

This system satisfies the following conservation properties:

$$\begin{aligned} &\text{Mass conservation: } \frac{\mathrm{d}}{\mathrm{d}t} \int\int F \,\mathrm{d}v \,\mathrm{d}x = 0, \\ &\text{Momentum conservation: } \frac{\mathrm{d}}{\mathrm{d}t} \left(\int u \,\mathrm{d}x + \epsilon \int\int vF \,\mathrm{d}v \,\mathrm{d}x \right) = 0, \\ &\text{Energy/Entropy dissipation: } \frac{\mathrm{d}}{\mathrm{d}t} \left(\int \frac{|u|^2}{2} \,\mathrm{d}x + \int\int (F \ln F + \frac{|v|^2}{2} F) \,\mathrm{d}v \,\mathrm{d}x \right) \\ &\quad + \frac{1}{\epsilon^2} \int\int \frac{|(\epsilon u - v)F - \nabla_v F|^2}{F} \,\mathrm{d}v \,\mathrm{d}x + \int |\nabla_x u|^2 \,\mathrm{d}x = 0. \end{aligned} \tag{1.3}$$

As $\epsilon \to 0$, it is shown in [10] that (1.1) has a hydrodynamic limit

$$\begin{cases} u_t + u \cdot \nabla_x u + \nabla_x p - \Delta_x u = 0, \\ \nabla_x \cdot u = 0, \\ \partial_t \rho + \nabla_x \cdot (u\rho - \nabla_x \rho) = 0, \end{cases} \tag{1.4}$$

with $\rho(x) = \int F(x,v)\,\mathrm{d}v$ being the particle density, which is self-consistent Navier-Stokes equations for u, and a convection-diffusion equation for ρ with drift velocity u.

Goudon et al. [9] proved the first existence result of (1.1), in the case of kinetic regime ($\epsilon = O(1)$) and initial data near the global equilibrium, which means that F is close enough to the global Maxwellian

$$\mu(v) = \frac{1}{(2\pi)^{3/2}|\mathbb{T}^3|} e^{-|v|^2/2}. \tag{1.5}$$

and u is close to 0, in some suitable Sobolev spaces. In fact their method also works for small ϵ. They first write

$$F = \mu + \sqrt{\mu} f. \tag{1.6}$$

Then (1.1) becomes the following system for (u, f):

$$\begin{cases} u_t + u \cdot \nabla_x u + \nabla_x p - \Delta_x u + u + \displaystyle\int \sqrt{\mu} u f \,\mathrm{d}v - \frac{1}{\epsilon} \int v \sqrt{\mu} f \,\mathrm{d}v = 0, \\ \nabla_x \cdot u = 0, \\ f_t + \dfrac{1}{\epsilon} v \cdot \nabla_x f + \dfrac{1}{\epsilon}(\nabla_v - \dfrac{v}{2})(uf) - \dfrac{1}{\epsilon} u \cdot v \sqrt{\mu} = \dfrac{1}{\epsilon^2}(\dfrac{-|v|^2}{4} + \dfrac{3}{2} + \Delta_v) f, \end{cases} \tag{1.7}$$

with initial data

$$u|_{t=0} = u_0, \quad f|_{t=0} = f_0. \tag{1.8}$$

They assume that (u_0, f_0), the perturbation of initial data, satisfies the conditions

$$\int u_0 \,\mathrm{d}x + \int\int v \sqrt{\mu} f_0 \,\mathrm{d}v\,\mathrm{d}x = 0, \quad \nabla_x \cdot u_0 = 0, \tag{1.9}$$

$$\int\int \sqrt{\mu} f_0 \,\mathrm{d}v\,\mathrm{d}x = 0, \tag{1.10}$$

which mean that the perturbation does not affect the total momentum and mass, and the perturbation of the fluid velocity is divergence-free. Then, combining with a relation for the mean fluid velocity

$$\bar{u}(t) = \frac{1}{|\mathbb{T}^3|} \int u(t,x)\,\mathrm{d}x, \tag{1.11}$$

$$\bar{u}_t + 2\bar{u} + \frac{1}{|\mathbb{T}^3|} \int\int \sqrt{\mu}(uf)\,\mathrm{d}v\,\mathrm{d}x = 0, \tag{1.12}$$

which is a consequence of (1.9), using energy estimates, they proved the decay of an energy functional, defined as the summation of some suitable Sobolev norms, under the assumption that it is small enough initially. Then, by using hypocoercivity arguments, they proved that the L^2 norms of u and f decay exponentially in time, under some smoothness assumptions.

On the numerical aspect, an Asymptotic-Preserving (AP) scheme was developed by Goudon et al. [12] for the model with the fine particle regime. The AP property, first introduced by Jin [16] for time-dependent kinetic problems, means that a numerical scheme for a kinetic model, as the Knudsen number ϵ goes to zero, automatically becomes a numerical scheme for the hydrodynamic limit of the kinetic model, with a numerical stability independent of ϵ. The AP property enables one to capture the hydrodynamic limit without resolving the small Knudsen number. Simply speaking, the AP scheme for this model uses a combination of the projection method for the Navier-Stokes equations and an implicit treatment of the stiff Fokker-Planck operator.

Most of the works on kinetic-fluid two-phase flow models are deterministic. However, there are many sources of uncertainties in these models. For example, the initial data and boundary data usually come from experiments, and thus have measurement error. Uncertainty could also arise from the modeling of drag forces, particle diffusions, etc. It is important to quantify these uncertainties, because such quantification can help us understand how the uncertainties affect the solution, and therefore make reliable predictions.

For simplicity, for the model (1.1) we only consider the uncertainty from initial data. To model the uncertainty, we use the same equations, but let the functions $u = u(t, x, z)$ and $F = F(t, x, v, z)$ depend on a random variable z, which lives in the random space I_z with probability distribution $\pi(z)\,\mathrm{d}z$. Then the uncertainty from initial data is described by letting the initial data u_0 and F_0 depend on z.

We summarize some popular numerical methods for uncertainty quantification (UQ) [6, 13, 24, 30, 31]: the first one is Monte-Carlo (MC) methods [25], which take random samples in I_z, solve the deterministic problem on these samples, and then get the statistical moments by taking the average on these samples. MC methods are half-order accurate for any dimensional random spaces, and thus they are not accurate enough for low dimensional random spaces, but very efficient for high-dimensional random spaces. The second method is stochastic collocation (sC) methods [2, 4, 26, 32], which take sample points on a well-designed grid (quadrature points, sparse grids, or by some optimization procedure), compute the deterministic solutions on the samples, and then reconstruct the solution in the whole random domain by some interpolation rules. SC methods can achieve good accuracy in low dimensional random spaces, but the efficiency drops as the dimension becomes high. The third method is stochastic Galerkin (sG) methods [4, 3, 33], which takes an orthonormal basis in the random domain, approximate the functions by a truncated Fourier series, and then obtain a deterministic system of equations on the Fourier coefficients via the Galerkin projection. SG methods are as accurate as sC methods for low dimensional random spaces, and behave better than sC for moderately high dimensional random spaces if one wants to achieve high accuracy [4].

For sG methods for kinetic equations with a hydrodynamic limit, it is important to have a property called 'stochastic asymptotic-preserving' (s-AP), first proposed by Jin et al. [21]. The s-AP property means that as the small parameter ϵ goes to zero, the sG method for the kinetic equation automatically becomes an sG method for the limiting hydrodynamic system. Similar to the AP property, the s-AP property enables one to choose all numerical parameters, including the number of basis functions K in polynomial chaos approximations, independent of ϵ. In [19] the authors proposed an s-AP method for the model with the fine particle regime. We followed the idea of the AP scheme in [12], and overcame the difficulty of the implicit treatment of the vectorized Fokker-Planck operator by proving a structure theorem of this operator.

In order to analyze the accuracy of the sC and sG methods, it is very important to analyze the regularity of the exact solution in the random space. In fact, in order to achieve a high accuracy order for the interpolations in sC, and the truncated series approximations in sG, one usually needs such regularity. For the sG methods, it is not straightforward to prove accuracy from the z-regularity, due to the Galerkin projection error. Instead, one has to derive the evolution equations for the error, and then conduct estimates based on the z-regularity of the exact solution. Recently there have been several attempts to prove the uniform-in-ϵ random space regularity for kinetic equations, including Jin et al. [17] for linear transport equations, Jin-Zhu [20] for the Vlasov-Poisson-Fokker-Planck equation, Jin-Liu [18] and Liu [23] for the linear semiconductor Boltzmann equation, and Li-Wang [22] for general linear kinetic equations that conserve mass. [17, 18, 23] also proves the spectral accuracy for the sG method.

In this paper, we first analyze the z-regularity of (1.7) for random initial data near the global equilibrium in some suitable Sobolev spaces (with derivatives with respect to x and z). We use energy estimates and hypocoercivity arguments similar to [9] on the z-derivatives of u and f. Our result implies that for near equilibrium initial data with regular dependence on x and z, the solution depends regularly on z for all time, and is insensitive to random perturbations on the initial data for large time. Then for the sG method, we consider the most popular choice of basis functions, the generalized polynomial chaos (gPC) [33], i.e., the orthonormal polynomials with respect to $\pi(z)\,\mathrm{d}z$. We write the equations for the gPC coefficients and do energy estimates, in which we manage to make this estimate independent of K, the number of basis functions. This difficulty will be explained in detail in the next paragraph. Finally we write the equations for the error of the gPC-sG method and do energy and hypocoercivity estimates. Our result implies that if the random initial data (u_0, f_0) is small enough in some suitable Sobolev spaces, then the gPC-sG method has spectral accuracy, uniformly in time and ϵ, and captures the exponential decay in time of the exact solution. An important feature of our results is that all the constants involved are *independent of* ϵ.

As mentioned in the previous paragraph, the biggest difficulty is that a naive energy estimates for the gPC coefficients require a small initial data condition depending on K, the number of basis functions, since the nonlinear terms in (1.7) produce a large number (K^3) of terms in the equations of the gPC coefficients. But it is desirable to have a small initial data condition independent of the numerical parameter K, which means that the accuracy results are true for this set of initial data, for all K. To overcome this difficulty, we introduce a weighted sum of the Sobolev norm of the gPC coefficients (Lemma 5.1), which enables us to combine some of the terms together as part of a convergent series, and control the nonlinear terms with an estimate independent of K.

This paper is organized as follows: in Section 2, we introduce some notations and state the main results; in Section 3 we prove the energy estimates for the z-derivatives of u and f; in Section 4 we use hypocoercivity arguments to prove the exponential decay of these derivatives; in Section 5 we prove the spectral accuracy of the sG method; in Section 6 we conclude the paper.

2 Notations and statements of main results

We follow the notation in [9].

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be a multi-index. Then define

$$\partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}. \tag{2.1}$$

We first introduce the norms in the (x, v) space. Denote the L^2 inner products on $\mathbb{R}^3$ and $\mathbb{T}^3 \times \mathbb{R}^3$ as

$$\langle f, g\rangle = \int fg \,\mathrm{d}x, \quad \text{or} \quad \langle f, g\rangle = \int\int fg \,\mathrm{d}v \,\mathrm{d}x, \tag{2.2}$$

and $\|\cdot\|_{L^2}$ the corresponding L^2 norms, and $\|\cdot\|_{H^s}$ the corresponding Sobolev norms (with respect to all possible x, v derivatives). Also define the partial Sobolev norm in the x direction

$$|f|_s^2 = \int \sum_{|\alpha|\le s} |\partial^\alpha f|^2 \,\mathrm{d}x \,\mathrm{d}v. \tag{2.3}$$

For simplicity we assume that the random variable z lives in a one-dimensional random space I_z. For functions $u = u(x, z)$, $f = f(x, v, z)$, denote the z-derivative of a function f of order γ by

$$f^\gamma = \partial_z^\gamma f. \tag{2.4}$$

Define the sum of Sobolev norms of the z derivatives by

$$\begin{aligned} \|u\|_{H^{s,r}}^2 &= \sum_{|\gamma|\le r} \|u^\gamma\|_{H^s}^2, \\ \|\bar{u}\|_r^2 &= \sum_{|\gamma|\le r} |\bar{u}^\gamma|^2, \\ |f|_{s,r}^2 &= \sum_{|\gamma|\le r} |f^\gamma|_s^2. \end{aligned} \tag{2.5}$$

Note that these norm sums are functions in z.

Then we introduce the inner products related to the hypocoercivity arguments. Define

$$\mathcal{K} = \nabla_v + \frac{v}{2}, \quad \mathcal{P} = v\cdot\nabla_x, \quad \mathcal{S}_i = [\mathcal{K}_i, \mathcal{P}] = \mathcal{K}_i\mathcal{P} - \mathcal{P}\mathcal{K}_i = \partial_{x_i}, \quad \mathcal{K}^* = -\nabla_v + \frac{v}{2}, \tag{2.6}$$

where $\mathcal{K}^*$ is the adjoint operator of $\mathcal{K}$.

Define

$$((f, g))_{s,r} = \sum_{|\gamma|\le r} \sum_{|\alpha|\le s} ((\partial^\alpha f^\gamma, \partial^\alpha g^\gamma)), \tag{2.7}$$

where

$$((f, g)) = 2\langle \mathcal{K}f, \mathcal{K}g\rangle + \epsilon\langle \mathcal{K}f, \mathcal{S}g\rangle + \epsilon\langle \mathcal{S}f, \mathcal{K}g\rangle + \epsilon^2\langle \mathcal{S}f, \mathcal{S}g\rangle. \tag{2.8}$$

We also define

$$[[f, g]] = \langle \mathcal{K}f, \mathcal{K}g\rangle + \epsilon^2\langle \mathcal{S}f, \mathcal{S}g\rangle + \langle \mathcal{K}^2 f, \mathcal{K}^2 g\rangle + \epsilon^2\langle \mathcal{K}\mathcal{S}f, \mathcal{K}\mathcal{S}g\rangle, \tag{2.9}$$

and similarly define $[[f, g]]_{s,r}$.

Finally we introduce the inner product in the (x, v, z) space:

$$\langle f, g\rangle_z = \int \langle f, g\rangle \pi(z) \,\mathrm{d}z, \tag{2.10}$$

and similarly define $((f,g))_z$, $((f,g))_{s,r,z}$, $[[f,g]]_z$, $[[f,g]]_{s,r,z}$ as the corresponding inner products integrated in z. We also define the norms in the (x,v,z) space:

$$
\begin{aligned}
\|u\|^2_{H^{s,r}_z} &= \int \|u\|^2_{H^{s,r}} \pi(z)\,\mathrm{d}z, \\
\|\bar{u}\|^2_{r,z} &= \int \|\bar{u}\|^2_{r} \pi(z)\,\mathrm{d}z, \\
|f|^2_{s,r,z} &= \int |f|^2_{s,r} \pi(z)\,\mathrm{d}z, \\
\|u\|_{W^{s,\infty}_z} &= \max_{|\alpha|\le s} \|\partial^\alpha u\|_{L^\infty_{x,z}}, \\
|f|_{W^{s,\infty}_z} &= \max_{|\alpha|\le s} \|\partial^\alpha f\|_{L^\infty_{x,z}(L^2_v)}.
\end{aligned}
\tag{2.11}
$$

Now we focus on the system (1.7) with the random variable z. In all of our results, the constants involved are *independent of* ϵ.

Our first main result is the following energy estimate assuming near equilibrium initial data:

Theorem 2.1. *Assume (u,f) solves (1.7) with initial data verifying (1.9). Fix a point z. Define the energy*

$$
E(t) = E_{s,r}(t) = \|u\|^2_{H^{s,r}} + |f|^2_{s,r} + \|\bar{u}\|^2_r,
\tag{2.12}
$$

with integers $s \ge 2$ and $r \ge 0$. Then there exists a constant $c_1 = c_1(s,r) > 0$, such that $E(0) \le c_1$ implies that $E(t)$ is non-increasing in t.

This theorem is proved by an energy estimate on $\partial^\alpha f^\gamma$. This theorem means that for initial data near the global equilibrium, in the sense that $E(0)$ is small, the solution depends regularly in z for all time and all ϵ, and the z-derivatives are bounded uniformly in t and ϵ.

Next, by a standard hypocoercivity argument, we strengthen the above theorem into the following one:

Theorem 2.2. *Assume (u,f) solves (1.7) with initial data verifying (1.9) and (1.10). There exists a constant $c_1'(s,r)$ such that, if we assume $s \ge 0$, $E_{s+3,r}(0) \le c_1'(s,r)$, and that $C^h_{s,r} = ((f,f))_{s,r}|_{t=0}$ (defined by (2.7)) is finite, then there exists a constant $\lambda > 0$ such that*

$$
E_{s,r}(t) \le C(E_{s,r}(0) + C^h_{s,r})e^{-\lambda t},
\tag{2.13}
$$

where $C = C(s,r)$.

This theorem implies that as long as the random perturbation (u_0, f_0) on the initial data is small in suitable Sobolev spaces and has vanishing total mass and momentum, the long-time behavior of the solution is not sensitive to the random initial data. The smallness condition is independent of ϵ.

We then introduce the gPC-sG method for the two-phase flow model (1.7). We start by taking the basis functions $\{\phi_k(z)\}_{k=1}^\infty$ as the gPC basis, i.e., the set of polynomials defined on I_z, orthonormal with respect to the given probability measure $\pi(z)\,\mathrm{d}z$, with ϕ_k being a polynomial of degree $k-1$.

We expand the functions u, f into

$$
u(t,x,z) = \sum_{k=1}^\infty u_k(t,x)\phi_k(z), \quad f(t,x,v,z) = \sum_{k=1}^\infty f_k(t,x,v)\phi_k(z),
\tag{2.14}
$$

and approximate them by truncated series up to order K:

$$u \approx u^K = \sum_{k=1}^{K} u_k \phi_k(z), \quad f \approx f^K = \sum_{k=1}^{K} f_k \phi_k(z). \tag{2.15}$$

Then substitute into (1.7) and conduct the Galerkin projection, one gets the following deterministic system for $(u_k, f_k)_{k=1}^K$:

$$\begin{cases} \partial_t u_k + (u \cdot \nabla_x u)_k + \nabla_x p - \Delta_x u_k + u_k + \int \sqrt{\mu}(uf)_k \,\mathrm{d}v - \int v\sqrt{\mu} f_k \,\mathrm{d}v = 0, \\ \nabla_x \cdot u_k = 0, \\ \partial_t f_k + v \cdot \nabla_x f_k + (\nabla_v - \frac{v}{2})(uf)_k - u_k \cdot v\sqrt{\mu} = (\frac{-|v|^2}{4} + \frac{3}{2} + \Delta_v) f_k, \end{cases} \tag{2.16}$$

with initial data

$$u_k|_{t=0} = (u_0)_k = \int u_0 \phi_k(z)\pi(z)\,\mathrm{d}z, \quad f_k|_{t=0} = (f_0)_k. \tag{2.17}$$

Here the gPC coefficient of a product is given by

$$(uw)_k = \sum_{i,j=1}^{K} S_{ijk} u_i w_j, \tag{2.18}$$

where

$$S_{ijk} = \int \phi_i \phi_j \phi_k \pi(z)\,\mathrm{d}z, \tag{2.19}$$

is the triple product coefficient.

We prove a similar energy estimate for (2.16):

Theorem 2.3. *Assume the technical condition*

$$\|\phi_k\|_{L^\infty} \le C k^p, \quad \forall k, \tag{2.20}$$

with a parameter $p > 0$. Let $q > p + 2$ and $s \ge 2$. Let (u_k, f_k), $k = 1, \ldots, K$, solve (2.16) with initial data verifying (1.9), and define the energy E^K by

$$E^K(t) = E^K_{s,q}(t) = \sum_{k=1}^{K} (\|k^q u_k\|^2_{H^s} + |k^q f_k|^2_s + |k^q \bar{u}_k|^2). \tag{2.21}$$

Then there exists a constant $c_2 = c_2(s,q) > 0$, independent of K, such that $E^K(0) \le c_2$ implies that $E^K(t)$ is decreasing in t.

This theorem is proved by the same type of energy estimate as Theorem 2.1, with the aid of a nonlinear estimate for gPC spectral convolution terms (Lemma 5.1). Next we give a sufficient condition on the initial data, under which the assumption $E^K(0) \le c_2$ in Theorem 2.3 holds:

Proposition 2.4. *With the same assumptions as Theorem 2.3, the condition $E^K_{s,q}(0) \le c_2(s,q)$ holds if $\|E_{s,r}(0)\|_{L^1_z} \le C c_2(s,q)$ with $r > q + \frac{1}{2}$, and $C = C(s,q,r)$.*

Notice that c_2 being independent of K is important, because it implies that the condition $E^K(0) \le c_2$ is in fact, in view of Proposition 2.4, a consequence of a smoothness condition on (u_0, f_0), for all K. This means for such initial data, the gPC-sG method is stable for all K.

Finally, by a combination of the above results, we obtain the spectral accuracy of the gPC-sG method, uniformly in t and ϵ, with a small initial data assumption on (u_0, f_0), independent of K and ϵ:

Theorem 2.5. *Let (u_k, f_k), $k = 1, \ldots, K$, solve (2.16) with initial data verifying (1.9)(1.10). There exists a constant $c_1''(s,r)$ such that the following holds: Assume $s \geq 2$, $r > p + \frac{5}{2}$, $\|E_{s+4,r}(0)\|_{L_z^\infty} \leq c_1''(s,r)$, and $C_{s+1,r}^h$ is finite. Then E^e, the energy of the gPC approximation error, defined by*

$$E^e = \|u^e\|_{H_z^s}^2 + |f^e|_{s,z}^2 + |\bar{u}^e|_{L_z^2}^2, \quad u^e = u - u^K, \quad f^e = f - f^K, \tag{2.22}$$

satisfies

$$E^e \leq \frac{C}{K^{2r}}, \tag{2.23}$$

for all time, i.e., the gPC-sG method has r-th order accuracy uniformly in time.

This theorem is proved by an energy estimate in the (x, v, z) space on (u^e, f^e) with the aid of the previous theorems.

Finally we prove that the error also decays exponentially in time, by a hypocoercivity argument:

Theorem 2.6. *Let (u_k, f_k), $k = 1, \ldots, K$, solves (2.16) with initial data verifying (1.9)(1.10). There exists a constant $c_2''(s,r)$ such that the following holds: Assume $s \geq 0$, $r > p + \frac{5}{2}$, $\|E_{s+7,r}(0)\|_{L_z^\infty} \leq c_2''(s,r)$, and $C_{s+4,r}^h$ is finite. Then there exists a constant $\lambda^e > 0$ such that*

$$E^e \leq Ce^{-\lambda^e t}. \tag{2.24}$$

These theorems imply that for random initial data near the global equilibrium, in the sense that (u_0, f_0) is small in some suitable Sobolev spaces, the gPC-sG method has spectral accuracy, uniformly in time and ϵ, and it captures the long-time behavior of (1.7) with random initial data.

3 Basic energy estimate: proof of Theorem 2.1

We first state some lemmas on nonlinear estimates. Denote the space of functions with finite $\|\cdot\|_{H^s}$, $|\cdot|_s$ norms as

$$H^s = \{u(x) : \|u\|_{H^s} < \infty\}, \quad \tilde{H}^s = \{f(x,v) : |f|_s < \infty\}. \tag{3.1}$$

The following lemma is from [9]:

Lemma 3.1. *Let $u = u(x) \in H^s, w = w(x) \in H^s, f = f(x,v) \in \tilde{H}^s$. Then for $s > 3/2$,*

$$\|uw\|_{H^s} \leq C\|u\|_{H^s}\|w\|_{H^s}, \tag{3.2}$$

$$|uf|_s \leq C\|u\|_{H^s}|f|_s, \tag{3.3}$$

where $C = C(s)$.

It follows that

Lemma 3.2. *Let $u = u(x,z) \in L_z^\infty(H^s), w = w(x,z) \in L_z^\infty(H^s), f = f(x,v,z) \in L_z^\infty(\tilde{H}^s)$. Let $|\gamma| \leq r$. Then for $s > 3/2$ and all z,*

$$\|(uw)^\gamma\|_{H^s} \leq C\|u\|_{H^{s,r}}\|w\|_{H^{s,r}}, \tag{3.4}$$

$$|(uf)^\gamma|_s \leq C\|u\|_{H^{s,r}}|f|_{s,r}, \tag{3.5}$$

where $C = C(s,r)$.

Proof. By the Leibniz rule,

$$(uw)^\gamma = \sum_{\beta=0}^{\gamma} \binom{\gamma}{\beta} u^\beta w^{\gamma-\beta}. \tag{3.6}$$

Then

$$\|(uw)^\gamma\|_{H^s} \le \sum_{\beta=0}^{\gamma} \binom{\gamma}{\beta} \|u^\beta w^{\gamma-\beta}\|_{H^s} \le C(s) \sum_{\beta=0}^{\gamma} \binom{\gamma}{\beta} \|u^\beta\|_{H^s} \|w^{\gamma-\beta}\|_{H^s} \le C(s,r)\|u\|_{H^{s,r}}\|w\|_{H^{s,r}}, \tag{3.7}$$

where the second inequality uses (3.2). This finishes the proof of (3.4). The proof of (3.5) is similar, in view of (3.3). □

And then a bilinear version follows:

Lemma 3.3. *Let $u = u(x,z) \in L_z^\infty(H^s), w = w(x,z) \in L_z^\infty(H^s), y = y(x,z) \in L_z^\infty(H^s), f = f(x,v,z) \in L_z^\infty(\tilde{H}^s), g = g(x,v,z) \in L_z^\infty(\tilde{H}^s)$. Let $|\gamma| \le r$, $|\alpha| \le s$. Then for $s > 3/2$ and all z,*

$$|\langle \partial^\alpha (uw)^\gamma, y^\gamma \rangle| \le C(\delta,s,r)\|u\|_{H^{s,r}}^2 \|w\|_{H^{s,r}}^2 + \delta \|y\|_{H^{0,r}}^2, \tag{3.8}$$

$$|\langle \partial^\alpha (uf)^\gamma, g^\gamma \rangle| \le C(\delta,s,r)\|u\|_{H^{s,r}}^2 |f|_{s,r}^2 + \delta |g|_{0,r}^2, \tag{3.9}$$

where δ is any positive number.

Proof. To prove (3.8),

$$\begin{aligned} |\langle \partial^\alpha (uw)^\gamma, y^\gamma \rangle| &\le \frac{1}{4\delta}\|\partial^\alpha (uw)^\gamma\|_{L^2}^2 + \delta \|y^\gamma\|_{L^2}^2 \le \frac{1}{4\delta}\|(uw)^\gamma\|_{H^s}^2 + \delta\|y\|_{H^{0,r}}^2 \\ &\le C(\delta,s,r)\|u\|_{H^{s,r}}^2\|w\|_{H^{s,r}}^2 + \delta\|y\|_{H^{0,r}}^2, \end{aligned} \tag{3.10}$$

where the first inequality uses Young's inequality, and the last inequality uses (3.4). The proof of (3.9) is similar. □

Proof of Theorem 2.1. Taking z-derivative of order γ and x-derivative of order α of (1.7), and taking z-derivative of order γ of (1.12) gives

$$\begin{aligned} &\partial_t \partial^\alpha u^\gamma + \partial^\alpha (u \cdot \nabla_x u)^\gamma + \nabla_x \partial^\alpha p^\gamma \underline{- \Delta_x \partial^\alpha u^\gamma} \underbrace{+ \partial^\alpha u^\gamma} + \int \sqrt{\mu}\, \partial^\alpha (uf)^\gamma \,\mathrm{d}v \underbrace{- \frac{1}{\epsilon} \int v \sqrt{\mu}\, \partial^\alpha f^\gamma \,\mathrm{d}v} = 0, \\ &\nabla_x \cdot \partial^\alpha u^\gamma = 0, \\ &\partial_t \partial^\alpha f^\gamma + \frac{1}{\epsilon} v \cdot \nabla_x \partial^\alpha f^\gamma + \frac{1}{\epsilon}(\nabla_v - \frac{v}{2})\partial^\alpha (uf)^\gamma \underbrace{- \frac{1}{\epsilon}\partial^\alpha u^\gamma \cdot v\sqrt{\mu}} = \underbrace{\frac{1}{\epsilon^2}(\frac{-|v|^2}{4} + \frac{3}{2} + \Delta_v)\partial^\alpha f^\gamma}, \\ &\partial_t \bar{u}^\gamma \underline{+ 2\bar{u}^\gamma} + \frac{1}{|\mathbb{T}^3|}\int\int \sqrt{\mu}(uf)^\gamma \,\mathrm{d}v \,\mathrm{d}x = 0. \end{aligned} \tag{3.11}$$

Now do L^2 estimate on each equation above (except the second one), i.e., multiply the first equation by $\partial^\alpha u^\gamma$ and integrate in x; multiply the third equation by $\partial^\alpha f^\gamma$ and integrate in (v,x); multiply the fourth equation by $\bar{u}^\gamma$. And then add the results together and sum over $|\gamma| \le r, |\alpha| \le s$. Then one gets the following equation:

$$\frac{1}{2}\partial_t E + G + B = 0, \tag{3.12}$$

where the energy E is given by (2.12). The good terms G are given by

$$G = \underline{G_1} + \underbrace{G_2} = \sum_{|\gamma|\le s} G_{1,\gamma} + \sum_{|\gamma|\le s} G_{2,\gamma}, \tag{3.13}$$

with

$$\begin{aligned} G_{1,\gamma} &= \|\nabla_x u^\gamma\|_{H^s}^2 + 2|\bar{u}^\gamma|^2 \ge C\|u^\gamma\|_{H^{s+1}}^2, \\ G_{2,\gamma} &= \left| u^\gamma\sqrt{\mu} - \frac{1}{\epsilon}\nabla_v f^\gamma - \frac{1}{\epsilon}\frac{v}{2} f^\gamma \right|_s^2, \end{aligned} \tag{3.14}$$

where the above inequality is by the Poincare-Wirtinger inequality. G_1 and G_2 come from the underlined terms and the underbraced terms in (3.11), respectively. To verify the G_2 term, we provide the following calculation:

$$\begin{aligned} &\langle \partial^\alpha u^\gamma, \partial^\alpha u^\gamma\rangle - \frac{1}{\epsilon}\langle v\sqrt{\mu}\partial^\alpha u^\gamma, \partial^\alpha f^\gamma\rangle - \frac{1}{\epsilon}\langle \partial^\alpha u^\gamma \cdot v\sqrt{\mu}, \partial^\alpha f^\gamma\rangle - \frac{1}{\epsilon^2}\langle (\frac{-|v|^2}{4} + \frac{3}{2} + \Delta_v)\partial^\alpha f^\gamma, \partial^\alpha f^\gamma\rangle \\ =&\langle \partial^\alpha u^\gamma\sqrt{\mu}, \partial^\alpha u^\gamma\sqrt{\mu}\rangle - 2\frac{1}{\epsilon}\langle \partial^\alpha u^\gamma\sqrt{\mu}, \frac{v}{2}\partial^\alpha f^\gamma\rangle - 2\frac{1}{\epsilon}\langle \partial^\alpha u^\gamma\sqrt{\mu}, \nabla_v\partial^\alpha f^\gamma\rangle \\ &+ \frac{1}{\epsilon^2}\langle \nabla_v\partial^\alpha f^\gamma + \frac{v}{2}\partial^\alpha f^\gamma, \nabla_v\partial^\alpha f^\gamma + \frac{v}{2}\partial^\alpha f^\gamma\rangle \\ =&\langle A_1, A_1\rangle - 2\langle A_1, A_3\rangle - 2\langle A_1, A_2\rangle + \langle A_2 + A_3, A_2 + A_3\rangle \\ =&\|A_1 - A_2 - A_3\|_{L^2}^2 = \left\| \partial^\alpha \left(u^\gamma\sqrt{\mu} - \frac{1}{\epsilon}\nabla_v f^\gamma - \frac{1}{\epsilon}\frac{v}{2} f^\gamma \right) \right\|_{L^2}^2, \end{aligned} \tag{3.15}$$

where we used integration by parts in v, $\nabla_v\sqrt{\mu} = -\frac{v}{2}\sqrt{\mu}$, and the notations

$$A_1 = \partial^\alpha u^\gamma\sqrt{\mu}, \quad A_2 = \frac{1}{\epsilon}\nabla_v\partial^\alpha f^\gamma, \quad A_3 = \frac{1}{\epsilon}\frac{v}{2}\partial^\alpha f^\gamma. \tag{3.16}$$

The bad terms B are given by

$$B = B_1 + B_2 + B_3 = \sum_{|\gamma|\le r, |\alpha|\le s} B_{1,\alpha,\gamma} + \sum_{|\gamma|\le r, |\alpha|\le s} B_{2,\alpha,\gamma} + \sum_{|\gamma|\le r} B_{3,\gamma}, \tag{3.17}$$

with

$$\begin{aligned} B_{1,\alpha,\gamma} &= \langle \partial^\alpha(u\cdot\nabla_x u)^\gamma, \partial^\alpha u^\gamma\rangle, \\ B_{2,\alpha,\gamma} &= \left\langle \partial^\alpha(uf)^\gamma, \partial^\alpha\left[u^\gamma\sqrt{\mu} - \frac{1}{\epsilon}\nabla_v f^\gamma - \frac{1}{\epsilon}\frac{v}{2}f^\gamma\right]\right\rangle, \\ B_{3,\gamma} &= \langle (uf)^\gamma, \bar{u}^\gamma\sqrt{\mu}\rangle, \end{aligned} \tag{3.18}$$

coming from the nonlinear terms.

By using Lemma 3.3, the bad terms are controlled by

$$\begin{aligned} |B_{1,\alpha,\gamma}| &\le C(\delta)\|u\|_{H^{s+1,r}}^2\|u\|_{H^{s,r}}^2 + \delta\|u\|_{H^{s,r}}^2 \le C(\delta)EG_1 + \delta G_1, \\ |B_{2,\alpha,\gamma}| &\le C(\delta)\|u\|_{H^{s,r}}^2|f|_{s,r}^2 + \delta\left|u^\gamma\sqrt{\mu} - \frac{1}{\epsilon}\nabla_v f^\gamma - \frac{1}{\epsilon}\frac{v}{2}f^\gamma\right|_s \le C(\delta)EG_1 + \delta G_2, \\ |B_{3,\gamma}| &\le C(\delta)\|u\|_{H^{s,r}}^2|f|_{s,r}^2 + \delta|\bar{u}^\gamma|^2 \le C(\delta)EG_1 + \delta G_1. \end{aligned} \tag{3.19}$$

In conclusion, we have the energy estimate

$$\frac{1}{2}\partial_t E \le -(1 - C(\delta)E - C\delta)G. \tag{3.20}$$

Take $\delta = \frac{1}{4C}$ where C is the constant in (3.20), and $c_1(s,r) = \frac{1}{4C(\delta)}$. Then we will show that $E(t) \leq c_1$ for all t. In fact, let

$$T^* = \sup\{\tilde{T} \geq 0 : \sup_{0\leq t<\tilde{T}} E(t) \leq c_1\}. \tag{3.21}$$

Then it follows that $E(t) \leq c_1$ for $0 \leq t \leq T^*$. Then by our choice of δ and c_1,

$$1 - C(\delta)E - C\delta \geq 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}, \tag{3.22}$$

and therefore (3.20) implies

$$\partial_t E + G \leq 0, \tag{3.23}$$

for $0 \leq t \leq T^*$. This prevents T^* from being finite. Thus we proved $E(t) \leq c_1$ for all t, and as a result, (3.23) holds for all t. Thus $E(t)$ is decreasing in t. □

4 Hypocoercivity estimates: proof of Theorem 2.2

We will use the following lemma, which is Proposition 4.2 in [9]:

Lemma 4.1. *There exisits a constant $C > 0$ such that for $f \in L^2$ orthogonal to $\sqrt{\mu}$, one has*

$$\|f\|_{L^2}^2 \leq C(\|\mathcal{K}f\|_{L^2}^2 + \|\mathcal{S}f\|_{L^2}^2). \tag{4.1}$$

We begin by proving the following lemma, which is indeed a modification of part of the proof of Proposition 4.1 in [9]:

Lemma 4.2. *For f and g orthogonal to $\sqrt{\mu}$,*

$$|((u \cdot \mathcal{K}^* f, g))| \leq C\frac{1}{\epsilon}\|u\|_{H^3}([[f,f]] + [[g,g]]). \tag{4.2}$$

Proof. Using the commutator relation

$$\mathcal{K}(u \cdot \mathcal{K}^* f) = \mathcal{K}^*(u \cdot \mathcal{K}f) + uf, \tag{4.3}$$

one gets

$$\begin{aligned}((u \cdot \mathcal{K}^* f, g)) =& 2\langle u \cdot \mathcal{K}f, \mathcal{K}^2 g\rangle + 2\langle uf, \mathcal{K}g\rangle + \epsilon\langle u \cdot \mathcal{K}f, \mathcal{K}\mathcal{S}g\rangle + \epsilon\langle uf, \mathcal{S}g\rangle \\ &+ \epsilon\langle \mathcal{S}(u \cdot f), \mathcal{K}^2 g\rangle + \epsilon^2\langle \mathcal{S}(u \cdot f), \mathcal{K}\mathcal{S}g\rangle \\ =& 2\langle u \cdot \mathcal{K}f, \mathcal{K}^2 g\rangle + 2\langle uf, \mathcal{K}g\rangle + \epsilon\langle u \cdot \mathcal{K}f, \mathcal{K}\mathcal{S}g\rangle + \epsilon\langle uf, \mathcal{S}g\rangle \\ &+ \epsilon\langle \mathcal{S}u \cdot f, \mathcal{K}^2 g\rangle + \epsilon\langle u \cdot \mathcal{S}f, \mathcal{K}^2 g\rangle + \epsilon^2\langle \mathcal{S}u \cdot f, \mathcal{K}\mathcal{S}g\rangle + \epsilon^2\langle u \cdot \mathcal{S}f, \mathcal{K}\mathcal{S}g\rangle.\end{aligned} \tag{4.4}$$

Now using the Cauchy-Schwarz inequality, Lemma 4.1, and the Sobolev inequality

$$\|u\|_{L^\infty} + \|\nabla_x u\|_{L^\infty} \leq C\|u\|_{H^3}, \tag{4.5}$$

on each term. we provide the details for two of them and omit the others:

$$\begin{aligned}\langle uf, \mathcal{K}g\rangle \leq& \|u\|_{L^\infty}\|f\|_{L^2}\|\mathcal{K}g\|_{L^2} \leq C\|u\|_{L^\infty}(\|\mathcal{K}f\|_{L^2} + \|\mathcal{S}f\|_{L^2})\|\mathcal{K}g\|_{L^2} \\ \leq& C\|u\|_{L^\infty}(\|\mathcal{K}f\|_{L^2}^2 + \|\mathcal{K}g\|_{L^2}^2 + \epsilon\|\mathcal{S}f\|_{L^2}^2 + \frac{1}{\epsilon}\|\mathcal{K}g\|_{L^2}^2), \\ \epsilon\langle uf, \mathcal{S}g\rangle \leq& \epsilon\|u\|_{L^\infty}\|f\|_{L^2}\|\mathcal{S}g\|_{L^2} \leq C\epsilon\|u\|_{L^\infty}(\|\mathcal{K}f\|_{L^2} + \|\mathcal{S}f\|_{L^2})\|\mathcal{S}g\|_{L^2} \\ \leq& C\epsilon\|u\|_{L^\infty}(\|\mathcal{K}f\|_{L^2}^2 + \|\mathcal{S}g\|_{L^2}^2 + \|\mathcal{S}f\|_{L^2}^2 + \|\mathcal{S}g\|_{L^2}^2).\end{aligned} \tag{4.6}$$

Then one gets the conclusion, in view of the definition of $[[\cdot,\cdot]]$. □

Now we prove the following lemma, which is an analog to Proposition 4.1 of [9]:

Lemma 4.3. *Let the assumptions of Theorem 2.2 be fulfilled. Then there exists a constant $c_1'(s,r) \le c_1(s+3,r)$ such that, if we assume that $E_{s+3,r}(0) \le c_1'(s,r)$ is small enough, then there exists a constant $\lambda_1 > 0$ such that*

$$\partial_t((f,f))_{s,r} + \lambda_1 \frac{1}{\epsilon^2}[[f,f]]_{s,r} \le C(\lambda_1)(\|u\|^2_{H^{s,r}} + \|\nabla_x u\|^2_{H^{s,r}} + \frac{1}{\epsilon^2}|\mathcal{K}f|^2_{s,r}). \tag{4.7}$$

Proof. One can write the evolution equation of $\partial^\alpha f^\gamma$ as

$$\partial_t \partial^\alpha f^\gamma + \frac{1}{\epsilon}\mathcal{P}\partial^\alpha f^\gamma + \frac{1}{\epsilon^2}\mathcal{K}^*\mathcal{K}\partial^\alpha f^\gamma = \frac{1}{\epsilon}\partial^\alpha u^\gamma \cdot v\sqrt{\mu} + \frac{1}{\epsilon}\sum_{0\le\eta\le\alpha}\sum_{0\le\beta\le\gamma}\binom{\gamma}{\beta}\binom{\alpha}{\eta}\partial^\eta u^\beta \cdot \mathcal{K}^*\partial^{\alpha-\eta} f^{\gamma-\beta}. \tag{4.8}$$

We will take the $((\cdot,\cdot))$ inner product of (4.8) with $\partial^\alpha f^\gamma$. For the linear terms, by the same argument as the proof of Proposition 4.1 of [9], one gets

$$\begin{aligned}
\frac{1}{\epsilon}((\mathcal{P}\partial^\alpha f^\gamma, \partial^\alpha f^\gamma)) =& 2\frac{1}{\epsilon}\langle \mathcal{S}\partial^\alpha f^\gamma, \mathcal{K}\partial^\alpha f^\gamma\rangle + \|\mathcal{S}\partial^\alpha f^\gamma\|^2_{L^2} \ge \frac{3}{4}\|\mathcal{S}\partial^\alpha f^\gamma\|^2_{L^2} - 4\frac{1}{\epsilon^2}\|\mathcal{K}\partial^\alpha f^\gamma\|^2_{L^2},\\
\frac{1}{\epsilon^2}((\mathcal{K}^*\mathcal{K}\partial^\alpha f^\gamma, \partial^\alpha f^\gamma)) =& 2\frac{1}{\epsilon^2}\|\mathcal{K}\partial^\alpha f^\gamma\|^2_{L^2} + 2\frac{1}{\epsilon^2}\|\mathcal{K}^2\partial^\alpha f^\gamma\|^2_{L^2} + \|\mathcal{S}\mathcal{K}\partial^\alpha f^\gamma\|^2_{L^2}\\
&+ \frac{1}{\epsilon}\langle \mathcal{K}\partial^\alpha f^\gamma, \mathcal{S}\partial^\alpha f^\gamma\rangle + 2\frac{1}{\epsilon}\langle \mathcal{K}^2\partial^\alpha f^\gamma, \mathcal{S}\mathcal{K}\partial^\alpha f^\gamma\rangle\\
\ge& \frac{3}{2}\frac{1}{\epsilon^2}\|\mathcal{K}\partial^\alpha f^\gamma\|^2_{L^2} + \frac{1}{2}\frac{1}{\epsilon^2}\|\mathcal{K}^2\partial^\alpha f^\gamma\|^2_{L^2} + \frac{1}{3}\|\mathcal{S}\mathcal{K}\partial^\alpha f^\gamma\|^2_{L^2} - \frac{1}{2}\|\mathcal{S}\partial^\alpha f^\gamma\|^2_{L^2},\\
\frac{1}{\epsilon}|((\partial^\alpha u^\gamma \cdot v\sqrt{\mu}, \partial^\alpha f^\gamma))| =& |2\frac{1}{\epsilon}\langle \mathcal{K}(\partial^\alpha u^\gamma \cdot v\sqrt{\mu}), \mathcal{K}\partial^\alpha f^\gamma\rangle + \langle \mathcal{K}(\partial^\alpha u^\gamma \cdot v\sqrt{\mu}), \mathcal{S}\partial^\alpha f^\gamma\rangle\\
&+ \langle \mathcal{S}(\partial^\alpha u^\gamma \cdot v\sqrt{\mu}), \mathcal{K}\partial^\alpha f^\gamma\rangle + \epsilon\langle \mathcal{S}(\partial^\alpha u^\gamma \cdot v\sqrt{\mu}), \mathcal{S}\partial^\alpha f^\gamma\rangle|\\
\le& \delta(\frac{1}{\epsilon^2}\|\mathcal{K}\partial^\alpha f^\gamma\|^2_{L^2} + \|\mathcal{S}\partial^\alpha f^\gamma\|^2_{L^2}) + C(\delta)(\|u\|^2_{H^{s,r}} + \|\nabla_x u\|^2_{H^{s,r}}).
\end{aligned} \tag{4.9}$$

For the nonlinear term (the summation), we apply Lemma 4.2 and get

$$\begin{aligned}
&\frac{1}{\epsilon}|((\partial^\eta u^\beta \cdot \mathcal{K}^*\partial^{\alpha-\eta} f^{\gamma-\beta}, \partial^\alpha f^\gamma))|\\
\le& C\frac{1}{\epsilon^2}\|\partial^\eta u^\beta\|_{H^3}([[\partial^{\alpha-\eta} f^{\gamma-\beta}, \partial^{\alpha-\eta} f^{\gamma-\beta}]] + [[\partial^\alpha f^\gamma, \partial^\alpha f^\gamma]])\\
\le& C\frac{1}{\epsilon^2}\|u\|_{H^{s+3,r}}[[f,f]]_{s,r},
\end{aligned} \tag{4.10}$$

where we used the fact that the x and z derivatives commute with the operators $\mathcal{K}$ and $\mathcal{S}$. With these estimates, we get

$$\begin{aligned}
&\frac{1}{2}\partial_t((\partial^\alpha f^\gamma, \partial^\alpha f^\gamma)) + \frac{1}{2}\frac{1}{\epsilon^2}\|\mathcal{K}^2\partial^\alpha f^\gamma\|^2_{L^2} + \frac{1}{3}\|\mathcal{S}\mathcal{K}\partial^\alpha f^\gamma\|^2_{L^2} + \frac{1}{4}\|\mathcal{S}\partial^\alpha f^\gamma\|^2_{L^2} - \frac{5}{2}\frac{1}{\epsilon^2}\|\mathcal{K}\partial^\alpha f^\gamma\|^2_{L^2}\\
&\le \delta(\frac{1}{\epsilon^2}\|\mathcal{K}\partial^\alpha f^\gamma\|^2_{L^2} + \|\mathcal{S}\partial^\alpha f^\gamma\|^2_{L^2}) + C(\delta)(\|u\|^2_{H^{s,r}} + \|\nabla_x u\|^2_{H^{s,r}}) + C\frac{1}{\epsilon^2}\|u\|_{H^{s+3,r}}[[f,f]]_{s,r}.
\end{aligned} \tag{4.11}$$

Then we choose $\delta = 1/8$ to absorb the term $\|\mathcal{S}\partial^\alpha f^\gamma\|^2_{L^2}$ on the RHS by the same term on the LHS. Summing over α, γ, one gets

$$\partial_t((f,f))_{s,r} + (\frac{1}{8} - C_1\|u\|_{H^{s+3,r}})\frac{1}{\epsilon^2}[[f,f]]_{s,r} \le C_2(\|u\|^2_{H^{s,r}} + \|\nabla_x u\|^2_{H^{s,r}} + \frac{1}{\epsilon^2}|\mathcal{K}f|^2_{s,r}), \tag{4.12}$$

where $C_1 = NC$, $C_2 = \max\{3, NC(\delta)\}$, N being the number of possible pairs (α, γ).

Thus if one chooses $c_1' = \min\{c_1(s+3, r), \frac{1}{16C_1}\}$, then by Theorem 2.1, $E_{s+3,r}(t)$ is decreasing, so $E_{s+3,r}(t) \le c_1'$ for all t. Thus $\|u\|_{H^{s+3,r}} \le E_{s+3,r} \le c_1'$ for all t, and one gets the conclusion, with $\lambda_1 = 1/16$.

□

Proof of Theorem 2.2. To obtain the energy decay estimate, we write

$$\begin{aligned} G &= \|\nabla_x u\|_{H^{s,r}}^2 + 2\|\bar{u}\|_r^2 + |u\sqrt{\mu} - \frac{1}{\epsilon}\mathcal{K}f|_{s,r}^2 \\ &\ge \|\bar{u}\|_r^2 + 2\lambda_2\|u\|_{H^{s+1,r}}^2 + |u\sqrt{\mu} - \frac{1}{\epsilon}\mathcal{K}f|_{s,r}^2 \\ &\ge \|\bar{u}\|_r^2 + \lambda_2\|u\|_{H^{s+1,r}}^2 + \frac{1}{2}|u\sqrt{\mu} - \frac{1}{\epsilon}\mathcal{K}f|_{s,r}^2 + \lambda_3\frac{1}{\epsilon^2}|\mathcal{K}f|_{s,r}^2, \end{aligned} \tag{4.13}$$

where $\lambda_3 = \min\{\frac{\lambda_2}{2}, \frac{1}{4}\}$. The first inequality is by the Poincare-Wirtinger inequality. The second inequality is because

$$\begin{aligned} |\frac{1}{\epsilon}\mathcal{K}f|_{s,r}^2 &= |(\frac{1}{\epsilon}\mathcal{K}f - u\sqrt{\mu}) + u\sqrt{\mu}|_{s,r}^2 \le 2(|\frac{1}{\epsilon}\mathcal{K}f - u\sqrt{\mu}|_{s,r}^2 + |u\sqrt{\mu}|_{s,r}^2) \\ &= 2(|\frac{1}{\epsilon}\mathcal{K}f - u\sqrt{\mu}|_{s,r}^2 + \|u\|_{H^{s,r}}^2). \end{aligned} \tag{4.14}$$

Thus, by adding to (3.23) some positive constant λ_4 (to be chosen) times (4.7), we have

$$\partial_t\tilde{E} + \tilde{G} \le \lambda_4\tilde{B}, \tag{4.15}$$

where

$$\tilde{E} = E + \lambda_4((f,f))_{s,r}, \quad \tilde{G} = G + \lambda_4\lambda_1\frac{1}{\epsilon^2}[[f,f]]_{s,r}, \tag{4.16}$$

$$\tilde{B} = C(\lambda_1)(\|u\|_{H^{s,r}}^2 + \|\nabla_x u\|_{H^{s,r}}^2 + \frac{1}{\epsilon^2}|\mathcal{K}f|_{s,r}^2). \tag{4.17}$$

It is clear from (4.13) that $\tilde{B} \le CG \le C\tilde{G}$. Thus by choosing $\lambda_4 = \min\{\frac{1}{2C}, 1\}$, C being the previous constant, we get

$$\partial_t\tilde{E} + \frac{1}{2}\tilde{G} \le 0. \tag{4.18}$$

Notice that Lemma 4.1 implies that

$$|f|_{s,r}^2 \le C(|\mathcal{K}f|_{s,r}^2 + |\mathcal{S}f|_{s,r}^2), \tag{4.19}$$

and by definition one also has

$$((f,f))_{s,r} \le C(|\mathcal{K}f|_{s,r}^2 + |\mathcal{S}f|_{s,r}^2) \le C\frac{1}{\epsilon^2}((f,f))_{s,r}. \tag{4.20}$$

Thus

$$\tilde{E} \le C(G + |f|_{s,r}^2) + \lambda_4((f,f))_{s,r} \le C(G + |\mathcal{K}f|_{s,r}^2 + |\mathcal{S}f|_{s,r}^2) \le C\tilde{G}. \tag{4.21}$$

This together with (4.18) implies

$$\tilde{E}(t) \le \tilde{E}(0)e^{-\lambda t}, \tag{4.22}$$

where $\lambda = \frac{1}{2C}$, C being the constant in (4.21).

Finally, the proof of Theorem 2.2 is finished by noticing that

$$E(t) \le \tilde{E}(t) \le \tilde{E}(0)e^{-\lambda t} \le (E(0) + C^h)e^{-\lambda t}. \tag{4.23}$$

□

5 Proof of spectral accuracy of the gPC-sG approximation

In order to prove the accuracy of the gPC-sG method, we first prove Theorem 2.3, which is an energy estimate for the gPC coefficients (u_k, f_k).

5.1 Estimate of the gPC coefficients: proof of Theorem 2.3

In order to prove the estimate for the gPC coefficients, we need an extra assumption on the basis functions.

We assume that

$$\|\phi_k\|_{L^\infty} \leq Ck^p, \quad \forall k, \tag{5.1}$$

for some positive constant p. Then it follows that

$$|S_{ijk}| \leq Ci^p, \tag{5.2}$$

since

$$|S_{ijk}| \leq \|\phi_i\|_{L^\infty} \langle |\phi_j|, |\phi_k| \rangle \leq \|\phi_i\|_{L^\infty} \|\phi_j\| \|\phi_k\| \leq Ci^p. \tag{5.3}$$

We mention some special cases where (5.1) is satisfied [28]. For the case $I_z = [-1,1]$ with uniform distribution, ϕ_k are the normalized Legendre polynomials, and (5.1) holds with $p = 1/2$. For the case $I_z = [-1,1]$ with the distribution $\pi(z) = \frac{2}{\pi\sqrt{1-z^2}}$, ϕ_k are the normalized Chebyshev polynomials, and (5.1) holds with $p = 0$.

Also, note that ϕ_k is a polynomial of degree $k-1$, orthogonal to all lower order polynomials. If $(i-1)+(j-1) < k-1$, then $S_{ijk} = 0$. Thus S_{ijk} may be nonzero only when the triangle inequality

$$i + j \geq k + 1, \tag{5.4}$$

holds.

Note that due to the symmetry in i, j, k of S_{ijk}, (5.2) and (5.4) also hold if i, j, k are permuted.

Then we have the following lemma, which is the key nonlinear estimate:

Lemma 5.1. *Assume condition (5.2). Let $q > p+2$. Let $s > \frac{3}{2}$, α be a multi-index with $|\alpha| \leq s$. Let $u_k = u_k(x) \in H^s, w_k = w_k(x) \in H^s, y_k = y_k(x) \in L^2, f_k = f_k(x,v) \in \tilde{H}^2, g_k = g_k(x,v) \in L^2$. Then*

$$\left| \sum_{k=1}^K k^{2q} \langle \partial^\alpha (uw)_k, y_k \rangle \right| \leq C(\delta) \sum_{i=1}^K \|i^q u_i\|_{H^s}^2 \sum_{j=1}^K \|j^q w_j\|_{H^s}^2 + \delta \sum_{k=1}^K \|k^q y_k\|_{L^2}^2, \tag{5.5}$$

$$\left| \sum_{k=1}^K k^{2q} \langle \partial^\alpha (uf)_k, g_k \rangle \right| \leq C(\delta) \sum_{i=1}^K \|i^q u_i\|_{H^s}^2 \sum_{j=1}^K |j^q f_j|_s^2 + \delta \sum_{k=1}^K \|k^q g_k\|_{L^2}^2, \tag{5.6}$$

where the constants are independent of K, and δ is any positive constant.

Proof. We focus on the proof of the first inequality, and the second one is similar (just use (3.3) instead of (3.2)). Note (by (3.2))

$$k^{2q} \|S_{ijk} \partial^\alpha (u_i w_j)\|_{L^2} \leq Ck^{2q} |S_{ijk}| \|u_i\|_{H^s} \|w_j\|_{H^s} = C\frac{k^{2q}}{i^q j^q} |S_{ijk}| \cdot \|i^q u_i\|_{H^s} \cdot \|j^q w_j\|_{H^s}. \tag{5.7}$$

First we consider the case $i \geq j$. Then since

$$i^q j^q \geq (\frac{k+1}{2})^q |S_{ijk}| j^{q-p}, \tag{5.8}$$

by (5.2) and (5.4), we conclude that

$$\frac{k^{2q}}{i^q j^q} |S_{ijk}| \leq C k^q j^{p-q}. \tag{5.9}$$

Thus if we write the $(uw)_k$ on the LHS of (5.5) as a summation in i, j by (2.18), the $i \geq j$ terms of can be estimated by

$$\begin{aligned}
&\left| \sum_{k=1}^{K} k^{2q} \sum_{i,j=1;\, i\geq j}^{K} \chi_{ijk} S_{ijk} \langle \partial^\alpha (u_i w_j), y_k \rangle \right| \\
\leq & \sum_{i,j,k=1;\, i\geq j}^{K} k^{2q} \|S_{ijk} \partial^\alpha (u_i w_j)\|_{L^2} \|y_k\|_{L^2} \chi_{ijk} \\
\leq & C \sum_{i,j,k=1;\, i\geq j}^{K} j^{p-q} \cdot \|i^q u_i\|_{H^s} \cdot \|j^q w_j\|_{H^s} \cdot \|k^q y_k\|_{L^2} \chi_{ijk} \\
\leq & C \sum_{i,j,k=1}^{K} j^{p-q} \cdot \|i^q u_i\|_{H^s} \cdot \|j^q w_j\|_{H^s} \cdot \|k^q y_k\|_{L^2} \chi_{ijk} \\
\leq & C(\delta) \sum_{i,j,k=1}^{K} j^{p-q} \cdot \|i^q u_i\|_{H^s}^2 \cdot \|j^q w_j\|_{H^s}^2 \chi_{ijk} + \delta \sum_{i,j,k=1}^{K} j^{p-q} \|k^q u_k\|_{L^2}^2 \chi_{ijk} \\
= & C(\delta) I + \delta II,
\end{aligned} \tag{5.10}$$

where the second inequality uses (5.9), and χ_{ijk} is the indicator function of the set of indexes (i, j, k) for which $S_{ijk} \neq 0$.

Now we claim that

$$I \leq 2 \sum_{i=1}^{K} \|i^q u_i\|_{H^s}^2 \cdot \sum_{j=1}^{K} \|j^q w_j\|_{H^s}^2. \tag{5.11}$$

In fact, fix i, then one can write

$$I = \sum_{i=1}^{K} \|i^q u_i\|_{H^s}^2 I_i, \quad I_i = \sum_{j,k=1}^{K} j^{p-q} \cdot \|j^q w_j\|_{H^s}^2 \chi_{ijk}. \tag{5.12}$$

Notice that $\chi_{ijk} = 1$ implies that $i - j + 1 \leq k \leq i + j - 1$, by (5.4). Thus in the last summation, there is at most $2j$ terms corresponding to a fixed j. Thus

$$I_i \leq 2 \sum_{j=1}^{K} j^{p-q+1} \|j^q w_j\|_{H^s}^2 \leq 2 \sum_{j=1}^{K} \|j^q w_j\|_{H^s}^2, \tag{5.13}$$

if $q > p + 1$. This proves (5.11).

II is controlled by

$$II \leq 2 \sum_{j=1}^{K} j^{p-q+1} \sum_{k=1}^{K} \|k^q y_k\|_{L^2}^2, \tag{5.14}$$

since for each fixed (j,k) there is at most $2j$ choices for i. Thus if $q>p+2$, one has

$$II \le C\sum_{k=1}^K \|k^q y_k\|_{L^2}^2,\quad C=2\sum_{j=1}^\infty j^{p-q+1}\le 2(1+(p-q+2)^{-1}). \tag{5.15}$$

Thus we conclude that the $i\ge j$ terms can be controlled by the RHS of (5.5) (with δ replaced by $C\delta$).

For the terms of the LHS of (5.5) with $i\le j$, we exchange the indexes i and j, and get the LHS of (5.10) with u and w exchanged. Thus one proceeds as before and get the same conclusion, since the RHS of (5.5) is invariant if u and w are exchanged. □

Remark 5.2. *The weight k^q appeared in the above lemma is essential. Suppose one uses a summation $\sum_{k=1}^K \langle \partial^\alpha (uw)_k, y_k\rangle$, then one ends up with the estimate*

$$\begin{aligned}\left|\sum_{k=1}^K \langle \partial^\alpha (uw)_k, y_k\rangle\right| =& \left|\sum_{i,j,k=1}^K S_{ijk}\langle \partial^\alpha (u_i w_j), y_k\rangle\right|\\ \le& \sum_{i,j,k=1}^K \min(i,j,k)^p[C(\delta)\|u_i\|_{H^s}^2\|w_j\|_{H^s}^2+\delta\|y_k\|_{L^2}^2]\\ \le& C(\delta)C_1(K)\sum_{i=1}^K\|u_i\|_{H^s}^2\sum_{j=1}^K\|w_j\|_{H^s}^2+\delta C_2(K)\sum_{k=1}^K\|y_k\|_{L^2}^2,\end{aligned} \tag{5.16}$$

where $C_1(K)=\sum_{k=1}^K k^p = O(K^{p+1})$, $C_2(K)=K\sum_{i=1}^K i^p = O(K^{p+2})$. Thus in this way one gets an estimate with the coefficient depending on K. If one uses this estimate to prove an analog of Theorem 2.3, then one will get a constant c_2 depending on K.

In view of Proposition 2.4, c_2 being independent of K implies that the conclusion of Theorem 2.3 holds if the initial data satisfies a smoothness condition independent of K. If c_2 depends on K, then the initial data needs to satisfy a K-dependent condition to make the conclusion of Theorem 2.3 true. This is not good, since it is desirable that the gPC-sG method is stable for a class of initial data, for all K.

Due to the similarity of Lemma 3.3 and Lemma 5.1, it is straightforward to modify the proof of Theorem 2.1 into a proof of Theorem 2.3:

Proof of Theorem 2.3. We take ∂^α on the first and third equations of (2.16) and do L^2 estimates, and do L^2 estimates directly on the fourth equation, and then sum over k and α with the k-th equation multiplied by k^{2q}. Then one gets

$$\frac{1}{2}\partial_t E^K + G^K + B^K = 0, \tag{5.17}$$

where

$$\begin{aligned}&E^K(t)=\sum_{k=1}^K(\|k^q u_k\|_{H^s}^2+|k^q f_k|_s^2+|k^q\bar u_k|^2),\\ &G^K=G_1^K+G_2^K=\sum_{k=1}^K(\|\nabla_x k^q u_k\|_{H^s}^2+2|k^q\bar u_k|^2)+\sum_{k=1}^K\left|k^q\left(u_k\sqrt{\mu}-\frac{1}{\epsilon}\nabla_v f_k-\frac{1}{\epsilon}\frac{v}{2}f_k\right)\right|_s^2,\\ &B^K=B_1^K+B_2^K+B_3^K=\sum_{|\alpha|\le s}B_{1,\alpha}^K+\sum_{|\alpha|\le s}B_{2,\alpha}^K+B_3^K,\end{aligned} \tag{5.18}$$

with

$$
\begin{aligned}
B_{1,\alpha}^K &= \sum_{k=1}^K k^{2q} \langle \partial^\alpha (u \cdot \nabla_x u)_k, \partial^\alpha u_k \rangle, \\
B_{2,\alpha}^K &= \sum_{k=1}^K k^{2q} \left\langle \partial^\alpha (uf)_k, \partial^\alpha \left[u_k \sqrt{\mu} - \frac{1}{\epsilon} \nabla_v f_k - \frac{1}{\epsilon} \frac{v}{2} f_k \right] \right\rangle, \\
B_3^K &= \sum_{k=1}^K k^{2q} \langle (uf)_k, \bar{u}_k \sqrt{\mu} \rangle.
\end{aligned} \tag{5.19}
$$

Now apply Lemma 5.1 to get

$$
\begin{aligned}
|B_{1,\alpha}^K| &\le C(\delta) \sum_{k=1}^K \|k^q u_k\|_{H^{s+1}}^2 \sum_{k=1}^K \|k^q u_k\|_{H^s}^2 + \delta \sum_{k=1}^K \|k^q u_k\|_{H^{s+1}}^2 \le C(\delta) E^K G_1^K + \delta G_1^K, \\
|B_{2,\alpha}^K| &\le C(\delta) \sum_{k=1}^K \|k^q u_k\|_{H^s}^2 \sum_{k=1}^K |k^q f_k|_s^2 + \delta G_2^K \le C(\delta) E^K G_1^K + \delta G_2^K, \\
|B_3^K| &\le C(\delta) \sum_{k=1}^K \|k^q u_k\|_{H^s}^2 \sum_{k=1}^K |k^q f_k|_s^2 + \sum_{k=1}^K \delta |k^q \bar{u}_k|^2 \le C(\delta) E^K G_1^K + \delta G_1^K.
\end{aligned} \tag{5.20}
$$

And then one concludes

$$
\frac{1}{2} \partial_t E^K \le -(1 - C(\delta) E^K - C\delta) G^K. \tag{5.21}
$$

Assuming $\delta = \frac{1}{4C}$ where C is the constant in (5.21), and $c_2(s,r) = \frac{1}{4C(\delta)}$, then by the same argument as in the proof of Theorem 2.1, if $E^K(0) \le c_2(s,r)$, then one has

$$
\partial_t E^K + G^K \le 0, \tag{5.22}
$$

and E^K is non-increasing.

□

Proof of Lemma 2.4. Note that u_k is the k-th gPC coefficient of the initial data u_0, and thus satisfies the spectral accuracy estimate

$$
|(u_k)_0| \le C \frac{\|u_0\|_{H_z^r}}{k^r}, \tag{5.23}
$$

at each fixed x. By integrating (5.23) in x and replacing u by $\partial^\alpha u$ and summing over α, one gets

$$
\|k^q (u_k)_0\|_{H^s} \le C k^{q-r} \|u_0\|_{H_z^{s,r}}. \tag{5.24}
$$

Thus if $r > q + \frac{1}{2}$, one has

$$
\sum_{k=1}^K \|k^q (u_k)_0\|_{H^s}^2 \le C \|u_0\|_{H_z^{s,r}}^2. \tag{5.25}
$$

Similar estimate holds for f and $\bar{u}$. Thus one has

$$
E_{s,q}^K(0) \le C \|E_{s,r}(0)\|_{L_z^1}, \tag{5.26}
$$

and the proof is finished. □

5.2 Accuracy analysis: proof of Theorem 2.5

Recall the reconstructed gPC solution

$$u^K = \sum_{k=1}^{K} u_k \phi_k(z). \tag{5.27}$$

Then at a fixed x point one has

$$\|u^K(x)\|_{L_z^2}^2 = \sum_{k=1}^{K} |u_k(x)|^2 \le \sum_{k=1}^{K} |k^q u_k(x)|^2. \tag{5.28}$$

Thus

$$\|u^K\|_{L_z^2}^2 \le E^K. \tag{5.29}$$

Similar estimates hold for f and $\bar{u}$ and their x derivatives.

Furthermore, with the assumption (5.1), one has the estimate

$$\|u^K(x)\|_{L_z^\infty}^2 \le C \left(\sum_{k=1}^{K} |u_k(x)| k^p \right)^2 \le C \left(\sum_{k=1}^{K} |k^q u_k(x)|^2 \right) \left(\sum_{k=1}^{K} k^{2(p-q)} \right) \le C \left(\sum_{k=1}^{K} |k^q u_k(x)|^2 \right), \tag{5.30}$$

since $q > p + 2$. Thus

$$\|u^K\|_{L_z^\infty(L^2)}^2 \le \|u^K\|_{L^2(L_z^\infty)}^2 \le CE^K. \tag{5.31}$$

Proof of Theorem 2.5. The gPC coefficients of the mean fluid velocity satisfies

$$\partial_t \bar{u}_k + 2\bar{u}_k + C \int\int \sqrt{\mu}(uf)_k \, \mathrm{d}v \, \mathrm{d}x = 0. \tag{5.32}$$

Denote the projection operator onto the span of $\{\phi_k\}_{k=1}^K$ by P_K. Multiplying (2.16) and (5.32) by $\phi_k(z)$ and summing in k, one gets the equations for (u^K, f^K)

$$\begin{aligned}
&\partial_t u^K + P_K(u^K \cdot \nabla_x u^K) + \nabla_x p^K - \Delta_x u^K + u^K + \int \sqrt{\mu} P_K(u^K f^K) \, \mathrm{d}v - \frac{1}{\epsilon} \int v \sqrt{\mu} f^K \, \mathrm{d}v = 0, \\
&\nabla_x \cdot u^K = 0, \\
&\partial_t f^K + \frac{1}{\epsilon} v \cdot \nabla_x f^K + \frac{1}{\epsilon}(\nabla_v - \frac{v}{2}) P_K(u^K f^K) - \frac{1}{\epsilon} u^K \cdot v\sqrt{\mu} = \frac{1}{\epsilon^2}(\frac{-|v|^2}{4} + \frac{3}{2} + \Delta_v) f^K, \\
&\partial_t \bar{u}^K + 2\bar{u}^K + \frac{1}{|\mathbb{T}^3|} \int\int \sqrt{\mu} P_K(u^K f^K) \, \mathrm{d}v \, \mathrm{d}x = 0.
\end{aligned} \tag{5.33}$$

Then subtracting from (1.7) and (1.12), one gets

$$\begin{aligned}
&\partial_t u^e + [(I - P_K)(u \cdot \nabla_x u) + P_K(u^e \cdot \nabla_x u + u^K \cdot \nabla_x u^e)] + \nabla_x p^e - \Delta_x u^e + u^e \\
&\qquad + \int \sqrt{\mu}[(I - P_K)(uf) + P_K(u^e f + u^K f^e)] \, \mathrm{d}v - \frac{1}{\epsilon} \int v \sqrt{\mu} f^e \, \mathrm{d}v = 0, \\
&\nabla_x \cdot u^e = 0, \\
&\partial_t f^e + \frac{1}{\epsilon} v \cdot \nabla_x f^e + \frac{1}{\epsilon}(\nabla_v - \frac{v}{2})[(I - P_K)(uf) + P_K(u^e f + u^K f^e)] \\
&\qquad - \frac{1}{\epsilon} u^e \cdot v\sqrt{\mu} = \frac{1}{\epsilon^2}(\frac{-|v|^2}{4} + \frac{3}{2} + \Delta_v) f^e, \\
&\partial_t \bar{u}^e + 2\bar{u}^e + \frac{1}{|\mathbb{T}^3|} \int\int \sqrt{\mu}[(I - P_K)(uf) + P_K(u^e f + u^K f^e)] \, \mathrm{d}v \, \mathrm{d}x = 0,
\end{aligned} \tag{5.34}$$

where (u^e, f^e) is the approximation error

$$u^e = u - u^K, \quad f^e = f - f^K. \tag{5.35}$$

Notice that (5.34) is linear in (u^e, f^e).

Now take ∂^α on (5.34) and do L^2 estimates in x, v, z. First notice that P_K commutes with x-derivatives, and has operator norm 1 on L^2_z. Thus one has

$$|\langle \partial^\alpha P_K(u^e \cdot \nabla_x u + u^K \cdot \nabla_x u^e), \partial^\alpha u^e \rangle_z| \le C(\|u\|_{W^{s+1,\infty}_z} + \|u^K\|_{W^{s,\infty}_z})\|u^e\|^2_{H^{s+1}_z}, \tag{5.36}$$

where the W norms mean the Sobolev norms with power index ∞. By estimating the terms $P_K(u^e f + u^K f^e)$ in the same manner, one gets the energy estimate

$$\frac{1}{2}\partial_t E^e \le -(\frac{2}{3} - CH)G^e + CS, \tag{5.37}$$

where

$$\begin{aligned}
E^e =& \|u^e\|^2_{H^s_z} + |f^e|^2_{s,z} + |\bar{u}^e|^2_{L^2_z},\\
G^e =& \|\nabla_x u^e\|^2_{H^s_z} + 2\|\bar{u}^e\|^2_{H^s_z} + \left| u^e\sqrt{\mu} - \frac{1}{\epsilon}\nabla_v f^e - \frac{1}{\epsilon}\frac{v}{2} f^e \right|^2_{s,z},\\
S =& (\|(I - P_K)(u \cdot \nabla_x u)\|^2_{H^s_z} + |(I - P_K)(uf)|^2_{s,z}),\\
H =& \|u\|_{W^{s+1,\infty}_z} + \|u^K\|_{W^{s,\infty}_z} + |f|_{W^{s,\infty}_z}.
\end{aligned} \tag{5.38}$$

First notice that by Sobolev embedding,

$$\|u\|_{W^{s+1,\infty}_z} \le C\|u\|_{L^\infty_z(H^{s+3})}, \quad |f|_{W^{s,\infty}_z} \le C\||f|_{s+2}\|_{L^\infty_z}, \tag{5.39}$$

and by (5.31)

$$\|u^K\|^2_{W^{s,\infty}_z} \le CE^K_{s+2,q}. \tag{5.40}$$

Thus H can be controlled by

$$H \le C(\|E_{s+3,0}\|_{L^\infty_z} + E^K_{s+2,q})^{1/2}. \tag{5.41}$$

In view of Lemma 2.4, for $r > p + \frac{5}{2}$ one has

$$H \le C\|E_{s+3,r}\|^{1/2}_{L^\infty_z}, \tag{5.42}$$

which implies that

$$CH \le \frac{1}{6}, \tag{5.43}$$

in (5.37) for all time if $\|E_{s+3,r}(0)\|_{L^\infty_z} \le c_1''(s,r) \le \min\{\frac{1}{4C}, c_1(s,r), c_2(s,q)\}$, in view of Theorem 2.1 and Theorem 2.3.

To estimate the source term S, notice that at each fixed x, v,

$$\|(I - P_K)\partial^\alpha(uf)(x,v)\|_{L^2_z} \le C\frac{\|\partial^\alpha(uf)(x,v)\|_{H^r_z}}{K^r}. \tag{5.44}$$

Integrate in x, v,

$$|(I - P_K)(uf)|_{s,z} \le C\frac{|uf|_{s,r,z}}{K^r}. \tag{5.45}$$

By Lemma 3.2,

$$|uf|_{s,r} \le C\|u\|_{H^{s,r}}|f|_{s,r} \le C(\|u\|^2_{H^{s,r}} + |f|^2_{s,r}). \tag{5.46}$$

Thus

$$\||uf|_{s,r}\|_{L^2_z} \le \||uf|_{s,r}\|_{L^\infty_z} \le C\|\|u\|^2_{H^{s,r}} + |f|^2_{s,r}\|_{L^\infty_z} \le C\|E_{s,r}\|_{L^\infty_z}. \tag{5.47}$$

Then by Theorem 2.2 (suppress the dependence on C^h), taking $c_1'' \le c_1'(s,r)$,

$$E_{s,r}(t) \le Ce^{-\lambda t}. \tag{5.48}$$

Thus one finally gets

$$|(I-P_K)(uf)|_{s,z} \le \frac{Ce^{-\lambda t}}{K^r}. \tag{5.49}$$

The term $\|(I-P_K)(u\cdot\nabla_x u)\|_{L^2}$ can be estimated similarly, by taking $c_1'' \le c_1'(s+1,r)$, and one gets

$$S \le \frac{Ce^{-2\lambda t}}{K^{2r}}. \tag{5.50}$$

In conclusion, we have the estimate

$$\partial_t E^e + G^e \le \frac{C}{K^r}e^{-\lambda t}. \tag{5.51}$$

Finally, combining (5.37), (5.43) and (5.50), noticing that $\int_0^\infty e^{-2\lambda t}\,\mathrm{d}t$ converges, one concludes that $E^e \le \frac{C}{K^{2r}}$ uniformly in time and ϵ.

□

5.3 Hypocoercivity estimates for the error: proof of Theorem 2.6

Proof of Theorem 2.6. In order to get a hypocoercivity estimate for (u^e, f^e), one writes the equation of $\partial^\alpha f^e$ as

$$\begin{aligned}\partial_t\partial^\alpha f^e + \frac{1}{\epsilon}\mathcal{P}\partial^\alpha f^e + \frac{1}{\epsilon^2}\mathcal{K}^*\mathcal{K}\partial^\alpha f^e &= \frac{1}{\epsilon}\partial^\alpha u^e\cdot v\sqrt{\mu}\\ &+ \frac{1}{\epsilon}[(I-P_K)\partial^\alpha(u\cdot\mathcal{K}^*f) + P_K\partial^\alpha(u^e\cdot\mathcal{K}^*f) + P_K\partial^\alpha(u^K\cdot\mathcal{K}^*f^e)].\end{aligned} \tag{5.52}$$

The linear terms can be handled in the same way as Lemma 4.2. The first nonlinear term is estimated by

$$\left|\frac{1}{\epsilon}((\,(I-P_K)\partial^\alpha(u\cdot\mathcal{K}^*f), f^e))_z\right| \le \frac{C}{K^r}\frac{1}{\epsilon^2}\|u\|_{L^\infty_z(H^{s+3,r})}([[f,f]]_{s,r,z} + [[f^e,f^e]]_{s,z}). \tag{5.53}$$

In fact, by modifying the proof of Lemma 4.2, one can get an expression like (4.4):

$$((\,(I-P_K)\partial^\alpha(u\cdot\mathcal{K}^*f), \partial^\alpha f^e))_z = 2\langle (I-P_K)\partial^\alpha(u\cdot\mathcal{K}f), \mathcal{K}^2\partial^\alpha f^e\rangle_z + \text{similar terms}. \tag{5.54}$$

The first term in (5.54) is estimated by

$$\begin{aligned}&|\langle (I-P_K)\partial^\alpha(u\cdot\mathcal{K}f), \mathcal{K}^2\partial^\alpha f^e\rangle_z|\\ \le&\|(I-P_K)\partial^\alpha(u\cdot\mathcal{K}f)\|_{L^2_z}\|\mathcal{K}^2\partial^\alpha f^e\|_{L^2_z} \le \frac{C}{K^r}|\partial^\alpha(u\cdot\mathcal{K}f)|_{0,r,z}\|\mathcal{K}^2 f^e\|_{H^s_z}\\ \le&\frac{C}{K^r}\max_{|\gamma|\le r, |\beta|\le s}\|\partial^\beta u^\gamma\|_{L^\infty_z}|\mathcal{K}f|_{s,r,z}\|\mathcal{K}^2 f^e\|_{H^s_z}\\ \le&\frac{C}{K^r}\|u\|_{L^\infty_z(H^{s+3,r})}|\mathcal{K}f|_{s,r,z}\|\mathcal{K}^2 f^e\|_{H^s_z},\end{aligned} \tag{5.55}$$

and other terms in (5.54) can be estimated similarly

The second nonlinear term in (5.52) is estimated by Lemma 4.2 as follows:

$$\begin{aligned}\left|\frac{1}{\epsilon}((P_K\partial^\alpha(u^e\cdot\mathcal{K}^*f),\partial^\alpha f^e))_z\right| &\le \left|\frac{1}{\epsilon}((\partial^\alpha(u^e\cdot\mathcal{K}^*f),\partial^\alpha f^e))_z\right| \\ &\le C\frac{1}{\epsilon^2}\|u^e\|_{L^\infty_z(H^{s+3})}(C(\delta)[[f,f]]_{s,z}+\delta[[f^e,f^e]]_{s,z}).\end{aligned} \tag{5.56}$$

The third nonlinear term is estimated by Lemma 4.2 as follows:

$$\left|\frac{1}{\epsilon}((P_K\partial^\alpha(u^K\cdot\mathcal{K}^*f^e),\partial^\alpha f^e))_z\right| \le \left|\frac{1}{\epsilon}((\partial^\alpha(u^K\cdot\mathcal{K}^*f^e),\partial^\alpha f^e))_z\right| \le C\frac{1}{\epsilon^2}\|u^K\|_{L^\infty_z(H^{s+3})}[[f^e,f^e]]_{s,z}. \tag{5.57}$$

Now by the assumption that $\|E_{s+3,r}(t)\|_{L^\infty_z}$ and $\|E^K_{s+3}(t)\|_{L^\infty_z}$ small enough at $t=0$ (which implies that they are small enough for all time, by Theorem 2.1 and Theorem 2.3), and as a result, $\|u\|_{L^\infty_z(H^{s+3,r})}$ and $\|u^K\|_{L^\infty_z(H^{s+3})}$ are small enough. By Theorem 2.5, $\|u^e\|_{L^\infty_z(H^{s+3})}$ is bounded by C. Then by choosing δ in (5.56) small enough, all the $[[f^e,f^e]]_{s,z}$ terms from the nonlinear terms can be absorbed by the corresponding term from the linear terms, and then one concludes the estimate

$$\partial_t((f^e,f^e))_{s,z}+\lambda^e_1\frac{1}{\epsilon^2}[[f^e,f^e]]_{s,z}\le C(\lambda^e_1)(\|u^e\|^2_{H^s_z}+\|\nabla_x u^e\|^2_{H^s_z}+\frac{1}{\epsilon^2}\|\mathcal{K}f^e\|^2_{H^s_z})+\frac{C}{K^r}\frac{1}{\epsilon^2}[[f,f]]_{s,r,z}. \tag{5.58}$$

Finally, similar to the proof of Theorem 2.2, by taking a suitable linear combination of (5.58)(5.51) and (4.18) integrated in z (where the appearance of (4.18) is to control the term $[[f,f]]_{s,r,z}$ in (5.58)), one gets

$$\partial_t\tilde{E}^e+\frac{1}{2}\tilde{G}^e\le\lambda^e_4\frac{C}{K^r}\frac{1}{\epsilon^2}[[f,f]]_{s,r,z}+\frac{C}{K^r}e^{-\lambda t}, \tag{5.59}$$

where

$$\tilde{E}^e=E^e+\lambda^e_4((f^e,f^e))_{s,z}+\frac{1}{K^r}\lambda^e_5\|\tilde{E}\|_{L^1_z}, \tag{5.60}$$

and

$$\tilde{G}^e=G^e+\lambda^e_4\lambda^e_1\frac{1}{\epsilon^2}[[f^e,f^e]]_{s,z}+\frac{1}{2K^r}\lambda^e_5\|\tilde{G}\|_{L^1_z}. \tag{5.61}$$

The choice of λ^e_4 is in the same way as the choice of λ_4. To choose λ^e_5, one wants the $\tilde{G}$ term to control the first RHS term in (5.59), and thus choose

$$\lambda^e_5=4\frac{C\lambda^e_4}{\lambda_4\lambda_1}, \tag{5.62}$$

where the C is the first constant in (5.59). Then

$$\partial_t\tilde{E}^e+\frac{1}{4}\tilde{G}^e\le\frac{C}{K^r}e^{-\lambda t}. \tag{5.63}$$

Then since $\tilde{E}^e\le C\tilde{G}^e$ (which can be proved similarly as the proof of $\tilde{E}\le C\tilde{G}$, see (4.21)), one concludes that

$$\tilde{E}^e\le\frac{C}{K^r}e^{-\lambda^e t}, \tag{5.64}$$

where $\lambda^e=\min\{\lambda,\frac{1}{4C}\}-\delta$ for some $\delta>0$ small enough, in view of the lemma below. □

Lemma 5.3. *Let* $\Phi = \Phi(t)$ *satisfy*

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} + a_1\Phi \le a_2 e^{-a_3 t}. \tag{5.65}$$

Then

$$\Phi(t) \le e^{-at}(\Phi(0) + a_2 C(\delta)), \tag{5.66}$$

with $a = \min\{a_1, a_3\} - \delta$*,* δ *being any positive constant.*

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{a_1 t}\Phi) \le a_2 e^{(a_1-a_3)t}, \tag{5.67}$$

$$e^{a_1 t}\Phi \le \Phi(0) + \int_0^t a_2 e^{(a_1-a_3)s}\,\mathrm{d}s, \tag{5.68}$$

$$\Phi(t) \le e^{-a_1 t}\Phi(0) + a_2\frac{e^{-a_3 t} - e^{-a_1 t}}{a_1 - a_3} = e^{-a_1 t}\Phi(0) + a_2 t e^{-\xi t}, \tag{5.69}$$

for some ξ between a_1 and a_3, by the mean value theorem. Then the conclusion follows since

$$te^{-\xi t} \le e^{-at}(te^{-\delta t}) \le C(\delta)e^{-at}, \tag{5.70}$$

where $C(\delta) = (\delta e)^{-1}$. □

6 Conclusion

In this paper we first prove the uniform regularity in the random space for a kinetic-fluid two-phase flow model with the light particle regime for random initial data near the global equilibrium, using an energy estimate in suitable Sobolev spaces. By hypocoercivity arguments we prove the energy $E(t)$ decays exponentially in time. This result implies that for random initial data near the global equilibrium, the long time behavior of the solution is insensitive to the random perturbation on initial data. Then we prove a result on the time decay of the solution of the generalized polynomial chaos stochastic Galerkin (gPC-sG) method, in which the requirement of the random initial data is independent of K, the number of basis functions. The key idea in this proof is the usage of E^K, a weighted sum of Sobolev norms of the gPC coefficients. Finally we prove the uniform spectral accuracy of the sG method for random initial data near the global equilibrium, by doing energy and hypocoercivity estimates on the sG error (u^e, f^e). All the constants involved in the results are *independent of* ϵ, the Knudsen number.

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