Sparse Stabilization of Dynamical Systems Driven by Attraction and Avoidance Forces ⋆

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Abstract:
We address dynamical systems of agents driven by attraction and repulsion forces, modelling cohesion and collision avoidance. When the total energy, which is composed of a kinetic part and a geometrical part describing the balance between attraction and repulsion forces, is below a certain threshold, then it is known that the agents will converge to a dynamics where mutual space confinement is guaranteed. In this paper we question the construction of a stabilization strategy, which requires the minimal amount of external intervention for nevertheless inducing space confinement, also when the initial energy threshold is violated. Our main result establishes that if the initial energy exceeds the threshold mainly because of its kinetic component, then a sparse control instantaneously applied with enough strength on the most rowdy agent, i.e., the one with maximal speed, will be able to steer in finite time the system to an energy level under the threshold.

Keywords: "We don't need no (sparse) control!"

1. INTRODUCTION

A popular model to describe the emergence of velocity consensus (flocking behavior) in a population of interacting and moving agents is the Cucker-Smale dynamical system, which was introduced in (Cucker and Smale, 2007). The behavior of the population is governed by the system of differential equations of the location $x_i(t)$ and the velocity $v_i(t)$ for every agent $i = 1, \ldots, N$:

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \sum_{j=1}^{N} a\left(\|x_i - x_j\|^2\right)(v_j - v_i), \quad i = 1, \ldots, N \end{cases}$$

where

$$a(r) = \frac{H}{(1+r)^\beta} \quad H > 0,\, \beta \geq 0,$$

describes the strength of the attractive force that one agents applies on another agent at the distance r. Agents, which follow these dynamics, will, under certain conditions on the initial configuration, exhibit flocking behaviour. Here flocking means that both velocities and inter-agent spacing tends to a certain consensus: $\lim_{t\to\infty} v_i(t) = \bar{v}$ and $\lim_{t\to\infty}(x_i(t) - x_j(t)) = \hat{x}_{ij}\ \forall\, i, j = 1, \ldots, N$.

Still this model is not well suited to describe physical agents, as the model does not force a spatial separation of agents, i.e., $x_i(t) - x_j(t) \neq 0$ for all $i, j = 1, \ldots, N$ and for all $t \geq 0$ is not enforced. To eliminate this shortcoming, in (Cucker and Dong, 2012) the authors modified the model as follows:

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = \underbrace{-b_i(t)v_i(t)}_{\text{damping}} \\ \qquad\quad + \underbrace{\sum_{j=1}^{N} a\left(\|x_i - x_j\|^2\right)(x_j - x_i)}_{\text{attraction force}} \\ \qquad\quad + \underbrace{\sum_{j=1}^{N} f\left(\|x_i - x_j\|^2\right)(x_i - x_j)}_{\text{repulsion force}}, \quad i = 1, \ldots, N \end{cases} \tag{1}$$

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This model now includes uniformly continuous, bounded damping functions $b_i : [0, \infty) \to [0, \Lambda]$, for $\Lambda > 0$. This damping term could be interpreted as a friction, which helps the system to stay confined, as we will show in details below. Furthermore it includes a locally Lipschitz continuous, nonincreasing repulsion function $f : (\delta, \infty) \to [0, \infty)$ which satisfies $\int_\delta^\infty f(r)dr < \infty$, for $\delta > 0$. This function can be seen as the repulsive force, which every agent uses on every other agent at distance r. Moreover, in contrast to the Cucker-Smale model, the attraction force acts in the direction of the difference of the locations of the agents and not between the velocities anymore.

To quantify the behavior of the system we introduce an energy, which consists of a kinetic part and a geometrical part, describing the balance between attraction and repulsion forces,

$$E(t) = E\left(x(t), v(t)\right) =$$

$$\underbrace{\sum_{i=1}^N \|v_i\|^2}_{\text{kinetic}} + \underbrace{\frac{1}{2}\sum_{i,j=1}^N \int_\delta^{\|x_i - x_j\|^2} a(r)dr + \frac{1}{2}\sum_{i,j=1}^N \int_{\|x_i-x_j\|^2}^\infty f(r)dr.}_{\text{geometric}}$$

Now, the core statement of the paper by Cucker and Dong is the following theorem, which gives conditions depending on the initial energy state for the system to tend to a consensus, defined as a uniform mutual confinement in space of the interacting agents.

Theorem 1. Consider the dynamics of a population of N agents under system (1) with initial positions satisfying $\|x_i(0) - x_j(0)\|^2 > \delta$ for all $i \neq j$ and initial energy state

$$E(0) < \frac{1}{2}\int_\delta^\infty f(r)dr.$$

Then, there exists a unique solution $(x(t), v(t))$ of system (1) with initial state $(x(0), v(0))$ and we have $b_i(t)v_i(t) \to 0$ as $t \to \infty$ for $1 \le i \le N$. In addition, assume that one of the following hypothesis holds:

(1) $\beta \le 1$,
(2) $\beta > 1$ and

$$E(0) < (N-1)\int_\delta^\infty a(r)dr.$$

Then, the population is cohesive and collision avoiding. More precisely, there exists two positive constants $B_0 > 0$ and $d_0 > \delta$ such that, for all $t \ge 0$, $d_0 \le \|x_i(t) - x_j(t)\|^2 \le B_0$, for all $1 \le i \neq j \le N$.

2. CONSENSUS CONTROL FOR THE CUCKER-DONG MODEL

Theorem 1 provides a sufficient condition for any population, modeled by a system like (1), to be cohesive and collision avoiding. However, it is also possible to produce counterexamples to consensus when such conditions are not fulfilled (Cucker and Dong, 2012). In the latter situation is therefore interesting to question whether by applying an external feedback control is possible to stabilize the dynamics to enter in finite time the *region of consensus* determined by condition (2) of Theorem 1; moreover, we shall also require that our control is "economic", i.e., it acts just on few agents of the system at each time. As we exclude the case (1) of Theorem 1 which would bring to unconditional consensus, we shall assume from now on $\beta > 1$. The strategy we will adopt is inspired by the one described in (Caponigro et al., 2012) for the Cucker-Smale model: we consider the system (1) and suppose that every agent may be subject to an external field $u_i : [0, +\infty) \to \mathbb{R}^d$, which are measurable functions for $i = 1, \dots, N$, satisfying the $\ell_1^N - \ell_2^d$-norm constraint

$$\sum_{i=1}^N \|u_i(t)\| \le M, \tag{2}$$

for every $t \ge 0$, for a given positive constant M. Thus, the time evolution of the state of each agent is given by

$$\begin{cases} \dot{\widetilde{x}}_i(t) = \widetilde{v}_i(t) \\ \dot{\widetilde{v}}_i(t) = -b_i(t)\widetilde{v}_i(t) + \sum_{j=1}^N a\left(\|\widetilde{x}_i - \widetilde{x}_j\|^2\right)(\widetilde{x}_j - \widetilde{x}_i) \\ \qquad + \sum_{j=1}^N f\left(\|\widetilde{x}_i - \widetilde{x}_j\|^2\right)(\widetilde{x}_i - \widetilde{x}_j) + u_i, \\ \qquad i = 1, \dots, N. \end{cases} \tag{3}$$

Our "economical criterion" is formulated in terms of the sparsity of the vector $u(t) = (u_1(t), \dots, u_N(t))$, i.e., for every $t \ge 0$ at most one of the entries of $u(t)$ is nonzero.

We begin with an estimate from above of the rate of decay of the energy function for the controlled system.

Lemma 2. For every $t \ge 0$,

$$\frac{d}{dt}E(\widetilde{x}(t), \widetilde{v}(t)) \le 2\sum_{i=1}^N \langle u_i(t), \widetilde{v}_i(t)\rangle \tag{4}$$

Proof: Let us compute

$$\begin{aligned} \frac{d}{dt}E(\widetilde{x}, \widetilde{v}) = {} & \frac{d}{dt}\sum_{i=1}^N \|\widetilde{v}_i\|^2 \\ & + \sum_{i,j=1}^N a(\|\widetilde{x}_i - \widetilde{x}_j\|^2)\langle \widetilde{x}_i - \widetilde{x}_j, \widetilde{v}_i - \widetilde{v}_j\rangle \\ & - \sum_{i,j=1}^N f(\|\widetilde{x}_i - \widetilde{x}_j\|^2)\langle \widetilde{x}_i - \widetilde{x}_j, \widetilde{v}_i - \widetilde{v}_j\rangle. \end{aligned} \tag{5}$$

The first term of the sum above is

$$
\begin{aligned}
\frac{d}{dt}\sum_{i=1}^{N}\|\widetilde{v}_i\|^2 &= 2\sum_{i=1}^{N}\langle\dot{\widetilde{v}}_i,\widetilde{v}_i\rangle \\
&= -2\sum_{i=1}^{N} b_i\|\widetilde{v}_i\|^2 \\
&\quad -\sum_{i,j=1}^{N} a(\|\widetilde{x}_i-\widetilde{x}_j\|^2)\langle\widetilde{x}_i-\widetilde{x}_j,\widetilde{v}_i-\widetilde{v}_j\rangle \\
&\quad +\sum_{i,j=1}^{N} f(\|\widetilde{x}_i-\widetilde{x}_j\|^2)\langle\widetilde{x}_i-\widetilde{x}_j,\widetilde{v}_i-\widetilde{v}_j\rangle \\
&\quad +2\sum_{i=1}^{N}\langle u_i,\widetilde{v}_i\rangle, \qquad (6)
\end{aligned}
$$

which, plugging (6) into (5), yields the desired inequality

$$
\begin{aligned}
\frac{d}{dt}E(\widetilde{x},\widetilde{v}) &= -2\sum_{i=1}^{N} b_i\|\widetilde{v}_i\|^2 + 2\sum_{i=1}^{N}\langle u_i,\widetilde{v}_i\rangle \\
&\le 2\sum_{i=1}^{N}\langle u_i,\widetilde{v}_i\rangle.
\end{aligned}
$$

□

Definition 3. Let $\varepsilon > 0$ and

$$
c_i(t) = \begin{cases} 1 & \text{if } i = k^\star \\ 0 & \text{if } i \ne k^\star \end{cases}
$$

where $k^\star$ is an index such that

$$
\|\widetilde{v}_{k^\star}(t)\| = \max_{i=1,\dots,N}\|\widetilde{v}_i(t)\|. \qquad (7)
$$

We define the sparse feedback control

$$
u_i(t) = -\alpha_i(t)\widetilde{v}_i(t), \text{ where } \alpha_i(t) = \frac{c_i(t)\varepsilon E(t)}{2\|\widetilde{v}_i(t)\|^2}. \qquad (8)
$$

Notice how the sparsity of $u(t)$ follows directly from the definition. As we are considering sparse feedback controls, which can change discontinuously in time, an appropriate definition of Filippov solution for the system (3) in terms of a differential inclusion would be formally required. For the sake of simplicity in this proceeding paper, we assume that a global solution of system (3) exists, which is piecewise C^1 in time (actually this would be legitimate if we considered, for instance, a sampling-and-hold approach as in Caponigro et al. (2012)). A more rigorous analysis will be presented in the extended journal paper (Bongini et al., 2013).

The following will be the main result of our work; with $\overline{v}(t)$ we denote the mean velocity of the system (3) at time t, i.e.,

$$
\overline{v}(t) = \frac{1}{N}\sum_{i=1}^{N}\widetilde{v}_i(t),
$$

and

$$
\vartheta := (N-1)\int_\delta^\infty a(r)dr,
$$

will stand for the energy threshold under which the system will successfully enter the consensus region.

Theorem 4. Suppose that $\|\overline{v}(0)\| \ge \eta > 0$, for η small enough, and that $E(0) > \vartheta$ (hence we are not in the consensus region). Then, provided the following conditions hold:

$$
\ln\left(\frac{E(0)}{\vartheta}\right) \le \frac{N}{2}\frac{(\|\overline{v}(0)\|^2-\eta^2)}{E(0)}, \qquad (9)
$$

$$
\frac{2N\Lambda E(0)\ln\left(\frac{E(0)}{\vartheta}\right)}{N(\|\overline{v}(0)\|^2-\eta^2) - 2E(0)\ln\left(\frac{E(0)}{\vartheta}\right)} \le \varepsilon \le \frac{2M\eta}{E(0)}, \qquad (10)
$$

the control designed in Definition 3 satisfies condition (2) and forces $E(t) < \vartheta$ as soon as $t > T^\star$, where

$$
T^\star = \frac{1}{\varepsilon}\ln\left(\frac{E(0)}{\vartheta}\right).
$$

Moreover (9) will only hold if the following relationship between the geometric part of the energy

$$
E_{geom}(0) := E(0) - \sum_{j=1}^{N}\|\widetilde{v}_j(0)\|^2
$$

and the energy threshold is satisfied:

$$
\frac{2\vartheta}{\sqrt{e}} \ge E_{geom}(0). \qquad (11)
$$

Remark 5. Before proving this result, we may want to comment its meaning. As the control we have chosen exclusively acts on velocity components because of (4), we immediately see that it will be affecting to our advantage only the kinetic part of the energy $E(t)$, while the geometric part is basically uncontrollable. For this reason, the condition (9) means that the initial energy $E(0)$ has to exceed the threshold ϑ only slightly and that this excess is mainly due to a large kinetic component (actually estimated from below by $N\|\overline{v}(0)\|^2 \le \sum_{j=1}^{N}\|\widetilde{v}_j(0)\|^2$), while the geometrical component $E_{geom}(0)$ has to be below the threshold $\frac{2\vartheta}{\sqrt{e}}$ as given by (11). Hence, if we apply a control with enough strength, as ensured by (10), we will be able to steer the system in finite time to a global energy level $E(t > T^\star)$ under the threshold ϑ, before the geometric part of the energy may start to affect the dynamics to our disadvantage.

In order to prove our theorem, let us start from some facts which follow from our choice of the control. For $\|\overline{v}(0)\| > \eta$, for technical reasons, we fix the following auxiliary time horizon

$$
T = \frac{N(\|\overline{v}(0)\|^2-\eta^2)}{(2N\Lambda+\varepsilon)E(0)}. \qquad (12)
$$

Lemma 6. For every $0 \le t \le T$,

$$
E(t) \le E(0)e^{-\varepsilon t}. \qquad (13)
$$

Proof: Indeed, the inequality (4) yields

$$
\begin{aligned}
\frac{d}{dt}E &\le 2\left\langle\frac{\varepsilon E}{2\|\widetilde{v}_{k^\star}\|^2}\widetilde{v}_{k^\star},\widetilde{v}_{k^\star}\right\rangle \\
&= \varepsilon E
\end{aligned}
$$

so, integrating between 0 and t we get the desired estimate.

□

Lemma 7. If $\|\overline{v}(0)\| \geq \eta > 0$ then we have

$$\|\widetilde{v}_{k^\star}(t)\| \geq \eta, \tag{14}$$

for all $t \in [0,T]$.

Proof: We shall derive an estimate from below of $\|\overline{v}(t)\|^2$.

$$\begin{aligned}
\frac{d}{dt}\|\overline{v}(t)\|^2 &= 2\left\langle \dot{\overline{v}}(t), \overline{v}(t)\right\rangle \\
&= -\frac{2}{N}\sum_{j=1}^{N}(b_j(t)+\alpha_j(t))\left\langle \widetilde{v}_j(t), \overline{v}(t)\right\rangle \\
&\geq -\frac{2}{N}\sum_{j=1}^{N}(b_j(t)+\alpha_j(t))\|\widetilde{v}_{k^\star}(t)\|^2 \\
&= -\frac{1}{N}\left[2\sum_{j=1}^{N} b_j(t)\|\widetilde{v}_{k^\star}(t)\|^2 + \varepsilon E(t)\right] \\
&\geq -\frac{1}{N}\left[2\sum_{j=1}^{N} b_j(t) + \varepsilon\right] E(t),
\end{aligned}$$

having used Cauchy-Schwarz in the first inequality and the maximality property (7) of the index $k^\star$. Now, using (13) and the fact that the b_j's are limited from above by Λ, we get

$$\begin{aligned}
-\frac{1}{N}\left[2\sum_{j=1}^{N} b_j(t) + \varepsilon\right] E(t) &\geq -\frac{1}{N}\left[2\sum_{j=1}^{N} b_j(t) + \varepsilon\right] E(0) \\
&\geq -\frac{1}{N}\left[2N\Lambda + \varepsilon\right] E(0).
\end{aligned}$$

Hence we obtain the inequality

$$\frac{d}{dt}\|\overline{v}(t)\|^2 \geq -\frac{1}{N}\left[2N\Lambda + \varepsilon\right] E(0),$$

which, integrated between 0 and t, yields

$$\|\overline{v}(t)\|^2 \geq -\frac{1}{N}\left[2N\Lambda + \varepsilon\right] E(0)t + \|\overline{v}(0)\|^2.$$

and thus,

$$\|\widetilde{v}_{k^\star}(t)\|^2 \geq \|\overline{v}(t)\|^2 \geq -\frac{1}{N}\left[2N\Lambda + \varepsilon\right] E(0)t + \|\overline{v}(0)\|^2,$$

again by maximality of the index $k^\star$.

Now, by taking T as in (12), we get

$$\|\widetilde{v}_{k^\star}(t)\| \geq \eta,$$

for all $t \in [0,T]$.

□

As an easy corollary of the lemma above, we get that our control satisfies condition (2), if ε is below a certain threshold.

Corollary 8. If $\varepsilon \leq \frac{2M\eta}{E(0)}$, then

$$\sum_{i=1}^{N}\|u_i(t)\| \leq M.$$

for all $t \in [0,T]$.

Proof: Clearly,

$$\begin{aligned}
\sum_{i=1}^{N}\|u_i(t)\| &= \sum_{i=1}^{N}\alpha_i(t)\|\widetilde{v}_i(t)\| \\
&= \frac{\varepsilon E(t)}{2\|\widetilde{v}_{k^\star}(t)\|} \\
&\leq \frac{\varepsilon E(0)}{2\eta} \\
&\leq M,
\end{aligned}$$

using Lemma 7.

□

We are now ready to prove our main result.

Proof of Theorem 4: We shall show that applying the control (8) will bring the group to the consensus region before the bound on the control (2) may get violated, hence, we shall actually prove that

$$T^\star \leq T.$$

But the latter condition is equivalent to

$$\frac{1}{\varepsilon}\ln\left(\frac{E(0)}{\vartheta}\right) \leq \frac{N(\|\overline{v}(0)\|^2 - \eta^2)}{(2N\Lambda+\varepsilon)E(0)}$$

which can be rewritten as

$$\begin{aligned}
&\varepsilon\left[N(\|\overline{v}(0)\|^2 - \eta^2) - 2E(0)\ln\left(\frac{E(0)}{\vartheta}\right)\right] \\
&\geq 2N\Lambda E(0)\ln\left(\frac{E(0)}{\vartheta}\right).
\end{aligned}$$

Since, by hypothesis, the quantity between square brackets is positive, we get the equivalent condition

$$\varepsilon \geq \frac{2N\Lambda E(0)\ln\left(\frac{E(0)}{\vartheta}\right)}{N(\|\overline{v}(0)\|^2 - \eta^2) - 2E(0)\ln\left(\frac{E(0)}{\vartheta}\right)},$$

which is actually true by assumption.

We shall now prove the necessary condition (11) for (9) to hold.

From condition (9) one immediately obtains

$$N\left\|\overline{v}(0)\right\|^2 - 2E(0)\ln\left(\frac{E(0)}{\vartheta}\right) \geq 0.$$

The initial total energy $E(0)$ can be split up in its kinetic part and its geometric part, which does not depend on the velocities: by triangle inequality, we can estimate the kinetic energy from below by

$$N\left\|\overline{v}(0)\right\|^2 \leq \sum_{j=1}^{N}\left\|\widetilde{v}_j(0)\right\|^2.$$

For the sake of simplicity, let us set $s := \sum_{j=1}^{N} \|\widetilde{v}_j(0)\|^2$. This gives us

$$\gamma(s) := s - 2(s + E_{geom}(0)) \ln\left(\frac{s + E_{geom}(0)}{\vartheta}\right) \geq 0 \quad (15)$$

We will now investigate the maximum of the left hand side in terms of s. For this we differentiate γ with respect to s, and we search for its null points

$$\frac{\partial \gamma}{\partial s} = -1 - 2 \ln\left(\frac{s + E_{geom}(0)}{\vartheta}\right) \stackrel{!}{=} 0,$$

as the function is actually concave

$$\frac{\partial^2 \gamma}{\partial s^2} = -\frac{2}{s + E_{geom}(0)} < 0.$$

The only root of the derivative is given by

$$s^* = \vartheta e^{-\frac{1}{2}} - E_{geom}(0).$$

Hance $\gamma(s^*) \geq \gamma(s) \geq 0$ from (15), and we obtain

$$2\vartheta e^{-\frac{1}{2}} - E_{geom}(0) \geq 0,$$

which, rearranged, gives (11).

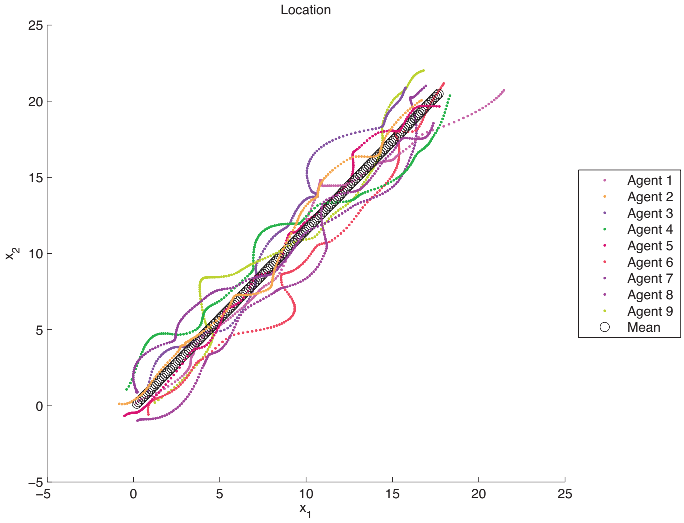
□

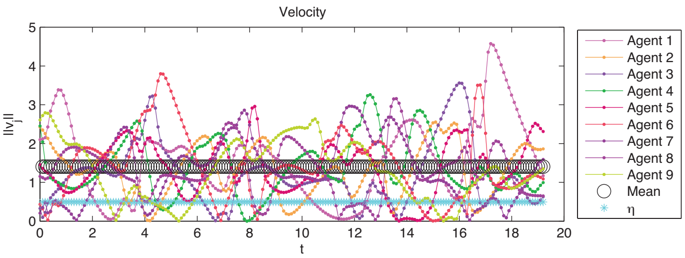
3. NUMERICAL SIMULATIONS

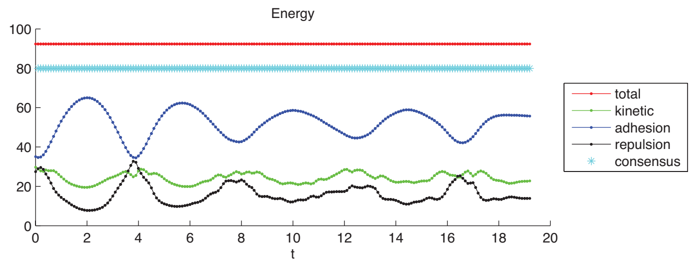
In this section we collect numerical simulations to illustrate the results of Theorem 4.

3.1 Without Control

First of all we show the dynamics of a system of 9 agents under attraction and repulsion forces, for which the initial energy $E(0)$ exceeds the threshold ϑ. While from this short-time simulation we cannot affirm that the system tends to lack of cohesion, we can nevertheless see that the total energy remains over the threshold, hence it will not give us any guarantees that in the future the cohesion of the group may get suddenly lost. The figures below are self-explanatory.





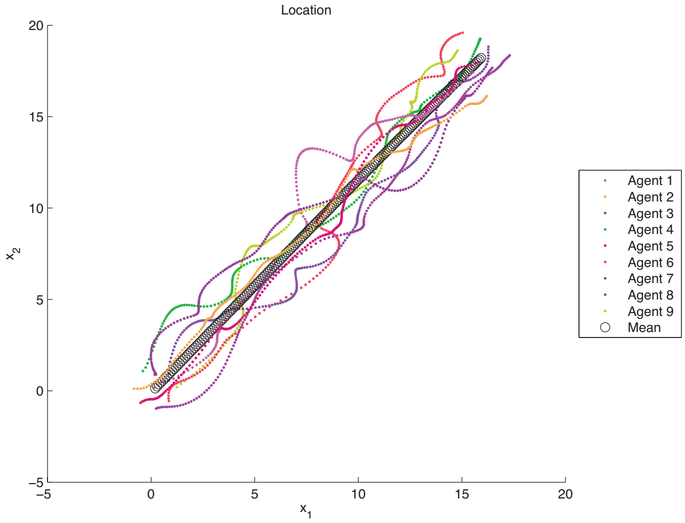


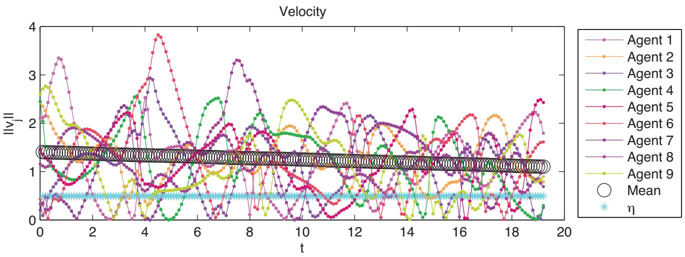
3.2 With Control

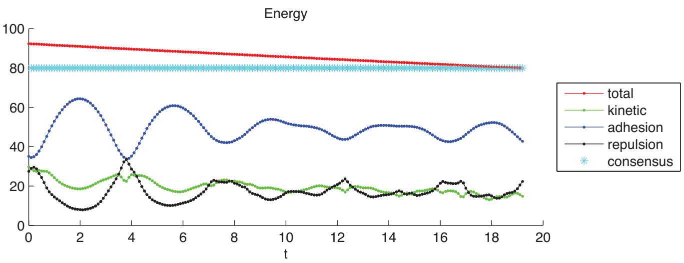
In the simulation reported in this section, and starting from the same initial conditions as the case above of the uncontrolled system, in particular $E(0) > \vartheta$, we apply now the control (8). The effect is of reducing the total energy, bringing it under the threshold in finite time, as predicted by Theorem 4. Notice that the condition

$$\|\bar{v}(t)\| > \eta,$$

for having a bounded control in the sense of (2) and Corollary 8, is actually satisfied with a large gap, denoting that our analysis is quite conservative in this case. We notice also that the control mainly affects the kinetic energy, which is substantially reduced, while the geometric part of the energy does not seem to be significantly influenced. Of course, after reaching the critical energy level $E(t > T^{\star}) < \vartheta$, we will have a guaranteed cohesion and avoidance properties of the dynamics as in Theorem 1.







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