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Analyticity and Decay Estimates of the Navier–Stokes Equations in Critical Besov Spaces

Hantaek Bae, Animikh Biswas, Eitan Tadmor

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Abstract

In this paper, we establish analyticity of the Navier–Stokes equations with small data in critical Besov spaces $\dot{B}^{\frac{3}{p}-1}_{p,q}$. The main method is Gevrey estimates, the choice of which is motivated by the work of Foias and Temam (Contemp Math 208:151–180, 1997). We show that mild solutions are Gevrey regular, that is, the energy bound $\|e^{\sqrt{t}\Lambda}v(t)\|_{E_p} < \infty$ holds in $E_p := \tilde{L}^{\infty}(0, T; \dot{B}^{\frac{3}{p}-1}_{p,q}) \cap \tilde{L}^{1}(0, T; \dot{B}^{\frac{3}{p}+1}_{p,q})$, globally in time for $p < \infty$. We extend these results for the intricate limiting case $p = \infty$ in a suitably designed E_∞ space. As a consequence of analyticity, we obtain decay estimates of weak solutions in Besov spaces. Finally, we provide a regularity criterion in Besov spaces.

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1. Introduction and Statement of Main Results

It is well-known that regular solutions of many dissipative equations, such as the Navier–Stokes (NS) equations, the Kuramoto–Sivashinsky equation, the surface

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quasi-geostrophic equation and the Smoluchowski equation are in fact analytic, in both space and time variables [4,14,18,36,49]. In fluid dynamics, the space analyticity radius has an important physical interpretation: at this length scale the viscous effects and the (nonlinear) inertial effects are roughly comparable. Below this length scale the Fourier spectrum decays exponentially [13,17,27,28]. In other words, the space analyticity radius yields a Kolmogorov type length scale encountered in turbulence theory. At a more practical level, this fact can be used to show that the finite dimensional Galerkin approximations converge exponentially quickly in these cases [12]. Other applications of analyticity radius occur in establishing sharp temporal decay rates of solutions in higher Sobolev norms [39], establishing geometric regularity criteria for the Navier–Stokes equations, and in measuring the spatial complexity of fluid flow [24,31,32].

In this paper, we study analyticity properties of the incompressible Navier–Stokes (NS) equations in $\mathbb{R}^3$. The system of equations is given by

$$v_t + v \cdot \nabla v - \mu \Delta v + \nabla p = 0, \tag{1.1a}$$

$$\nabla \cdot v = 0, \tag{1.1b}$$

where v is the velocity field, p is the pressure, and $\mu > 0$ is the viscosity coefficient, which for simplicity we set as $\mu = 1$.

Since $\nabla \cdot v = 0$, we can rewrite the momentum equation (1.1a) by projecting it onto the divergence-free space. Let $\mathbb{P} = Id - \nabla(-\Delta)^{-1} div$ be the orthogonal projection of L^2 on divergence-free vector fields. By applying $\mathbb{P}$ to (1.1a), we obtain

$$v_t + \mathbb{P}\nabla \cdot (v \otimes v) - \Delta v = 0. \tag{1.2}$$

Formally, we can express a solution v of (1.2) in the integral form:

$$v(t) = e^{t\Delta} v_0 - \int_0^t \left[e^{(t-s)\Delta} \mathbb{P}\nabla \cdot (v \otimes v)(s) \right] ds. \tag{1.3}$$

Any solution satisfying this integral equation is called a *mild solution*. We can find it by using a fixed point argument for the function $v \mapsto F(v)$, where

$$F(v)(t) = e^{t\Delta} v_0 - \int_0^t \left[e^{(t-s)\Delta} \mathbb{P}\nabla \cdot (v \otimes v)(s) \right] ds.$$

The invariant space for solving this integral equation corresponds to a scaling invariance property of the equation. Assume that (v, p) solves (1.1). Then, the same is true for rescaled functions:

$$v_\lambda(t, x) = \lambda v(\lambda^2 t, \lambda x), \quad p_\lambda(t, x) = \lambda^2 p(\lambda^2 t, \lambda x), \quad \lambda > 0. \tag{1.4}$$

Under these scalings, L^3, $\dot{H}^{\frac{1}{2}}$, $\dot{W}^{\frac{3}{p}-1,p}$ and $\dot{B}^{\frac{3}{p}-1}_{p,q}$ are *critical spaces* for initial data ($t = 0$), that is, the corresponding norms are invariants under these scalings. One can find various well-posedness results for small data in these critical spaces in [6,7,9,21,29,30,40].

The goal of this paper is threefold:

(i) analyticity of mild solutions in critical Besov spaces,
(ii) decay of Besov norms of weak solutions, and
(iii) a new regularity condition in Besov spaces.

More details on these three topics will be presented in Sects. 1.1, 1.2, and 1.3, respectively.

1.1. Analyticity of Mild Solutions

Let us begin with analyticity results of this paper. Compared to previous works by [15,23,37] (using iterative derivative estimates), [4] (l_p space on $\mathbb{T}^3$), and [25,26] (complexified equations), we are able to establish analyticity of the Navier–Stokes equations by obtaining *Gevrey estimates* in Besov spaces $\dot{B}^{\frac{3}{p}-1}_{p,q}$. Specifically, we will show that a solution $v(t) \in \dot{B}^{\frac{3}{p}-1}_{p,q}$ satisfies

$$\sup_{t>0} \|e^{\sqrt{t}\Lambda} v(t)\|_{\dot{B}^{\frac{3}{p}-1}_{p,q}} < \infty,$$

where Λ is the Fourier multiplier whose symbol is given by $|\xi|_1 = \sum_{i=1}^{3} |\xi_i|$. We emphasize that here $\Lambda \equiv \Lambda_1$ is quantified by the l^1 norm rather than the usual l^2 norm associated with $\Lambda_2 := (-\Delta)^{\frac{1}{2}}$. This approach enables one to avoid cumbersome recursive estimation of higher order derivatives.

In order to explain the main idea, we define $V(t) = e^{\sqrt{t}\Lambda} v(t)$. Then, $V(t)$ satisfies the following equation:

$$\begin{aligned} V(t) &= e^{\sqrt{t}\Lambda + t\Delta} v_0 - \int_0^t \Big[e^{[\sqrt{t}\Lambda + (t-s)\Delta]} \mathbb{P}\nabla \cdot (e^{-\sqrt{s}\Lambda} V(s) \otimes e^{-\sqrt{s}\Lambda} V(s)) \Big] ds \\ &= e^{\sqrt{t}\Lambda + t\Delta} v_0 \\ &\quad - \int_0^t \Big[e^{[(\sqrt{t}-\sqrt{s})\Lambda + (t-s)\Delta]} \mathbb{P}\nabla \cdot e^{\sqrt{s}\Lambda} (e^{-\sqrt{s}\Lambda} V(s) \otimes e^{-\sqrt{s}\Lambda} V(s)) \Big] ds. \end{aligned}$$

Since $e^{\sqrt{t}|\xi|_1}$ is dominated by $e^{-t|\xi|^2}$ for $|\xi| \gg 1$, the behavior of the linear term, $e^{\sqrt{t}\Lambda + t\Delta} v_0$, closely resembles that of $v(t)$. The estimates of the nonlinear term are similar to those of $v(t)$ due to the nice boundedness property of the bilinear operator B_{s}:

$$B_{\mathrm{s}}(f, g) = e^{\sqrt{s}\Lambda} \big(e^{-\sqrt{s}\Lambda} f(s) e^{-\sqrt{s}\Lambda} g(s) \big).$$

As noticed from the above argument, the existence result of $v(t)$ is crucial in establishing Gevrey regularity. Thus, in Sects. 3 (for $p < \infty$) and 4 (for $p = \infty$), we will first show the existence of a mild solution and then proceed to explain how to modify the existence proof to obtain Gevrey regularity.

Compared to previous works in [19,39], which defined Gevrey norms of the form $\|e^{\sqrt{t}\Lambda_2}v(t)\|_X$ with a L^2 based space Sobolev space X, the use of Λ instead of Λ_2 is fundamental for the estimates of B_s in L^p based function spaces. A similar approach using Λ in L^p-spaces was taken by [33]. However, our results cover a larger Besov class of initial data in the full range of $p, q \in [1, \infty]$, and the same method can be applied to other dissipative equations; see for example [1,2].

We now present our existence/analyticity results for $p < \infty$ and $p = \infty$ separately. For notational simplicity, we will suppress the dependence of norms (defined below) on q.

1.1.1. The Case $p < \infty$ The existence of global-in-time solutions for small data in (the homogeneous Besov space) $\dot{B}^{\frac{3}{p}-1}_{p,q}$ for $p < \infty$ was proved by CHEMIN [9]. The result indicates a gain of two derivatives from the maximal regularity of the heat kernel, which is realized in terms of the function space E_p,

$$E_p := \left\{u \in \mathscr{S}' : \|u\|_{E_p} = \|u\|_{\tilde{L}^\infty_t \dot{B}^{\frac{3}{p}-1}_{p,q}} + \|u\|_{\tilde{L}^1_t \dot{B}^{\frac{3}{p}+1}_{p,q}} < \infty\right\},$$

where

$$\|u\|_{\tilde{L}^\infty_t \dot{B}^{\frac{3}{p}-1}_{p,q}} = \left(\sum_{j\in\mathbb{Z}} 2^{j(\frac{3}{p}-1)q} \|\triangle_j u\|^q_{L^\infty_t L^p}\right)^{\frac{1}{q}},$$

and

$$\|u\|_{\tilde{L}^1_t \dot{B}^{\frac{3}{p}+1}_{p,q}} = \left(\sum_{j\in\mathbb{Z}} 2^{j(\frac{3}{p}+1)q} \|\triangle_j u\|^q_{L^1_t L^p}\right)^{\frac{1}{q}},$$

with the usual change for $q = \infty$ (here and below, $L^q_t X$ denotes $L^q([0, \infty); X)$).

Theorem 1 (Existence [9]). *Let* $1 \leqq p < \infty$, $1 \leqq q \leqq \infty$ *and* $v_0 \in \dot{B}^{\frac{3}{p}-1}_{p,q}$. *There exists a constant* $\epsilon_0 > 0$ *such that for all* $v_0 \in \dot{B}^{\frac{3}{p}-1}_{p,q}$ *with* $\|v_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,q}} \leqq \epsilon_0$, *the NS equations* (1.2) *admit a global-in-time solution* $v \in E_p$. *Moreover, if* $q < \infty$, $v \in C([0, \infty); \dot{B}^{\frac{3}{p}-1}_{p,q})$.

The first result of this paper is showing that solutions of Theorem 1 are, in fact, analytic in the following sense.

Theorem 2 (Analyticity). *There exists a positive constant* $\epsilon_0 > 0$ *such that for all* $v_0 \in \dot{B}^{\frac{3}{p}-1}_{p,q}$ *with* $\|v_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,q}} \leqq \epsilon_0$, *the NS equations* (1.2) *admit a solution* $v \in E_p$ *such that* $e^{\sqrt{t}\Lambda}v \in E_p$.

1.1.2. The Case $p = \infty$ The case of $\dot{B}^{-1}_{\infty,q}$ data, corresponding to $p = \infty$, is much harder because the Navier–Stokes equations are ill-posed in $\dot{B}^{-1}_{\infty,q}$ for $q > 2$ [5,22,51]. To circumvent the difficulty in this case, we prove the existence of solutions subject to a restricted class of initial data in $\dot{B}^{-1}_{\infty,q} \cap \dot{B}^{0}_{3,\infty}$, $1 \leqq q < \infty$. The corresponding function space is defined as follows.

$$E_\infty := \left\{ u \in \mathscr{S}' : \|u\|_{E_\infty} = \|u\|_{\tilde{L}^\infty_t \dot{B}^0_{3,\infty}} + \sup_{t>0} \left[\|u(t)\|_{\dot{B}^{-1}_{\infty,q}} + t^{\frac{3}{4}} \|u(t)\|_{\dot{B}^{\frac{1}{2}}_{\infty,q}} \right] \right\}.$$

Compared to the time-integrated gain of two derivatives we had in the case $p < \infty$, here we have pointwise-in-time gain of regularity of order $\frac{3}{2}$, which is realized in E_∞.

Theorem 3 (Existence). *Let* $1 \leqq q < \infty$ *and* $v_0 \in \dot{B}^{-1}_{\infty,q} \cap \dot{B}^0_{3,\infty}$. *There exists a constant* $\epsilon_0 > 0$ *such that for all* $v_0 \in \dot{B}^{-1}_{\infty,q} \cap \dot{B}^0_{3,\infty}$ *with* $\|v_0\|_{\dot{B}^{-1}_{\infty,q}} + \|v_0\|_{\dot{B}^0_{3,\infty}} \leqq \epsilon_0$, *the NS equations* (1.2) *admit a global-in-time solution* $v \in E_\infty$.

Remark 1. We note that one can replace $\dot{B}^0_{3,\infty}$ with other auxiliary spaces, $\dot{B}^{\frac{3}{p}-1}_{p,\infty}$, $3 \leqq p < \infty$ in Theorem 1. We choose the former because it is closely related to spaces appearing in the regularity criterion [16] with initial data in L^3. Also, we choose $q < \infty$ to avoid the embedding $\dot{B}^0_{3,\infty} \subset \dot{B}^{-1}_{\infty,\infty}$.

Once we show the existence of a solution of (1.3) in E_∞, we can show the following analyticity result along the lines of the proof of Theorem 2 and Theorem 3.

Theorem 4 (Analyticity). *There exists a positive constant* $\epsilon_0 > 0$ *such that for all* $v_0 \in \dot{B}^{-1}_{\infty,q} \cap \dot{B}^0_{3,\infty}$ *with* $\|v_0\|_{\dot{B}^{-1}_{\infty,q}} + \|v_0\|_{\dot{B}^0_{3,\infty}} \leqq \epsilon_0$, *the NS equations* (1.2) *admit a solution* $v \in E_\infty$ *such that* $e^{\sqrt{t}\Lambda} v \in E_\infty$.

1.2. Decay of Weak Solution

As an application of the analyticity of solutions addressed above, we will estimate decay rates of weak solutions. Most of the decay results have been based on L^2 estimates, as one can see in [38,39,41–45,50]. Here, we obtain the decay of weak solutions in Besov spaces $\dot{B}^{\frac{3}{p}-1}_{p,q}$ for all p. Before presenting our result, we recall the usual notion of a weak solution, $v(t) \in L^\infty_t L^2 \cap L^2_t \dot{H}^1$, satisfying (1.2) in the sense of distributions and the additional energy inequality,

$$\|v(t)\|^2_{L^2} + \int_0^t \|\nabla v(s)\|^2_{L^2} ds \leqq \|v_0\|^2_{L^2}.$$

The fundamental theorem of Leray [35] states the existence of such weak solutions for initial data $v_0 \in L^2$ with $\nabla \cdot v_0 = 0$.

We now briefly explain the main idea of the decay estimates: (i) The energy inequality implies that $\liminf_{t\to\infty} \|v(t)\|_{\dot{H}^1} = 0$. If we show that Besov norms

$\|v(t)\|_{\dot{B}^{\frac{3}{p}-1}_{p,q}}$ can be controlled by the $\dot{H}^1$ norm, then $\|v(t_0)\|_{\dot{B}^{\frac{3}{p}-1}_{p,q}}$ becomes sufficiently small after a certain transient time t_0. (ii) Theorem 2 and Theorem 4 tell us that if initial data are sufficiently small in critical spaces $\dot{B}^{\frac{3}{p}-1}_{p,q}$, then the solution satisfies the estimate $\|e^{\sqrt{t}\Lambda}v(t)\|_{\dot{B}^{\frac{3}{p}-1}_{p,q}}$ globally in time. Combining these two observations, we will show the following decay estimates of weak solutions in Sect. 5.

Theorem 5 (Decay). *Let v be a weak solution of the three dimensional Navier–Stokes equations, subject to initial data $v_0 \in L^2$ and, in addition $\omega_0 = \nabla \times v_0 \in L^1$ in case $1 < p < 2$. Then, there exists a time $t_0 > 0$ such that $\|v(t_0)\|_{\dot{B}^{\frac{3}{p}-1}_{p,q}}$ becomes sufficiently small so that Theorem 2 or Theorem 4 holds for $p < \infty$ and, respectively, $p = \infty$. Moreover, the following decay estimate holds for $\zeta > 0$ with $C_\zeta = \|\Lambda_2^\zeta e^{-\Lambda}\|_{L^1}$,*

$$\|\Lambda_2^\zeta v(t)\|_{\dot{B}^{\frac{3}{p}-1}_{p,q}} \leqq C_\zeta \|v(t_0)\|_{\dot{B}^{\frac{3}{p}-1}_{p,q}} (t-t_0)^{-\frac{\zeta}{2}}, \quad \begin{cases} q \geqq 2 & \text{for} \quad p \geqq 2, \\ q \geqq \frac{p}{p-1} & \text{for} \quad 1 < p < 2. \end{cases}$$

Remark 2. This decay rate reflects the usual parabolicity of the Navier-Stokes equations in the sense of PETROWSKY [48]. By the Sobolev embedding, one can extend previous L^2-based decay results [38,39,41–45,50] to obtain decay rates in L^p-based spaces for $p > 2$. However, the decay rates in Theorem 5 for $p < 2$ are new.

Remark 3. The case $p = q = 2$ corresponds to the upper bound of the decay rate in [39]. In that paper, two additional assumptions were made: (i) there exist positive real numbers M_1 and γ which may depend on v_0 such that $\|v(t)\|^2_{L^2} \leqq \dfrac{M_1}{(1+t)^\gamma}$ for all $t \geqq 0$; and (ii) for $r \geqq \dfrac{3}{2}$, $\liminf_{t\to\infty} \|v(t)\|_{H^r} < \infty$. In contrast, the decay estimate asserted in Theorem 5 was obtained without these additional assumptions.

1.3. Regularity Condition

As a related subject, we will provide a Serrin-type regularity criterion in Besov spaces. The Serrin criterion [46] says that the Leray weak solution v is smooth for $t \in (0, T]$ if $v \in L^r(0, T; L^s)$, with $\dfrac{2}{r} + \dfrac{3}{s} = 1$. We would like to show a new regularity criterion in Besov spaces with less regularity.

Theorem 6. *There exists a smooth solution of the Navier–Stokes equation on the time interval $[0, T]$ for smooth initial data v_0 if, on any time interval $[T-t, T]$,*

$$(T-t)^{\frac{1}{q}} \|v(t)\|_{\dot{B}^\sigma_{p,\infty}} \begin{cases} \leqq C, \ t < T, \\ \to 0, \ t \mapsto T, \end{cases} \qquad (1.5)$$

$$\frac{2}{q} + \frac{3}{p} - \sigma = 1, \quad 3 \leqq p \leqq \infty, \quad 2 < q < \infty.$$

Remark 4. Theorem 6 generalizes the Serrin's regularity condition and other related results in two aspects: $(i)\,\sigma$ can be negative and (ii) for $\sigma = 0$, $\|v(t)\|_{\dot{B}^0_{p,\infty}} \leqq \|v(t)\|_{L^p}$.

Remark 5. Our result should be compared with the regularity criterion [10] obtained in $L^q(0, T; B^{\frac{2}{q}-1}_{\infty,\infty})$ for $2 < q < \infty$. Whereas their result is better than our result in the spatial variables by $\dot{B}^{\frac{2}{q}-1}_{\infty,\infty} \subset B^{\frac{2}{q}-1}_{\infty,\infty}$ for a negative regularity index $\frac{2}{q} - 1$, our result improves the criterion in the time variable because the time singularity $(T-t)^{-\frac{1}{q}}$ is measured in the weak L^q space, which contains the L^q space. Moreover, our proof is much simpler. However, our method does not cover several known results due to the missing end point $q = \infty$: for example, $L^\infty(0, T; L^3)$ in [16] and $C((0, T]; B^{-1}_{\infty,\infty})$ in [10].

2. Notations: the Littlewood–Paley Decomposition and Paraproducts

We begin with some notations. $L^p(0, T; X)$ denotes the Banach space of Bochner measurable functions f from $(0, T)$ to X endowed with either the norm $\Big(\int_0^T \|f(\cdot, t)\|_X^p dt\Big)^{\frac{1}{p}}$ for $1 \leqq p < \infty$ or $\sup_{0 \leqq t \leqq T} \|f(\cdot, t)\|_X$ for $p = \infty$. For $T = \infty$, we use $L^p_t X$ instead of $L^p(0, \infty; X)$. For a sequence $\{a_j\}_{j\in\mathbb{Z}}$, $\{a_j\}_{l^q} := \big(\sum_{j\in\mathbb{Z}} |a_j|^q\big)^{\frac{1}{q}}$, with the usual change for $q = \infty$. Finally, $A \lesssim B$ means there is a constant C such that $A \leqq CB$.

Next, we provide notation and definitions in the Littlewood–Paley theory. We take a couple of smooth functions (χ, φ) supported on $\{\xi : |\xi| \leqq 1\}$ with values in $[0, 1]$ such that for all $\xi \in \mathbb{R}^d$,

$$\chi(\xi) + \sum_{j=0}^{\infty} \psi(2^{-j}\xi) = 1, \quad \psi(\xi) = \varphi(\frac{\xi}{2}) - \varphi(\xi),$$

and we denote $\psi(2^{-j}\xi)$ by $\psi_j(\xi)$. The homogeneous dyadic blocks and lower frequency cut-off functions are defined by

$$\triangle_j u = 2^{jd} \int_{R^d} h(2^j y) u(x-y) dy, \quad S_j u = 2^{jd} \int_{R^d} \tilde{h}(2^j y) u(x-y) dy, \tag{2.1}$$

with $h = \mathscr{F}^{-1}\psi$ and $\tilde{h} = \mathscr{F}^{-1}\chi$. Then, we can define the *homogeneous Littlewood–Paley decomposition* by

$$u = \sum_{j\in\mathbb{Z}} \triangle_j u \quad \text{in} \quad \mathscr{S}'_h, \tag{2.2}$$

where $\mathscr{S}_h'$ is the space of tempered distributions u such that $\lim_{j\to-\infty} S_j u = 0$ in $\mathscr{S}'$. Using this decomposition, we define stationary/ time dependent *homogeneous Besov spaces* as follows:

$$\dot{B}^s_{p,q} = \Big\{ f \in \mathscr{S}_h' \ ; \ \|f\|_{\dot{B}^s_{p,q}} := \Big(\sum_{j\in\mathbb{Z}} 2^{jsq} \|\triangle_j f\|^q_{L^p} \Big)^{\frac{1}{q}} < \infty \Big\}, \tag{2.3a}$$

$$L^r(0,T;\dot{B}^s_{p,q}) = \Big\{ f \in \mathscr{S}_h' \ ; \ \|f\|_{L^r(0,T;\dot{B}^s_{p,q})} := \Big\| \Big(\sum_{j\in\mathbb{Z}} 2^{jsq} \|\triangle_j f\|^q_{L^p} \Big)^{\frac{1}{q}} \Big\|_{L^r(0,T)} < \infty \Big\}, \tag{2.3b}$$

$$\tilde{L}^r(0,T;\dot{B}^s_{p,q}) = \Big\{ f \in \mathscr{S}_h' \ ; \ \|f\|_{\tilde{L}^r(0,T;\dot{B}^s_{p,q})} := \Big(\sum_{j\in\mathbb{Z}} 2^{jsq} \|\triangle_j f\|^q_{L^r(0,T;L^p)} \Big)^{\frac{1}{q}} < \infty \Big\}, \tag{2.3c}$$

with the usual modification for $q = \infty$. According to the Minkowski inequality, we have

$$\begin{aligned} \|f\|_{\tilde{L}^r(0,T;\dot{B}^s_{p,q})} &\leqq \|f\|_{L^r(0,T;\dot{B}^s_{p,q})} \quad \text{if} \quad r \leqq q, \\ \|f\|_{\tilde{L}^r(0,T;\dot{B}^s_{p,q})} &\geqq \|f\|_{L^r(0,T;\dot{B}^s_{p,q})} \quad \text{if} \quad r \geqq q. \end{aligned} \tag{2.4}$$

The concept of *paraproduct* enables us to deal with the interaction of two functions in terms of low or high frequency parts, [8]. For two tempered distributions f and g,

$$\begin{aligned} &fg = T_f g + T_g f + R(f,g), \\ &T_f g = \sum_{i\leqq j-2} \triangle_i f \triangle_j g = \sum_{j\in\mathbb{Z}} S_{j-1} f \triangle_j g, \quad R(f,g) = \sum_{|j-j'|\leqq 1} \triangle_j f \triangle_{j'} g. \end{aligned} \tag{2.5}$$

Then, up to finitely many terms,

$$\triangle_j(T_f g) = S_{j-1} f \triangle_j g, \quad \triangle_j R(f,g) = \sum_{k\geqq j-2} \triangle_k f \triangle_k g. \tag{2.6}$$

In Sect. 3, we will use the following decomposition:

$$fg = \sum_{j\in\mathbb{Z}} S_j f \triangle_j g + \sum_{j\in\mathbb{Z}} S_j g \triangle_j f. \tag{2.7}$$

Again, up to finitely many terms, we have

$$\triangle_j(fg) = \sum_{k\geqq j-2} S_k f \triangle_k g + \sum_{k\geqq j-2} \triangle_k f S_k g. \tag{2.8}$$

Finally, we recall a few inequalities which will be used in the sequel.

Bernstein's Inequality
For $1 \leqq p \leqq q \leqq \infty$ and $k \in \mathbb{N}$,

$$\sup_{|\alpha|=k} \|\partial^\alpha \triangle_j f\|_{L^p} \simeq 2^{jk} \|\triangle_j f\|_{L^p}, \quad \|\triangle_j f\|_{L^q} \lesssim 2^{jd(\frac{1}{p}-\frac{1}{q})} \|\triangle_j f\|_{L^p}. \tag{2.9}$$

Localization of the Heat Kernel

$$\|e^{t\Delta} \triangle_j f\|_{L^p} \lesssim e^{-t2^{2j}} \|\triangle_j f\|_{L^p}, \tag{2.10}$$

where the constants involved in the above relations $\simeq$ and $\lesssim$ are independent of f and j.

3. The Case $p < \infty$: Proof of Theorem 1 (Existence) and Theorem 2 (Analyticity)

In this section, we prove Theorem 2, analyticity of the Navier–Stokes equations with small initial data in $\dot{B}^{\frac{3}{p}-1}_{p,q}$, with $p < \infty$. The proof is based on an adequate modification of the proof of Theorem 1. Therefore, we begin with the detailed existence proof of [9].

3.1. Proof of Theorem 1

We recall the definition of the function space E_p,

$$E_p = \left\{ u \in \mathscr{S}' : \|u\|_{E_p} = \|u\|_{\tilde{L}^\infty_t \dot{B}^{\frac{3}{p}-1}_{p,q}} + \|u\|_{\tilde{L}^1_t \dot{B}^{\frac{3}{p}+1}_{p,q}} < \infty \right\}. \tag{3.1}$$

We construct a solution in the integral form: $v(t) = e^{t\Delta} v_0 - \mathscr{B}(v, v)$, where the bilinear form $\mathscr{B}$ is

$$\mathscr{B}(u, v) = \int_0^t \left[e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (u \otimes v)(s) \right] ds. \tag{3.2}$$

We need to show only that $\mathscr{B}$ maps $E_p \times E_p$ to E_p. We first decompose the product $u \otimes v$ as paraproduct (2.7). Then, $\mathscr{B}$ can be decomposed as $\mathscr{B}(u, v) = \mathscr{B}_1(u, v) + \mathscr{B}_2(u, v)$, where

$$\mathscr{B}_1(u, v) = \sum_{j \in \mathbb{Z}} \mathscr{B}(S_j u \otimes \triangle_j v), \quad \mathscr{B}_2(u, v) = \sum_{j \in \mathbb{Z}} \mathscr{B}(S_j v \otimes \triangle_j u).$$

We now estimate $\mathscr{B}_1$ in E_p. We apply $\triangle_j$ to $\mathscr{B}_1$ and take the L^p norm. By Bernstein's inequality (2.9) and localization of the heat kernel as (2.10), we have

$$\begin{aligned}\|\triangle_j \mathscr{B}_1(u,v)\|_{L^p} &\lesssim \sum_{k\geqq j-2} 2^j \int_0^t \Big[e^{-(t-s)2^{2k}} \|S_k u(s) \triangle_k v(s)\|_{L^p}\Big] ds \\ &\leqq \sum_{k\geqq j-2} 2^j \int_0^t \Big[e^{-(t-s)2^{2k}} \|S_k u(s)\|_{L^\infty} \|\triangle_k v(s)\|_{L^p}\Big] ds.\end{aligned} \tag{3.3}$$

By Bernstein's inequality (2.9),

$$\|S_k u\|_{L^\infty} \leqq \sum_{l\leqq k-1} \|\triangle_l u\|_{L^\infty} \lesssim \sum_{l\leqq k-1} 2^{l\frac{3}{p}} \|\triangle_l u\|_{L^p} = \sum_{l\leqq k-1} 2^l 2^{l(\frac{3}{p}-1)} \|\triangle_l u\|_{L^p}.$$

Therefore, we can replace the right-hand side of (3.3) by

$$\begin{aligned}&\|\triangle_j \mathscr{B}_1(u,v)\|_{L^p} \\ &\lesssim \sum_{k\geqq j-2} 2^j \int_0^t \Big[e^{-(t-s)2^{2k}} \Big(\sum_{l\leqq k-1} 2^l 2^{l(\frac{3}{p}-1)} \|\triangle_l u(s)\|_{L^p}\Big) \|\triangle_k v(s)\|_{L^p}\Big] ds.\end{aligned} \tag{3.4}$$

By taking the L^∞ norm of (3.4) in time with the aid of Young's inequality in time, we obtain

$$\begin{aligned}\|\triangle_j \mathscr{B}_1(u,v)\|_{L_t^\infty L^p} &\lesssim \sum_{k\geqq j-2} 2^j \Big[\sum_{l\leqq k-1} 2^l 2^{l(\frac{3}{p}-1)} \|\triangle_l u\|_{L_t^\infty L^p}\Big] \|\triangle_k v\|_{L_t^1 L^p} \\ &\lesssim \|u\|_{\tilde{L}_t^\infty \dot{B}_{p,q}^{\frac{3}{p}-1}} \sum_{k\geqq j-2} 2^j 2^k \|\triangle_k v\|_{L_t^1 L^p} \\ &\leqq \|u\|_{E_p} \sum_{k\geqq j-2} 2^j 2^{-k(\frac{3}{p})} 2^{k(\frac{3}{p}+1)} \|\triangle_k v\|_{L_t^1 L^p}.\end{aligned} \tag{3.5}$$

We multiply (3.5) by $2^{j(\frac{3}{p}-1)}$. Then,

$$2^{j(\frac{3}{p}-1)} \|\triangle_j \mathscr{B}_1(u,v)\|_{L_t^\infty L^p} \lesssim \|u\|_{E_p} \sum_{k\geqq j-2} 2^{-(k-j)(\frac{3}{p})} 2^{k(\frac{3}{p}+1)} \|\triangle_k v\|_{L_t^1 L^p}. \tag{3.6}$$

Since $\frac{3}{p} > 0$, we can use Young's inequality to estimate the right-hand side of (3.6) with respect to l^q. Namely, we let $a_j = 2^{-j(\frac{3}{p})}$ and $b_j = 2^{j(\frac{3}{p}+1)} \|\triangle_j v\|_{L_t^1 L^p}$ and apply Young's inequality to $\sum_{k\geqq j-2} a_{k-j} b_k$ to obtain

$$\|\mathscr{B}_1(u,v)\|_{\tilde{L}_t^\infty \dot{B}_{p,q}^{\frac{3}{p}-1}} \lesssim \|u\|_{E_p} \|v\|_{\tilde{L}_t^1 \dot{B}_{p,q}^{\frac{3}{p}+1}} \leqq \|u\|_{E_p} \|v\|_{E_p}. \tag{3.7}$$

Similarly, we can obtain the L^1 in time estimation of $\mathscr{B}_1(u,v)$. By taking the L^1 norm in time to (3.4) and applying Young's inequality in time,

$$\|\triangle_j \mathscr{B}_1(u,v)\|_{L^1_t L^p} \lesssim \|u\|_{E_p} 2^{-2j} \sum_{k \geqq j-2} 2^j 2^{-k(\frac{3}{p})} 2^{k(\frac{3}{p}+1)} \|\triangle_k v\|_{L^1_t L^p}. \quad (3.8)$$

We multiply (3.8) by $2^{j(\frac{3}{p}+1)}$. Then,

$$2^{j(\frac{3}{p}+1)} \|\triangle_j \mathscr{B}_1(u,v)\|_{L^1_t L^p} \lesssim \|u\|_{E_p} \sum_{k \geqq j-2} 2^{-(k-j)(\frac{3}{p})} 2^{k(\frac{3}{p}+1)} \|\triangle_k v\|_{L^1_t L^p},$$

from which we have

$$\|\mathscr{B}_1(u,v)\|_{\tilde{L}^1_t \dot{B}^{\frac{3}{p}+1}_{p,q}} \lesssim \|u\|_{E_p} \|v\|_{E_p}. \quad (3.9)$$

Therefore, we conclude that $\|\mathscr{B}_1(u,v)\|_{E_p} \lesssim \|u\|_{E_p}\|v\|_{E_p}$. By the symmetry of the paraproduct of $\mathscr{B}(u,v)$, we finally have

$$\|\mathscr{B}(u,v)\|_{E_p} \lesssim \|u\|_{E_p} \|v\|_{E_p}, \quad (3.10)$$

which completes the proof.

Remark 6. We will need to apply Young's inequality to sequences several times to estimate sequences which have convolution structure. Since the structure of sequences appearing later (for example, (4.5), (4.9), (4.10) and (6.3)) is exactly of the form used to obtain (3.7), we will apply Young's inequality to sequences without defining $\{a_j\}$ and $\{b_j\}$ each time.

3.2. Preliminaries

The proof of Theorem 2 in this section and likewise, Theorem 4 in Sect. 4, requires a couple of elementary inequalities which are summarized in the following two lemmas.

Lemma 1. *Consider the operator* $E := e^{-[\sqrt{t-s}+\sqrt{s}-\sqrt{t}]\Lambda}$ *for* $0 \leqq s \leqq t$. *Then* E *is either the identity operator or is an* L^1 *kernel whose* L^1 *norm is bounded independent of* s, t.

Proof. Clearly, $a := \sqrt{t-s} + \sqrt{s} - \sqrt{t}$ is non-negative for $s \leqq t$. In case $a = 0$, $E = e^{-a\Lambda}$ is the identity operator, while if $a > 0$, $E = e^{-a\Lambda}$ is a Fourier multiplier with symbol $\widehat{E}(\xi) = \prod_{i=1}^d e^{-a|\xi_i|}$. Thus, the kernel of E is given by the product of one-dimensional Poisson kernels $\prod_{i=1}^d \frac{a}{\pi(a^2+x_i^2)}$. The L^1 norm of this kernel is bounded by a constant independent of a. □

Lemma 2. *The operator* $E = e^{\frac{1}{2}a\Delta+\sqrt{a}\Lambda}$ *is a Fourier multiplier which maps boundedly* $L^p \mapsto L^p$, $1 < p < \infty$, *and its operator norm is uniformly bounded with respect to* $a \geqq 0$.

Proof. When $a = 0$, E is the identity operator. When $a > 0$, then E is a Fourier multiplier with symbol $\widehat{E}(\xi) = e^{-\frac{1}{2}|\sqrt{a}\xi|^2+|\sqrt{a}\xi|_1}$. Since $\widehat{E}(\xi)$ is uniformly bounded for all ξ and decays exponentially for $|\xi| \gg 1$, the claim follows from Hormander's multiplier theorem, e.g., [47]. □

3.3. *Proof of Theorem 2*

We are now ready to prove Theorem 2. For notational simplicity, we define a function space F_p such that

$$F_p = \Big\{ v(t) \in E_p \ \ ; \ \ e^{\sqrt{t}\Lambda} v(t) \in E_p \Big\}.$$

We need to show only that $\mathscr{B}$ in (3.2) is bounded from $F_p \times F_p$ to F_p. To this end, we first apply $e^{\sqrt{t}\Lambda}$ to $\mathscr{B}$ in (3.2).

$$e^{\sqrt{t}\Lambda}\mathscr{B}(u,v) = e^{\sqrt{t}\Lambda}\int_0^t \Big[e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes v)(s)\Big]ds. \tag{3.11}$$

Let $U(s) = e^{\sqrt{s}\Lambda}u$, $V(s) = e^{\sqrt{s}\Lambda}v$, with $U, V \in E_p$. Then,

$$e^{\sqrt{t}\Lambda}\mathscr{B}(u,v) = e^{\sqrt{t}\Lambda}\int_0^t \Big[e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(e^{-\sqrt{s}\Lambda}U\otimes e^{-\sqrt{s}\Lambda}V)(s)\Big]ds. \tag{3.12}$$

We rewrite (3.12) as

$$\begin{aligned} &e^{\sqrt{t}\Lambda}\mathscr{B}(u,v) \\ &= \int_0^t \Big[e^{(\sqrt{t}-\sqrt{s})\Lambda}e^{\frac{1}{2}(t-s)\Delta}e^{\frac{1}{2}(t-s)\Delta}e^{\sqrt{s}\Lambda}\mathbb{P}\nabla\cdot(e^{-\sqrt{s}\Lambda}U\otimes e^{-\sqrt{s}\Lambda}V)(s)\Big]ds. \end{aligned} \tag{3.13}$$

We apply $\triangle_j$ to (3.13) and take the L^p norm. By Lemmas 1 and 2, we have

$$\begin{aligned} &\|\triangle_j e^{\sqrt{t}\Lambda}\mathscr{B}(u,v)\|_{L^p} \\ &\lesssim \int_0^t \Big[e^{-\frac{1}{2}(t-s)2^{2j}}2^j \big\| e^{\sqrt{s}\Lambda}\triangle_j\big(e^{-\sqrt{s}\Lambda}U\otimes e^{-\sqrt{s}\Lambda}V)(s)\big)\big\|_{L^p}\Big]ds. \end{aligned} \tag{3.14}$$

To deal with the right-hand side of (3.14), we decompose the product $e^{-\sqrt{s}\Lambda}U\otimes e^{-\sqrt{s}\Lambda}V$ as

$$\begin{aligned} &e^{-\sqrt{s}\Lambda}U\otimes e^{-\sqrt{s}\Lambda}V \\ &= \sum_{j\in\mathbb{Z}}\big(e^{-\sqrt{s}\Lambda}S_jU\big)\otimes\big(e^{-\sqrt{s}\Lambda}\triangle_jV\big) + \sum_{j\in\mathbb{Z}}\big(e^{-\sqrt{s}\Lambda}\triangle_jU\big)\otimes\big(e^{-\sqrt{s}\Lambda}S_jV\big). \end{aligned}$$

Then,

$$\begin{aligned} &\|\triangle_j e^{\sqrt{t}\Lambda}\mathscr{B}(u,v)\|_{L^p} \\ &\leqq \int_0^t \sum_{k\geqq j-2}\Big[e^{-\frac{1}{2}(t-s)2^{2j}}2^j \big\| e^{\sqrt{s}\Lambda}\big(e^{-\sqrt{s}\Lambda}S_kU\otimes e^{-\sqrt{s}\Lambda}\triangle_kV)(s)\big\|_{L^p}\Big]ds \\ &\quad + \int_0^t \sum_{k\geqq j-2}\Big[e^{-\frac{1}{2}(t-s)2^{2j}}2^j \big\| e^{\sqrt{s}\Lambda}\big(e^{-\sqrt{s}\Lambda}\triangle_kU\otimes e^{-\sqrt{s}\Lambda}S_kV)(s)\big\|_{L^p}\Big]ds. \end{aligned} \tag{3.15}$$

To estimate the right-hand side of (3.15), we introduce the bilinear operators B_t of the form

$$B_t(f,g) = e^{\sqrt{t}\Lambda}(e^{-\sqrt{t}\Lambda} f e^{-\sqrt{t}\Lambda} g)$$
$$= \int_{R^3}\int_{R^3} e^{ix\cdot(\xi+\eta)} e^{\sqrt{t}(|\xi+\eta|_1-|\xi|_1-|\eta|_1)} \hat{f}(\xi)\hat{g}(\eta)d\xi d\eta.$$

Recall that for a vector $\xi = (\xi_1, \xi_2, \xi_3)$, we denoted $\|\xi\|_1 = \sum_{i=1}^{3} |\xi_i|$. As one can see below, this l^1 version of Λ, instead of the usual l^2 version of Λ, is crucial for estimating B_t. For $\xi = (\xi_1, \xi_2, \xi_3)$, $\eta = (\eta_1, \eta_2, \eta_3)$, we now split the domain of integration of the above integral into sub-domains depending on the sign of ξ_j, η_j and $\xi_j + \eta_j$. In order to do so, we introduce the operators acting on one variable (see page 253 in [34]) by

$$K_1 f = \frac{1}{2\pi}\int_0^\infty e^{\imath x\xi}\hat{f}(\xi)\,d\xi, \quad K_{-1} f = \frac{1}{2\pi}\int_{-\infty}^0 e^{\imath x\xi}\hat{f}(\xi)\,d\xi.$$

Let the operators $L_{a,1}$ and $L_{a,-1}$ be defined by

$$L_{a,1} f = f, \quad L_{a,-1} f = \frac{1}{2\pi}\int_{\mathbb{R}} e^{\imath x\xi} e^{-2a|\xi|}\hat{f}(\xi)\,d\xi.$$

For $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3) \in \{-1, 1\}^3$, denote the operator

$$Z_{a,\boldsymbol{\alpha},\boldsymbol{\beta}} = K_{\beta_1} L_{t,\alpha_1\beta_1} \otimes \cdots \otimes K_{\beta_3} L_{t,\alpha_3\beta_3} \text{ and } K_{\boldsymbol{\alpha}} = K_{\alpha_1} \otimes K_{\alpha_2} \otimes K_{\alpha_3}.$$

The above tensor product means that the j-th operator in the tensor product acts on the j-th variable of the function $f(x_1, x_2, x_3)$. A tedious (but elementary) calculation now yields the following identity:

$$B_t(f,g) = \sum_{(\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma})\in\{-1,1\}^{3\times 3}} K_{\alpha_1} \otimes K_{\alpha_2} \otimes K_{\alpha_3} \big(Z_{t,\boldsymbol{\alpha},\boldsymbol{\beta}} f Z_{t,\boldsymbol{\alpha},\boldsymbol{\gamma}} g\big). \quad (3.16)$$

We now note that the operators $K_{\boldsymbol{\alpha}}$, $Z_{a,\boldsymbol{\alpha},\boldsymbol{\beta}}$ defined above, being linear combinations of Fourier multipliers (including Hilbert transform) and the identity operator, commute with Λ. Moreover, they are bounded linear operators on L^p, $1 < p < \infty$ and the corresponding operator norm of $Z_{t,\boldsymbol{\alpha},\boldsymbol{\beta}}$ is bounded independent of $t \geqq 0$. By taking the L^p norm to (3.16), we have

$$\|B_t(f,g)\|_{L^p} \lesssim \|Z_{t,\boldsymbol{\alpha},\boldsymbol{\beta}} f Z_{t,\boldsymbol{\alpha},\boldsymbol{\gamma}} g\|_{L^p}.$$

We now apply this argument to the right-hand side of (3.15). Then,

$$\begin{aligned}\|e^{\sqrt{t}\Lambda}\mathscr{B}(u,v)\|_{L^p} \lesssim &\int_0^t \sum_{k\geqq j-2}\Big[e^{-\frac{1}{2}(t-s)2^{2j}}2^j\|ZS_kU\otimes Z\triangle_k V\|_{L^p}\Big]ds\\ &+\int_0^t \sum_{k\geqq j-2}\Big[e^{-\frac{1}{2}(t-s)2^{2j}}2^j\|Z\triangle_k U\otimes ZS_k V\|_{L^p}\Big]ds,\end{aligned} \tag{3.17}$$

where we denote Z without indices $t,\boldsymbol{\alpha},\boldsymbol{\beta}$ for the notational simplicity. By Bernstein's inequality (2.9), we have

$$\begin{aligned}\|ZS_ku\|_{L^\infty} &\leqq \sum_{l\leqq k-1}\|Z\triangle_l u\|_{L^\infty} \lesssim \sum_{l\leqq k-1}\|\triangle_l Zu\|_{L^\infty}\\ &\lesssim \sum_{l\leqq k-1}2^{l\frac{3}{p}}\|\triangle_l Zu\|_{L^p} \lesssim \sum_{l\leqq k-1}2^l 2^{l(\frac{3}{p}-1)}\|\triangle_l u\|_{L^p},\end{aligned}$$

where we use the fact that Z commutes with $\triangle_l$ and the boundedness of Z on L^p. Therefore, we can follow the lines from (3.4) to (3.10) in proof of Theorem 1 to obtain

$$\|\mathscr{B}(u,v)\|_{F_p} \lesssim \|U\|_{E_p}\|V\|_{E_p} \leqq \|u\|_{F_p}\|v\|_{F_p}, \tag{3.18}$$

which completes the proof.

4. The Case $p=\infty$: Proof of Theorem 3 (Existence) and Theorem 4 (Analyticity)

We now show the well-posedness and analyticity for the Navier–Stokes equations of the limiting case $p=\infty$. To this end, we recall the definition of the space E_∞

$$E_\infty := \left\{u\in\mathscr{S}' : \|u\|_{E_\infty} = \|u\|_{\tilde{L}^\infty_t\dot{B}^0_{3,\infty}} + \sup_{t>0}\Big[\|v(t)\|_{\dot{B}^{-1}_{\infty,q}} + t^{\frac{3}{4}}\|v(t)\|_{\dot{B}^{\frac{1}{2}}_{\infty,q}}\Big]\right\}.$$

As one can see below, we need to obtain two additional estimates, $t^{\frac{1}{4}}\|v(t)\|_{\dot{B}^{-\frac{1}{2}}_{\infty,q}}$ and $t^{\frac{1}{2}}\|v(t)\|_{\dot{B}^0_{\infty,q}}$. (See (4.10), (4.14) and (4.17)). These terms can be obtained by interpolating $\|v(t)\|_{\dot{B}^{-1}_{\infty,q}}$ and $t^{\frac{3}{4}}\|v(t)\|_{\dot{B}^{\frac{1}{2}}_{\infty,q}}$, but we proceed with the proof by establishing bounds in K_∞ whose norm is given by

$$\begin{aligned}&\|v\|_{K_\infty}\\ &= \sup_{t>0}\Big[\|v(t)\|_{\dot{B}^{-1}_{\infty,q}} + t^{\frac{1}{4}}\|v(t)\|_{\dot{B}^{-\frac{1}{2}}_{\infty,q}} + t^{\frac{1}{2}}\|v(t)\|_{\dot{B}^0_{\infty,q}} + t^{\frac{3}{4}}\|v(t)\|_{\dot{B}^{\frac{1}{2}}_{\infty,q}}\Big],\end{aligned} \tag{4.1}$$

to avoid complicated expressions coming from the interpolation. The time weights appearing in (4.1) will be introduced below through the Gaussian bound,

$$|\xi|^a e^{-t|\xi|^2} \lesssim t^{-\frac{a}{2}}. \tag{4.2}$$

In addition, we will use the following lemma repeatedly in the proof of Theorem 3.

Lemma 3. *For any* $0 < a < 1$ *and* $0 < b < 1$,

$$\int_0^t \Big[(t-s)^{-a} s^{-b}\Big] ds \lesssim t^{1-a-b}.$$

The result follows by decomposing the time integral into two parts,

$$\begin{aligned}\int_0^t \Big[(t-s)^{-a} s^{-b}\Big] ds &= \int_0^{\frac{t}{2}} \Big[(t-s)^{-a} s^{-b}\Big] ds + \int_{\frac{t}{2}}^t \Big[(t-s)^{-a} s^{-b}\Big] ds \\ &\lesssim t^{-a} \int_0^{\frac{t}{2}} s^{-b} ds + t^{-b} \int_{\frac{t}{2}}^t (t-s)^{-a} ds.\end{aligned}$$

4.1. Proof of Theorem 3

As we did in the proof Theorem 2, we need to show that $\mathscr{B}$ maps $E_\infty \times E_\infty$ to E_∞. Since we already estimated the bilinear term $\mathscr{B}$ in $\dot{B}^0_{3,\infty}$ in Theorem 1, we need to show only that $\mathscr{B}$ maps from $E_\infty \times E_\infty$ to K_∞. We decompose $u \otimes v$ as (2.5). Then,

$$\begin{aligned}\mathscr{B}(u,v) &= \int_0^t \Big[\nabla e^{(t-s)\Delta}\mathbb{P}(T_u \otimes v + T_v \otimes u + R(u \otimes v))\Big] ds \\ &:= \mathscr{B}_1(u,v) + \mathscr{B}_2(u,v) + \mathscr{B}_3(u,v).\end{aligned} \tag{4.3}$$

Estimation of $\mathscr{B}_3(u,v)$

We estimate $\mathscr{B}_3$ first, where we need the auxiliary norm $\|u\|_{\tilde{L}^\infty_t \dot{B}^0_{3,\infty}}$. By Bernstein's inequality (2.9) and localization of the heat kernel (2.10),

$$\begin{aligned}&\|\triangle_j \mathscr{B}_3(u,v)(t)\|_{L^\infty} \lesssim 2^j \|\triangle_j \mathscr{B}_3(u,v)(t)\|_{L^3} \\ &\lesssim \int_0^t \Big[2^{2j} e^{-(t-s)2^{2j}} \sum_{k \geqq j-2} \|\triangle_k u(s)\|_{L^3} \|\triangle_k v(s)\|_{L^\infty}\Big] ds\end{aligned}$$

$$\lesssim \|u\|_{\tilde{L}_t^\infty \dot{B}^0_{3,\infty}} \int_0^t \left[2^{\frac{3j}{2}} e^{-(t-s)2^{2j}} \sum_{k \geqq j-2} 2^{\frac{j-k}{2}} 2^{\frac{k}{2}} \|\triangle_k v(s)\|_{L^\infty} \right] ds$$

$$\leqq \|u\|_{E_\infty} \int_0^t \left[2^{\frac{3j}{2}} e^{-(t-s)2^{2j}} \sum_{k \geqq j-2} 2^{\frac{j-k}{2}} 2^{\frac{k}{2}} \|\triangle_k v(s)\|_{L^\infty} \right] ds. \tag{4.4}$$

We will repeatedly use (4.4) to estimate $\mathscr{B}_3(u, v)$ in K_∞. To estimate $\mathscr{B}_3(u, v)$ in $L_t^\infty \dot{B}^{-1}_{\infty,q}$, we multiply (4.4) by 2^{-j}. Then,

$$\begin{aligned} & 2^{-j} \|\triangle_j \mathscr{B}_3(u, v)(t)\|_{L^\infty} \\ & \lesssim \|u\|_{E_\infty} \int_0^t \left[2^{\frac{j}{2}} e^{-(t-s)2^{2j}} \sum_{k \geqq j-2} 2^{\frac{j-k}{2}} 2^{\frac{k}{2}} \|\triangle_k v(s)\|_{L^\infty} \right] ds \qquad (4.5) \\ & \lesssim \|u\|_{E_\infty} \int_0^t \left[(t-s)^{-\frac{1}{4}} s^{-\frac{3}{4}} \sum_{k \geqq j-2} 2^{\frac{j-k}{2}} s^{\frac{3}{4}} 2^{\frac{k}{2}} \|\triangle_k v(s)\|_{L^\infty} \right] ds. \end{aligned}$$

By taking the l^q norm to (4.5) with the aid of Young's inequality (as mentioned in Remark 6), we have

$$\|\mathscr{B}_3(u, v)(t)\|_{\dot{B}^{-1}_{\infty,q}} \lesssim \|u\|_{E_\infty} \left[\sup_{t>0} t^{\frac{3}{4}} \|v(t)\|_{\dot{B}^{\frac{1}{2}}_{\infty,q}} \right] \int_0^t \left[(t-s)^{-\frac{1}{4}} s^{-\frac{3}{4}} \right] ds.$$

Therefore, Lemma 3 implies that

$$\|\mathscr{B}_3(u, v)(t)\|_{\dot{B}^{-1}_{\infty,q}} \lesssim \|u\|_{E_\infty} \|v\|_{E_\infty}. \tag{4.6}$$

We will do the same calculation to estimate $\mathscr{B}_3$ for the next two terms in (4.1) without details.

$$\begin{aligned} \|\mathscr{B}_3(u, v)(t)\|_{\dot{B}^{-\frac{1}{2}}_{\infty,q}} & \lesssim \|u\|_{E_\infty} \left[\sup_{t>0} t^{\frac{3}{4}} \|v(t)\|_{\dot{B}^{\frac{1}{2}}_{\infty,q}} \right] \int_0^t \left[(t-s)^{-\frac{1}{2}} s^{-\frac{3}{4}} \right] ds \qquad (4.7) \\ & \lesssim \frac{\|u\|_{E_\infty} \|v\|_{E_\infty}}{t^{\frac{1}{4}}}. \\ \|\mathscr{B}_3(u, v)(t)\|_{\dot{B}^{0}_{\infty,q}} & \lesssim \|u\|_{E_\infty} \left[\sup_{t>0} t^{\frac{3}{4}} \|v(t)\|_{\dot{B}^{\frac{1}{2}}_{\infty,q}} \right] \int_0^t \left[(t-s)^{-\frac{3}{4}} s^{-\frac{3}{4}} \right] ds \qquad (4.8) \\ & \lesssim \frac{\|u\|_{E_\infty} \|v\|_{E_\infty}}{t^{\frac{1}{2}}}. \end{aligned}$$

To estimate $t^{\frac{3}{4}}\|\mathscr{B}_3(u,v)(t)\|_{\dot{B}^{\frac{1}{2}}_{\infty,q}}$, we need to divide the time integration into two parts.

$$\begin{aligned}
&2^{\frac{j}{2}}\|\triangle_j\mathscr{B}_3(u,v)(t)\|_{L^\infty}\\
&\lesssim 2^{\frac{3j}{2}}\int_0^{\frac{t}{2}}\Big[2^j e^{-(t-s)2^{2j}}\sum_{k\geqq j-2}\|\triangle_k u(s)\|_{L^\infty}\|\triangle_k v(s)\|_{L^3}\Big]ds\\
&\quad+2^{\frac{j}{2}}\int_{\frac{t}{2}}^{t}\Big[2^j e^{-(t-s)2^{2j}}\sum_{k\geqq j-2}\|\triangle_k u(s)\|_{L^\infty}\|\triangle_k v(s)\|_{L^\infty}\Big]ds\\
&=I_j+II_j.
\end{aligned}$$

We begin with I_j,

$$\begin{aligned}
I_j&=\int_0^{\frac{t}{2}}\Big[2^{2j}e^{-(t-s)2^{2j}}2^{\frac{j}{2}}\sum_{k\geqq j-2}\|\triangle_k u(s)\|_{L^3}\|\triangle_k v(s)\|_{L^\infty}\Big]ds\\
&\lesssim\int_0^{\frac{t}{2}}\Big[\frac{1}{t-s}\sum_{k\geqq j-2}\|\triangle_k u(s)\|_{L^3}2^{\frac{j-k}{2}}2^{\frac{k}{2}}\|\triangle_k v(s)\|_{L^\infty}\Big]ds\\
&\leqq\|u\|_{L^\infty_t\dot{B}^0_{3,\infty}}\int_0^{\frac{t}{2}}\Big[\frac{1}{t-s}\sum_{k\geqq j-2}2^{\frac{j-k}{2}}2^{\frac{k}{2}}\|\triangle_k v(s)\|_{L^\infty}\Big]ds\\
&\leqq\|u\|_{E_\infty}\int_0^{\frac{t}{2}}\Big[\frac{1}{t-s}\sum_{k\geqq j-2}2^{\frac{j-k}{2}}2^{\frac{k}{2}}\|\triangle_k v(s)\|_{L^\infty}\Big]ds.
\end{aligned}$$

Using Young's inequality, we obtain that

$$\begin{aligned}
\big\{I_j\big\}_{l^q}&\lesssim\|u\|_{E_\infty}\Big[\sup_{t>0}t^{\frac{3}{4}}\|v(t)\|_{\dot{B}^{\frac{1}{2}}_{\infty,q}}\Big]\int_0^{\frac{t}{2}}\Big[(t-s)^{-1}s^{-\frac{3}{4}}\Big]ds\\
&\lesssim\frac{\|u\|_{E_\infty}\|v\|_{E_\infty}}{t^{\frac{3}{4}}}.
\end{aligned}\tag{4.9}$$

Next, we estimate II_j.

$$\begin{aligned}
II_j&\lesssim\int_{\frac{t}{2}}^{t}\Big[(t-s)^{-\frac{1}{2}}\sum_{k\geqq j-2}2^{\frac{j-k}{2}}2^{\frac{k}{2}}\|\triangle_k u(s)\|_{L^\infty}\|\triangle_k v(s)\|_{L^\infty}\Big]ds\\
&=\int_{\frac{t}{2}}^{t}\Big[(t-s)^{-\frac{1}{2}}s^{-\frac{3}{4}}s^{-\frac{1}{2}}\sum_{k\geqq j-2}2^{\frac{j-k}{2}}s^{\frac{1}{2}}\|\triangle_k u(s)\|_{L^\infty}s^{\frac{3}{4}}2^{\frac{k}{2}}\|\triangle_k v(s)\|_{L^\infty}\Big]ds,
\end{aligned}$$

from which we have

$$\{II_j\}_{l^q} \lesssim \Big[\sup_{t>0} t^{\frac{1}{2}} \|v(t)\|_{\dot{B}^0_{\infty,q}}\Big]\Big[\sup_{t>0} t^{\frac{3}{4}} \|v(t)\|_{\dot{B}^{\frac{1}{2}}_{\infty,q}}\Big] \int_{\frac{t}{2}}^{t} \Big[(t-s)^{-\frac{1}{2}} s^{-\frac{3}{4}} s^{-\frac{1}{2}}\Big] ds \lesssim \frac{\|u\|_{E_\infty}\|v\|_{E_\infty}}{t^{\frac{3}{4}}}. \tag{4.10}$$

By (4.9) and (4.10), we have

$$\|\mathscr{B}_3(u,v)(t)\|_{\dot{B}^{\frac{1}{2}}_{\infty,q}} \lesssim \frac{\|u\|_{E_\infty}\|v\|_{E_\infty}}{t^{\frac{3}{4}}}. \tag{4.11}$$

Therefore, by (4.6), (4.7), (4.8), and (4.11),

$$\|\mathscr{B}_3(u,v)\|_{K_\infty} \lesssim \|u\|_{E_\infty}\|v\|_{E_\infty}. \tag{4.12}$$

Estimation of $\mathscr{B}_1(u,v)$ and $\mathscr{B}_2(u,v)$

Now, we estimate $\mathscr{B}_1(u,v)$. By applying $\triangle_j$ to $\mathscr{B}_1(u,v)$ and taking the L^∞ norm, we have

$$\|\triangle_j \mathscr{B}_1(u,v)(t)\|_{L^\infty} \lesssim \int_0^t \Big[2^j e^{-(t-s)2^{2j}} \|S_j u(s)\|_{L^\infty} \|\triangle_j v(s)\|_{L^\infty}\Big] ds.$$

We note that we used the auxiliary norm $\|u\|_{\dot{B}^0_{3,\infty}}$ to replace $\|\triangle_k u\|_{L^\infty}$ by $\|\triangle_k u\|_{L^3}$ to gain one derivative to estimate $\mathscr{B}_3(u,v)$. Here, we can gain one derivative from $S_j u$ to estimate $\mathscr{B}_1$ and $\mathscr{B}_2(u,v)$ as follows.

$$\|S_j u\|_{L^\infty} \lesssim \sum_{l=-\infty}^{j} \|\triangle_l u\|_{L^\infty} = \sum_{l=-\infty}^{j} 2^l 2^{-l} \|\triangle_l u\|_{L^\infty} \lesssim 2^j \|u\|_{\dot{B}^{-1}_{\infty,q}} \leqq 2^j \|u\|_{E_\infty}. \tag{4.13}$$

We will use this property to estimate $\mathscr{B}_1(u,v)$ for the first three terms in (4.1). We begin with the estimation of $\|\mathscr{B}_1(u,v)\|_{L^\infty_t \dot{B}^{-1}_{\infty,q}}$.

$$\begin{aligned} 2^{-j}\|\triangle_j \mathscr{B}_1(u,v)(t)\|_{L^\infty} &\lesssim \|u\|_{E_\infty} \int_0^t \Big[2^j e^{-(t-s)2^{2j}} \|\triangle_j v(s)\|_{L^\infty}\Big] ds \\ &\lesssim \|u\|_{E_\infty} \int_0^t \Big[(t-s)^{-\frac{1}{2}} \|\triangle_j v(s)\|_{L^\infty}\Big] ds \\ &= \|u\|_{E_\infty} \int_0^t \Big[(t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} s^{\frac{1}{2}} \|\triangle_j v(s)\|_{L^\infty}\Big] ds, \end{aligned}$$

which implies, by Lemma 3, that

$$\begin{aligned}\|\mathscr{B}_1(u,v)(t)\|_{\dot{B}^{-1}_{\infty,q}} &\lesssim \|u\|_{E_\infty}\Big[\sup_{t>0} t^{\frac{1}{2}}\|v(t)\|_{\dot{B}^{0}_{\infty,q}}\Big]\int_0^t \Big[(t-s)^{-\frac{1}{2}}s^{-\frac{1}{2}}\Big]ds \\ &\lesssim \|u\|_{E_\infty}\|v\|_{E_\infty}. \qquad (4.14)\end{aligned}$$

Again, we will use the same calculation to estimate $\mathscr{B}_1(u,v)$ for the following two terms without details.

$$\begin{aligned}\|\mathscr{B}_1(u,v)(t)\|_{\dot{B}^{-\frac{1}{2}}_{\infty,q}} &\lesssim \|u\|_{E_\infty}\Big[\sup_{t>0} t^{\frac{3}{4}}\|v(t)\|_{\dot{B}^{\frac{1}{2}}_{\infty,q}}\Big]\int_0^t \Big[(t-s)^{-\frac{1}{2}}s^{-\frac{3}{4}}\Big]ds \\ &\lesssim \frac{\|u\|_{E_\infty}\|v\|_{E_\infty}}{t^{\frac{1}{4}}}. \qquad (4.15)\end{aligned}$$

$$\begin{aligned}\|\mathscr{B}_1(u,v)(t)\|_{\dot{B}^{0}_{\infty,q}} &\lesssim \|u\|_{E_\infty}\Big[\sup_{t>0} t^{\frac{3}{4}}\|v(t)\|_{\dot{B}^{\frac{1}{2}}_{\infty,q}}\Big]\int_0^t \Big[(t-s)^{-\frac{3}{4}}s^{-\frac{3}{4}}\Big]ds \\ &\lesssim \frac{\|u\|_{E_\infty}\|v\|_{E_\infty}}{t^{\frac{1}{2}}}. \qquad (4.16)\end{aligned}$$

To estimate $t^{\frac{3}{4}}\|\mathscr{B}_1(u,v)(t)\|_{\dot{B}^{\frac{1}{2}}_{\infty,q}}$, we need to divide the time integration into two parts.

$$\begin{aligned}2^{\frac{j}{2}}\|\triangle_j\mathscr{B}_1(u,v)(t)\|_{L^\infty} &\lesssim \int_0^{\frac{t}{2}} \Big[2^{\frac{j}{2}}2^j e^{-(t-s)2^{2j}}\|S_j u(s)\|_{L^\infty}\|\triangle_j v(s)\|_{L^\infty}\Big]ds \\ &\quad + \int_{\frac{t}{2}}^{t} \Big[2^{\frac{j}{2}}2^j e^{-(t-s)2^{2j}}\|S_j u(s)\|_{L^\infty}\|\triangle_j v(s)\|_{L^\infty}\Big]ds \\ &= III_j + IV_j.\end{aligned}$$

We begin with III_j. By (4.13),

$$III_j \lesssim \|u\|_{E_\infty}\int_0^{\frac{t}{2}} \Big[2^{2j}e^{-(t-s)2^{2j}}2^{\frac{j}{2}}\|\triangle_j v(s)\|_{L^\infty}\Big]ds.$$

Thus, we have

$$\begin{aligned}\big\{III_j\big\}_{l^q} &\lesssim \|u\|_{E_\infty}\Big[\sup_{t>0} t^{\frac{3}{4}}\|v(t)\|_{\dot{B}^{\frac{1}{2}}_{\infty,q}}\Big]\int_0^{\frac{t}{2}} \Big[(t-s)^{-1}s^{-\frac{3}{4}}\Big]ds \\ &\lesssim \frac{\|u\|_{E_\infty}\|v\|_{E_\infty}}{t^{\frac{3}{4}}}.\end{aligned}$$

To estimate IV_j, we slightly change the estimation of $\|S_j u\|_{L^\infty}$ as follows:

$$\begin{aligned}\|S_j u(s)\|_{L^\infty} &\lesssim \sum_{l=-\infty}^{j} \|\triangle_l u(s)\|_{L^\infty} = \sum_{l=-\infty}^{j} 2^{\frac{l}{2}} s^{-\frac{1}{4}} 2^{-\frac{l}{2}} s^{\frac{1}{4}} \|\triangle_l u(s)\|_{L^\infty} \\ &\lesssim 2^{\frac{j}{2}} s^{-\frac{1}{4}} \Big[\sup_{s>0} s^{\frac{1}{4}} \|u(s)\|_{\dot{B}^{-\frac{1}{2}}_{\infty,q}}\Big] \leqq 2^{\frac{j}{2}} s^{-\frac{1}{4}} \|u\|_{E_\infty}.\end{aligned} \tag{4.17}$$

Therefore,

$$\begin{aligned}IV_j &\lesssim \|u\|_{E_\infty} \int_{\frac{t}{2}}^{t} \Big[2^{\frac{3j}{2}} e^{-(t-s)2^{2j}} s^{-\frac{1}{4}} 2^{\frac{j}{2}} \|\triangle_j v(s)\|_{L^\infty}\Big] ds \\ &\lesssim \|u\|_{E_\infty} \int_{\frac{t}{2}}^{t} \Big[(t-s)^{-\frac{3}{4}} s^{-\frac{1}{4}} 2^{\frac{j}{2}} \|\triangle_j v(s)\|_{L^\infty}\Big] ds,\end{aligned}$$

which implies that

$$\begin{aligned}\{IV\}_{l^q} &\lesssim \|u\|_{E_\infty} \Big[\sup_{t>0} t^{\frac{3}{4}} \|v(t)\|_{\dot{B}^{\frac{1}{2}}_{\infty,q}}\Big] \int_{\frac{t}{2}}^{t} \Big[(t-s)^{-\frac{3}{4}} s^{-\frac{1}{4}} s^{-\frac{3}{4}}\Big] ds \\ &\lesssim \frac{\|u\|_{E_\infty} \|v\|_{E_\infty}}{t^{\frac{3}{4}}}.\end{aligned}$$

By (4.17) and (4.18), we have

$$\|\mathscr{B}_1(u, v)(t)\|_{\dot{B}^{\frac{1}{2}}_{\infty,q}} \lesssim \frac{\|u\|_{E_\infty} \|v\|_{E_\infty}}{t^{\frac{3}{4}}}. \tag{4.18}$$

Therefore, By (4.14), (4.15), (4.16), and (4.18),

$$\|\mathscr{B}_1(u, v)\|_{K_\infty} \lesssim \|u\|_{E_\infty} \|v\|_{E_\infty}. \tag{4.19}$$

Since $\mathscr{B}_2(u, v)$ is of the form of $\mathscr{B}_1(u, v)$ by changing the role of u and v, we have

$$\|\mathscr{B}_2(u, v)\|_{K_\infty} \lesssim \|u\|_{E_\infty} \|v\|_{E_\infty}. \tag{4.20}$$

Combining (4.12), (4.19), and (4.20), we finally have

$$\|\mathscr{B}(u, v)\|_{K_\infty} \lesssim \|u\|_{E_\infty} \|v\|_{E_\infty}, \tag{4.21}$$

which completes the proof of Theorem 3.

4.2. *Proof of Theorem 4*

We can prove Theorem 4 along the lines the proof of Theorem 2 and Theorem 3. We define a function space F_∞ such that

$$F_\infty = \left\{ v(t) \in E_\infty \;\; ; \;\; e^{\sqrt{t}\Lambda} v(t) \in E_\infty \right\}.$$

We need to show only that $\mathscr{B}$ in (3.2) is bounded from $F_\infty \times F_\infty$ to F_∞. Let $U(s) = e^{\sqrt{s}\Lambda} u$, $V(s) = e^{\sqrt{s}\Lambda} v$, with $U, V \in E_\infty$. By replacing the L^p norm by the L^∞ norm in (3.14), we have

$$\begin{aligned} &\|\triangle_j e^{\sqrt{t}\Lambda} \mathscr{B}(u, v)\|_{L^\infty} \\ &\lesssim \int_0^t \left[e^{-\frac{1}{2}(t-s)2^{2j}} 2^j \left\| e^{\sqrt{s}\Lambda} \triangle_j \left(e^{-\sqrt{s}\Lambda} U \otimes e^{-\sqrt{s}\Lambda} V \right)(s) \right\|_{L^\infty} \right] ds. \end{aligned} \tag{4.22}$$

Then, we decompose $\left(e^{-\sqrt{s}\Lambda} U \otimes e^{-\sqrt{s}\Lambda} V \right)$ as

$$T_{e^{-\sqrt{s}\Lambda} U} \otimes e^{-\sqrt{s}\Lambda} V + T_{e^{-\sqrt{s}\Lambda} V} \otimes e^{-\sqrt{s}\Lambda} U + R(e^{-\sqrt{s}\Lambda} U \otimes e^{-\sqrt{s}\Lambda} V)$$

and follow the calculations in the proof of Theorem 3. In general, $K_{\boldsymbol{\alpha}}$ and $Z_{t,\boldsymbol{\alpha},\boldsymbol{\beta}}$ do not map L^∞ to L^∞. However, these operators are bounded in L^∞ when localized in dyadic blocks in the Fourier spaces. Therefore,

$$\|e^{\sqrt{t}\Lambda} \mathscr{B}(u, v)\|_{K_\infty} \lesssim \|U\|_{E_\infty} \|V\|_{E_\infty} \leqq \|u\|_{F_\infty} \|v\|_{F_\infty}. \tag{4.23}$$

Since we already obtained the bound of $\|e^{\sqrt{t}\Lambda} \mathscr{B}(u, v)\|_{L^\infty_t \dot{B}^0_{3,\infty}}$ in Sect. 3, we finally have

$$\|\mathscr{B}(u, v)\|_{F_\infty} \lesssim \|u\|_{F_\infty} \|v\|_{F_\infty}.$$

This completes the proof of Theorem 4.

5. Proof of Theorem 5: Decay of Besov Norms

In this section, we will obtain various decay estimates of weak solutions of the Navier–Stokes equations in Besov spaces. We need the following lemma to proceed.

Lemma 4. *The Fourier multipliers corresponding to the symbols $m(\xi) = |\xi|^\zeta e^{-\sqrt{t}|\xi|_1}$ are given by convolution with corresponding kernel k, which is an L^1 function with $\|k\|_{L^1} \leqq \frac{C_\zeta}{t^{\zeta/2}}$.*

Proof. For a proof of the L^1 bound on k, we apply lemma 2.1 in [34] under the scaling: $\xi \mapsto t^{\frac{1}{2}} \xi$. □

Theorem 2 and Theorem 4 tell us that if initial data are sufficiently small in critical spaces $\dot{B}^{\frac{3}{p}-1}_{p,q}$, then the solution is globally in the Gevrey class. Then, lemma 4 allows us to obtain the following time decay of (homogeneous) Besov norms:

$$\|\Lambda_2^{\zeta} v(t)\|_{\dot{B}^{\frac{3}{p}-1}_{p,q}} = \|\Lambda_2^{\zeta} e^{-\sqrt{t}\Lambda} e^{\sqrt{t}\Lambda} v(t)\|_{\dot{B}^{\frac{3}{p}-1}_{p,q}} \leqq C_{\zeta} t^{-\frac{\zeta}{2}} \|e^{\sqrt{t}\Lambda} v(t)\|_{\dot{B}^{\frac{3}{p}-1}_{p,q}}, \quad \zeta > 0,$$

where we recall that $\Lambda_2 = (-\Delta)^{\frac{1}{2}}$. If we can show that a solution $v(t)$ satisfies

$$\liminf_{t\to\infty} \|v(t)\|_{\dot{B}^{\frac{3}{p}-1}_{p,q}} = 0, \tag{5.1}$$

then due to Theorem 2 and Theorem 4, after a certain transient time t_0, we have

$$\sup_{t>t_0} \|e^{\sqrt{t}\Lambda} v(t)\|_{\dot{B}^{\frac{3}{p}-1}_{p,q}} < \infty.$$

Consequently, we obtain

$$\|\Lambda_2^{\zeta} v(t)\|_{\dot{B}^{\frac{3}{p}-1}_{p,q}} \leqq C_{\zeta} \|v(t_0)\|_{\dot{B}^{\frac{3}{p}-1}_{p,q}} (t-t_0)^{-\frac{\zeta}{2}}, \quad \zeta > 0, \tag{5.2}$$

where $\|v(t_0)\|_{\dot{B}^{\frac{3}{p}-1}_{p,q}}$ is sufficiently small to apply Theorem 2 or Theorem 4.

5.1. Proof of Theorem 5

We only need to show (5.1). For $v_0 \in L^2$, we have the following energy inequality:

$$\|v(t)\|_{L^2}^2 + \int_0^t \|\nabla v(s)\|_{L^2}^2 ds \leqq \|v_0\|_{L^2}^2.$$

This implies that

$$\sup_{t>0} \|v(t)\|_{L^2}^2 \leqq \|v_0\|_{L^2}^2, \quad \liminf_{t\to\infty} \|v(t)\|_{\dot{H}^1} = 0. \tag{5.3}$$

In order to obtain the second relation in (5.3), for $\epsilon > 0$ arbitrary, choose t large so that $\frac{1}{t}\|v_0\|_{L^2}^2 < \frac{\epsilon}{4}$. We note that the energy inequality yields $\frac{1}{t}\int_0^t \|\nabla v(s)\|_{L^2}^2 \leqq \frac{1}{t}\|v_0\|_{L^2}^2$. This immediately implies that there exists $t_0 \in (0, t)$ such that $\|\nabla v(t_0)\|_{L^2}^2 < \epsilon$. Due the uniform bound on $\|v(t)\|_{L^2}$, it also follows that $\liminf_{t\to\infty} \|u(t)\|_{\dot{H}^{\beta}} = 0$ for $0 < \beta \leqq 1$.

For $p = q = 2$, $\dot{B}^{\frac{1}{2}}_{2,2} = \dot{H}^{\frac{1}{2}}$. Thus by (5.2), we have

$$\|\Lambda_2^{\zeta} v(t)\|_{\dot{H}^{\frac{1}{2}}} \leqq C_{\zeta} \|v(t_0)\|_{\dot{H}^{\frac{1}{2}}} (t-t_0)^{-\frac{\zeta}{2}}, \quad \zeta > 0, \tag{5.4}$$

where $\|v(t_0)\|_{\dot{H}^{\frac{1}{2}}}$ is sufficiently small to apply Theorem 2.

For $p > 2$ and $q \geqq 2$, the embedding $\dot{H}^{\frac{1}{2}} \subset \dot{B}^{\frac{3}{p}-1}_{p,q}$ implies that $\|v(t)\|_{\dot{B}^{\frac{3}{p}-1}_{p,q}} \to 0$ as $t \to \infty$. Therefore,

$$\|\Lambda_2^{\zeta} v(t)\|_{\dot{B}^{\frac{3}{p}-1}_{p,q}} \leqq C_{\zeta} \|v(t_0)\|_{\dot{B}^{\frac{3}{p}-1}_{p,q}} (t-t_0)^{-\frac{\zeta}{2}}, \quad \zeta > 0, \tag{5.5}$$

where $\|v(t_0)\|_{\dot{B}^{\frac{3}{p}-1}_{p,q}}$ is sufficiently small to apply Theorem 2 for $p < \infty$ or Theorem 4 for $p = \infty$ and $q < \infty$.

To deal with the case $p < 2$, we will use the vorticity $\omega = \nabla \times v$. From the vorticity equation, $\omega_t + v \cdot \nabla\omega - \Delta\omega = \omega\nabla v$, we have ([11])

$$\|\omega(t)\|_{L^1} \leqq \|v_0\|^2_{L^2} + \|\omega_0\|_{L^1}. \tag{5.6}$$

We will use this L^1 vorticity bound to estimate the decay rate in Besov spaces. By the interpolation of the L^1 norm and the L^2 norm of $\triangle_j v(t)$, we have

$$\|\triangle_j v(t)\|_{L^p} \leqq C\|\triangle_j v(t)\|^{\alpha}_{L^2} \|\triangle_j v\|^{1-\alpha}_{L^1}, \quad \alpha = \frac{2p-2}{p}. \tag{5.7}$$

We multiply (5.7) by $2^{j(\frac{3}{p}-1+\zeta)}$. Then,

$$\begin{aligned} 2^{j(\frac{3}{p}-1+\zeta)} \|\triangle_j v(t)\|_{L^p} &\leqq C 2^{j(\frac{1}{p}+\zeta)} \|\triangle_j v(t)\|^{\alpha}_{L^2} \big(2^j \|\triangle_j v(t)\|_{L^1}\big)^{1-\alpha} \\ &\leqq C 2^{j(\frac{1}{p}+\zeta)} \|\triangle_j v(t)\|^{\alpha}_{L^2} \big(\|\triangle_j w(t)\|_{L^1}\big)^{1-\alpha} \\ &\leqq C 2^{j(\frac{1}{p}+\zeta)} \|\triangle_j v(t)\|^{\alpha}_{L^2}, \end{aligned} \tag{5.8}$$

where we use the fact that

$$\|\triangle_j \nabla v\|_{L^1} \leqq C\|\triangle_j w\|_{L^1} \leqq C\|w\|_{L^1}, \quad C \ \text{ independent of } \ j.$$

By taking the l^q norm to (5.8), we have

$$\|\Lambda_2^{\zeta} v(t)\|_{\dot{B}^{\frac{3}{p}-1}_{p,q}} \leqq C\Big(\sum_j 2^{jq(\frac{1}{p}+\zeta)} \|\triangle_j v(t)\|^{\alpha q}_{L^2}\Big)^{\frac{1}{q}}. \tag{5.9}$$

We take q_0 such as $\alpha q_0 = 2$. Then for $q \geqq q_0 = \frac{p}{p-1} > 2$, we have

$$\|\Lambda_2^{\zeta} v(t)\|_{\dot{B}^{\frac{3}{p}-1}_{p,q}} \leqq C\|v(t)\|^{\frac{2}{q}}_{\dot{H}^{\frac{q}{2}(\frac{1}{p}+\zeta)}}. \tag{5.10}$$

Since $\frac{q}{2}(\frac{1}{p}+\zeta) = \frac{1}{2(p-1)} + \frac{q\zeta}{2} > \frac{1}{2}$ for $p < 2$, the right-hand side of (5.10) goes to 0 as $t \to \infty$ by (5.4). Therefore,

$$\|\Lambda_2^{\zeta} v(t)\|_{\dot{B}^{\frac{3}{p}-1}_{p,q}} \leqq C_{\zeta} \|v(t_0)\|_{\dot{B}^{\frac{3}{p}-1}_{p,q}} (t-t_0)^{-\frac{\zeta}{2}}, \quad \zeta > 0, \tag{5.11}$$

where $\|v(t_0)\|_{\dot{B}^{\frac{3}{p}-1}_{p,q}}$ is sufficiently small to apply Theorem 2. This completes the proof of Theorem 5.

Remark 7. By using the relation between Besov spaces and Triebel-Lizorkin spaces $\dot{F}^s_{p,q}$

$$\dot{B}^s_{p,p} \subset \dot{F}^s_{p,p},$$

we can obtain the decay of weak solutions in Sobolev spaces $\dot{W}^{s,p}$ with $p < 2$. (For the definition of Triebel-Lizorkin spaces and embedding properties, see [20]). By taking $p = q < 2$,

$$\|e^{\sqrt{t}\Lambda} v(t)\|_{\dot{F}^{\frac{3}{p}-1}_{p,p}} \lesssim \|v(t_0)\|_{\dot{B}^{\frac{3}{p}-1}_{p,p}}.$$

Since $l^p \subset l^2$ for $p < 2$,

$$\|e^{\sqrt{t}\Lambda} v(t)\|_{\dot{F}^{\frac{3}{p}-1}_{p,2}} \lesssim \|v(t_0)\|_{\dot{B}^{\frac{3}{p}-1}_{p,p}},$$

which is equivalent to the estimate solutions in the potential function $\dot{W}^{s_0,p}$ such that

$$\|e^{\sqrt{t}\Lambda} v(t)\|_{\dot{W}^{s_0,p}} \lesssim \|v(t_0)\|_{\dot{B}^{\frac{3}{p}-1}_{p,q}}, \quad s_0 = \frac{3}{p} - 1 > 0.$$

Therefore,

$$\|\Lambda_2^{\zeta} v(t)\|_{L^p} \leqq C_\zeta \|v(t_0)\|_{\dot{B}^{\frac{3}{p}-1}_{p,p}} (t - t_0)^{-\frac{\zeta - s_0}{2}}, \quad \zeta > s_0,$$

where $\|v(t_0)\|_{\dot{B}^{\frac{3}{p}-1}_{p,p}}$ is sufficiently small to apply Theorem 2.

6. Proof of Theorem 6: Regularity Condition in Besov Spaces

In order to prove Theorem 6, we need to show only that $\|\nabla v(t)\|_{L^\infty}$ appearing in the blowup criterion [3] can be controlled by (1.5). To this end,we will estimate ∇v in Besov spaces $\dot{B}^{\frac{3}{p}}_{p,1}$, which is contained in the L^∞. As before, we express v in the integral form:

$$v(t) = e^{t\Delta} v_0 - \int_0^t \Big[e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (v \otimes v)(s) \Big] ds. \tag{6.1}$$

By applying $\triangle_j$ to (6.1) and taking the L^p norm, we have

$$\begin{aligned}
\|\triangle_j v(t)\|_{L^p} &\lesssim e^{-t2^{2j}} \|\triangle_j v_0\|_{L^p} + \int_0^t \Big[2^j e^{-(t-s)2^{2j}} \|\triangle_j (v \otimes v)(s)\|_{L^p} \Big] ds \\
&\lesssim e^{-t2^{2j}} \|\triangle_j v_0\|_{L^p} + \int_0^t 2^j \Big[e^{-(t-s)2^{2j}} \Big(\|S_j v(s)\|_{L^\infty} \|\triangle_j v(s)\|_{L^p}
\end{aligned}$$

$$+ \sum_{k \geqq j-2} 2^{k\frac{3}{p}} \|\triangle_k v(s)\|_{L^p} \|\triangle_k v(s)\|_{L^p} \Big) \Big] ds$$
$$= e^{-t2^{2j}} \|\triangle_j v_0\|_{L^p} + I + II,$$

where we use the decomposition $v \otimes v = 2T_v \otimes v + R(v \otimes v)$ at the second inequality.

Estimation of I

By Bernstein's inequality (2.9),

$$\|S_j v(s)\|_{L^\infty} \lesssim \sum_{l=-\infty}^{j} 2^{l\frac{3}{p}} 2^{-\sigma l} 2^{\sigma l} \|\triangle_l v(s)\|_{L^p} \lesssim 2^{j(\frac{3}{p}-\sigma)} \|v(s)\|_{\dot{B}^\sigma_{p,\infty}},$$

where we use the condition $\frac{3}{p} - \sigma = 1 - \frac{2}{q} > 0$ for $q > 2$. Then,

$$I(t) \lesssim \int_0^t \Big[2^{j(1+\frac{3}{p}-\sigma)} e^{-(t-s)2^{2j}} \|v(s)\|_{\dot{B}^\sigma_{p,\infty}} \|\triangle_j v(s)\|_{L^p} \Big] ds.$$

Therefore,

$$\begin{aligned}
\|(I)\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}} &\lesssim \int_0^t \Big[(t-s)^{-\frac{1}{2}(1+\frac{3}{p}-\sigma)} s^{-\frac{1}{q}} s^{\frac{1}{q}} \|v(s)\|_{\dot{B}^\sigma_{p,\infty}} \|v(s)\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}} \Big] ds \\
&\lesssim \sup_{0<\tau<t} \Big[\tau^{\frac{1}{q}} \|v(\tau)\|_{\dot{B}^\sigma_{p,\infty}} \|v(\tau)\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}} \Big] \int_0^t \Big[(t-s)^{-\frac{1}{2}(1+\frac{3}{p}-\sigma)} s^{-\frac{1}{q}} \Big] ds \\
&\lesssim \sup_{0<\tau<t} \Big[\tau^{\frac{1}{q}} \|v(\tau)\|_{\dot{B}^\sigma_{p,\infty}} \|v(\tau)\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}} \Big], \qquad (6.2)
\end{aligned}$$

where we use the condition $1 + \frac{3}{p} - \sigma = 2 - \frac{2}{q} < 2$ for $q < \infty$, and $\frac{1}{2}\left(1 + \frac{3}{p} - \sigma\right) + \frac{1}{q} = 1$ to apply Lemma 3.

Estimation of II

$$\begin{aligned}
II(t) &\lesssim \int_0^t \Big[2^j e^{-(t-s)2^{2j}} \sum_{k \geqq j-2} 2^{-k} 2^{-k\sigma} 2^{k\sigma} \|\triangle_k v(s)\|_{L^p} 2^{k(\frac{3}{p}+1)} \|\triangle_k v(s)\|_{L^p} \Big] ds \\
&\lesssim \int_0^t \Big[2^{-j\sigma} e^{-(t-s)2^{2j}} \|v(s)\|_{\dot{B}^\sigma_{p,\infty}} \sum_{k \geqq j-2} 2^{(j-k)} 2^{k(\frac{1}{p}+1)} \|\triangle_k v(s)\|_{L^p} \Big] ds,
\end{aligned}$$

from which we obtain

$$\|II(t)\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}} \lesssim \int_0^t \left[(t-s)^{-\frac{1}{2}(1+\frac{3}{p}-\sigma)} s^{-\frac{1}{q}} s^{\frac{1}{q}} \|v(s)\|_{\dot{B}^{\sigma}_{p,\infty}} \|v(s)\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}}\right] ds$$

$$\lesssim \sup_{0<\tau<t} \left[\tau^{\frac{1}{q}} \|v(\tau)\|_{\dot{B}^{\sigma}_{p,\infty}} \|v(\tau)\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}}\right] \int_0^t \left[(t-s)^{-\frac{1}{2}(1+\frac{3}{p}-\sigma)} s^{-\frac{1}{q}}\right] ds$$

$$\lesssim \sup_{0<\tau<t} \left[\tau^{\frac{1}{q}} \|v(\tau)\|_{\dot{B}^{\sigma}_{p,\infty}} \|v(\tau)\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}}\right]. \tag{6.3}$$

By (6.2) and (6.3),

$$\|v(t)\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}} \lesssim \|v_0\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}} + \sup_{0<\tau<t} \left[\tau^{\frac{1}{q}} \|v(\tau)\|_{\dot{B}^{\sigma}_{p,\infty}} \|v(\tau)\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}}\right]. \tag{6.4}$$

We translate the time interval from $[0, t]$ to $[T-a, T]$. Then,

$$\|v(T)\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}}$$
$$\lesssim \|v(T-a)\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}} + \sup_{0<\tau<a} \left[(\tau)^{\frac{1}{q}} \|v(t+\tau)\|_{\dot{B}^{\sigma}_{p,\infty}} \|v(t+\tau)\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}}\right]. \tag{6.5}$$

Therefore, $\|v(t)\|_{\dot{B}^{\frac{3}{p}+1}_{p,1}}$ does not blow up at T as long as (1.5) holds. This completes the proof of Theorem 6.

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Center for Scientific Computation and Mathematical Modeling,
University of Maryland,
College Park, MD 20742, USA.
e-mail: hbae@cscamm.umd.edu

and

Department of Mathematics and Statistics,
University of North Carolina,
Charlotte, NC 28223, USA.
e-mail: abiswas@uncc.edu

and

Department of Mathematics, Center for Scientific Computation
and Mathematical Modeling (CSCAMM),
Institute for Physical Science & Technology (IPST),
University of Maryland,
College Park, MD 20742, USA.
e-mail: tadmor@cscamm.umd.edu

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