CONSTRUCTION OF BOUNDED SOLUTIONS OF $div\,\mathbf{u} = f$ IN CRITICAL SPACES

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Abstract. We construct uniformly bounded solutions of the equation $div\,\mathbf{u} = f$ for arbitrary data f in the critical spaces $L^d(\Omega)$, where Ω is a domain of $\mathbb{R}^d$. This question was addressed by Bourgain & Brezis, [BB2003], who proved that although the problem has a uniformly bounded solution, it is critical in the sense that there exists no linear solution operator for general L^d-data. We first discuss the validity of this existence result under weaker conditions than $f \in L^d(\Omega)$, and then focus our work on constructive processes for such uniformly bounded solutions. In the $d = 2$ case, we present a direct one-step explicit construction, which generalizes for $d > 2$ to a $(d-1)$-step construction based on induction. An explicit construction is also proposed for compactly supported data in $L^{d,\infty}(\Omega)$. We finally present constructive approaches based on optimization of a certain loss functional adapted to the problem. This approach provides a two-step construction in the $d = 2$ case. This optimization is used as the building block of a hierarchical multistep process introduced in [Tad2016] that converges to a solution in more general situations.

Dedicated to our friend and colleague Wolfgang Dahmen on his 75th birthday

Contents

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1. Introduction

Let Y denote a Banach space of functions defined on a d-dimensional domain $\Omega \subset \mathbb{R}^d$, where $d \geqslant 2$. We are concerned with the existence and construction of uniformly bounded solutions u to the equation,

$$div\, \mathbf{u} = f, \tag{1.1}$$

whenever $f \in Y$. Namely, we ask whether there exists a $\gamma > 0$ such that for every $f \in Y$ there exists a solution $\mathbf{u} = (u_1, \ldots, u_d) \in L^\infty(\Omega)$ to (1.1) such that

$$\|\mathbf{u}\|_{L^\infty} \leqslant \gamma \|f\|_Y. \tag{1.2}$$

Here for a vector valued function $\mathbf{v} = (v_1, \ldots, v_d)$ such that each v_i belongs to a function space X, we use the simpler notation $\mathbf{v} \in X$ and $\|\mathbf{v}\|_X$ instead of X^d. We say the space Y is *admissible* if for all $f \in Y$, (1.1) admits a solution such that (1.2) holds.

There exists of course infinitely many solutions to (1.1) since as soon as one exists, we can add to it a null divergence function, for example a constant. The most natural candidate for a solution $\mathbf{u}$ when given f is to solve Laplace's equation with data f and then to take $\mathbf{u}$ as the gradient of the solution. More precisely, we introduce

$$\psi(x) = \nabla\phi(x) = \frac{C_d}{|x|^d}x, \quad x \in \mathbb{R}^d,$$

where ϕ is the fundamental solution of the Laplacian on $\mathbb{R}^d$, and define the so-called Helmholtz solution as

$$\mathbf{u}(x) = \mathbf{u}_{\rm Hel}(x) = \int_\Omega f(x)\psi(x-y)\mathrm{d}y = \tilde{f} * \psi(x),$$

where $\tilde{f}(x) = f(x)$ for $x \in \Omega$ and $\tilde{f}(x) = 0$ when $x \notin \Omega$. Note that $\mathbf{u}_{\rm Hel}$ depends linearly on f. When Ω is a bounded domain, it is readily seen that $\mathbf{u}_{\rm Hel}$ is a uniformly bounded solution of (1.1) for $f \in L^p(\Omega)$ whenever $p > d$, and therefore the spaces $Y = L^p$, $p > d$ are all admissible.

The question of whether $Y = L^d$ is admissible was addressed in the seminal work of Bourgain & Brezis [BB2003]. Their work studies the particular case where $\Omega = \mathbb{T}^d$ is the d-dimensional torus, which leads to assume in addition that $\int_{\mathbb{T}^d} f = 0$. They proved that the problem (1.1) is *critical* in the sense that it admits bounded solutions, but there is no *linear* solution operator from Y to L^∞. In particular, one cannot invoke the Helmholtz solution. We say a space Y is critical if it is admissible but there is no linear mapping taking $f \in Y$ into a solution $\mathbf{u} \in L^\infty$ of (1.2).

The main interest of the present paper is two-fold. We first ask which of the classical function spaces Y are admissible. Secondly, we are interested in explicit constructions of solutions to the Bourgain-Brezis problem. In particular, can we explicitly construct nonlinear mappings solving (1.2) when Y is critical. In section 2 we discuss theoretical aspects of the problem. We recall certain known results of Meyer which show that for $\Omega = \mathbb{R}^d$, the space G of all f that admit a solution $\mathbf{u} \in L^\infty$ to (1.2) is the dual space $W^{1,1}_{\rm hom}$ which is defined as the closure of the smooth test functions for the total variation. Therefore, any admissible space Y must be a subspace of G. In particular, Y is admissible if and only if $W^{1,1}_{\rm hom}$ embeds into Y^*. In particular, we show that not only is L^d admissible but also the larger space weak-L^d, i.e. $L^{d,\infty}(\Omega)$, is admissible, as well as even weaker Morrey spaces.

In section 3, we present explicit constructions of bounded solutions for certain critical admissible spaces Y. We give a one step formula in the $d = 2$ case with L^2-data, and we treat the case $d > 2$ with L^d-data by a $(d-1)$-step construction based on induction [1] We end this section by a construction for

[1] We recently learned that this type of construction was independently derived by D. Stolyarov [Sto2024], however unpublished

$L^{d,\infty}$-data assuming in addition compact support. The reader may find these constructions interesting for their own sake. In section 4, we propose variational-based approaches for the constructions of bounded solutions. In the case $d = 2$, this approach delivers the solution in two steps. More generally, we use this optimization as the building block of a hierarchical decomposition that was used in [Tad2016] to construct solutions to the Bourgain-Brezis with L^d data by a limiting process. We use this multi-step hierarchical approach to construct solutions for more general data.

2. Theory

2.1. Existence of bounded solutions for L^d-data.

Let $\Omega \subset \mathbb{R}^d$. The space

$$G = G(\Omega) := \{f = div(\mathbf{u}) \;:\; \mathbf{u} \in L^\infty(\Omega)\},$$

of distributions u whose divergence is uniformly bounded has been studied in various contexts, in particular image processing and nonlinear PDE's.

As noted in [Mey2002], for the case $\Omega = \mathbb{R}^d$, the space $G(\mathbb{R}^d)$ is the dual of the homogeneous space $W^{1,1}_{\rm hom}(\mathbb{R}^d)$. The latter is defined as is the completion of the space of test functions $\mathcal{D}(\mathbb{R}^d)$ for the total variation, which defines a norm on this space.

Let us recall that the total variation of $v \in BV(\Omega)$ is defined as

$$|v|_{TV} := \sup_{\mathbf{w}} \int_\Omega v\, div\, \mathbf{w},$$

where the supremum is taken over all $\mathbf{w} = (w_1, \ldots, w_d) \in \mathcal{D}(\Omega)$ such that $\|\mathbf{w}\|_{L^\infty} = \sup_{x\in\Omega} |\mathbf{w}(x)|_2 \leqslant 1$. An equivalent quantity is defined in terms of finite difference:

$$|v|_{TV} \sim \sup_{h>0} h^{-1} \sup_{|y|\leqslant h} \|v - v(\cdot - y)\|_{L^1(\Omega_h)},$$

where $\Omega_h := \{x \in \Omega \;:\; \mathrm{dist}(x, \partial\Omega) > h\}$. When $\nabla v \in L^1$, in particular when $v \in W^{1,1}_{\rm hom}$, one simply has

$$|v|_{TV} = \|\nabla v\|_{L^1}.$$

Let us stress that $W^{1,1}_{\rm hom}(\mathbb{R}^d)$ is strictly smaller than $BV(\mathbb{R}^d)$.

Therefore, every f in a function space Y of locally integrable functions defined on $\mathbb{R}^d$ admits the representation $f = div\,\mathbf{u}$ with a uniformly bounded $\mathbf{u}$ satisfying the bound (1.2) if and only if for any test function $g \in \mathcal{D}(\mathbb{R}^d)$ one has

$$\Big|\int_{\mathbb{R}^d} fg\Big| \leqslant \gamma \|f\|_Y |g|_{TV}. \tag{2.1}$$

Note that this is equivalent to the condition that

$$\Big|\int_E f\Big| \leqslant \gamma \|f\|_Y \mathrm{per}(E), \tag{2.2}$$

for all open sets E of finite perimeter. Indeed the above is obtained from (2.1) by taking $g = \varphi_\epsilon * \chi_E$ where φ_ϵ is a mollifier and letting $\epsilon \to 0$. But from the coarea formula

$$|g|_{TV} = \int_{-\infty}^{+\infty} \mathrm{per}(E_t)\mathrm{d}t, \quad E_t := \{x \;:\; g(x) > t\}, \tag{2.3}$$

see [EG1992], it also implies (2.1) for any $g \in \mathcal{D}(\mathbb{R}^d)$. Note that here, the perimeter $\mathrm{per}(E)$ coincides with the Hausdorff measure $\mathcal{H}^{d-1}(\partial E)$ only for sufficiently nice sets (for example with Jordan domains with rectifiable boundaries). More generally it should be be defined as $|\chi_E|_{TV}$ or equivalently as $\mathcal{H}^{d-1}(\partial E^*)$ where ∂E^* is the so-called reduced boundary as introduced by de Giorgi.

Remark 2.1. The co-area formula also shows that (2.2) actually implies the validity of (2.1) for any $g \in BV(\mathbb{R}^d)$. Therefore any $f \in G$ that is in addition locally integrable is also an element of the dual of BV. We give further in Remark 2.7 an example of a distribution that belongs to G but are not in the dual of BV.

For $Y = L^d(\mathbb{R}^d)$, the validity of (2.1) is ensured by the Sobolev embedding of $BV(\mathbb{R}^d)$ into $L^{d'}(\mathbb{R}^d)$ where $\frac{1}{d} + \frac{1}{d'} = 1$. Since for a general domain Ω, we can trivially extend any $f \in L^d(\Omega)$ by 0 to obtain a function of $L^d(\mathbb{R}^d)$ with the same norm, this implies that $Y = L^d(\Omega)$ is admissible.

In [BB2003], the same result is given in the periodic context, where $\Omega = \mathbb{T}^d$ is the d-dimensional torus. In this case, $Y = L^d(\Omega)$ is modified into

$$Y = L^d_{\#}(\mathbb{T}^d) = \left\{ f \in L^d(\mathbb{T}^d) \ : \ \int_\Omega f = 0 \right\}$$

As shown in Proposition 2 therein, there exists no linear solution operator $f \in L^d_{\#}(\mathbb{T}^d) \mapsto \mathbf{u} \in L^\infty(\mathbb{T}^d)$. Indeed, restricting attention to the simpler case of the two-dimensional torus, if $K : L^2_{\#}(\mathbb{T}^2) \mapsto L^\infty(\mathbb{T}^2)$ would be such a linear solution operator so that $div\, K = \mathbb{I}$ is the identity, then so is

$$\widetilde{K} := \int\limits_{y \in \mathbb{T}^2} \tau_{-y} K \tau_y \mathrm{d}y,$$

which averages K over all 2D translations τ_y. Now $\widetilde{K}$ is translation invariant and it has a symbol, $\Lambda(n) := (\lambda_1(n), \lambda_2(n))$ such that $\widetilde{K}(e^{in\cdot x}) = \Lambda(n) e^{in\cdot x}$. Since $\widetilde{K}$ is assumed to boundedly map L^2 to L^∞, one should have $(\Lambda(n))_{n\in\mathbb{Z}^2} \in \ell^2(\mathbb{Z}^2)$. However, since $div\, \widetilde{K} = div\, K = \mathbb{I}$, that is $n \cdot \Lambda(n) = 1$, this implies $|\Lambda(n)|_2 \geqslant \dfrac{1}{|n|_2}$ which is a contradiction to $(\Lambda(n))_{n\in\mathbb{Z}^2} \in \ell^2(\mathbb{Z}^2)$.

The lack of linearity is attributed to the general fact that the problem of solving $\mathcal{L}\mathbf{u} = f$ with $\mathbf{u} \in X$ is critical if $\mathrm{Ker}(\mathcal{L})$ has no complement in X [BB2007, Aji2009]. This is the case of div in L^∞. One of the main themes in [BB2003] is the existence of solution with further $W^{1,d}$-regularity, similar to the Helmholtz solution $\mathbf{u}_{\mathrm{Hel}}$ that cannot be ensured to be uniformly bounded since it depends linearly on f.

2.2. **Existence of solutions for $L^{d,\infty}$ data.** One first observation is that the Bourgain-Brezis problem has also a positive answer for the larger Lorentz space $Y = L^{d,\infty}(\Omega)$. Recall that a measurable function f is in $L^{d,\infty}(\Omega)$ if and only if $|\{x \in \Omega : |f(x)| > t\}| \leqslant C^d t^{-d}$, $t > 0$, and the smallest C for which this holds is its $L^{d,\infty}(\Omega)$ norm.

Theorem 2.2. *There exists a constant $\gamma = \gamma_d$ such that for any $f \in L^{d,\infty}(\Omega)$, there exists $\mathbf{u} \in L^\infty(\Omega)$ satisfying*

$$div\, \mathbf{u} = f, \quad \|\mathbf{u}\|_{L^\infty} \leqslant \gamma \|f\|_{L^{d,\infty}(\Omega)}. \tag{2.4}$$

Proof. By definition of $Y = L^{d,\infty}(\Omega)$, one has for any $f \in Y$ and measurable E,

$$\int_E |f| = \int_{t>0} |\{x \in E \ : \ |f(x)| > t\}| \mathrm{d}t \leqslant \int_{t>0} \min\{|E|, \|f\|^d_{L^{d,\infty}} t^{-d}\} \mathrm{d}t \leqslant \frac{d}{d-1} \|f\|_{L^{d,\infty}} |E|^{\frac{d-1}{d}}.$$

Therefore (2.2) follows by application of the isoperimetric inequality with $\gamma_d = \frac{d}{d-1} K_d$ where K_d is the isoperimetry constant [Fed1969, 3.2.43]. □

Remark 2.3. An equivalent proof consists in establishing that $BV(\mathbb{R}^d)$ has continuous embedding in the dual Lorentz space $Y^* = L^{d',1}(\mathbb{R}^d)$, that is

$$\|g\|_{L^{d',1}(\mathbb{R}^d)} \leqslant \beta_d |g|_{TV}. \tag{2.5}$$

This readily follows by using the expression of the $L^{d,1}(\mathbb{R}^d)$ norm through the distribution function

$$\|g\|_{L^{d',1}(\mathbb{R}^d)} = d' \int_0^\infty |\{x \in \Omega \,:\, |g(x)| > t\}|^{\frac{d-1}{d}}\,\mathrm{d}t,$$

see e.g. [BC2011], and invoking the co-area formula and isoperimetric inequality to bound this quantity by the total variation of g.

Remark 2.4. The property assering that

$$\int_E |f| \leqslant C|E|^{\frac{d-1}{d}}$$

holds for all measurable sets E is actually equivalent to the membership of f in $L^{d,\infty}(\Omega)$. The smallest C for which this is valid gives an equivalent norm for $L^{d,\infty}$. We shall use this norm in going forward in this paper.

2.3. **Beyond $L^{d,\infty}$.** What is the largest Banach space Y of Borel measures μ which can be expressed as divergences of uniformly bounded $\mathbf{u}$? Placing ourselves in $\Omega = \mathbb{R}^d$, we know that such a Y should be embedded in $G(\mathbb{R}^d)$ the dual of $W^{1,1}_{\rm hom}(\mathbb{R}^d)$, that is, for all $\mu \in Y$ and $g \in W^{1,1}_{\rm hom}(\mathbb{R}^d)$ one has

$$\int_{\mathbb{R}^d} g\,\mathrm{d}\mu \leqslant \gamma \|\mathrm{d}\mu\|_Y |g|_{TV}. \tag{2.6}$$

Using the co-area formula, this is ensured in particular if

$$|\mu(E)| \leqslant \gamma \|\mu\|_Y \mathrm{per}(E) \tag{2.7}$$

for all sets $E \subset \mathbb{R}^d$ of finite perimeter.

Let us introduce the linear space of measures $S^d(\mathbb{R}^d)$ that satisfy the condition

$$|\mu|(B) \leqslant CR^{d-1}, \quad R > 0, \tag{2.8}$$

for all balls B or radius R, equipped with the norm

$$\|\mu\|_{S^d} := \sup R^{1-d} |\mu|(B), \tag{2.9}$$

where the supremum is taken over all balls B. For a general domain Ω, we define $S^d(\Omega)$ in a similar manner, replacing B by $B \cap \Omega$, and observe that any measure in this space has its extension by 0 contained in $S^d(\mathbb{R}^d)$ with a smaller or equal norm.

Remark 2.5. General conditions of the form $\mu(B) \leqslant CR^s$ were introduced by Otto Frostman in the study of fractional dimension of sets. In dimension $d = 2$, for positive measures, the specific condition $\mu(B) \leqslant CR$ was studied by Guy David for Dirac measures on a curve Γ. He proved that this condition is equivalent to the Ahlfors regularity of Γ and to the boundedness of the Cauchy integral operator acting on $L^2(\Gamma, \mu)$.

A distinction should be made between $S^d(\Omega)$ and the Morrey space $M^d(\Omega)$ that consists of all locally integrable f such that, for all ball B of radius R,

$$\int_{B \cap \Omega} |f| \leqslant CR^{d-1}, \tag{2.10}$$

with norm defined in a similar manner. This space is included in $S^d(\Omega)$ with equal norm when μ is of the form $f\,dx$, but the inclusion is strict: consider for example μ to be the Dirac measure on a segment of the plane in the case $d = 2$. In view of Remark 2.4, we have

$$L^{d,\infty} \subset M^d \subset S^d,$$

and these inclusions are strict. The following result that follows the arguments from [Mey2002] and [PT2017], shows that the Bourgain-Brezis problem has also a positive answer for $Y = M^d(\Omega)$ and $Y = S^d(\Omega)$.

Theorem 2.6. *There exists a constant $\gamma = \gamma_d$ such that the following holds. For any $f \in M^d(\Omega)$, there exists $\mathbf{u} \in L^\infty(\Omega)$ satisfying*

$$\text{(2.11)} \qquad div\,\mathbf{u} = f, \quad \|\mathbf{u}\|_{L^\infty} \leqslant \gamma\|f\|_{M^d}.$$

For any $\mu \in S^d(\Omega)$, there exists $\mathbf{u} \in L^\infty(\Omega)$ satisfying

$$\text{(2.12)} \qquad div\,\mathbf{u} = \mu, \quad \|\mathbf{u}\|_{L^\infty} \leqslant \gamma\|\mu\|_{S^d}.$$

Proof. Without loss of generality we work on $\Omega = \mathbb{R}^d$. As a first step, we use the boxing inequality [PT2008, Theorem 2.11] that states that any open set $E \subset \mathbb{R}^d$ of finite perimeter can be covered by balls B_j of radius R_j such that

$$\text{(2.13)} \qquad \sum_j R_j^{d-1} \leqslant C\text{per}(E),$$

where the constant C depends only on d. This shows that for any $f \in M^d(\mathbb{R}^d)$, we have

$$\int_E |f| dx \leqslant C\|f\|_{M^d}\text{per}(E).$$

which implies (2.2) and therefore proving (2.11).

Similarly, for any $\mu \in S^d(\mathbb{R}^d)$, we have

$$|\mu|(E) \leqslant C\|\mu\|_{S^d}\text{per}(E).$$

For any test function $g \in \mathcal{D}(\mathbb{R}^d)$, we write $g = g_+ - g_-$ and

$$\Big|\int_{\mathbb{R}^d} g d\mu\Big| \leqslant \int_{\mathbb{R}^d} g_+ d|\mu| + \int_{\mathbb{R}^d} g_- d|\mu|.$$

For the first term, we have

$$\int_{\mathbb{R}^d} g_+ d|\mu| = \int_0^\infty |\mu|(E_t) dt \leqslant C\|\mu\|_{S^d} \int_0^\infty \text{per}(E_t)$$

where $E_t := \{x \,:\, g(x) > t\}$. With a similar treatment of the second term and using the co-area formula, we reach

$$\Big|\int_{\mathbb{R}^d} g d\mu\Big| \leqslant C\|\mu\|_{S^d}|g|_{TV}$$

which shows that μ belongs to the space $G(\mathbb{R}^d)$ and thereby proves (2.12). □

Remark 2.7. We stress that, in contrast to the functions of $M^d(\mathbb{R}^d)$, the measures of $S^d(\mathbb{R}^d)$ belong to $G(\mathbb{R}^d)$ but not to the dual of $BV(\mathbb{R}^d)$. This is due to the fact that the trace of a BV function on a $d-1$ dimensional surface could be meaningless. For example if μ is the Dirac measure on a segment of the $2d$ plane, we cannot apply it to the BV function $g = \chi_Q$ where Q is a square that admits this segment as one of its side.

Remark 2.8. As pointed out in [Mey2002] in the case $d = 2$, a positive measure belongs to $G(\mathbb{R}^d)$ if and only if it belongs to $S^d(\mathbb{R}^d)$. Indeed, on the one hand the above result shows that $S^d(\mathbb{R}^d)$ is contained in $G(\mathbb{R}^d)$. On the other hand, if μ is a positive measure that is contained in $G(\mathbb{R}^d)$ then to any ball $B = B(x_0, R)$ we associate the $W^{1,1}$ function

$$g(x) = \max\left\{0, 2 - \frac{|x - x_0|}{R}\right\}$$

Since μ is positive and belongs to $G(\mathbb{R}^d)$, and since g is positive and larger than 1 on B, we find that

$$\mu(B) \leqslant \int g \mathrm{d}\mu \leqslant C|g|_{TV}.$$

On the other hand, it is easily checked that $|g|_{TV} \leqslant C_d R^{d-1}$ where C_d only depend of d, therefore proving that $\mu \in S^d(\mathbb{R}^d)$.

3. Explicit constructions of bounded solutions

3.1. **A one-step explicit construction for L^2-data.** What follows is probably the simplest and most instructive construction of bounded solutions to the Bourgain-Brezis problem (1.1), at least in the d =-case with $Y = L^2(\Omega)$. Again, without loss of generality we will work on $\Omega = \mathbb{R}^2$.

For any $(x, y) \in \mathbb{R}^2$ and any fixed $f \in L^2(\mathbb{R}^2)$, we define

$$V^2(x) := \int_{-\infty}^{+\infty} |f(x,y)|^2 \,\mathrm{d}y, \quad H^2(y) := \int_{-\infty}^{+\infty} |f(x,y)|^2 \,\mathrm{d}x \tag{3.1}$$

and

$$\alpha(x,y) := \frac{V(x)}{H(y) + V(x)}, \quad \beta(x,y) := \frac{H(y)}{H(y) + V(x)}. \tag{3.2}$$

We then consider the splitting $f = f_1 + f_2$ where

$$f_1(x,y) := \alpha(x,y) f(x,y), \quad \text{and} \quad f_2(x,y) := \beta(x,y) f(x,y), \tag{3.3}$$

and we define

$$u_1(x,y) := \int_{-\infty}^{x} f_1(s,y)\mathrm{d}s \quad \text{and} \quad u_2(x,y) := \int_{-\infty}^{y} f_2(x,t)\mathrm{d}t \tag{3.4}$$

Therefore $\mathbf{u} = (u_1, u_2)$ satisfies $div\,\mathbf{u} = f_1 + f_2 = f$ and it remains to check that u is uniformly bounded. Let us bound $|u_1(x,y)|$ for any arbitrary but fixed $(x,y) \in \mathbb{R}^2$. If $H(y) = 0$, then obviously $u_1(x,y) = 0$ for all x so we assume $H(y) \neq 0$. Then, for any x we have

$$\begin{aligned}
|u_1(x,y)| &\leqslant \int_{-\infty}^{\infty} |f_1(s,y)|\,\mathrm{d}s \leqslant \int_{-\infty}^{\infty} |f(s,y)| \frac{V(s)}{H(y)}\,\mathrm{d}s \\
&\leqslant H(y)^{-1} \left(\int_{-\infty}^{\infty} |f(s,y)|^2\,\mathrm{d}s\right)^{1/2} \left(\int_{-\infty}^{\infty} V(s)^2\,\mathrm{d}s\right)^{1/2} \\
&= \left(\int_{-\infty}^{\infty} V(s)^2\,\mathrm{d}s\right)^{1/2} = \|f\|_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

In a similar way, we obtain the bound $\|u_2\|_{L^\infty} \leqslant \|f\|_{L^2(\mathbb{R}^2)}$, and so $\mathbf{u}$ is a bounded solution to the Bourgain-Brezis problem for f.

Remark 3.1. The above splitting of f into f_1 and f_2 is designed to ensure that the univariate primitive u_1 and u_2 are uniformly bounded. An interesting variant consists in taking

$$f_1(x,y) := f(x,y)\chi_{\{H(y)\leqslant V(x)\}} \quad \text{and} \quad f_2(x,y) := f(x,y)\chi_{\{V(x)<H(y)\}} \tag{3.5}$$

for which it is easily checked that the solution u also has each component u_1 and u_2 uniformly bounded by $\|f\|_{L^2}$. The extra feature of this choice is that f_1 and f_2 have disjoint supports. As we discuss next, in the more general d-dimensional case with $Y = L^d(\mathbb{R}^d)$, it is possible to explicitly construct a splitting $f = f_1 + \cdots + f_d$ also with disjoint supports and such that the univariate primitive u_j of f_j with respect to x_j are uniformly bounded.

3.2. **A $(d-1)$-step explicit construction for L^d-data.** We now consider the general d-dimensional case with data $f \in L^d(\mathbb{R}^d)$. Given any such f, we construct d pairwise disjoint sets $\Omega_j = \Omega_j(f)$ with $\mathbb{R}^d = \bigcup_{j=1}^d \Omega_j$, so that the functions $f_j := f\chi_{\Omega_j}$ satisfy $f = f_1 + \cdots + f_d$ as well as

$$\int_{\mathbb{R}} |f_j(x_1,\dots,x_{j-1},s,x_{j+1},\dots,x_d)|\,\mathrm{d}s \leqslant \|f\|_{L^d(\mathbb{R}^d)}, \quad j = 1,\dots,d. \tag{3.6}$$

In turn the functions

$$u_j(x_1,\dots,x_d) := \int_{-\infty}^{x_j} f_j(x_1,\dots,x_{j-1},s,x_{j+1},\dots,x_d)\,\mathrm{d}s, \quad j = 1,\dots,d, \tag{3.7}$$

satisfy $\|u_j\|_{L^\infty(\mathbb{R}^d)} \leqslant \|f\|_{L^d(\mathbb{R}^d)}$. Hence, $\mathbf{u} = (u_1,\dots,u_d)$ is a solution to (1.1) with $\|\mathbf{u}\|_{L^\infty(\mathbb{R}^d)} \leqslant \|f\|_{L^d(\mathbb{R}^d)}$.

The construction proceeds by induction on d. When $d = 1$, given any $f \in L^1(\mathbb{R})$, we define $\Omega_1 = \mathbb{R}$ in which case the above claim is obvious. We assume we have shown how to construct such sets $\Omega_1(g)\dots,\Omega_{d-1}(g)$ whenever $g \in L^{d-1}(\mathbb{R}^{d-1})$ and give the construction of $\Omega_1(f),\dots,\Omega_d(f)$ whenever $f \in L^d(\mathbb{R}^d)$. Without loss of generality, we can assume $\|f\|_{L^d(\mathbb{R}^d)} = 1$.

We write any vector $x \in \mathbb{R}^d$ as (x_1, y) where $y = (x_2,\dots,x_d) \in \mathbb{R}^{d-1}$ and define the thresholds

$$t(y)^{d-1} := \int_{\mathbb{R}} |f(x_1,y)|^d \mathrm{d}x_1, \quad y \in \mathbb{R}^{d-1}.$$

Let

$$\Omega_1 := \Omega_1(f) := \{x = (x_1,y) \in \mathbb{R}^d : |f(x_1,y)| \geqslant \tau(y)\}, \tag{3.8}$$

and Ω_1' be its complement in $\mathbb{R}^d$. We define

$$f_1 := f\chi_{\Omega_1} \quad \text{and} \quad g := f - f_1 = f\chi_{\Omega_1'}. \tag{3.9}$$

This determines u_1 and for any $x = (x_1,y) \in \mathbb{R}^d$, we have

$$\int_{\mathbb{R}} |f_1(s,y)|\mathrm{d}s \leqslant \int_{\mathbb{R}} |f(s,y)| \left(\frac{|f(s,y)|}{t(y)}\right)^{d-1} \mathrm{d}s = t(y)^{1-d} \int_{\mathbb{R}} |f(s,y)|^d \mathrm{d}s = 1.$$

This shows that (3.6) holds for f_1 and $\|u_1\|_{L^\infty(\mathbb{R})} \leqslant 1$ as desired.

We proceed to construct the set $\Omega_2,\dots,\Omega_d$. For any fixed $x_1 \in \mathbb{R}$, we consider the function $g(x_1,y)$ as a function of $y \in \mathbb{R}^{d-1}$. We have

$$\int_{\mathbb{R}^{d-1}} |g(x_1,y)|^{d-1}\mathrm{d}y \leqslant \int_{\mathbb{R}^{d-1}} t(y)^{d-1}\mathrm{d}y = \|f\|_{L^d(\mathbb{R}^d)}^d = 1. \tag{3.10}$$

From the induction hypothesis, we can apply our construction to $g(x_1,\cdot)$ which is a function of $d-1$ variables $y = (y_2,\dots,y_d)$. This gives $d-1$ disjoint sets $\Omega_j(x_1)$ for $j = 2,\dots,d$ whose union is $\mathbb{R}^{d-1}$ and for which the above results are valid. Therefore $g(x_1,\cdot)$. is split into functions $g_j(x_1,\cdot) = g(x_1,\cdot)\chi_{\Omega_j(x_1)}$ which satisfy

$$\int_{\mathbb{R}} |g_j(x_1, y_2, \dots, y_{j-1}, s, y_{j+1}, \dots, y_d)|\,\mathrm{d}s \leqslant 1, \quad j = 2, \dots, d. \tag{3.11}$$

We now define

$$\Omega_j = \Omega_j(f) := \{(x_1, y)) : \ x_1 \in \mathbb{R},\ y \in \Omega_j(x_1)\}, \quad j = 2, \dots, d.$$

This completes the definition of the sets Ω_j and the functions f_j and u_j. We are left to check (3.6) for $j \neq 1$. Since $f_j(x_1, y) = g_j(x_1, y)$, it suffices to write

$$\int_{\mathbb{R}} |f_j(x_1, \dots, x_{j-1}, s, x_{j+1}, \dots, x_d)|\,\mathrm{d}s = \int_{\mathbb{R}} |g_j(x_1, y_2, \dots, y_{j-1}, s, y_{j+1}, \dots, y_{d-1})|\,\mathrm{d}s \leqslant 1,$$

This establishes the properties we want of our construction for d dimensions.

3.3. **A constructive decomposition for $L^{d,\infty}$.** We know from the theoretical results of §2, that the space $Y = L^{d,\infty}(\Omega)$ is admissible whenever $\Omega \subset \mathbb{R}^d$ is measurable. In this section, for any $\tau > 1$ and any bounded measurable set $\Omega \subset \mathbb{R}^d$, we give an algorithm that takes any $f \in L^{d,\infty}(\Omega)$ and constructs a solution $\mathbf{u}$ to (1.1) such that $\|\mathbf{u}\|_{L^\infty} \leqslant \tau \|f\|_{L^{d,\infty}}$. For the sake of notational simplicity, we detail the construction in the case $d = 2$ and sketch its generalization to $d > 2$.

We fix Ω and f in going forward. We assume without loss of generality that $\Omega = [-R, R]^2$, for some $R > 0$ and $\|f\|_{L^{2,\infty}(\Omega)} = 1$. We define f to be zero outside of Ω. We construct a disjoint splitting $\Omega = \Omega_1 \cup \Omega_2$ and $f_j := f\chi_{\Omega_j}$, $j = 1, 2$, and take

$$u_1(x, y) := \int_{-\infty}^{x} f_1(s, y)\,\mathrm{d}s, \quad u_2(x, y) := \int_{-\infty}^{y} f_2(x, s)\,\mathrm{d}s. \tag{3.12}$$

Thus $div(\mathbf{u}) = f$ when $\mathbf{u} := (u_1, u_2)$, and the only issue will be to show that

$$\int_{-\infty}^{\infty} |f_1(s, y)|\,\mathrm{d}s \leqslant \tau, \quad \int_{-\infty}^{\infty} |f_2(x, s)|\,\mathrm{d}s \leqslant \tau, \tag{3.13}$$

for all $x, y \in \mathbb{R}$.

Let us first note that $f \in L^1(\mathbb{R}^2)$ and

$$M := \|f\|_{L^1(\mathbb{R}^2)} = \int_{\Omega} |f| \leqslant |\Omega|^{1/2}, \tag{3.14}$$

where we used Remark 2.4. We define the horizontal line $L_H(y) := \{(x, y) : \ x \in [-R, R]\}$ at level y and the vertical line $L_V(x) := \{(x, y) : \ y \in [-R, R]\}$ at level x. For any measurable function g that is supported on Ω, we define the energies

$$E_H(g, y) := \int_{L_H(y)} |g(x, y)|\,\mathrm{d}x,\ y \in [-R, R], \quad E_V(g, x) := \int_{L_V(x)} |g(x, y)|\,\mathrm{d}y,\ x \in [-R, R], \tag{3.15}$$

which may be infinite.

Here is the first step of our construction. Let $A := \{y \in [-R, R] : \ E_H(f, y) \leqslant \tau\}$ and $A' := \{y \in [-R, R] : E_H(f, y) > \tau\}$ and

$$\Omega_H := \{(x, y) \in \mathbb{R}^2 : E_H(f, y) \leqslant \tau\} = \bigcup_{y \in A} L_H(y).$$

We define $f_1 := f$ on Ω_H and $f_2 := 0$ on Ω_H. Notice that we have

$$\int_{-\infty}^{x} |f_1(s, y)|\,\mathrm{d}s \leqslant \tau, \quad y \in A. \tag{3.16}$$

This means that (3.13) is satisfied for $y \in A$.

We proceed to the second step of our construction. Let $B := \{x \in [-R,R] : E_V(f,x) \leqslant \tau\}$ and $B' := \{x \in [-R,R] : E_V(f,x) > \tau\}$ and

$$\Omega_V := \{(x,y) \in \Omega : E_V(f,x) \leqslant \tau\} = \bigcup_{x \in B} L_V(x).$$

We define $f_2 = f$ on $\Omega_V \setminus \Omega_H$ and f_1 is defined to be zero on this set. We have

$$\int_{-\infty}^{y} |f_2(x,s)|\,\mathrm{d}s \leqslant \tau, \quad x \in B. \tag{3.17}$$

Thus far, we have defined f_1 and f_2 on $\Omega_H \cup \Omega_V$. Let $\Omega' := \Omega \setminus (\Omega_H \cup \Omega_V)$. The important thing to notice is that Ω' is gotten from Ω by removing horizontal and vertical strips. The following lemma shows that we have removed a significant portion of Ω in this construction.

Lemma 1. The measure of Ω' satisfies

$$|\Omega'| \leqslant \tau^{-2}|\Omega|. \tag{3.18}$$

Proof: To prove this claim, we observe that

$$\Omega' := \{(x,y) \in \Omega : E_H(f,y) \geqslant \tau \text{ and } E_V(f,x) \geqslant \tau\} = A' \times B'.$$

Let $a := |A'|$ and $b := |B'|$ be the univariate Lebesgue measure of these sets. Then, we have

$$a\tau \leqslant \int_{\Omega} |f(x,y)|\,\mathrm{d}x\mathrm{d}y \leqslant |\Omega|^{1/2}. \tag{3.19}$$

A similar argument gives $b\tau \leqslant |\Omega|^{1/2}$. Hence, we have

$$|\Omega'| = ab \leqslant \tau^{-2}|\Omega|, \tag{3.20}$$

which proves the lemma. □

After applying the first step of our construction, we have defined f_1 and f_2 outside of Ω'. Let $\Omega^1 := \Omega'$ and let us repeat our construction for the set Ω^1 in place of Ω. This gives a new set $\Omega^2 := [\Omega^1]' \subset \Omega^1$ and thereby give the definitions of f_1, f_2 outside of Ω^2. The new residual set Ω^2 satisfies $|\Omega^2| \leqslant \tau^{-2}|\Omega^1| \leqslant \tau^{-4}|\Omega|$. Iterating this procedure gives in the limit a definition of f_1 and f_2 on all of Ω except for a set of measure zero. One easily checks that (3.13) holds. For example to check this for f_1, we note that if f_1 is defined to be nonzero on on a line $L_H(y)$ then at the (first) step k where it is defined to be nonzero it is completely defined on this line and (3.13) holds on this line.

The generalization of the construction to $d > 2$ with $\Omega = [-R,R]^d$ is based on a similar process assuming $\|f\|_{L^{d,\infty}(\Omega)} = 1$: introducing the energies

$$E_i(f, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d) := \int_{-R}^{R} |f(x_1, \dots, x_d)| dx_1$$

and comparing them with the level τ, we reccursively define $f_1, f_2, \dots, f_d$ on $\Omega \setminus \Omega'$, where

$$\Omega' = \{(x_1, \dots, x_d) \in \Omega \ : \ E_i(f, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d) \geqslant \tau, \ i = 1, \dots, d\}.$$

In order to conclude, we prove a contraction property similar to Lemma 1. For each $i = 1, \dots, d$, we denote by $\Omega'_i \in [-R,R]^d$ the projection of Ω on the $d-1$-hyperplane orthogonal to the x_i axis, and invoke the Loomis-Whitney inequality

$$|\Omega'|^{d-1} \leqslant \prod_{i=1}^{d} |\Omega'_i|, \tag{3.21}$$

see [LW1949]. Obviously we have

$$\Omega_i' \subset \{(x_1,\dots,x_{i-1},x_{i+1},\dots,x_d) \in [-R,R]^{d-1} \ : \ E_i(f,x_1,\dots,x_{i-1},x_{i+1},\dots,x_d) \geqslant \tau\},$$

and therefore

$$\tau|\Omega_i'| \leqslant \int_\Omega |f(x_1,\dots,x_d)| \leqslant |\Omega|^{\frac{d-1}{d}}.$$

Combining with (3.21), we obtain the contraction property $|\Omega'| \leqslant \tau^{-d}|\Omega|$ allowing us to conclude in a similar manner.

Remark 3.2. Note that the final result is independent of the support $[R,R]^d$, but the above construction does not seem to generalize in a straighforward manner to $\Omega = \mathbb{R}^d$ which is left as an open problem.

4. Variational-based constructions

4.1. **Minimization problems.** One natural way of approaching bounded solutions to (1.1) for data $f \in Y(\Omega)$ is to consider the minimization of functionals of the form

$$\mathcal{V}(\mathbf{u}) = \mathcal{V}_\lambda(\mathbf{u}) = \|\mathbf{u}\|_{L^\infty} + \lambda\|f - div\,\mathbf{u}\|_Y^p \tag{4.1}$$

for some fixed $p \geqslant 1$. Indeed, one intuition is that if a uniformly bounded solution to (1.1) exists, the minimizer of $\mathcal{V}_\lambda$ should tend to the solution $\mathbf{u}$ of (1.1) with minimal L^∞ norm as $\lambda \to \infty$.

We shall first see that in the case of $Y = L^2$ and $p = 2$, it is possible to avoid letting $\lambda \to \infty$ through a two-step constructive approach. We then discuss more general situations where we can construct a uniformly bounded solution $\mathbf{u}$ by a hierarchical decompositions based on iterated minimizations of the above functional.

Before going further let us observe that the existence of a minimizer for $\mathcal{V}$ can be derived by the elementary arguments under a mild assumption on the space Y.

Lemma 2. Assume that $Y = Z'$ is a dual space of distribution, so that

$$\|v\|_Y := \max\{\langle v,\varphi\rangle_{Y,Z} \ : \ \varphi \in \mathcal{D}(\Omega),\ \|\varphi\|_Z \leqslant 1\},$$

then there exists a minimizer $\mathbf{u}_1$ of $\mathcal{V}$

Proof. Consider a minimizing sequence $\mathbf{u}^n$, therefore such that $\|\mathbf{u}^n\|_{L^\infty}$ and $\|div\,\mathbf{u}^n\|_Y$ are are uniformly bounded. Then, up to a subsequence extraction, we have the following properties :

(i) Both $\|f - div\,\mathbf{u}^n\|_{L^2}$ and $\|\mathbf{u}^n\|_{L^\infty}$ have limits A and B such that $A+\lambda B$ is the infimum of $\mathcal{V}$.
(ii) $\mathbf{u}^n$ converges in the L^∞ weak-$*$ sense to some $\mathbf{u}_1 \in L^\infty$.
(iii) $div\,\mathbf{u}^n$ converges in the Y weak-$*$ sense to the (weak) divergence $div\,\mathbf{u}_1 \in L^2$.

From this and the properties of weak lower semi-continuity of norms, it readily follows that $\mathbf{u}_1$ is a minimizer of $\mathcal{V}$. □

Existence is therefore ensured for reflexive spaces Y such as $L^d(\Omega)$ in the d-dimensional case, but also for $Y = L^{d,\infty}$ which we have seen earlier to be an admissible choice for the existence of uniformly bounded property. We stress that uniqueness of minimizers, is in general not ensured, however in the case where Y is strictly convex, such as L^d, we find that $div\,(\mathbf{u}_1)$ is unique.

Note that the minimization of $\mathcal{V}$ may be computationally intensive, depending on the form of the Y norm. In the particular case of $Y = L^d$, it can be computed by solving relatively simple Euler-Lagrange equations, see [TT2011].

4.2. **A two-step approach for L^2-data.** Consider the case of a domain $\Omega \subset \mathbb{R}^2$ and $Y = L^2(\Omega)$. With the choise $p = 2$, the functional of interest is therefore

$$\mathcal{V}(\mathbf{u}) = \|\mathbf{u}\|_{L^\infty} + \lambda\|f - div\,\mathbf{u}\|_{L^2}^2 \tag{4.2}$$

An interesting property of the minimizers is given by the following.

Lemma 3. Fix $\lambda > 0$ and let $r_\lambda = f - div\,\mathbf{u}_\lambda$ be the residual of the equation (1.1) for a minimizer $\mathbf{u}_\lambda$ of (4.2). Then r_λ belongs to $BV(\Omega)$ with

$$|r_\lambda|_{TV} \leqslant \frac{1}{2\lambda}. \tag{4.3}$$

Proof. For any $\mathbf{z} \in \mathcal{D}(\Omega)$ and $\epsilon > 0$, we have

$$\begin{aligned} \mathcal{V}(\mathbf{u}_\lambda) = \|\mathbf{u}_\lambda\|_{L^\infty} + \lambda\|f - div\,\mathbf{u}_\lambda\|_{L^2}^2 &\leqslant \|\mathbf{u}_\lambda + \epsilon\mathbf{z}\|_{L^\infty} + \lambda\|f - div\,(\mathbf{u}_\lambda + \epsilon\mathbf{z})\|_{L^2}^2 \\ &\leqslant \|\mathbf{u}_\lambda\|_{L^\infty} + \epsilon\|\mathbf{z}\|_{L^\infty} + \lambda\|r_\lambda\|_{L^2}^2 - 2\lambda\epsilon\int_\Omega r_\lambda div\,\mathbf{z} + o(\epsilon). \\ &= \mathcal{V}(\mathbf{u}_\lambda) + \epsilon\|\mathbf{z}\|_{L^\infty} - 2\lambda\epsilon\int_\Omega r_\lambda div\,\mathbf{z} + o(\epsilon) \end{aligned}$$

and by letting $\epsilon \downarrow 0$ we find that

$$\int_\Omega r_\lambda div\,\mathbf{z} \leqslant \frac{1}{2\lambda}\|\mathbf{z}\|_{L^\infty}, \tag{4.4}$$

for all $\mathbf{z} \in \mathcal{D}(\Omega)$, which shows that $r_\lambda \in BV(\Omega)$ with bound (4.3) for its total variation. □

Note that we also have the trivial bounds

$$\|r_\lambda\|_{L^2} \leqslant \|f\|_{L^2} \quad \text{and} \quad \|\mathbf{u}\|_{L^\infty} \leqslant \lambda\|f\|_{L^2}^2, \tag{4.5}$$

by comparing $\mathcal{V}(\mathbf{u}_\lambda)$ with $\mathcal{V}(0)$. However Lemma 3 shows a "regularization effect" $f \in L^2 \mapsto r_\lambda \in BV$. As noted in Remark 2.3, the space BV has a continuous embedding in the Lorentz space $L^{2,1}$ which is strictly smaller than L^2.

This effect leads us to a direct construction of a bounded solution. Without loss of generality, we again work on $\Omega = \mathbb{R}^2$, and denote by $\mathbf{u}_1$ the minimizer and $r_1 = f - div\,\mathbf{u}_1$ the residual, when taking the particular value $\lambda := \|f\|_{L^2}^{-1}$. Using both (4.3) and (2.5), we have on the one hand

$$\|r_1\|_{L^{2,1}} \leqslant \beta_2|r_1|_{TV} \leqslant \frac{\beta_2}{2\lambda} = \frac{\beta_2}{2}\|f\|_{L^2}; \tag{4.6}$$

and on the other hand

$$\|\mathbf{u}_1\|_{L^\infty} \leqslant \|f\|_{L^2}, \tag{4.7}$$

in view of the second bound in (4.5). We then write $r_1 = div\,\mathbf{u}_2$, where

$$\mathbf{u}_2 := \psi * r_1 = \frac{1}{2\pi}\frac{x}{|x|^2} * r_1, \tag{4.8}$$

is the Helmoltz solution for the data r_1. Since $\psi \in L^{2,\infty}$ and $r_1 \in L^{2,1}$, it is readily seen that $\mathbf{u}_2$ is uniformly bounded by

$$\|\mathbf{u}_2\|_{L^\infty} \leqslant \|\psi\|_{L^{2,\infty}}\|r_1\|_{L^{2,1}} \leqslant C\|f\|_{L^2}, \quad C := \frac{\beta_2}{2}\|\psi\|_{L^{2,\infty}}. \tag{4.9}$$

Thus we end up with

$$\mathbf{u}_{2\text{step}} := \mathbf{u}_1 + \mathbf{u}_2, \tag{4.10}$$

as a two-step construction of a uniformly bounded solution to (1.1) which satisfies (1.2) with $\gamma = 1 + C$.

Remark 4.1. A similar regularization effect takes place in the $d > 2$ case for data $f \in L^d$

$$f \in L^d(\Omega) \mapsto r_1 \in L^{d,d-1}(\Omega).$$

However, since $\psi \in L^{d,\infty}$, this is not enough to derive a similar two-step construction by applying the Helmoltz solver to the residual. Instead, this will be addressed by the multi-step construction in the next section below.

Figure 4.1, quoted from [TT2011, Section 6], shows the two-step solution of the example due to L. Nirenberg, [BB2003, Remark 7], which demonstrates the unboundedness of $\|\mathbf{u}_{\text{Hel}}\|_{L^\infty}$ solved for $\mathbf{u} \in L^2_\#([-1,1]^2)$ with periodic boundary conditions, given by

$$f = \Delta v \qquad v(x_1, x_2) := x_1 |\log|x||^{1/3} \zeta(|x|), \qquad \zeta(r) = \chi_{(-1,1)} e^{-\frac{1}{1-r^2}}. \tag{4.11}$$

In this case, Helmholtz solution, $\mathbf{u}_{\text{Hel}} = \nabla V$, has a fractional logarithmic growth at the origin, which should be contrasted with the hounded two-step constructed solution shown in figure 4.1. Table 4.1 reports that the ratio between $N \times N$ grid discretization of $\|\mathbf{u}^N_{2\text{step}}\|_{L^\infty}$ and $\|f^N\|_{L^2}$ remains bounded when N is large, in contrast to the computed solution of Helmholtz.

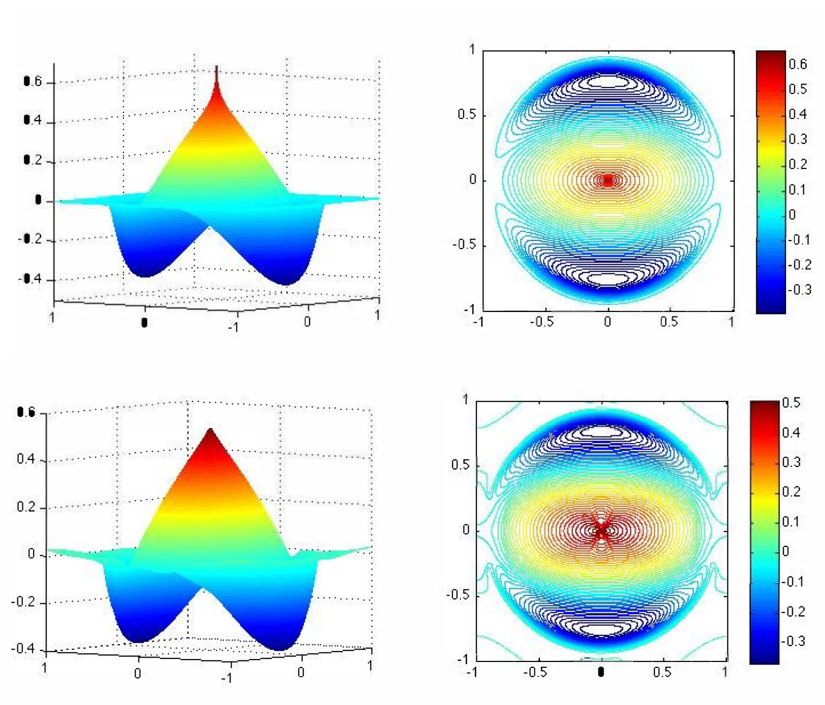

FIGURE 4.1. Solution of Bourgain-Brezis problem with 2D data in (4.11).
Helmholtz solution, $\mathbf{u}_{\text{Hel}}$ (top), vs. two-step solution, $\mathbf{u}_{2\text{step}}$ (bottom).

4.3. **Hierarchical constructions for data in Fréchet differentiable spaces.** We now work with general data $f \in Y(\mathbb{R}^d)$. In this section, we use a hierarchical approach to construct uniformly bounded solution, under the assumption that the Y norm is Frechet differentiable.

Assumption (Fréchet differentiablity). *The Y-norm is* Fréchet differentiable*, namely — there exists $\phi : Y \to Y'$ such that*

$$\|v + \epsilon w\|_Y = \|v\|_Y + \epsilon \langle \phi(v), w \rangle + o(\epsilon) \ \text{ for all } \ v, w \in Y, \ \ v \neq 0. \tag{4.12}$$

The $N\times N$ grid	50×50	100×100	200×200	400×400	800×800
$\dfrac{\|\mathbf{u}^{1,N}_{\rm Hel}\|_{L^\infty}}{\|f^N\|_{L^2}}$	0.2295	0.2422	0.2540	0.2650	0.2752
$\dfrac{\|\mathbf{u}^{1,N}_{\rm 2step}\|_{L^\infty}}{\|f^N\|_{L^2}}$	0.2096	0.2128	0.2144	0.2151	0.2154

TABLE 4.1. L^∞ norm of numerical solutions for different grids: Helmholtz vs. the two-step solution of (4.11) for different grids.

As an immediate consequence of this assumption, for any $p>1$, the application $v\mapsto \|v\|_Y^p$ is also Fréchet differentiable and its derivative is given by

$$\phi_p(v) := p\|v\|_Y^{p-1}\phi(v).$$

As we discuss further, spaces admitting Fréchet differentiable norms are for example $Y=L^d$ as well as $Y=L^{d,q}$ when $1<q<\infty$, but not $Y=L^{d,\infty}$.

Let us now consider for a $p>1$ the general functional $\mathcal{V}_\lambda$ of (4.1), assuming as before that Y is a dual space. In the sequel we use the following two results which makes use of ϕ_p. The first is the generalization of Lemma 3 that shows a regularization effect, now on $\phi_p(r_\lambda)$ where $r_\lambda = f-div\,\mathbf{u}_\lambda$ is the residual.

Lemma 4. If $\mathbf{u}_\lambda\in L^\infty$ is a minimizer of (4.1) with residual $r_\lambda = f-div\,\mathbf{u}_\lambda\in Y$ then

$$|\phi_p(r_\lambda)|_{TV}\leqslant \frac{1}{\lambda}. \tag{4.13}$$

Proof. For any test function $\mathbf{z}\in\mathcal{D}(\Omega)$, we have

$$\begin{aligned}\mathcal{V}_\lambda(\mathbf{u}_\lambda)=\|\mathbf{u}_\lambda\|_{L^\infty}+\lambda\|f-div\,\mathbf{u}_\lambda\|_Y^p &\leqslant \|\mathbf{u}_\lambda+\epsilon\mathbf{z}\|_{L^\infty}+\lambda\|f-div\,(\mathbf{u}_\lambda+\epsilon\mathbf{z})\|_Y^p\\ &\leqslant \|\mathbf{u}_\lambda\|_{L^\infty}+|\epsilon|\cdot\|\mathbf{z}\|_{L^\infty}+\lambda\|r_\lambda\|_Y^p-\lambda\epsilon\langle\phi_p(r_\lambda),div\,\mathbf{z}\rangle+o(\epsilon).\\ &= \mathcal{V}_\lambda(\mathbf{u}_\lambda)+|\epsilon|\cdot\|\mathbf{z}\|_{L^\infty}-\lambda\epsilon\langle\phi_p(r_\lambda),div\,\mathbf{z}\rangle+o(\epsilon),\end{aligned}$$

and by letting $\epsilon\downarrow 0$ we find $|\phi_p(r_\lambda)|_{TV}=\displaystyle\sup_{0\neq\mathbf{z}\in\mathcal{D}(\Omega)}\frac{\int_\Omega\phi_p(r_\lambda)\,div\,\mathbf{z}}{\|\mathbf{z}\|_{L^\infty}}\leqslant\frac{1}{\lambda}.$ □

Remark 4.2. Note that when $\lambda<\frac{1}{|\phi_p(f)|_{TV}}$, we then have a trivial minimizer $\mathbf{u}_\lambda=0$ and $r_\lambda=f$. Lemma 4 is relevant when λ is large enough

$$\lambda>\frac{1}{|\phi_p(f)|_{TV}}. \tag{4.14}$$

Then $\mathbf{u}_\lambda\neq 0$, and (4.13) asserts the BV regularity of $\phi_p(r_\lambda)$. In fact, for large enough λ, one has the equality $|\phi_p(r_\lambda)|_{TV}=1/\lambda$, and the minimizer $\mathbf{u}_\lambda$ with residual r_λ is characterized as an exteremal pair in the sense that

$$\int div\mathbf{u}_\lambda\phi_p(r_\lambda)=|\mathbf{u}_\lambda|_\infty|\phi_p(r_\lambda)|_{TV}=\frac{|\mathbf{u}_\lambda|_\infty}{\lambda}, \tag{4.15}$$

see [Mey2002, Theorem 3],[Tad2016, Lemma A.3].

We also need a second priori estimate which will be useful as a *closure bound* for the iterative procedure of hierarchical construction described below.

Lemma 5. Assume that Y has a Fréchet differentiable norm and that BV is embedded in Y' in the sense that

$$\|v\|_{Y'} \leqslant \beta |v|_{TV}, \quad v \in BV(\mathbb{R}^d). \tag{4.16}$$

Then, the following a priori bound holds

$$\|v\|_Y^{p-1} \leqslant \eta |\phi_p(v)|_{TV}, \qquad v \in Y, \tag{4.17}$$

with $\eta = \beta/p$.

Proof. Fixing $v \in Y$ and comparing the first order terms in

$$(1+\epsilon)^p \|v\|_Y^p = \|v + \epsilon v\|_Y^p = \|v\|_Y^p + \epsilon \langle \phi_p(v), v\rangle + o(\epsilon),$$

implies $p\|v\|_Y^p = \langle \phi_p(v), v\rangle$. Therefore

$$p\|v\|_Y^p \leqslant \|\phi_p(v)\|_{Y'} \|v\|_Y \leqslant \beta |\phi_p(v)|_{TV} \|v\|_Y,$$

which yields (4.17). □

Remark 4.3. Recalling that $\phi_p(v) := p\|v\|_Y^{p-1}\phi(v)$, (4.17) amounts to the lower bound, $|\phi(v)|_{TV} \geqslant \frac{1}{\beta}$, which indicates that the Fréchet differential must be oscillatory. Combining with lemma 4 we find

$$\frac{2}{\beta}\|r_\lambda\|_Y \leqslant 2\|r_\lambda\|_Y |\phi(r_\lambda)|_{TV} = |\phi_2(r_\lambda)|_{TV} \leqslant \frac{1}{\lambda}. \tag{4.18}$$

This means that given $f \in Y$ then minimization of the functional (4.2) yields a residual $r_\lambda \in Y$ with the closure bound $\|r_\lambda\|_Y \leqslant \eta/\lambda$. In section 4.5 below we show that in case of $Y = L^d$, the mapping $f \mapsto r$ is not only bounded but in fact, it is Hölder on L^d.

Following [Tad2016], we fix the value $p = 2$ and for a given sequence $(\lambda_j)_{j\geqslant 1}$, we define iteratively $\mathbf{u}_j$ as the minimizer of

$$\mathcal{V}_j(\mathbf{u}) = \|\mathbf{u}\|_{L^\infty} + \lambda_j \|r_{j-1} - div\,\mathbf{u}\|_Y^2, \tag{4.19}$$

and $r_j = r_{j-1} - div\,\mathbf{u}_j$. The following result shows that, with a proper choice of λ_j, the sum of the $\mathbf{u}_j$ admits a limit which is our desired uniformly bounded solution to (1.1).

Theorem 4.4. *Consider a Fréchet differentiable space Y such that (4.16) holds and set $\lambda_j = \lambda_1 2^{j-1}$, where $\lambda_1 := \frac{2\eta}{\|f\|_Y}$. Then, for any given $f \in Y$, the sum of the $\mathbf{u}_j$ converges in L^∞ to a limit $\mathbf{u} = \sum_{j=1}^\infty \mathbf{u}_j$ which is solution to (1.1) and satisfies*

$$\|\mathbf{u}\|_{L^\infty} \leqslant 2\beta \|f\|_Y. \tag{4.20}$$

Proof. Comparing $\mathbf{u}_j$ with the 0 solution, we find that

$$\|\mathbf{u}_j\|_{L^\infty} \leqslant \mathcal{V}_j(0) = \lambda_j \|r_{j-1}\|_Y^2,$$

and in particular

$$\|\mathbf{u}_1\|_{L^\infty} \leqslant \lambda_1 \|f\|_Y^2.$$

On the other hand, by the closure bound (4.18) we find that

$$\|r_{j-1}\|_Y \leqslant \eta |\phi_2(r_{j-1})|_{TV} \leqslant \frac{\eta}{\lambda_{j-1}}. \tag{4.21}$$

Therefore, for $j \geqslant 2$

$$\|\mathbf{u}_j\|_{L^\infty} \leqslant \eta^2 \frac{\lambda_j}{\lambda_{j-1}^2} = \frac{8\eta^2}{\lambda_1} 2^{-j}$$

and $\sum_{j=1}^{\infty} \mathbf{u}_j$ thus converges to a uniformly bounded limit with

$$\|\mathbf{u}\|_{L^\infty} \leqslant \lambda_1 \|f\|_Y^2 + \frac{4\eta^2}{\lambda_1} = 4\eta\|f\|_Y = 2\beta\|f\|_Y,$$

where we have used the chosen value of λ_1. In addition, a telescoping sum of $r_j = r_{j-1} - div\,\mathbf{u}_j$ yields $f = div(\sum_{k=1}^{j} \mathbf{u}_k) + r_j$ and the residual r_j tends to 0 in Y. This proves that $div(\mathbf{u}) = f$. □

Remark 4.5. Theorem 4.4 extends the hierarchical construction of Bourgain-Brezis problem in [Tad2016] for $Y = L^d$-data with $p = d$. In fact, the choice of the parameter $p > 1$ need not be tied to Y, which led to the simpler choice of $p = 2$ in (4.19).

Remark 4.6. The closure (4.17) implies that our choice of λ_1 is admissible in the sense that (4.14) holds,

$$\lambda_1 = \frac{2\eta}{\|f\|_Y} \geqslant \frac{2\eta}{\eta|\phi_2(f)|_{TV}} > \frac{1}{|\phi_2(f)|_{TV}}.$$

In other words — already the first variational iteration produces a non-trivial minimizer, $\mathbf{u}_1 \neq 0$. In fact, one can underestimate $\lambda_1 < \beta/\|f\|_Y$ in case β in (4.16) is not accessible, and yet the variational iterations become effective after the first $\log(\beta)$ iterations with zero minimizers.

Example 4.7. Inspired by [Mey2002], we demonstrate the hierarchical constriction of theorem 4.4 in the two-dimensional example of $f = \alpha\chi_R$, where α is a fixed constant and χ_R is the characteristic function of the ball of radius R. Of course, in this case of BV function we can simply solve $\mathbf{u} = \nabla\Delta^{-1} f = \alpha\frac{x}{2}\chi_R$. The minimization with $\lambda > 1/(4\pi\alpha)$ yields

$$f = div\mathbf{u}_\lambda + r_\lambda, \quad \mathbf{u}_\lambda = (\alpha-\beta)\nabla\Delta^{-1}\chi_R = (\alpha-\beta)\frac{x}{2}\chi_R \ \text{ and } \ r_\lambda = \beta\chi_R \ \text{ with } \ \beta := \frac{1}{4\pi\lambda}.$$

This is verified by checking that $|r_\lambda|_{TV} = \beta 2\pi R = \frac{1}{2\lambda}$, and the extremal property (see remark 4.2),

$$\int div\mathbf{u}_\lambda r_\lambda = (\alpha-\beta)\frac{R}{4\lambda} = |\mathbf{u}_\lambda|_\infty |r_\lambda|_{TV}, \quad |\mathbf{u}_\lambda|_\infty = (\alpha-\beta)\frac{R}{2}.$$

Iterating we find (for $\alpha > 8\pi$)

$$f = \sum_{j=1}^{\infty} \mathbf{u}_j, \qquad \mathbf{u}_j = \begin{cases} \left(\alpha - \frac{1}{8\pi}\right)\frac{x}{2}\chi_R & j = 1, \\ \\ \frac{1}{4\pi 2^j}\frac{x}{2}\chi_R, & j \geqslant 2. \end{cases}$$

The above theorem can be used to construct uniformly bounded solutions to (1.1) for data f in Lorentz spaces $Y = L^{d,q}(\mathbb{R}^d)$ when $1 < q < \infty$. Indeed, since $L^{d,q}$ is reflexive, they qualify as Asplund spaces with Fréchet differentiable norm, see [Asp1968] or [Phe1993, thm 2.12]. Except for the case $q = d$ corresponding to the space $Y = L^d$, the Fréchet derivative of the $L^{d,q}$ does not have a simple explicit form, however we note that this is not required for defining the hierarchical solution. Other applications of hierarchical constructions in inverse problems that arise in image processing can be found in [MNR2019].

In contrast, the space $Y = L^{d,\infty}(\mathbb{R}^d)$ does not have a Fréchet differentiable norm, as seen by the following counter-example due Luc Tatar, [Tar2011]. The purpose is to show that there exists f, g and $\alpha > 0$ such that

$$\|f + \epsilon g\|^p_{L^{p,\infty}} \geqslant \|f\|^p_{L^{p,\infty}} + \alpha|\epsilon| \tag{4.22}$$

proving that $\|\cdot\|^p_{L^{p,\infty}}$ is not Fréchet — not even Gateaux differentiable. To this end one restrict attention to the unit interval $(0,1)$. Set $f(x) := x^{-\frac{1}{p}}$ and

$$g(x) := \sum_{k=0}^{\infty} (-1)^k 2^{\frac{k}{p}} \mathbb{1}_{\mathbb{I}_k}(x), \qquad \mathbb{I}_k = (2^{-(k+1)}, 2^{-k}).$$

The second rearrangement of f is given by $f^{**}(t) = \dfrac{p}{p-1} t^{-\frac{1}{p}}$, $0 < t < 1$. Since $|g(x)| \leqslant f(x)$ it follows that $F := f + \epsilon g \geqslant 0$ for $|\epsilon| < 1$, and hence

$$\begin{aligned}
\|f + \epsilon g\|_{L^{p,\infty}} &\geqslant t^{\frac{1}{p}} F^{**}(t) = t^{-1+\frac{1}{p}} \int_0^t F^*(s) \mathrm{d}s \\
&\geqslant t^{-1+\frac{1}{p}} \int_0^t F(s) \mathrm{d}s = t^{-1+\frac{1}{p}} \int_0^t f^*(s) \mathrm{d}s + \epsilon t^{-1+\frac{1}{p}} \int_0^t g(s) \mathrm{d}s \\
&= \|f\|_{L^{p,\infty}} + \epsilon t^{-1+\frac{1}{p}} \int_0^t g(s) \mathrm{d}s \ \text{ for all } \ t < 1.
\end{aligned} \tag{4.23}$$

It remains to lower bound the term on the right. We compute

$$\int_{\mathbb{I}_k} g(s) \mathrm{d}s = (-1)^k 2^{\frac{k}{p}} |\mathbb{I}_k| = (-1)^k \rho^k, \qquad \rho := 2^{-1+\frac{1}{p}} < 1.$$

It follows that $\displaystyle\int_0^{2^{-k}} g(s) \mathrm{d}s = \frac{(-1)^k \rho^k}{1+\rho}$ implying that $\displaystyle\epsilon t^{-1+\frac{1}{p}} \int_0^t g(s) \mathrm{d}s_{|t=2^{-k}} = \frac{(-1)^k \epsilon}{1+\rho}$, and (4.22) follows from (4.23) at $t = 2^{-k}$ with $(-1)^k \epsilon > 0$ and $\alpha = 1/(1+\rho)$.

4.4. **A hierarchical construction for general data.** Finally, we work with general data $f \in Y(\Omega)$ for some $\Omega \in \mathbb{R}^d$, without making assumption on the Fréchet differentiability of the Y norm, but instead taking as a prior assumption that Y is a space such that uniformly bounded solution to (1.1) exist with the bound (1.2) for some $\gamma > 0$.

We also assuming that Y is a dual space so that there exists a minimizer to the functional F having the general form (4.1). Here we use the exponent $p = 1$ so that

$$\mathcal{V}(\mathbf{u}) = \mathcal{V}_\lambda(\mathbf{u}) = \|\mathbf{u}\|_{L^\infty} + \lambda \|div\,\mathbf{u} - f\|_Y \tag{4.24}$$

For a value of λ to be fixed later, we denote by $\mathbf{u}_1$ is the minimizer and $r_1 = f - div\,\mathbf{u}_1$ the residual. In addition to the trivial bound

$$\|r_1\|_Y \leqslant \|f\|_Y \quad \text{and} \quad \|\mathbf{u}_1\|_{L^\infty} \leqslant \lambda \|f\|_Y, \tag{4.25}$$

that are obtained by comparison of $\mathcal{V}(\mathbf{u}_1)$ and $\mathcal{V}(0)$, we can also compare $\mathcal{V}(\mathbf{u}_1)$ with $\mathcal{V}(\mathbf{u})$ where $\mathbf{u}$ is a solution to (1.1) satisfying the bound (1.2). It follows that

$$\lambda \|r_1\|_Y \leqslant \|\mathbf{u}_j\|_{L^\infty} \leqslant \gamma \|f\|_Y. \tag{4.26}$$

Therefore, taking $\lambda > \gamma$, we obtain a contraction property

$$\|r_1\|_Y \leqslant \rho \|f\|_Y, \tag{4.27}$$

with $\rho = \gamma/\lambda < 1$.

This suggests a hierarchical construction that was proposed in a more general context in [Tad2016]: with this fixed value of λ, we define iteratively $\mathbf{u}_j$ as the minimizer of

$$\mathcal{V}_j(\mathbf{u}) = \|\mathbf{u}\|_{L^\infty} + \lambda\|r_{j-1} - div\,\mathbf{u}\|_Y, \tag{4.28}$$

and $r_j = r_{j-1} - div\,\mathbf{u}_j$. By recursive application of the above contraction principle, it follows that

$$\|r_j\|_Y \leqslant \rho\|r_{j-1}\|_Y \leqslant \cdots \leqslant \rho^j\|f\|_Y. \tag{4.29}$$

as well as

$$\|\mathbf{u}_j\| \leqslant \gamma\|r_{j-1}\|_Y \leqslant \gamma\rho^{j-1}\|f\|_Y. \tag{4.30}$$

From this it follows that the hierarchical construction

$$\mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_j + \cdots, \tag{4.31}$$

converges to a uniformly bounded $\mathbf{u}$ that satisfies the equation (1.1) and the bound

$$\|\mathbf{u}\|_{L^\infty} \leqslant \frac{\gamma}{1-\rho}\|f\|_Y. \tag{4.32}$$

4.5. **Hölder continuity.** A main ingredient in the hierarchical decomposition is to secure that the mapping $T : f \mapsto r$ is bounded on Y. This is carried out in lemma 4 and 5 and clarified in remark 4.3. We show that the (nonlinear) mapping T is Hölder on $Y = L^d$. Indeed, given $f \in Y$ and $f' \in Y$ with the corresponding minimizing pairs $(\mathbf{u}_\lambda, r_\lambda)$ and $(\mathbf{u}'_\lambda, r'_\lambda)$. These pairs are characterized by the extremal property (4.15). In particular

$$\int div\,\mathbf{u}_\lambda\phi_p(r'_\lambda) \leqslant |\mathbf{u}_\lambda|_\infty|\phi_p(r'_\lambda)|_{TV} = \frac{|\mathbf{u}_\lambda|_\infty}{\lambda} = \int div\,\mathbf{u}_\lambda\phi_p(r_\lambda)$$

$$\int div\,\mathbf{u}'_\lambda\phi_p(r_\lambda) \leqslant |\mathbf{u}'_\lambda|_\infty|\phi_p(r_\lambda)|_{TV} = \frac{|\mathbf{u}'_\lambda|_\infty}{\lambda} = \int div\,\mathbf{u}'_\lambda\phi_p(r'_\lambda).$$

This implies the monotonicity

$$0 \leqslant \langle div\,\mathbf{u}_\lambda - div\,\mathbf{u}'_\lambda, \phi_p(r_\lambda) - \phi_p(r'_\lambda)\rangle,$$

or, given that $div\,\mathbf{u} = f - r_\lambda$ and $div\,\mathbf{u}' = f' - r'_\lambda$,

$$\langle r_\lambda - r'_\lambda, \phi_p(r_\lambda) - \phi_p(r'_\lambda)\rangle \leqslant \langle f - f', \phi_p(r_\lambda) - \phi_p(r'_\lambda)\rangle. \tag{4.33}$$

We now specify the case $Y = L^d$ with Fréchet differential (to simplify we switch from $p = 2$ to $p = d$) $\phi_d(r) = d\,\mathrm{sgn}(r)|r|^{d-1}$, for which

$$(r-s)(\phi_d(r) - \phi_d(s)) \geqslant d|r-s|^d, \qquad \phi_d(r) = d\,\mathrm{sgn}(r)|r|^{d-1}.$$

Then, the left of (4.33) admits the lower-bound

$$\langle r_\lambda - r'_\lambda, \phi_d(r_\lambda) - \phi_d(r'_\lambda)\rangle \geqslant d\|r_\lambda - r'_\lambda\|^d_{L^d}.$$

On the other hand, since $\|\phi_p(r_\lambda)\|_{L^{d'}} \leqslant \beta|\phi_p(r_\lambda)|_{TV} \leqslant \beta/\lambda$, the right of (4.33) admits the upper-bound

$$\langle f - f', \phi_p(r_\lambda) - \phi_p(r'_\lambda)\rangle \leqslant \|\phi_d(r_\lambda) - \phi_d(r'_\lambda)\|_{L^{d'}}\|f - f'\|_{L^d} \leqslant \frac{2\beta}{\lambda}\|f - f'\|_{L^d},$$

and we conclude $d\|r_\lambda - r'_\lambda\|^d_{L^d} \leqslant \frac{2\beta}{\lambda}\|f - f'\|_{L^d}$. Thus, the mapping $T : f \mapsto r_\lambda$ is Hölder of order $\theta = 1/d$:

$$\|Tf - Tf'\|_{L^d} \leqslant C_\lambda\|f - f'\|^\theta_{L^d}, \qquad C_\lambda = \left(\frac{2\beta}{d\lambda}\right)^\theta$$

clearly, by successive application of T with an increasing sequence of λ's, one recovers the solution of Bourgain-Brezis with L^d-data. Note that when Y is a Hilbert space, e.g., in the two-dimensional case, then T is in fact Lipschitz on L^2.

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