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# An improved local blow-up condition for Euler–Poisson equations with attractive forcing

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## ABSTRACT

We improve the recent result of Chae and Tadmor (2008) [10] proving a one-sided threshold condition which leads to a finite-time breakdown of the Euler–Poisson equations in arbitrary dimension  $n$ .

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## 1. Introduction

The pressure-less Euler–Poisson (EP) equations in dimension  $n \geq 1$  are

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0 \quad (1.1a)$$

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = k \nabla \Delta^{-1}(\rho - c), \quad (1.1b)$$

governing the unknown density  $\rho = \rho(t, x) : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}_+$  and velocity  $\mathbf{u} = \mathbf{u}(t, x) : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}^n$  subject to initial conditions  $\rho(0, x) = \rho_0(x)$  and  $\mathbf{u}(0, x) = \mathbf{u}_0(x)$ . They involve two constants: (i) a fixed background state  $c \geq 0$  – typical cases include the case of zero background,  $c = 0$ , or the case of a nonzero background given by the average mass,  $c = \int \rho(t, x) dx = \int \rho_0(x) dx$ ; and (ii) a constant  $k$  which parameterizes the repulsive  $k > 0$  or attractive  $k < 0$  forcing, governed by the Poisson potential  $\Delta^{-1}(\rho - c)$ . The EP system appears in numerous applications including semiconductors and plasma physics ( $k > 0$ ) and the collapse of stars due to self gravitation ( $k < 0$ ) [1–4]. In particular, the pressureless EP model becomes relevant in interstellar clouds where gravitational

forces dominate pressure gradient, [5], for example, or in the context of the Euler–Monge–Ampère systems and their quasi-neutral limits to the incompressible Euler equations [6].

This paper is restricted to the attractive case,  $k < 0$ . We begin by setting  $c = 1$ ,  $k = -1$  in (1.1a), (1.1b) to arrive at the unit-free EP system,

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \quad (1.2a)$$

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \Delta^{-1}(\rho - 1). \quad (1.2b)$$

Our discussion remains valid for the general physical parameters  $c \geq 0$ ,  $k < 0$  upon a simple rescaling and limiting arguments, outlined in Corollary 1.1 below for  $c > 0$  and Corollary 1.2 for the case of zero background  $c = 0$ .

We are concerned here with the persistence of  $C^1$  regularity for solutions of the attractive EP system. Our Main theorem reveals a pointwise criterion on the initial data, a so-called critical threshold criterion [7–9], that leads to finite time blow-up of  $\nabla \mathbf{u}$ . It quantifies the balance between the two term  $\operatorname{div} \mathbf{u}$  and  $\rho$ , which govern two competing mechanisms that dictate the  $C^1$  regularity of EP flows. Our result also stands out as a generalization of several existing results [7,10,11,9] for which further discussion is given after the Main theorem and its corollary.

**Main Theorem 1.1.** Consider the  $n$ -dimensional, attractive Euler–Poisson system (1.2a), (1.2b) subject to initial data  $\rho_0, \mathbf{u}_0$ . Then, the solution will lose  $C^1$  regularity at a finite time  $t = t_c < \infty$ , if there exists a non-vacuum initial state  $\rho_0(\bar{x}) > 0$  with vanishing initial

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vorticity,  $\nabla \times \mathbf{u}_0(\bar{x}) = 0$ , at some  $\bar{x} \in \mathbb{R}^n$  such that the following sup-critical condition is fulfilled,

$$\operatorname{div} \mathbf{u}_0(\bar{x}) < \operatorname{sgn}(\rho_0(\bar{x}) - 1) \sqrt{nF(\rho_0(\bar{x}))}, \quad (1.3a)$$

where

$$F(\rho) := \begin{cases} 1 + \frac{2\rho}{n-2} - \frac{n\rho^{2/n}}{n-2}, & n \neq 2, \\ 1 - \rho + \rho \ln \rho, & n = 2. \end{cases} \quad (1.3b)$$

In particular,  $\min_x \operatorname{div} \mathbf{u}(t, x) \rightarrow -\infty$  and  $\max_x \rho(t, x) \rightarrow \infty$  as  $t \uparrow t_c$ .

**Proof.** Combine Lemmas 3.1 and 4.2, while noting that the curve

$$\operatorname{div} \mathbf{u} = \operatorname{sgn}(\rho - 1) \sqrt{nF(\rho)},$$

is the separatrix along the boundary of the blow-up region  $\Omega = \Omega_1 \cup \Omega_2$  defined in (4.3) and illustrated in Fig. 4.1.  $\square$

We note in passing that, by classical arguments, the force-free Euler system  $\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = 0$  exhibits finite time blow-up if and only if there exists at least one negative eigenvalue of  $\nabla \mathbf{u}_0(\bar{x})$ . In the above theorem, however, finite-time blow-up can occur solely depending on the initial profile of  $\operatorname{div} \mathbf{u}_0$  and  $\rho_0$  regardless of individual eigenvalues of  $\nabla \mathbf{u}_0$ .

We also note that, by rescaling  $\rho$  to  $\rho/c$ ,  $x$  to  $\sqrt{-kc}x$  and  $t$  to  $\sqrt{-kc}t$ , the Main theorem immediately applies to the EP system (1.1a), (1.1b) with physical parameters. Since the EP system with  $k < 0$  models the collapse of interstellar cloud, the following corollary reveals a pointwise condition for mass concentration,  $\rho \rightarrow \infty$ , which interestingly precludes the birth of new stars.

**Corollary 1.1.** Consider the Euler–Poisson system (1.1a), (1.1b) with  $c > 0$ ,  $k < 0$  subject to initial data  $\rho_0, \mathbf{u}_0$ . Then, the solution will lose  $C^1$  regularity at a finite time  $t_c < \infty$ , if there exists a non-vacuum initial state  $\rho_0(\bar{x}) > 0$  with a vanishing initial vorticity,  $\nabla \times \mathbf{u}_0(\bar{x}) = 0$ , such that the super-critical condition is fulfilled,

$$\operatorname{div} \mathbf{u}_0(\bar{x}) < \operatorname{sgn}(\rho_0(\bar{x}) - c) \sqrt{-nk c F\left(\frac{\rho_0(\bar{x})}{c}\right)} \quad (1.4)$$

where  $F(\cdot)$  is given in (1.3b). In particular,  $\min_x \operatorname{div} \mathbf{u}(t, x) \rightarrow -\infty$  and  $\max_x \rho(t, x) \rightarrow \infty$  as  $t \uparrow t_c$ .

In the limiting regime as  $c \rightarrow 0+$ , condition (1.4) converges to a super-critical condition which is summarized by the following result, the proof of which is given in Section 5.

**Corollary 1.2.** Consider the  $n$ -dimensional Euler–Poisson system (1.1a), (1.1b) with  $c = 0$ ,  $k < 0$  subject to initial data  $\rho_0, \mathbf{u}_0$ . Assume a vanishing initial vorticity everywhere,  $\nabla \times \mathbf{u}_0 \equiv 0$ . Then, the solution will lose  $C^1$  regularity at a finite time  $t_c < \infty$ , if either (i)  $n = 1, 2$  or (ii)  $n \geq 3$  and there exists a non-vacuum initial state  $\rho_0(\bar{x}) > 0$  such that

$$\operatorname{div} \mathbf{u}_0(\bar{x}) < \sqrt{-\frac{2nk\rho_0(\bar{x})}{n-2}}, \quad n \geq 3. \quad (1.5)$$

In other words, the pressureless and vorticity-free one- and two-dimensional attractive Euler–Poisson systems with zero background ( $c = 0$ ), inevitably collapse to singularity at a finite time. On the other hand, the complete characterization of finite-time breakdown in higher dimensions remains open, even for  $c = 0$ .

The concept of Critical Threshold and associated methodology is originated and developed in a series of papers by Engelberg, Liu and Tadmor [7], Liu and Tadmor [9,8] and more. It first appears in [7] regarding pointwise criteria for  $C^1$  solution regularity of

1D EP system. The key argument in that paper is based on the convective derivative along particle paths  $' = \partial_t + \mathbf{u} \cdot \nabla$ . It makes it possible to obtain a 2-by-2 ODE system for  $u_x$  and  $\rho$  along particle paths – the so-called Lagrangian formulation. Phase plane analysis is then employed to study the finiteness of the ODE solutions and therefore  $C^1$  regularity of the PDE solution. Similar results stay valid for Euler–Poisson systems with geometric symmetry in higher dimensions [3,8]. To treat genuinely multi-D cases, Liu and Tadmor introduce in [8] the method of spectral dynamics which relies on the ODE system governing eigenvalues of

$$M := \nabla \mathbf{u},$$

which is the velocity gradient matrix, along particle paths. They identify if-and-only-if, pointwise conditions for global existence of  $C^1$  solutions to restricted Euler–Poisson systems. Chae and Tadmor [10] further extend the Critical Threshold argument to multi-D full Euler–Poisson systems (1.2a), (1.2b) with attractive forcing  $k < 0$ . Their result, however, offers a blow-up region  $\nabla \times \mathbf{u}_0 = 0$ ,  $\operatorname{div} \mathbf{u}_0 < -\sqrt{-nkc}$  which is only a subset of the blow-up region in (1.4). This subset is to the left of the solid line  $d \leq d^- := -\sqrt{-nkc}$  depicted in Fig. 4.1. Finally, a recent paper by Tadmor and Wei [12] reveals the critical threshold phenomena in the 1D Euler–Poisson system with pressure.

When tracking other results on the well-posedness of Euler–Poisson equations, we find them commonly relying on (the vast family of) energy methods and thus fundamentally differ from our pointwise results obtained via the Lagrangian approach. With a repulsive force  $k > 0$ , we refer to [13,14] for the global existence of classical solutions with small data and [15] for the nonexistence of global solutions. With attractive force  $k < 0$ , see [1] for local regularity of classical solutions and [16,17] for nonexistence results. Discussions on weak solutions of Euler–Poisson systems can be found in e.g. [18–20]. We also refer to [21–25] and references therein for steady-state solutions. The study of the Euler–Poisson system with damping relaxation can be found in e.g. [26–28].

The rest of this paper is organized as follows. In Section 2, we follow the idea of [10] to derive along particle paths an ODE system governing the dynamics of eigenvalues for  $S := \frac{1}{2}(M + M^T)$ . This is a variation of the spectral dynamics for  $M$  introduced in [8]. We then derive in Section 3 a closed  $2 \times 2$  ODE system (3.1) at the cost of turning one equation into inequality. By the comparison principle, this inequality is in favor of blow-up. Thus, with the inequality sign being replaced with an equality sign, a modified ODE system is used to yield sub-solutions and to study a blow-up scenario for the original system. Section 4, devoted to the modified system, reveals the Critical Threshold for such a system. Consequently, a pointwise blow-up condition for the original system is identified. Finally, in Section 5 we prove Corollary 1.2 regarding the Euler–Poisson system with zero background using techniques developed in previous sections.

## 2. Spectral dynamics

We examine the gradient matrix  $M = \nabla \mathbf{u}$  and its symmetric part,  $S = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$ . Both matrices are used to study the spectral dynamics of Euler systems (see e.g. [8] for  $M$  and [10] for  $S$ ). The relation between the spectra of  $M$  and  $S$  is described in the following.

**Proposition 2.1.** Let  $\{\lambda_M\}$  denote the eigenvalues of  $M$  and  $\{\lambda_S\}$  for  $S$ . Then

$$\sum_{\lambda_M} \lambda_M = \sum_{\lambda_S} \lambda_S = \operatorname{div} \mathbf{u}, \quad (2.1)$$

$$\sum_{\lambda_M} \lambda_M^2 = \sum_{\lambda_S} \lambda_S^2 - \frac{1}{2}|\boldsymbol{\omega}|^2. \quad (2.2)$$

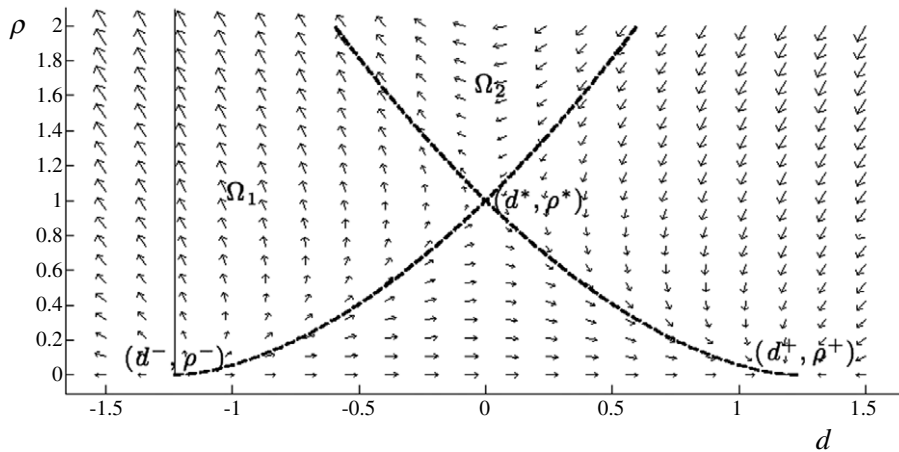


Fig. 4.1. Phase plane of (4.1) with blow-up region  $\Omega_1 \cup \Omega_2$  which extends the Chae–Tadmor region [10]  $d \leq d^-$ .

Here,  $\omega$  is the  $\frac{n(n-1)}{2}$  vorticity vector which consists of the off-diagonal entries of  $A := \frac{1}{2}(\nabla \mathbf{u} - (\nabla \mathbf{u})^\top)$ .

**Proof.** Use identity  $M = S + A$  and the skew-symmetry of  $A$ ,

$$\sum_{\lambda_M} \lambda_M = \text{tr}(M) = \text{tr}(S + A) = \text{tr}(S) = \sum_{\lambda_S} \lambda_S.$$

Squaring the last identity we have  $M^2 = S^2 + A^2 + AS + SA$  and therefore,

$$\sum_{\lambda_M} \lambda_M^2 = \text{tr}(M^2) = \text{tr}(S^2 + A^2 + AS + SA) = \sum_{\lambda_S} \lambda_S^2 + \text{tr}(A^2).$$

Note that  $AS + SA$  is skew-symmetric and thus traceless. A simple calculation yields  $\text{tr}(A^2) = -\frac{1}{2}|\omega|^2$ .  $\square$

Following [8], we turn to study the dynamics of  $M$  along particle paths. Take the gradient of (1.2b) to find

$$M' + M^2 \equiv M_t + \mathbf{u} \cdot \nabla M + M^2 = -R(\rho - 1), \quad (2.3)$$

where  $R$  stands for the Riesz matrix,  $R = \{R_{ij}\} := \{\partial_{x_i x_j} \Delta^{-1}\}$ .

The trace of (2.3) then yields that the divergence,  $d := \text{tr}(M)$ , is governed by

$$d' = -\sum_{\lambda_M} \lambda_M^2 - (\rho - 1),$$

and in view of (2.2),

$$d' = -\sum_{\lambda_S} \lambda_S^2 + \frac{1}{2}|\omega|^2 - (\rho - 1). \quad (2.4)$$

We now make the first observation regarding the invariance of the vorticity  $\omega$ : taking the skew-symmetric part of the  $M$ -equation (2.3),

$$A' + AS + SA = 0. \quad (2.5)$$

It follows that if the initial vorticity vanishes,  $\omega_0(\bar{x}) \mapsto \nabla \times \mathbf{u}_0(\bar{x}) = 0$ , then by (2.5),  $\omega \mapsto \nabla \times \mathbf{u}$  vanishes along the particle path which emanates from  $\bar{x}$ . This allows us to decouple the vorticity and divergence dynamics, and (2.4) implies

$$d' = -\sum_{\lambda_S} \lambda_S^2 - (\rho - 1), \quad \nabla \times \mathbf{u} = 0. \quad (2.6)$$

Finally, we use Cauchy–Schwartz  $\sum \lambda_S^2 \geq \frac{1}{n}(\sum \lambda_S)^2 = \frac{1}{n}d^2$  and the fact that all  $\lambda_S$  are real (due to the symmetry of  $S$ ), to deduce the inequality,

$$d' \leq -\frac{1}{n}d^2 - (\rho - 1). \quad (2.7a)$$

This, together with the mass equation (1.2a) which can be written along particle path

$$\rho' = -d\rho, \quad (2.7b)$$

give us the desired closed system which dominates  $(\rho, d)$  along particle paths.

**Remark 2.1.** The approach pursued in this paper will be based on the inequality (2.7a) and is therefore limited to derivation of a finite time breakdown. To argue the global regularity, one needs to study the underlying equality (2.6), and to this end, to study the trace  $\sum \lambda_S^2$ . In the two-dimensional case, for example, one can use  $\sum \lambda_S^2 = d^2/2 + \eta^2/2$  to replace (2.7a) with

$$d' = -\frac{1}{2}d^2 - \frac{1}{2}\eta^2 - (\rho - 1), \quad \eta := \lambda_{S,2} - \lambda_{S,1}.$$

In this framework, global 2D regularity is dictated by the dynamics of the spectral gap,  $\eta = \lambda_{S,2} - \lambda_{S,1}$ , which in turn requires the dynamics of the Riesz transform  $R(\rho - 1)$ .

### 3. A comparison principle with a majorant system

The blow-up analysis, driven by the inequalities (2.7),

$$d' \leq -\frac{1}{n}d^2 - (\rho - 1), \quad (3.1a)$$

$$\rho' = -d\rho. \quad (3.1b)$$

is carried out by standard comparison with the majorant system

$$e' = -\frac{1}{n}e^2 - (\zeta - 1), \quad (3.2a)$$

$$\zeta' = -e\zeta. \quad (3.2b)$$

The following proposition guarantees the monotonicity of the solution operator associated with (3.1).

**Lemma 3.1.** The following monotone relation between system (3.1) and system (3.2) is invariant forward in time,

$$\begin{cases} d(0) < e(0) \\ 0 < \zeta(0) < \rho(0) \end{cases} \text{ implies } \begin{cases} d(t) < e(t) \\ 0 < \zeta(t) < \rho(t) \end{cases} \text{ for } t \geq 0, \quad (3.3)$$

as long as all solutions remain finite on the time interval  $[0, t]$ .

**Proof.** Invariance of positivity of  $\zeta$  is a direct consequence of (3.2b) and finiteness of  $e$ . The rest can be proved by contradiction. Suppose  $t_1$  is the earliest time when (3.3) is violated. Then,

$$\begin{aligned} \zeta(t_1) &= \zeta(0) \exp\left(-\int_0^{t_1} e(t)dt\right) < \rho(0) \exp\left(-\int_0^{t_1} d(t)dt\right) \\ &= \rho(t_1). \end{aligned} \quad (3.4)$$

Therefore, we are left with only one possibility, namely,  $e(t_1) = d(t_1)$ . Subtracting (3.1a) from (3.2a),

$$(e - d)' \geq -\frac{1}{n}(e^2 - d^2) - (\zeta - \rho), \tag{3.5}$$

and by (3.4), we find that at  $t = t_1$ ,

$$\text{RHS of (3.5)}|_{t=t_1} = 0 - [\zeta(t_1) - \rho(t_1)] > 0.$$

However, this contradicts the negativity of the expression on the left of (3.5), since  $e(t) - d(t) > 0$  for all  $t < t_1$  and vanishes at  $t = t_1$  which imply that

$$\text{LHS of (3.5)}|_{t=t_1} = (e(t_1) - d(t_1))' \leq 0. \quad \square$$

In the next section, we employ phase plane analysis on the modified system (3.2). When translated in terms of the original system (3.1), however, such analysis can only yield blow-up results and is insufficient for global existence results. In other words, estimate (3.3) is only useful for proving  $d \searrow -\infty$ , the key mechanism for blow-up of  $C^1$  solutions.

#### 4. Stability analysis of the majorant system

We shall prove the blow-up of the majorant system (3.2),  $e(t) \rightarrow -\infty$  as  $t \uparrow t_c$ , which in turn, by Lemma 3.1 implies  $d(t) \rightarrow -\infty$ . Abusing notations, we express the majorant system in terms of the original variables  $(e, \zeta) \mapsto (d, \rho)$ :

$$d' = -\frac{1}{n}d^2 - (\rho - 1), \tag{4.1a}$$

$$\rho' = -d\rho. \tag{4.1b}$$

The (in-)stability analysis of (4.1) hinges on the path invariants of this system. To this end, we use the same  $q$ -transformation employed in [29,9]: setting  $q := d^2$  and differentiate along the path  $\{(t, X(a, t)) \mid X_t(a, t) = u(t, X(a, t)), X(a, 0) = a\}$ , we find

$$\frac{dq}{dq} = 2d \frac{d'}{d^2} = \frac{2}{n}q + 2 \left(1 - \frac{1}{\rho}\right),$$

which yields

$$\frac{d}{d\rho} \left( q\rho^{-\frac{2}{n}} \right) = 2(1 - \rho^{-1})\rho^{-\frac{2}{n}}.$$

Upon integration, we arrive at the following key observation.

**Lemma 4.1.** *The majorant system (4.1) is equipped with the path invariant,*

$$I(d(t), \rho(t)) = I(d_0, \rho_0),$$

along each path  $(t, x(t))$  initiated with a non-vacuum state  $(d_0, \rho_0 > 0)$ . Here,

$$\begin{aligned} I(d, \rho) &:= d^2 \rho^{-\frac{2}{n}} - 2 \int_1^\rho (1 - r^{-1})r^{-\frac{2}{n}} dr \\ &= \rho^{-\frac{2}{n}} (d^2 - nF(\rho)), \end{aligned} \tag{4.2}$$

where  $F(\cdot)$  is specified in (1.3b).

It is a simple calculation to show that the majorant system (4.1) admits three distinct critical points (see Fig. 4.1):

$$(d^*, \rho^*) := (0, 1), \quad (d^\pm, \rho^\pm) := (\pm\sqrt{n}, 0).$$

and that  $(0, 1)$  is a saddle point,  $(-\sqrt{n}, 0)$  a nodal source and  $(\sqrt{n}, 0)$  a nodal sink. The separatrix is given by the zero level set  $I(d, \rho) = 0$ . Moreover, the right branch of the separatrix,  $d = \sqrt{nF(\rho)}$  connects critical points  $(0, 1)$  and  $(\sqrt{n}, 0)$  while the left branch,  $d = -\sqrt{nF(\rho)}$  connects  $(0, 1)$  and  $(-\sqrt{n}, 0)$ .

By inspection of the phase plane in Fig. 4.1, we postulate the following invariant region of finite-time blow-up for the modified system (4.1),

$$\Omega = \Omega_1 \cup \Omega_2 = \{(d, \rho) \mid d < \text{sgn}(\rho - 1)\sqrt{nF(\rho)}\} \tag{4.3a}$$

where

$$\Omega_1 := \{(d, \rho) \mid I(d, \rho) > 0 \text{ and } d < 0 \text{ and } \rho > 0\}, \tag{4.3b}$$

$$\Omega_2 := \{(d, \rho) \mid I(d, \rho) < 0 \text{ and } \rho > 1\}. \tag{4.3c}$$

**Lemma 4.2.** *Consider the modified system (4.1), equipped with initial data  $(d_0, \rho_0)$ . If  $(d_0, \rho_0) \in \Omega$ , then  $\text{div } \mathbf{u} \rightarrow -\infty$  and  $\rho \rightarrow \infty$  at a finite time.*

**Proof.** We begin by recalling (1.3b), consult (4.2),

$$F(\rho) = \frac{2}{n}\rho^{\frac{2}{n}} \int_1^\rho (1 - r^{-1})r^{-\frac{2}{n}} dr.$$

Clearly,  $F(1) = F'(1) = 0$  and a simple calculation shows that  $F''(\rho) = \frac{2}{n}\rho^{\frac{2}{n}-2}$ , which implies that  $F(\rho)$  is a strictly convex function of positive  $\rho$  and attains its only minimum at  $\rho = 1$ ,

$$F(\rho) \geq F(1) = 0. \tag{4.4}$$

We shall also utilize the invariance of (4.2)

$$d^2 - nF(\rho) = \rho^{\frac{2}{n}}I_0, \quad I_0 = I(d_0, \rho_0). \tag{4.5}$$

We now turn to discuss the two possible blow-up scenarios, depending whether the initial data  $(d_0, \rho_0)$  belong to the blow-up regions  $\Omega_1$  or  $\Omega_2$  given in (4.3).

Case #1. Assume that  $(d_0, \rho_0) \in \Omega_1$  so that the invariant  $I$  remains a positive constant

$$I > 0.$$

In this case,  $d$  remains negative, for otherwise, setting  $d = 0$  in (4.5) would result in  $F(\rho) = -\rho^{\frac{2}{n}}I/n < 0$ , violating (4.4). Thus, (4.5) and (4.4) yield an upper bound,

$$d \leq -\rho^{\frac{1}{n}}\sqrt{I}.$$

Then, by (4.1b), we have a Riccati type of equation  $\rho' \geq \sqrt{I}\rho^{1+\frac{1}{n}}$  for which the solution exhibits blow-up  $\rho \rightarrow +\infty$  and the divergence  $d = \text{div } \mathbf{u}$  approaches  $-\infty$  at a finite time due to (4.5).

Case #2. Assume that  $(d_0, \rho_0) \in \Omega_2$  so that the invariant  $I$  remains a negative constant

$$I < 0.$$

In this case,  $\rho - 1$  remains positive, for otherwise setting  $\rho = 1$  in (4.5) would result in  $F(1) = (d^2 - I)/n > 0$  in contradiction to (4.4). Now, for  $\rho > 1$  we have

$$F(\rho) = \frac{2}{n}\rho^{2/n} \int_1^\rho \left(1 - \frac{1}{r}\right) \frac{1}{r^{2/n}} dr \leq \frac{2}{n}\rho^{2/n}(\rho - 1).$$

This together with (4.5) yield

$$\frac{2}{n}\rho^{2/n}(\rho - 1) \geq F(\rho) = \frac{1}{n}(d^2 - \rho^{2/n}I) \geq -\frac{1}{n}\rho^{2/n}I$$

and the lower bound,  $\rho - 1 \geq -I/2$  follows. Thus, by (4.1a), we end up with a Riccati type of equation

$$d' \leq -\frac{d^2}{n} + \frac{I}{2}.$$

Since the invariant  $I$  remains a negative constant, the solution exhibits blow-up  $d = \text{div } \mathbf{u} \rightarrow -\infty$  at a finite time even if initially  $d_0 > 0$ . The density  $\rho$  also approaches  $\infty$  in finite time due to (4.5).  $\square$

The last step of proving the Main theorem is just to combine the comparison principle in Lemma 3.1 with the above lemma. We notice that  $\Omega$  is an open set and thus given any initial data  $(d_0, \rho_0) \in \Omega$  for the original system, we can always find  $\varepsilon > 0$  and initial data  $(d_0 + \varepsilon, \rho_0 - \varepsilon) \in \Omega$  for the modified system. This



latter initial data will lead to a finite time blow-up of the modified system and therefore, by Lemma 3.1, initial data  $(d_0, \rho_0) \in \Omega$  will lead to finite time blow-up of the original system.

**5. Critical threshold for zero background**

We now turn to the attractive Euler–Poisson system (1.1a), (1.1b) with zero background  $c = 0$  and prove Corollary 1.2. For simplicity, we only show the case with  $k = -1$  since a straightforward rescaling argument,  $x \rightarrow \sqrt{-k}x$  and  $t \rightarrow \sqrt{-k}t$ , will cover the case for general  $k < 0$ .

**Proof of Corollary 1.2.** Following the same calculation that leads to the majorant system (4.1a), (4.1b), we arrive at a similar ODE system for the case  $c = 0, k = -1$ ,

$$d' = -\frac{1}{n}d^2 - \rho, \tag{5.1a}$$

$$\rho' = -d\rho. \tag{5.1b}$$

Then, as an analogue to the invariant (4.2), we find the corresponding invariant,

$$I(d, \rho) := d^2 \rho^{-\frac{2}{n}} - 2 \int_a^\rho r^{-\frac{2}{n}} dr.$$

By choosing the constant

$$a = \begin{cases} +\infty, & n = 1, \\ 1, & n = 2, \\ 0, & n \geq 3, \end{cases}$$

we have

$$I(d, \rho) = \begin{cases} d^2 \rho^{-2} + 2\rho^{-1}, & n = 1 \\ d^2 \rho^{-1} - 2 \ln \rho, & n = 2 \\ d^2 \rho^{-\frac{2}{n}} - \frac{2n}{n-2} \rho^{1-\frac{2}{n}}, & n \geq 3. \end{cases} \tag{5.2}$$

Using the positivity of  $\rho$  and  $d^2$  in (5.2), we have  $I \geq 2\rho^{-1} > 0$  for  $n = 1$  and  $I \geq -2 \ln \rho$  for  $n = 2$ . Both estimates imply that  $\rho$  is bounded from below by a positive constant. In the case of  $n \geq 3$ , the sup-critical condition (1.5) implies  $I < 0$ . Thus, by (5.2), we have  $0 > I \geq -\frac{2n}{n-2} \rho^{1-\frac{2}{n}}$  which, again, implies  $\rho$  is greater than a positive constant.

Therefore, by (5.1a),  $d$  satisfies a differential inequality

$$d' \leq -\frac{d^2 + \alpha}{n}$$

with positive constant  $\alpha$ . Obviously,  $d(t)$  approaches  $-\infty$  at a time no later than  $\frac{n\pi}{2\sqrt{\alpha}}$ .  $\square$

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