# APPROXIMATE PERIODIC SOLUTIONS for the RAPIDLY ROTATING SHALLOW-WATER and RELATED EQUATIONS

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We study the stabilizing effect of rotational forcing in the nonlinear setting of two-dimensional shallow-water equations. The pressureless version of these equations admit global smooth solutions for a large set of sub-critical initial configurations. But what happens with more realistic models, in the presence of pressure? It is shown that when rotational forcing dominates the pressure, it prolongs the life-span of such sub-critical solutions, for a time period $\ln(1/\delta) \gg 1$ dictated by the ratio $\delta$ =Rossby number/squared Froude number. Our study reveals a "nearby" periodic-in-time approximate solution in the small $\delta$ regime, upon which hinges the long time existence of the exact smooth solution. These results are in agreement with the close-to-periodic dynamics observed in the "near inertial oscillation" (NIO) regime which follows oceanic storms.

*Keywords*: shallow-water equations, rapid rotation, pressureless equations, critical threshold, two-dimensional Euler equations, long-time existence

## 1. Introduction and statement of the main results

We investigate smooth solutions of two-dimensional systems of nonlinear Eulerian equations driven by pressure and rotational forces. It is well known that in the absence of rotation, these equations admit finite-time breakdown[12]: for generic smooth initial conditions, the corresponding solutions lose $C^1$-smoothness in a finite time due to shock formation. The presence of rotational forces, however, has a stabilizing effect[13] : the pressureless version of these equations admit global smooth solutions for a large set of so-called sub-critical initial configurations. It is therefore a natural extension to study the balance between the regularizing effects of rotation vs. destabilizing mechanisms such as nonlinear advection and pressure. In this paper we prove the long-time existence of rapidly rotating flows governed by the shallow-water and more general Eulerian equations. We show that the solution for such equations are characterized by the existence of "near-by" periodic flows for long time periods. Thus, rotation *prolongs* the life-span of smooth solutions over increasingly long time periods, which grows longer as the rotation forces become more dominant over pressure. Our novel approach[4] employs iterative approximations and nonlinear analysis of fast manifold of the flow.

Our model problem is the Rotational Shallow Water (RSW) equations. This system of equations models large scale geophysical motions in a thin layer of fluid under the influence of the Coriolis rotational forcing, e.g. [14, §3.3], [7, §2.1],

$$\partial_t h + \nabla \cdot (h\mathbf{u}) = 0 \tag{1.1a}$$

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + g\nabla h - f\mathbf{u}^\perp = 0. \tag{1.1b}$$

It governs the unknown velocity field $\mathbf{u} := \big(u^{(1)}(t,x,y), u^{(2)}(t,x,y)\big)$ and height $h := h(t,x,y)$, where $g$ and $f$ stand for the gravitational constant and the Coriolis frequency. Recall that equation (1.1a) observes the conservation of mass and equations (1.1b) describe balance of momentum by the pressure gradient, $g\nabla h$, and rotational forcing, $f\mathbf{u}^\perp := f\big(u^{(2)}, -u^{(1)}\big)$. For convenience, we rewrite the system (1.1) in terms of rescaled, nondimensional variables,

$$\partial_t h + \mathbf{u}\cdot\nabla h + \left(\frac{1}{\sigma} + h\right)\nabla\cdot\mathbf{u} = 0. \tag{1.2a}$$

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{\sigma}\nabla h - \frac{1}{\tau}J\mathbf{u} = 0,. \tag{1.2b}$$

Here $\sigma$ and $\tau$ are respectively the Froude number measuring the inverse pressure forcing and the Rossby number measuring the inverse rotational forcing. Here and below, we use $J$ to denote the $2\times 2$ rotation matrix $J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

To trace the behavior of its solutions, we approximate (1.2a), (1.2b) with the successive iterations,

$$\partial_t h_j + \mathbf{u}_{j-1}\cdot\nabla h_j + \left(\frac{1}{\sigma} + h_j\right)\nabla\cdot\mathbf{u}_{j-1} = 0, \qquad j = 2, 3, \ldots \quad (1.3a)$$

$$\partial_t \mathbf{u}_j + \mathbf{u}_j\cdot\nabla\mathbf{u}_j + \frac{1}{\sigma}\nabla h_j - \frac{1}{\tau}J\mathbf{u}_j = 0, \qquad j = 1, 2, \ldots, \quad (1.3b)$$

subject to initial conditions, $h_j(0,\cdot) = h_0(\cdot)$ and $\mathbf{u}_j(0,\cdot) = \mathbf{u}_0(\cdot)$. This approximation simplifies the original coupled mass and momentum equations. That is, given $j$, (1.3) are only weakly coupled through the dependence of $\mathbf{u}_j$ on $h_j$. Moreover, for $\sigma \gg \tau$, the momentum equations (1.2b) are "approximately decoupled" from the mass equation (1.2a) since rotational forcing is substantially dominant over pressure forcing. Therefore a first approximation of constant height function will enforce this decoupling, serving as the starting point of the above iterative scheme,

$$h_1 \equiv \text{constant}. \quad (1.4a)$$

This together with $j = 1$ in (1.3b) leads to the first approximate velocity field, $\mathbf{u}_1$, as the solution of the *pressureless equations*,

$$\partial_t \mathbf{u}_1 + \mathbf{u}_1\cdot\nabla\mathbf{u}_1 - \frac{1}{\tau}J\mathbf{u}_1 = 0, \quad \mathbf{u}_1(0,\cdot) = \mathbf{u}_0(\cdot). \quad (1.4b)$$

Here we note by passing that (1.4b) is a genuinely nonlinear system that accommodates possibly singular parameter $\frac{1}{\tau} \gg 1$. Liu and Tadmor[13] have shown that there is a large generic set of so-called sub-critical initial configurations, $\mathbf{u}_0$, for which the pressureless equations (1.4b) admit *global* smooth solutions. Moreover, the pressureless velocity $\mathbf{u}_1(t,\cdot)$ is in fact $2\pi\tau$-periodic in time. In Section 2, we complete Liu and Tadmor's regularity result with a new argument on the time-periodicity of $\mathbf{u}_1$ outlined in Ref. 4.

Having the pressureless solution, $(h_1, \mathbf{u}_1)$ in (1.4) as a first approximation for the RSW solution $(h, \mathbf{u})$, we used it to construct an improved approximation of the RSW equations, $(h_2, \mathbf{u}_2)$, which solves an "adapted" version of the second iteration ($j = 2$) of (1.3). The regularity and periodicity of $(h_2, \mathbf{u}_2)$ hinges on an important lemma presented in Section 3; it states that *any* conservative scalar $\phi$ advected by the pressureless field $\mathbf{u}_1$,

$$\partial_t \phi + \nabla\cdot(\mathbf{u}_1\phi) = 0, \quad (1.5)$$

is time-periodic and hence globally smooth. In particular, setting $j = 2$ in (1.3a) implies that $\phi = \dfrac{1}{\sigma} + h_2$ is $2\pi\tau$-time periodic and consequently, we show that the linearized solution, $\mathbf{u}_2$, subject to sub-critical initial data $\mathbf{u}_0$ retains the same time periodicity. It follows that the approximate solution $(h_2, \mathbf{u}_2)$ is *globally* smooth.

Next, in Section 4 we turn to estimate the deviation between the solution $(h_2, \mathbf{u}_2)$ of the linearized RSW system and the solution $(h, \mathbf{u})$ of the full RSW system. To this end, we introduce a new non-dimensional parameter

$$\delta := \frac{\tau}{\sigma^2},$$

measuring the *relative* strength of rotation vs. the pressure forcing. We assume that rotation is the dominant forcing in the sense that $\delta \ll 1$. Using the standard energy method[8] we show in Theorem 4.1 and its corollary that, starting with $H^m$ initial data, the RSW solution $\big(h(t,\cdot), \mathbf{u}(t,\cdot)\big)$ remains sufficiently close to $\big(h_2(t,\cdot), \mathbf{u}_2(t,\cdot)\big)$ in the sense that,

$$\|h(t,\cdot) - h_2(t,\cdot)\|_{H^{m-3}} + \|\mathbf{u}(t,\cdot) - \mathbf{u}_2(t,\cdot)\|_{H^{m-3}} \lesssim \delta \frac{e^{C_0 t}}{(1 - e^{C_0 t}\delta)^2},$$

where constant $C_0 = \widehat{C}_0(m, |\nabla \mathbf{u}_0|_\infty, |h_0|_\infty) \cdot \|\mathbf{u}_0, h_0\|_m$. In particular, we conclude that for a large set of sub-critical initial data, the RSW equations (1.2) admit smooth, "approximate periodic" solutions for large time, $t \lesssim 1 + \ln(\delta^{-1})$, in the rotationally dominant regime $\delta \ll 1$. Therefore, strong rotation stabilizes the flow by imposing approximate periodicity to the flow, which in turn postpones finite time breakdown of classical solutions to long time.

A physically relevant example consistent with our results is found in the so-called "near-inertial oscillation" (NIO) regime, which is observed during the days that follow oceanic storms; see, e.g., Ref. 17. These NIOs are triggered when storms pass by and only a thin layer of the oceans is reactive, corresponding to $\delta \ll 1$. Specifically, upon using the multilayer model ( [14, §6.16]) with Rossby number $\tau \sim \mathcal{O}(0.1)$ and Froude number $\sigma \sim \mathcal{O}(1)$ we find $\delta \sim 0.1$, which yields the existence of a smooth, "approximate periodic" solution for $t \sim 2$ days. We note that the counterclockwise rotation of cyclonic storms on the Northern Hemisphere produce negative vorticity, which is a preferred scenario of the subcritical condition (2.2).

Several other related results are mentioned in the last Section 5. We make generalization to Euler systems describing the isentropic gasdynamics and ideal gasdynamics in Ref. 4. Then we give a brief discussion on the complementary large $\delta \gg 1$ regime that will be the main focus in the upcoming papers[2,3] .

Our results confirm the stabilization effect of rotation in the nonlinear setting, when it interacts with the slow components of the system, which otherwise tend to destabilize the dynamics. The study of such interaction is essential to the understanding of rotating dynamics, primarily to geophysical flows. For a state-of-the-art of the mathematical theory for such flows, the readers are referred to the recent book of Chemin et. al.[1] and the references therein. We conclude this section by a very brief literature review followed by comparison of our approach vs. the existing ones. Often referred to as "fast wave analysis", "fast wave filtering" or "separation of fast and slow dynamics", the theoretical foundation of previous works can be found in papers such as Schochet[15] , which can be further traced back to the earlier works of Klainerman and Majda[9] and Kreiss[10] (see also[16]). The key idea is to identify the limiting system as some singular parameters (in the RSW case, $\tau$ and $\sigma$) approach zero, which filters out fast scales. The full system is then approximated to a first order, by this slowly evolving limiting system. The novelty of our approach, on the other hand, is to adopt the *rapidly* oscillating and fully *nonlinear* pressureless system as a first approximation and then consider the full system as a perturbation of this fast scale. This enables us to preserve both slow and fast dynamics, and especially, the rotation-induced time periodicity. The approach also enables us to avoid the constraint on absolute smallness of parameters, e.g. $\tau \sim \sigma \ll 1$ (see Ref. 5) and to only require relative smallness of the ratio $\delta = \tau/\sigma^2 \ll 1$.

## 2. First approximation– the pressureless system

We consider the pressureless system

$$\partial_t \mathbf{u}_1 + \mathbf{u}_1 \cdot \nabla \mathbf{u}_1 - \frac{1}{\tau} J \mathbf{u}_1 = 0, \tag{2.1}$$

subject to initial condition $\mathbf{u}_1(0, \cdot) = \mathbf{u}_0(\cdot)$. We begin by recalling the main theorem in Ref. 13 regarding the global regularity of the pressureless equations (2.1).

**Theorem 2.1.** *Consider the pressureless equations (2.1) subject to $C^1$-initial data $\mathbf{u}_1(0, \cdot) = u_0(\cdot)$. Then, the solution $\mathbf{u}_1(t, \cdot)$ stays $C^1$ for all time if and only if the initial data satisfy the critical threshold condition,*

$$\tau\omega_0(x) + \frac{\tau^2}{2}\eta_0^2(x) < 1 \quad \text{for all} \quad x \in \mathbb{R}^2. \tag{2.2}$$

*Here, $\omega_0(x) = -\nabla \times \mathbf{u}_0(x) = \partial_y u_0 - \partial_x v_0$ is the initial vorticity and $\eta_0(x) := \lambda_1 - \lambda_2$ is the (possibly complex-valued) spectral gap associated with the eigenvalues of gradient matrix $\nabla \mathbf{u}_0(x)$. Moreover, these globally smooth solutions, $\mathbf{u}_1(t, \cdot)$, are $2\pi\tau$-periodic in time.*[4]

Liu and Tadmor[13] gave two different proofs of (2.2) based on the spectral dynamics of $\lambda_j(\nabla \mathbf{u})$ and on the flow map associated with (2.1). We follow yet another proof[4] which utilizes the Ricatti-type equation satisfied by the gradient matrix $M =: \nabla \mathbf{u}_1$,

$$M' + M^2 = \tau^{-1} J M.$$

Here, $' := \partial_t + \mathbf{u}_1 \cdot \nabla$ is the Lagrange derivative along particle trajectory induced by $\mathbf{u}_1$. Starting with $M_0 = M(t_0, x_0)$, the solution of this Ricatti equation along the corresponding trajectory is given by

$$M = e^{tJ/\tau} \left( I + \tau^{-1} J \left( I - e^{tJ/\tau} \right) M_0 \right)^{-1} M_0,$$

and a straightforward calculation based on the Cayley-Hamilton Theorem (for computing the inverse of a matrix) shows that

$$\max_{t,x} |\nabla \mathbf{u}_1| = \max_{t,x} |M| = \max_{t,x} \left| \frac{\text{polynomial}(\tau, e^{tJ/\tau}, \nabla \mathbf{u}_0)}{(1 - \tau\omega_0 - \frac{\tau^2}{2}\eta_0^2)_+} \right|. \tag{2.3}$$

The critical threshold (2.2) follows from the requirement that the determinant at the denominator will not vanish: it follows that there exists a critical Rossby number, $\tau_c := \tau_c(\nabla \mathbf{u}_0)$, such that the pressureless solution $\mathbf{u}_1(t, \cdot)$ remains smooth for all time whenever $\tau \in (0, \tau_c)$. Observe that the critical threshold, $\tau_c$ *need not* be small, and in fact, $\tau_c = \infty$ for rotational initial data such that $\eta_0^2 < 0, \omega_0 < \sqrt{-2\eta_0^2}$. We shall always limit ourselves, however, to a finite value of the critical threshold, $\tau_c$.

Moreover, the periodicity of $\mathbf{u}_1$ follows upon integration of $\mathbf{u}_1{}' = \frac{1}{\tau} J u$ and $x' = \mathbf{u}_1$ along particle trajectories $\Gamma_0$. It turns out both $x(t)$ and $\mathbf{u}_1(t, x(t))$ are $2\pi\tau$ periodic, which clearly implies that $\mathbf{u}_1(t, \cdot)$ shares the same periodicity as well.

## 3. Second approximation – advection by the pressureless velocity

Once we establish the global properties of the pressureless velocity $\mathbf{u}_1$, it can be used as the starting point for second iteration of (1.3). We begin with the approximate height, $h_2$, governed by (1.3a),

$$\partial_t h_2 + \mathbf{u}_1 \cdot \nabla h_2 + \left( \frac{1}{\sigma} + h_2 \right) \nabla \cdot \mathbf{u}_1 = 0, \qquad h_2(0, \cdot) = h_0(\cdot). \tag{3.1}$$

The periodicity of $\mathbf{u}_1$, see Section 2, imposes the same periodicity on passive scalars transported by such $\mathbf{u}_1$'s.

**Lemma 3.1.** *Let scalar function $\phi$ be governed by*

$$\partial_t \phi + \nabla \cdot (\mathbf{u}_1 \phi) = 0 \tag{3.2}$$

*where* $\mathbf{u}_1(t,\cdot)$ *is a globally smooth,* $2\pi\tau$*-periodic solution of the pressureless equations (2.1). Then* $\phi(t,\cdot)$ *is also* $2\pi\tau$*-periodic.*

To prove lemma 3.1, observe that the very same equation (3.2) governs the dynamics of relative vorticity, $\omega_1 := \nabla\times\mathbf{u}_1 + \frac{1}{\tau}$. Consequently, a straightforward calculation reveals that the ratio $\omega_1/\phi$ remains invariant along particle trajectories, $(\partial_t + \mathbf{u}_1\cdot\nabla)(\omega_1/\phi) = 0$. It follows that since $\mathbf{u}_1$ and $\omega_1$ are $2\pi\tau$-periodic, then so is $\phi$. We note in passing that the same argument applies for ratios involving the absolute height, $(h_2 + \frac{1}{\sigma})/\phi$ and the local deformation $\det(\nabla\mathbf{u}_1)/\phi$; the invariance of the latter along particle trajectories is classical.[14]

To continue with the second approximation, we turn to the approximate momentum equation (1.3b) with $j = 2$.

$$\partial_t\mathbf{u}_2 + \mathbf{u}_2\cdot\nabla\mathbf{u}_2 + \frac{1}{\sigma}\nabla h_2 - \frac{1}{\tau}\mathbf{u}_2 = 0. \tag{3.3}$$

A *splitting* approach leads to a simplified linearization of (3.3) which is "close" to (3.3) and still maintains the nature of our methodology in Ref. 4.

$$\mathbf{u}_2 := \mathbf{u}_1 + \frac{\tau}{\sigma}J(I - e^{tJ/\tau})\nabla h_2(t,\cdot). \tag{3.4a}$$

A straightforward computation shows that this velocity field, $\mathbf{u}_2$, satisfies the following approximate momentum equation,

$$\partial_t\mathbf{u}_2 + \mathbf{u}_1\cdot\nabla\mathbf{u}_2 + \frac{1}{\sigma}\nabla h_2 - \frac{1}{\tau}\mathbf{u}_2{}^{\perp} = R \tag{3.4b}$$

where

$$\begin{aligned} R &:= \frac{\tau}{\sigma}J(I - e^{tJ/\tau})(\partial_t + \mathbf{u}_1\cdot\nabla)\nabla h_2(t,\cdot) \\ \text{(by (3.1))} \qquad &= -\frac{\tau}{\sigma}J(I - e^{tJ/\tau})\left[(\nabla\mathbf{u}_1)^{\top}\nabla h_2 + \nabla((\frac{1}{\sigma} + h_2)\nabla\cdot\mathbf{u}_1)\right]. \end{aligned} \tag{3.4c}$$

## 4. Approximate periodicity – error estimate by the energy method

The $C^1$ regularity and time-periodicity results established in Theorem 2.1, Lemma 3.1 and equations (3.4) allow us to study Sobolev $H^m$ regularity of the approximate systems. In particular, it suffices to study the local regularity of periodic solutions within one period. Once this is done, we can proceed to use standard energy arguments (see e.g. Ref. 8,9,11) to estimate the errors $\mathbf{u} - \mathbf{u}_2$ and $h - h_2$. We refer the reader to Ref.[4] for detailed proofs, noting that the residual term $R$ in equation (3.4b), (3.4c) is of scale $\delta = \tau/\sigma^2$.

**Theorem 4.1.** *Consider the rotational shallow water (RSW) equations on a fixed 2D torus,*

$$\partial_t h + \mathbf{u} \cdot \nabla h + \left(\frac{1}{\sigma} + h\right) \nabla \cdot \mathbf{u} = 0 \tag{4.1a}$$

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\sigma} \nabla h - \frac{1}{\tau} J \mathbf{u} = 0 \tag{4.1b}$$

*subject to initial data* $(h_0, \mathbf{u}_0) \in H^m(\mathcal{T}^2)$ *with* $m > 5$ *and* $\alpha_0 := \min(1 + \sigma h_0(\cdot)) > 0$. *Let*

$$\delta = \frac{\tau}{\sigma^2}$$

*denote the ratio between the Rossby number* $\tau$ *and the squared Froude number* $\sigma$*, and assume the subcritical condition* $\tau \leq \tau_c$ *so that (2.2) holds. Assume* $\sigma \leq 1$ *for substantial amount of pressure forcing in (4.1b). Then, there exists a constant* $C_0$*, depending only on* $m$*,* $\tau_c$*,* $\alpha_0$ *and in particular depending linearly on* $\|(h_0, \mathbf{u}_0)\|_m$*, such that the RSW equations admit a smooth, "almost periodic" solution in the sense that there exists a near-by* $2\pi\tau$*-periodic solution,* $(h_2(t, \cdot), \mathbf{u}_2(t, \cdot))$ *such that*

$$\|p(t, \cdot) - p_2(t, \cdot)\|_{m-3} + \|\mathbf{u}(t, \cdot) - \mathbf{u}_2(t, \cdot)\|_{m-3} \lesssim \delta \frac{e^{C_0 t}}{1 - e^{C_0 t}\delta}, \tag{4.2}$$

*where* $p$ *is the "normalized height" such that* $1 + \frac{1}{2}\sigma p = \sqrt{1 + \sigma h}$ *and correspondingly* $p_2$ *satisfies* $1 + \frac{1}{2}\sigma p_2 = \sqrt{1 + \sigma h_2}$.

It follows that the life span of the RSW solution, $t \lesssim t_\delta := 1 + \ln(\delta^{-1})$ is prolonged due to the rapid rotation $\delta \ll 1$, and in particular, it tends to infinity when $\delta \to 0$. We close this section by noting that one may recover[4] the regularity of the original height $h$ from the "normalized height" $p$.

## 5. Related works

We extend Theorem 4.1 to rotational two-dimensional Euler equations for isentropic gas and, more generally, full Euler equations for ideal gasdynamics. The same methodology applies and leads the existence of a near-by approximate solution for a time period $\sim \ln(1/\delta)$, where

$$\delta = \frac{\tau}{\sigma^2} \ll 1$$

involves the usual Rossby number $\tau$ but $\sigma$ now stands for the Mach number, replacing the Froude number in Theorem 4.1. This generalization reflects the fact that the RSW equations are a special case of general 2D Euler equations.

Another extension of Theorem 4.1 deals with $\delta \gg 1$, where rotational forcing become mild or even weak, relative to pressure forcing. In such regimes, the new parameter $\delta$ still plays a central role in determining the dynamics of the flow. More precisely, in Ref. 2,3, we prove that

**Theorem 5.1.** *Consider the same equations and assumptions as in Theorem 4.1 except setting the domain as the whole* $\mathbb{R}^2$ *space. Let* $\kappa := \max\{\delta^{-1}, \sigma\} \lesssim 1$. *Then there exists an incompressible flow* $\mathbf{u}_{inc}$ *such that for some finite time* $T'$ *and some* $m' < m, p > 2, q > 2, \alpha' > 0$,

$$||\mathbf{u} - \mathbf{u}_{inc}||_{L^p\left([0,T'];W^{m',q}\right)} \lesssim \kappa^{\alpha'} \tag{5.1}$$

The idea is to employ fast wave analysis for nonlinear hyperbolic PDEs with Strichartz type estimates to reveal an approximate incompressible flow. Our argument is highlighted with an invariant-based analysis on the nonlinear interactions of fast waves. This enables us to consider the *two* fast scales associated with $\sigma$ and $\tau$ and therefore to generalize previous results that rely on only one fast scale.[6]

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