



Suppressing Chemotactic Blow-Up Through a Fast Splitting Scenario on the Plane

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Abstract

We revisit the question of global regularity for the Patlak–Keller–Segel (PKS) chemotaxis model. The classical 2D parabolic–elliptic model blows up for initial mass $M > 8\pi$. We consider a more realistic scenario which takes into account the flow of the ambient environment induced by *harmonic* potentials, and thus retain the identical elliptic structure as in the original PKS. Surprisingly, we find that already the simplest case of *linear* stationary vector field, Ax^\top , with large enough amplitude A , prevents chemotactic blow-up. Specifically, the presence of such an ambient fluid transport creates what we call a ‘fast splitting scenario’, which competes with the focusing effect of aggregation so that ‘enough mass’ is pushed away from concentration along the x_1 -axis, thus avoiding a finite time blow-up, at least for $M < 16\pi$. Thus, the enhanced ambient flow doubles the amount of allowable mass which evolve to global smooth solutions.

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1. Introduction

The Patlak–Keller–Segel (PKS) model describes the time evolution of colony of bacteria with density $n(x, t)$ subject to two competing mechanisms — aggregation triggered by the concentration of chemo-attractant driven by velocity field $\mathbf{u} := \nabla(-\Delta)^{-1}n(\cdot, t)$, and diffusion due to run-and-tumble effects,

$$n_t + \nabla \cdot (n\mathbf{u}) = \Delta n, \quad \mathbf{u} := \nabla(-\Delta)^{-1}n.$$

We focus on the two-dimensional case where $(-\Delta)^{-1}n$ takes the general form of a fundamental solution together with an arbitrary harmonic function

$$(-\Delta)^{-1}n(x, t) = (K * n)(x, t) + H(x, t), \quad K(x) := -\frac{1}{2\pi} \log |x|, \quad \Delta H(\cdot, t) \equiv 0.$$

The resulting PKS equation then reads

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\nabla c) + \mathbf{b} \cdot \nabla n = \Delta n, \quad c = K * n, \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad (1.1)$$

subject to prescribed initial conditions $n(x, 0) = n_0(x)$. Here the divergence free vector field $\mathbf{b}(\cdot)$ represents the environment of a background fluid transported with velocity $\mathbf{b}(x, t) := \nabla H(x, t)$. When $\mathbf{b} \equiv 0$, the system is the classical parabolic-elliptic PKS equation modeling chemotaxis in a *static* environment [21,27]. We recall the large literature on the static case $\mathbf{b} = 0$, referring the interested reader to the review [19] and the follow-up representative works [3–8,10–15,20,28,29]. It is well-known that the large-time behavior of the static case (1.1) _{$\mathbf{b}=0$} depends on whether the initial total mass $M := |n_0|_1$ crosses the critical threshold of 8π : the equation admits global smooth solution in the sub-critical case $M < 8\pi$ and it experiences a finite time blow-up if $M > 8\pi$ [7] (when $M = 8\pi$, aggregation and diffusion exactly balance each other and solutions with finite second moments form Dirac mass as time approach infinity [8]).¹

In this paper we study a more realistic scenario of the PKS model (1.1), where we take into account an ambient environment due to the fluid transport by vector field $\mathbf{b}(\cdot, t)$. Surprisingly, we find that already the simplest case of linear stationary vector field, $\mathbf{b} = A(-x_1, x_2)$, corresponding to $H(x) = \frac{1}{2}A(x_2^2 - x_1^2)$, prevents chemotactic blow-up for $M < 16\pi$. As we shall see, the presence of such an ambient fluid transport creates what we call a ‘fast splitting scenario’ which competes with the focusing effect of aggregation so that ‘enough mass’ is able to escape a finite time blow-up, at least for $M < 16\pi$. This scenario is likely to be enhanced even further when larger amount of mass can be transported by a more pronounced ambient field $\mathbf{b}(x, t) = \nabla H(\cdot, t) \gg |x|^q$ at $|x| \gg 1$.

We mention two other scenarios of non-static PKS with strong enough transport preventing blow-up for $M > 8\pi$. In [23] the authors exploit the relaxation enhancing of a vector field \mathbf{b} with a large enough amplitude in order to enforce

¹ We let $|x|$ denote the ℓ^2 -size of vector x and let $|f|_p$ denote the L^p -norm of a vector function $f(\cdot)$.

global smooth solutions. Here regularity follows due to a *mixing property* over \mathbb{T}^2 and \mathbb{T}^3 . In [2] it was shown how to exploit the enhanced dissipation effect of non-degenerate *shear flow* with large enough amplitude, [1], in order to suppress the blow up in (1.1) on $\mathbb{T}^2, \mathbb{T}^3, \mathbb{T} \times \mathbb{R}, \mathbb{T} \times \mathbb{R}^2$. It is worth mentioning that the model (1.1) is one among many attempts to take into account the underlying fluid transport effect, see, e.g. [16, 18, 24–26].

1.1. A Fast Splitting Scenario

Here, we exploit yet another mechanism that suppress the possible chemotactic blow up of the equation (1.1), where the underlying fluid flow splits the population of bacteria with density n exponentially fast, resulting in several isolated subgroups with mass smaller than the critical 8π . In this manner, an initial total mass greater than 8π is able to escape the finite-time concentration of Dirac mass. This provides a first no blow-up scenario over \mathbb{R}^2 , at least for M up to 16π .

We now fix the vector field driving a hyperbolic flow as the strain flow in [22]):

$$\mathbf{b}(x) := A(-x_1, x_2). \tag{1.2}$$

Our aim is to show that a large enough amplitude, $A \gg 1$, guarantees the global existence of solution of PKS (1.1) subject to initial mass $M < 16\pi$. Intuitively, the large enough amplitude $A \gg 1$ is required so that the ambient field $A(-x_1, x_2)$ ‘pushes away’ highly concentrated mass near the x_1 -axis, namely, $\int_{|x_2| \leq \epsilon} n_0(x) dx \gg 1$. With this we can state the main theorem of the paper.

Theorem 1.1. Consider the PKS equation (1.1), (1.2) subject to initial data, $n_0 \in H^s (s \geq 2)$ with total mass, $M := |n_0|_1 < 16\pi$, such that $(1 + |x|^2)n_0 \in L^1(\mathbb{R}^2)$ and $n_0 \log n_0 \in L^1(\mathbb{R}^2)$. Assume n_0 is symmetric about the x_1 -axis, and that the “y-component” of its center of mass in the upper half plane

$$y_+(t) := \frac{1}{M_+} \int_{x_2 \geq 0} n(x, t)x_2 dx, \quad M_+ := \int_{x_2 \geq 0} n(x, t) dx \equiv \frac{M}{2},$$

is not too close to the x_1 -axis in the sense that

$$y_+^2(0) > \frac{2}{M_+} V_+(0), \quad V_+(t) := \int_{x_2 \geq 0} n(x, t)|x_2 - y_+|^2 dx. \tag{1.3}$$

Then there exists a large enough amplitude, $A = A(M, y_+(0), V_+(0))$, such that the weak solution of (1.1), (1.2) exists for all time and the free energy

$$E[n](t) := \int (n \log n - \frac{1}{2}cn - H(x)n(x, t)) dx, \quad H(x) = \frac{A}{2}(x_2^2 - x_1^2), \tag{1.4}$$

satisfies the dissipation relation

$$E[n](t) + \int_0^t \int_{\mathbb{R}^2} n |\nabla \log n - \nabla c - \mathbf{b}|^2 dx ds \leq E[n_0]. \tag{1.5}$$

We conclude the introduction with a series of remarks.

Remark 1.1. (Why large enough stationary field prevents second-moment collapse) Our main theorem extends the amount of critical mass, so that global regularity of (1.1), (1.3) prevails for $M < 16\pi$, provided A is large enough. To gain further insight why large enough A will prevent blow-up, we recall that the blow up phenomena in the static case, $\mathbf{b} \equiv 0$ is deduced from the time evolution of the second moment, $V(t) := \int_{\mathbb{R}^2} n(x, t)|x|^2 dx$. Indeed, a straightforward computation yields, $\dot{V}(t) = 4M\left(1 - \frac{M}{8\pi}\right) < 0$, which implies that the positive $V(t)$ decreases to zero in a finite time and hence rules out existence of global classical solutions for $M > 8\pi$. In contrast, the second moment of our non-static PKS equation (1.1), (1.2), does not decrease to zero if A is chosen large enough. This is the content of our next lemma.

Lemma 1.1. Let $n(x, t)$ be the solution of (1.1) with vector field $\mathbf{b}(x) = A(-x_1, x_2)$, subject to initial data n_0 such that $W_0 := \int_{\mathbb{R}^2} n_0(x)(x_2^2 - x_1^2) dx$ is strictly positive. Then, if A is chosen large enough, the second moment of the (classical) solution $V(t) = \int_{\mathbb{R}^2} n(x, t)|x|^2 dx$ increases in time.

Proof. First, the time evolution of $V(t)$ can be calculated as follows:

$$\begin{cases} \frac{d}{dt} V = 4M\left(1 - \frac{M}{8\pi}\right) + 2 \int x \cdot \mathbf{b} n(x, t) dx \\ = 4M\left(1 - \frac{M}{8\pi}\right) + 2AW, \quad W(t) = \int_{\mathbb{R}^2} (-x_1^2 + x_2^2)n(x, t) dx. \end{cases} \tag{1.6}$$

Next, we compute the time evolution of $W(t)$:

$$\begin{aligned} \frac{d}{dt} W &= \int (-x_1^2 + x_2^2) \nabla \cdot (\nabla n - \nabla cn - \mathbf{b}n) dx \\ &= -\frac{1}{2\pi} \iint n(x, t)(-2x_1, 2x_2) \cdot \frac{x - y}{|x - y|^2} n(y, t) dx dy + 2AV \\ &= -\frac{1}{2\pi} \iint \frac{-(x_1 - y_1)^2 + (x_2 - y_2)^2}{(x_1 - y_1)^2 + (x_2 - y_2)^2} n(x, t)n(y, t) dx dy + 2AV, \end{aligned}$$

where the last step follows by symmetrization. Since the first term on the right is bounded from below by $-\frac{1}{2\pi}M^2$, we have

$$\frac{d}{dt} W \geq -\frac{1}{2\pi}M^2 + 2AV. \tag{1.7}$$

Finally, notice that since W_0 (and hence V_0) are assumed strictly positive, we can choose A large such that

$$AV_0 - \frac{1}{4\pi}M^2 \geq 0, \quad AW_0 + 2M\left(1 - \frac{M}{8\pi}\right) \geq 0. \tag{1.8}$$

Combining (1.8), (1.6) and (1.7) yields that $W(t) > 0, V(t) > 0$. \square

This shows that collapse may be prevented if A is large enough. Indeed, our theorem 1.1 shows that collapse is avoided for large enough A : specifically (consult (3.4) below), $A = M_+ \delta^{-2}$ with small enough δ such that

$$\delta \leq (R - 1) \sqrt{\frac{2V_+(0)}{M_+}}, \quad R^2 := M_+ \frac{y_+^2(0)}{2V_+(0)}.$$

Observe that by (1.3) $R > 1$, which enables the choice of $\delta \ll 1$ and hence $A \gg 1$ for the global regularity of (1.1), (1.3).

We conjecture that a similar fast splitting scenario holds with even higher-degree harmonics, $\mathbf{b}(x, t) = \nabla H(\cdot, t) \gg |x|^q$ at $|x| \gg 1$: the higher the degree q is, the larger amount of critical mass is expected to be ‘pushed away’ from the origin in different directions, with even higher thresholds than 16π . A main difficulty, however, is the lack of closure to control the second-order moments in these higher-order cases.

Remark 1.2. (On the free energy) We note that when $\mathbf{b} = 0$, $E[n]$ becomes the classical dissipative free energy

$$\mathcal{F} = \int_{\mathbb{R}^2} n \log n dx - \frac{1}{2} \int_{\mathbb{R}^2} n c dx. \tag{1.9}$$

Due to the importance of the property (1.5), a weak solution of (1.1) satisfying (1.5) will be called a *free energy solution*. One of the important properties of the PKS equation (1.1) with background flow velocity (1.2) is the dissipation of its free energy $E[n]$. The formal computation, indicating the energy dissipation in non-static smooth solutions, is the content of our last lemma in this section.

Lemma 1.2. Consider the PKS equation (1.1) with background fluid velocity (1.2). If the solution is smooth enough, the free energy $E[n](t)$ is decreasing.

Proof. The time evolution of the free energy (1.4) can be computed in terms of the potential $H = \frac{1}{2} A(x_2^2 - x_1^2)$,

$$\begin{aligned} \frac{d}{dt} E[n](t) &= \int n_t (\log n - c - H) dx \\ &= - \int n (\nabla \log n - \nabla c - \mathbf{b}) \cdot (\nabla \log n - \nabla c - \nabla H) dx \\ &= - \int n |\nabla \log n - \nabla c - \mathbf{b}|^2 dx \leq 0. \end{aligned}$$

This completes the proof of the lemma. \square

Remark 1.3. (Smoothness) Arguing along the lines [17], one can prove that the free energy solution admits higher-order integrability and consequently retains Sobolev smoothness for all positive time, $n \in C_c^\infty((0, T]; C_x^\infty)$ for all $T < \infty$, thus our global weak solution is in fact a global strong solution.

Remark 1.4. (*Stability*) We note that by symmetry, our 16π -threshold stated in theorem 1.1 is evenly divided between the upper- and lower-halves of the plane, each should contain at most 8π mass. We expect that this 16π threshold remains valid even for small *asymmetric* perturbations, as long as the mass in each half is kept *separately* below the 8π threshold.

Our paper is organized as follows. In Section 2, we introduce the regularized problem to (1.1) which leads to the local existence results. In Section 3, we prove the main theorem, and in the appendix, we give detailed proofs of the results stated in Section 2.

2. Local Existence

2.1. Weak Formulation

It is standard to understand the Keller–Segel equation (1.1) with background fluid velocity (1.2) in the following weak formulation:

Definition 2.1. (weak formulation) The function n is the weak solution of (1.1) if for $\forall \varphi \in C_c^\infty(\mathbb{R}_+^2)$, the following holds:

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^2} \varphi n dx &= \int_{\mathbb{R}^2} \Delta \varphi n dx \\ &- \frac{1}{4\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{(\nabla \varphi(x) - \nabla \varphi(y)) \cdot (x - y)}{|x - y|^2} n(x, t) n(y, t) dx dy \\ &+ \int_{\mathbb{R}^2} \nabla \varphi \cdot \mathbf{b} n dx. \end{aligned} \tag{2.1}$$

Taking advantage of the assumed symmetry across the x_1 -axis, one can further simplify the notion of a weak solution by restricting attention to the upper half plane, $\mathbb{R}_+^2 = \{(x_1, x_2) \mid x_2 \geq 0\}$.

Theorem 2.1. *The function n is a weak solution of (1.1) if $n_+ := n \mathbf{1}_{x_2 \geq 0}$ satisfies for $\forall \varphi \in C_c^\infty(\mathbb{R}_+^2)$,*

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}_+^2} \varphi n_+ dx &= \int_{\mathbb{R}_+^2} \Delta \varphi n_+ dx \\ &- \frac{1}{4\pi} \int_{\mathbb{R}_+^2 \times \mathbb{R}_+^2} \frac{(\nabla \varphi(x) - \nabla \varphi(y)) \cdot (x - y)}{|x - y|^2} n_+(x, t) n_+(y, t) dx dy \\ &+ \int_{\mathbb{R}_+^2} \nabla c_- \cdot \nabla \varphi n_+ dx + \int_{\mathbb{R}_+^2} \nabla \varphi \cdot \mathbf{b} n_+ dx. \end{aligned} \tag{2.2}$$

Here

$$\begin{aligned} \nabla c_-(x) &:= - \int_{(\mathbb{R}_+^2)^c} \frac{x - y}{2\pi |x - y|^2} n_-(y) dy \\ &= - \int_{(\mathbb{R}_+^2)^c} \frac{x - y}{2\pi |x - y|^2} n_+(-y) dy, \quad n_- := n \mathbf{1}_{x_2 \leq 0}. \end{aligned}$$

Proof. Rewrite (2.1) as follows:

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}_+^2} \varphi n_+ dx &= \int_{\mathbb{R}_+^2} \Delta \varphi n_+ dx \\ &\quad - \frac{1}{4\pi} \int_{\mathbb{R}_+^2 \times \mathbb{R}_+^2} \frac{(\nabla \varphi(x) - \nabla \varphi(y)) \cdot (x - y)}{|x - y|^2} n_+(x, t) n_+(y, t) dx dy \\ &\quad - \frac{1}{2\pi} \int_{\mathbb{R}_+^2 \times \mathbb{R}^2} \frac{\nabla \varphi(x) \cdot (x - y)}{|x - y|^2} n_+(x, t) n_-(y, t) dx dy \\ &\quad + \int_{\mathbb{R}_+^2} \nabla \varphi \cdot \mathbf{b} n_+ dx. \end{aligned}$$

The third term can be rewritten as $\int_{\mathbb{R}_+^2} \nabla c_- \cdot \nabla \varphi n_+(x) dx$, and we get (2.2). \square

2.2. Regularized Equation and Local Existence Theorems

In this section we introduce the local existence theorem and the blow up criterion for the Keller–Segel equation with advection. The theorems are standard, so the proofs are postponed to the appendix. The interested reader are referred to the papers [7, 8] for further details.

In order to prove the local existence theorem and the blow up criterion for the Keller–Segel system with advection (1.1), we regularize the system as follows:

$$\frac{\partial n^\epsilon}{\partial t} + \nabla \cdot (n^\epsilon \nabla c^\epsilon) + \mathbf{b} \cdot \nabla n^\epsilon = \Delta n^\epsilon, \quad c^\epsilon := K^\epsilon * n, \quad x \in \mathbb{R}^2, t > 0, \quad (2.3)$$

with the regularized kernel, K^ϵ , given by

$$K^\epsilon(z) := K^1\left(\frac{z}{\epsilon}\right) - \frac{1}{2\pi} \log \epsilon, \quad K^1(z) := \begin{cases} -\frac{1}{2\pi} \log |z|, & \text{if } |z| \geq 4, \\ 0, & \text{if } |z| \leq 1. \end{cases} \quad (2.4)$$

Noting that $|\nabla K^\epsilon(z)| \leq C_\epsilon$ for all $z \in \mathbb{R}^2$, it follows that the solutions to the equation (2.3) exist for all time. The proof is similar to the corresponding proof in the classical case. We refer the interested reader to the paper [7] for more details.

Before stating the local existence theorems, we introduce the entropy of the solution

$$S[n] := \int_{\mathbb{R}^2} n \log n dx. \quad (2.5)$$

Now the local existence theorems are expressed as follows:

Proposition 2.1. (Local Existence Criterion) *Assume that $|\mathbf{b}|(x) \leq C|x|, \forall x \in \mathbb{R}^2$. Suppose $\{n^\epsilon\}_{\epsilon \geq 0}$ are the solutions of the regularized equation (2.3) on $[0, T^*)$. If $\{S[n^\epsilon](t)\}_\epsilon$ is bounded from above uniformly in ϵ and in $t \in [0, T^*)$, then the cluster points of $\{n^\epsilon\}_{\epsilon \rightarrow 0}$, in a suitable topology, are non-negative weak solutions of the PKS system with advection (1.1) on $[0, T^*)$ and satisfies the relation (1.5).*

Proposition 2.2. (Maximal Free-energy Solutions) *Assume the boundedness of the vector field $|\mathbf{b}|(x) \leq C|x|$ and the integrability of initial data $(1 + |x|^2)n_0 \in L^1_+(\mathbb{R}^2)$, $n_0 \log n_0 \in L^1(\mathbb{R}^2)$. Then there exists a maximal existence time $T^* > 0$ of a free energy solution to the PKS system with advection (1.1), (1.5). Moreover, if $T^* < \infty$ then*

$$\lim_{t \rightarrow T^*} \int_{\mathbb{R}^2} n \log n dx = \infty.$$

We conclude that if the entropy $S[n](t) = \int n \log n$ is bounded, then the free energy solution of (1.1) exists locally. Moreover, if $S[n](t) < \infty$ for all $t < \infty$, the solution exists for all time.

3. Proof of the Main Results

3.1. The Three-Step ‘Battle-Plan’

We proceed in three steps. The *first step* carried in Section 3.2 below, is to control cell density distribution. From the last section, we see that an entropy bound is essential for derivation of local existence theorems for the PKS equation (1.1), (1.2). To this end, information about the distribution of cell density is crucial. The following lemma is the key to the proof of the main results. It shows that mass cannot concentrate along the the x_1 -axis, since we can find a thin enough strip along the x_1 -axis with controlled amount of mass.

For the rest of this section we fixed a small parameter $0 < \eta \ll 1$ which will quantify the sharp estimates in the sequel.

Lemma 3.1. *Suppose a sufficiently smooth n_0 is symmetric about the x_1 axis and assume that*

$$R^2 := M_+ \frac{y_+^2(0)}{2V_+(0)} > 1. \tag{3.1}$$

Fix a small enough $0 < \eta \ll 1$. Then there exists $\delta = \delta(y_+(0), V_+(0), M, \eta)$ such that if we choose $A > \frac{M_+}{\delta^2}$, the smooth solutions to the regularized (2.3) $_\epsilon$ satisfy, uniformly for small enough ϵ ,

$$\int_{|x_2| \leq 2\delta} n^\epsilon(x, t) dx \leq \frac{(1 + \eta)^2}{2R^2} M. \tag{3.2}$$

Condition (3.2) implies, at least for $M < 16\pi$, that the mass inside that δ -strip is less than 8π . On the other hand, it indicates the reason for the limitation $R > 1$: for if $R < 1$, then the bound (3.2) would allow a concentration of mass $\frac{M}{2R^2} \geq 8\pi$ inside the strip $|x_2| \leq 2\delta$, which in turn leads to a finite-time blow-up.

The proof of lemma 3.1 is based on the following simple observation. Given f with \mathbb{R}^2_+ -center of mass at (\cdot, y_f) and variation $V_f = \int |x_2 - y_f|^2 f(x) dx$, we

find that its total mass *outside* the strip $\mathcal{S}[y_f, r] := \{(x_1, x_2) \mid |x_2 - y_f| \leq r\}$ with radius $r = R\sqrt{2V_f/M_f}$, does not exceed

$$\begin{aligned} \int_{|x_2 - y_f| > r} f(x) dx &= \int_{|x_2 - y_f| > r} f(x) \frac{|x_2 - y_f|^2}{|x_2 - y_f|^2} dx \\ &\leq \frac{M_f}{2R^2 V_f} \int f(x) |x_2 - y_f|^2 dx = \frac{M_f}{2R^2}. \end{aligned}$$

If we can find the δ such that our target strip $\mathcal{S}_\delta := \{|x_2| \leq 2\delta\}$ is lying below and *outside* the strip $\mathcal{S}[y_f, r]$, then the total mass in the strip \mathcal{S}_δ would be smaller than $\frac{1}{2R^2} M_f$. When $n^\epsilon(x, t)$ takes the role of $f(x)$ with $(y_f, V_f) \mapsto (y_+(t), V_+(t))$, the aim is to bound the strip $\mathcal{S}[y_+(t), r(t)]$ with radius $r(t) = R\sqrt{2V_+(t)/M_+}$ away from a fixed strip \mathcal{S}_δ . To this end we collect the necessary estimates on $y_+(t), V_+(t)$ and complete the proof of the lemma in Section 3.2.

The *second step*, carried in Section 3.3, is to prove the main theorem under a *constrained setup*: equipped with lemma 3.1 we can control the entropy and prove a weaker form of our main theorem for any $M < 16\pi$ (which is still larger than the 8π barrier), under the constraint that the mass M concentrates far enough from the x_1 -axis. This is quantified in our next theorem. We recall that a small parameter $0 < \eta \ll 1$ was already quantified in lemma 3.1.

Theorem 3.1. *Consider the PKS equation (1.1) with background fluid velocity (1.2) subject to $H^s (s \geq 2)$ initial data, symmetric about the x_1 -axis, with mass $M = |n_0|_1 < 16\pi$, and bounded second moment $(1 + |x|^2)n_0 \in L^1(\mathbb{R}^2)$. If $R^2 = M_+ \frac{y_+^2(0)}{2V_+(0)}$ is large enough so that*

$$R^2 > \frac{(1 + \eta)^2 M}{16\pi - M}, \tag{3.3}$$

then there exists a large enough $A = A(M, y_+(0), V_+(0), \eta)$, such that the free energy solution to PKS (1.1), (1.2) exists for all time.

Observe that R is a dimensionless parameter and R being large indicates that most of the mass concentrates away from the ‘critical’ strip, \mathcal{S}_δ along the x_1 -axis, namely — either the center of mass y_+ is far enough and/or the mass variation V_+/M_+ is small enough. Either way, we find that for any $M < 16\pi$, if R is large enough so that (3.3) holds, then (3.1) yields global existence. Although theorem 3.1 is not as sharp as the main theorem, its proof is more illuminating and can be extended easily to prove the main theorem for the ‘limiting case’ of any $M < 16\pi$. We therefore include its proof in Section 3.3.

Finally, the *third step* carried in Section 3.4 presents the proof of the main theorem 1.1.

We turn to a detailed discussion of the three steps.

3.2. Step 1—Control of the Cell Density Distribution

As pointed out before, the proof involves the calculation of $y_+(t)$ and $V_+(t)$, summarized in the following two lemmas. Here and below, we let $A \lesssim B$ denote the relation $A \leq CB$ with a constant C which is independent of δ .

Lemma 3.2. *Consider the regularized PKS equation (2.3) with background fluid velocity (1.2). Assume that the initial center of mass $y_+(0)$ is bounded away from the x_1 -axis in the sense that (3.1) holds. Then for any sufficiently small $\delta > 0$, there exists a large enough amplitude of the ambient vector field, $A = \delta^{-2}M_+$, and a constant C (independent of δ), such that $y_+(0) - C\delta > 0$ and the time evolution of (2.3), (1.2) pushes the center of mass, $y_+(t)$, away from the critical strip, namely*

$$y_+(t) \geq [y_+(0) - C\delta] e^{At}, \quad A = \frac{M_+}{\delta^2}. \tag{3.4}$$

Lemma 3.3. *Consider the regularized PKS equation (2.3) with background fluid velocity (1.2). Assume that the initial variation around the center of mass $V_+(0)$ is not too large in the sense that (3.1) holds. Then for any sufficiently small $\delta > 0$, there exists a constant $C = C(V_+(0))$ such that the variation $V_+(t)$ remains bounded from above,*

$$V_+(t) \leq [CM_+\delta + V_+(0)] e^{2At}. \tag{3.5}$$

We note that all the calculations made below should be carried out at the level of the regularized equation (2.3), but for the sake of simplicity, we proceed at the formal level using the weak formulation (2.2). We explicitly point when there is a technical subtlety in the derivation due to difference between the regularized and weak formations.

We begin with the proof of Lemma 3.2. To calculate the dynamics of the center of mass, one needs to formally test the equation with respect to $\varphi(x) = x_2$. To stay away from the critical strip \mathcal{S}_δ , however, we introduce an approximate test function² $\varphi \approx x_2$ with a cut-off away from \mathcal{S}_δ

$$\varphi := \begin{cases} x_2 & x_2 \in (2\delta, \infty), \\ 0 & x_2 \in (-\infty, \delta), \\ \text{smooth} & x_2 \in (\delta, 2\delta). \end{cases}$$

Note that there exists a constant C_φ such that $|\varphi| \leq 2\delta$, $\forall x_2 \leq 2\delta$ and $|\nabla\varphi| + \delta|\nabla^2\varphi| \leq C_\varphi$. Here and below, we use C_φ to denote φ -dependent constants that otherwise are independent of δ .

Moreover, replacing $\int_{\mathbb{R}_+^2} x_2 n dx$ with $\int_{\mathbb{R}_+^2} \varphi n dx$, we lose information on the stripe $\{(x_1, x_2) | |x_2| \leq 2\delta\}$; however, the contribution of this part is small in the sense that

$$\left| \int_{0 \leq x_2 \leq 2\delta} \varphi n_+ dx - \int_{0 \leq x_2 \leq 2\delta} x_2 n_+ dx \right| \leq 4M_+\delta. \tag{3.6}$$

² To be precise, one should use here the usual argument of a further smooth cut-off for large φ at $|x_1| + x_2 \gg 1$, and recover the result uniformly with respect to that cut-off parameter.

Next, one can use φ and the weak formulation (2.2) to extract information about y_+ :

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}_+^2} \varphi n_+ dx &= \int_{\mathbb{R}_+^2} \Delta \varphi n_+ dx \\ &\quad - \frac{1}{4\pi} \int_{\mathbb{R}_+^2 \times \mathbb{R}_+^2} \frac{(\nabla \varphi(x) - \nabla \varphi(y)) \cdot (x - y)}{|x - y|^2} n_+(x, t) n_+(y, t) dx dy \\ &\quad + \int_{\mathbb{R}_+^2} n_+ \nabla c_- \cdot \nabla \varphi dx + \int_{\mathbb{R}_+^2} \nabla \varphi \cdot \mathbf{b} n_+ dx \\ &= I + II + III + IV. \end{aligned} \tag{3.7}$$

Now we estimate the right hand side of (3.7) term by term. The first and second terms are relatively easy to control from above,

$$|I| = \left| \int_{\mathbb{R}_+^2} \Delta \varphi n_+ dx \right| \leq \frac{C_\varphi M_+}{\delta}, \tag{3.8a}$$

$$|II| \leq \frac{1}{4\pi} |\nabla^2 \varphi|_\infty \int \int_{\mathbb{R}_+^2 \times \mathbb{R}_+^2} n_+(x) n_+(y) dx dy \leq \frac{1}{4\pi} \frac{C_\varphi}{\delta} M_+^2. \tag{3.8b}$$

Next we upper-bound the third term in (3.7) in the following way:

$$|III| = \left| \int_{\mathbb{R}_+^2} \partial_{x_2} c_- n_+ \partial_{x_2} \varphi dx \right| \leq \frac{C_\varphi}{2\pi} \frac{M_+}{\delta} \int_{\mathbb{R}_+^2} n_+ dx \leq \frac{C_\varphi}{2\pi} \frac{M_+^2}{\delta}. \tag{3.8c}$$

This follows from the pointwise bound $|\partial_{x_2} c_-(x)| \leq \frac{1}{2\pi} \frac{M_+}{|x_2|}$ shown below, and the fact that $\text{supp}(\varphi)$ is δ away from the x_1 -axis, and hence $\frac{1}{|x_2|} \leq \frac{1}{\delta}$,

$$\begin{aligned} |\partial_{x_2} c_-(x)| &= \frac{1}{2\pi} \left| \int_{(\mathbb{R}_+^2)^c} \frac{(x - y)_2}{|x - y|^2} n_-(y) dy \right| \\ &\leq \frac{1}{2\pi} \frac{1}{|x_2|} \int_{(\mathbb{R}_+^2)^c} \frac{|x_2|}{|x_2| + |y_2|} n_-(y) dy \leq \frac{1}{2\pi} \frac{M_-}{|x_2|} = \frac{1}{2\pi} \frac{M_+}{|x_2|}. \end{aligned} \tag{3.9}$$

Finally, we need to address additional transport term IV in (3.7) to compete with the focusing effect. Recall that $\mathbf{b} = A(-x_1, x_2)$ with $A = \frac{M_+}{\delta^2}$. First we write IV down explicitly:

$$IV = \int \mathbf{b} \cdot n_+ \nabla \varphi dx = \frac{M_+}{\delta^2} \int_{\mathbb{R}_+^2} x_2 \partial_{x_2} \varphi n_+ dx.$$

Next we replace the right hand side by $\int \varphi n_+ dx$. Due to the fact that $x_2 \partial_{x_2} \varphi = x_2 = \varphi$ for $x_2 > 2\delta$, the error introduced in this process originates from the thin 2δ -strip

$$\left| \int_{\delta < x_2 < 2\delta} (x_2 \partial_{x_2} \varphi - \varphi) n_+ dx \right| \leq \int_{\delta < x_2 < 2\delta} |x_2 \partial_{x_2} \varphi - \varphi| n_+ dx \leq C_\varphi \delta M_+,$$

and we conclude that

$$IV \geq \frac{M_+}{\delta^2} \left(\int_{\mathbb{R}_+^2} \varphi n_+ dx - C_\varphi \delta M_+ \right) \geq \frac{M_+}{\delta^2} \int_{\mathbb{R}_+^2} \varphi n_+ dx - \frac{C_\varphi M_+^2}{\delta}. \quad (3.10)$$

Combining equation (3.7) with the upper-bounds (3.8) and the lower-bound (3.10) while recalling the definition of A (3.4), we arrive at

$$\frac{d}{dt} \int_{\mathbb{R}_+^2} \varphi n_+ dx \geq -C_\varphi \frac{M_+^2}{\delta} + A \int_{\mathbb{R}_+^2} \varphi n_+ dx,$$

which implies that

$$\int_{\mathbb{R}_+^2} \varphi n_+ dx \geq \left(\int_{\mathbb{R}_+^2} \varphi n_0 dx - C_\varphi M_+ \delta \right) e^{At}. \quad (3.11)$$

Finally, we calculate the center of mass of the upper half plane using the lower bound (3.11) and the error control (3.6)

$$\begin{aligned} y_+(t) &= \frac{1}{M_+} \left(\int_{0 \leq x_2 \leq 2\delta} x_2 n_+ dx - \int_{0 \leq x_2 \leq 2\delta} \varphi n_+ dx + \int_{\mathbb{R}_+^2} \varphi n_+ dx \right) \\ &\geq -\frac{1}{M_+} \left| \int_{0 \leq x_2 \leq 2\delta} x_2 n_+ dx - \int_{0 \leq x_2 \leq 2\delta} \varphi n_+ dx \right| + \frac{1}{M_+} \int_{\mathbb{R}_+^2} \varphi n_+ dx \\ &\geq -4\delta + \frac{1}{M_+} \left(\int_{\mathbb{R}_+^2} \varphi n_0 dx - C_\varphi M_+ \delta \right) e^{At} \\ &\geq (y_+(0) - C_\varphi \delta) e^{At}. \end{aligned}$$

This completes the proof of lemma 3.2. \square

Remark 3.1. The only difference in estimating the regularized solutions (2.3) vs. the formal calculation we have done above is in terms II and III. In the calculation for the (2.3), we will need the estimate

$$|\nabla K^\epsilon(z)| \leq \frac{1}{2\pi|z|}, \quad \forall z \in \mathbb{R}^2.$$

Here we show how to get a similar estimate for term II in (3.7) for the regularized equation (2.3):

$$\begin{aligned} II &= \left| \iint_{\mathbb{R}_+^2 \times \mathbb{R}_+^2} \nabla \varphi(x) \nabla_x [K^\epsilon(|x-y|)] n^\epsilon(y) n^\epsilon(x) dx dy \right| \\ &= \frac{1}{2} \left| \iint_{\mathbb{R}_+^2 \times \mathbb{R}_+^2} \frac{(\nabla \varphi(x) - \nabla \varphi(y)) \cdot (x-y)}{|x-y|} \nabla K^\epsilon(|x-y|) n^\epsilon(x) n^\epsilon(y) dx dy \right| \\ &\leq \frac{1}{4\pi} \left| \iint_{\mathbb{R}_+^2 \times \mathbb{R}_+^2} \frac{|\nabla \varphi(x) - \nabla \varphi(y)|}{|x-y|} n^\epsilon(x) n^\epsilon(y) dx dy \right| \\ &\leq \frac{1}{4\pi} |\nabla^2 \varphi|_\infty M_+^2 \leq \frac{1}{4\pi} \frac{C_\varphi}{\delta} M_+^2. \end{aligned}$$

The treatment of term *III* is similar to the one we gave above.

Next we address the proof of Lemma 3.3. The main goal is to calculate time evolution of the variation

$$V_+(t) := \int_{\mathbb{R}_+^2} |x_2 - y_+(t)|^2 n(x, t) dx.$$

We again use *C* to denote constants which may change from line to line but are independent of δ .

The first obstacle is that we cannot choose $|x_2 - y_+|^2$ as a test function due to the fact that $y_+(t)$ depends on the solution. However, by the definition of y_+ we can expand the V_+ -integrand, ending up with the usual

$$V_+(t) = \int_{\mathbb{R}_+^2} |x_2|^2 n_+(x, t) dx - M_+ y_+^2(t). \tag{3.12}$$

Since we already know y_+ , it is enough to calculate the $\int_{\mathbb{R}_+^2} |x_2|^2 n(x, t) dx$. For simplicity, we plug $|x_2|^2$ inside the weak formulation (2.2) and (3.12) to get the time evolution of V_+ . Of course, what one really does is to use a test function to approximate the $|x_2|^2$. Furthermore, when we use the weak formulation, we formally integrated by part twice, but since the value and the first derivative of the function $|x_2|^2$ are zero on the boundary, we will not create extra dangerous boundary term.

First combining (2.2) and (3.12) yields

$$\begin{aligned} \frac{d}{dt} V_+ &= \frac{d}{dt} \int_{\mathbb{R}_+^2} |x_2|^2 n_+(x, t) dx - M_+ \frac{d}{dt} y_+^2(t) \\ &= \int_{\mathbb{R}_+^2} \Delta \varphi n_+ dx \\ &\quad - \frac{1}{4\pi} \int_{\mathbb{R}_+^2 \times \mathbb{R}_+^2} \frac{(\nabla \varphi(x) - \nabla \varphi(y)) \cdot (x - y)}{|x - y|^2} n_+(x, t) n_+(y, t) dx dy \\ &\quad + \int_{\mathbb{R}_+^2} \nabla c_- n_+ \nabla \varphi dx + \int_{\mathbb{R}_+^2} \nabla \varphi \cdot \mathbf{b} n_+ dx - \frac{d}{dt} (M_+ (y_+)^2) \\ &= I + II + III + IV - M_+ \frac{d}{dt} y_+^2(t). \end{aligned} \tag{3.13}$$

Next we estimate every term on the right hand side of (3.13). The first two terms are estimated as follows:

$$|I| = \left| \int_{\mathbb{R}_+^2} \Delta(x_2^2) n_+ dx \right| = 2M_+, \tag{3.14}$$

and

$$|II| = \left| -\frac{1}{4\pi} \int_{\mathbb{R}_+^2 \times \mathbb{R}_+^2} \frac{(\nabla(x_2^2) - \nabla(y_2^2)) \cdot (x - y)}{|x - y|^2} n_+(x, t) n_+(y, t) dx dy \right|$$

$$\leq \frac{1}{4\pi} \int_{\mathbb{R}_+^2 \times \mathbb{R}_+^2} 2n_+(x)n_+(y)dx dy \leq \frac{1}{2\pi} M_+^2. \tag{3.15}$$

Now for the third term in (3.13), we use the previous upper-bound (3.9), $|\partial_{x_2} c_-(x)| \leq \frac{1}{2\pi} \frac{M_+}{|x_2^2|}$, which yields

$$|III| = \left| \int_{\mathbb{R}_+^2} \partial_{x_2} c_{-n_+} 2x_2 dx \right| \leq \frac{1}{2\pi} \int_{\mathbb{R}_+^2} 2|x_2| \frac{M_+}{|x_2|} n_+ dx \leq \frac{1}{\pi} M_+^2. \tag{3.16}$$

Note that for the term *II* and *III*, we only estimate them formally above, one can prove the estimates explicitly using the same techniques as the one in Remark 3.1. For the *IV* term in (3.13), we use the (3.12) again to obtain

$$|IV| = \left| \int_{\mathbb{R}_+^2} \nabla x_2^2 \cdot \mathbf{b} n_+ dx \right| = 2A \left| \int_{\mathbb{R}_+^2} x_2^2 n_+ dx \right| = 2A(V_+ + M_+ y_+^2). \tag{3.17}$$

Collecting equation (3.13) and all the estimates (3.14), (3.15), (3.16) and (3.17) above, we have the following differential inequality:

$$\frac{d}{dt} \left(\frac{1}{M_+} V_+(t) + y_+^2(t) \right) \leq C(1 + M_+) + 2A \left(\frac{1}{M_+} V_+(t) + y_+^2(t) \right),$$

which yields

$$\frac{1}{M_+} V_+(t) + y_+^2(t) \leq C \left(1 + \frac{1}{M_+} \right) \delta^2 e^{2At} + \frac{1}{M_+} V_+(0) e^{2At} + y_+^2(0) e^{2At}. \tag{3.18}$$

Combining Lemma 3.2 with (3.18) yields

$$\begin{aligned} \frac{1}{M_+} V_+(t) + \left[(y_+(0) - C\delta) e^{At} \right]^2 &\leq \frac{1}{M_+} V_+(t) + y_+^2(t) \\ &\leq C \left(1 + \frac{1}{M_+} \right) \delta^2 e^{2At} \\ &\quad + \frac{1}{M_+} V_+(0) e^{2At} + y_+^2(0) e^{2At}. \end{aligned}$$

By collecting similar terms, we finally have

$$\frac{1}{M_+} V_+(t) \leq \left[2C\delta y_+(0) + C \left(1 + \frac{1}{M_+} \right) \delta^2 + \frac{1}{M_+} V_+(0) \right] e^{2At}, \tag{3.19}$$

which completes the proof of Lemma 3.3. □

Equipped with the estimate on $V_+(t)$, we can now conclude the proof of Lemma 3.1.

Proof. (Lemma 3.1) Once $0 < \eta \ll 1$ was fixed, we can clearly choose a small enough δ such that by (3.5) $_\delta$, there holds

$$V_+(t) \leq (1 + \eta)V_+(0)e^{2At}. \tag{3.20}$$

Now recalling that $R = y_+(0)\sqrt{\frac{M_+}{2V_+(0)}} > 1$, then we can use (3.4), (3.20) and further choose δ small enough to get

$$\begin{aligned} y_+(t) - \frac{R}{1 + \eta}\sqrt{\frac{2V_+(t)}{M_+}} &\geq \left[y_+(0) - C\delta - \frac{R}{\sqrt{1 + \eta}}\sqrt{\frac{2V_+(0)}{M_+}} \right] e^{At} \\ &= \left[\left(1 - \frac{1}{\sqrt{1 + \eta}} \right) y_+(0) - C\delta \right] e^{At} \\ &\geq 2\delta e^{At} \geq 2\delta. \end{aligned}$$

Thus, the ‘thin’ δ -strip along the x_1 -axis, $\mathcal{S}_\delta := \{(x_1, x_2) | 0 \leq x_2 \leq 2\delta\}$, lies *outside* the strip centered around $y_+(t)$, uniformly in time,

$$\mathcal{S}_\delta \subset \{(x_1, x_2) \mid |x_2 - y_+(t)| > R_\eta(t)\}, \quad R_\eta(t) := \frac{R}{1 + \eta}\sqrt{\frac{2V_+(t)}{M_+}}.$$

It follows that thanks to our choice of δ , the mass inside the δ -strip \mathcal{S}_δ does not exceed

$$\begin{aligned} \int_{\mathcal{S}_\delta} n_+(x, t) dx &\leq \int_{\mathbb{R}_+^2 \cap \{|x_2 - y_+| > R_\eta\}} n_+(x, t) dx \\ &\leq \int_{\mathbb{R}_+^2 \cap \{|x_2 - y_+| > R_\eta\}} n_+(x, t) \frac{|x_2 - y_+|^2}{|x_2 - y_+|^2} dx \\ &\leq \frac{(1 + \eta)^2}{R^2 2V_+/M_+} \int_{\mathbb{R}_+^2} n_+(x, t) |x_2 - y_+|^2 dx = \frac{(1 + \eta)^2}{R^2 2V_+/M_+} V_+ \\ &\leq \frac{(1 + \eta)^2}{2R^2} M_+. \end{aligned}$$

By symmetry, the mass inside the symmetric δ -strip, $\{(x_1, x_2) \mid |x_2| \leq 2\delta\}$ is smaller than $\frac{(1 + \eta)^2}{2R^2} 2M_+ = \frac{(1 + \eta)^2}{2R^2} M$, uniformly in time, which completes the proof of Lemma 3.1. \square

Remark 3.2. One can do a similar computation to get the evolution for the higher moment estimates $\int_{\mathbb{R}_+^2} n(x, t) |x|^{2k} dx$, and derive similar results to Lemma 3.1.

3.3. Step 2—Proof of the Main Theorem Under the Constrained Setup (3.3)

With the Lemma 3.1 at our disposal, we can now turn to the proof of theorem 3.1 along the lines of [8]. Note that the actual calculation are to be carried out with the regularized solutions n^ϵ of (2.3), though for the sake of simplicity, we only do the formal calculation on $n(x) = n(\cdot, t)$.

The key is to use the logarithmic Hardy–Littlewood–Sobolev inequality to get a bound on the entropy $S[n]$.

Theorem 3.2. (Logarithmic Hardy–Littlewood–Sobolev Inequality) [9] *Let f be a nonnegative function in $L^1(\mathbb{R}^2)$ such that $f \log f$ and $f \log(1 + |x|^2)$ belong to $L^1(\mathbb{R}^2)$. If $\int_{\mathbb{R}^2} f dx = M$, then*

$$\int_{\mathbb{R}^2} f \log f dx + \frac{2}{M} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x)f(y) \log |x - y| dx dy \geq -C(M) \tag{3.21}$$

with $C(M) := M(1 + \log \pi - \log M)$.

Remark 3.3. It is pointed out in [8] that by multiplying f by indicator functions, one can prove that the inequality (3.21) remains true with \mathbb{R}^2 replaced by any bounded domains $\mathcal{D} \subset \mathbb{R}^2$.

The idea of the proof goes as follows. By observing that the mass in the upper half plane and lower half plane are subcritical ($|n_\pm|_1 < 8\pi$), we plan to use the logarithmic Hardy–Littlewood–Sobolev inequality on these sub-domains to get uniform bound on the entropy. However, without extra information concerning the cell density distribution, naive application of logarithmic Hardy–Littlewood–Sobolev inequality fails. For this approach to work, the density distribution constraint required is that the cells in the upper and lower half plane are well-separated by a ‘cell clear strip’ in which the total number of cells is sufficiently small. The strip is constructed through applying Lemma 3.1. Combining the logarithmic Hardy–Littlewood–Sobolev inequality and the cell separation constraint, we can use a ‘total entropy reconstruction’ trick introduced in [8] to obtain entropy bound. Now let’s start the whole proof.

Proof of Theorem 3.1. According to propositions 2.1, 2.2, the life-span of free-energy solution is determined by the finite bound on the entropy $S[n](t)$. To this end, we decompose the free energy into three parts,

$$\begin{aligned} E[n] &\equiv \left(1 - \frac{K}{8\pi}\right) \int_{\mathbb{R}^2} n \log n dx \\ &\quad + \frac{1}{8\pi} \left(K \int_{\mathbb{R}^2} n \log n dx \right. \\ &\quad \left. + 2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} n(x)n(y) \log |x - y| dx dy \right) - \int_{\mathbb{R}^2} H n dx \\ &=: \left(1 - \frac{K}{8\pi}\right) S[n] + \mathcal{I}_1 - \mathcal{I}_2. \end{aligned} \tag{3.22}$$

Recall that since the free energy $E[n](t)$ in (1.4) is decreasing,

$$\left(1 - \frac{K}{8\pi}\right) S[n] \equiv E[n] - \mathcal{I}_1 + \mathcal{I}_2 \leq E[n_0] - \mathcal{I}_1 + \mathcal{I}_2,$$

then the desired entropy bound and hence a finite entropy solution follow provided we show the existence of a constant $K < 8\pi$, for which $\mathcal{I}_1 = \mathcal{I}_1(K)$ is bounded from below and \mathcal{I}_2 is bounded from above. Our main task is estimating $\mathcal{I}_1(K)$ from below, which is further decomposed into three terms

$$\begin{aligned} \mathcal{I}_1(K) &= K \int_{\mathbb{R}^2} n \log^+ n dx + 2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} n(x)n(y) \log |x - y| dx dy \\ &\quad - K \int_{\mathbb{R}^2} n \log^- n dx \\ &=: K\mathcal{I}_{11} + \mathcal{I}_{12} + K\mathcal{I}_{13}. \end{aligned}$$

To estimate the various terms, we first construct the 'cell clear strip', Γ_0 shown in Fig. 1,

$$\Gamma_+ := \{x_2 \mid x_2 > 2\delta\}, \quad \Gamma_- := \{x_2 \mid x_2 < -2\delta\}, \quad \Gamma_0 := \{x_2 \mid |x_2| \leq 2\delta\}. \quad (3.23)$$

Thus, region Γ_+ contains points in the upper half plane which are 2δ away from the x_1 axis, whereas region Γ_- contains points in the lower half plane with the same property. Region Γ_0 is a closed 2δ -neighborhood of the x_1 -axis. The δ neighborhood of the Γ_+, Γ_- region is denoted as follows:

$$\Gamma_+^{(\delta)} = \{x_2 \mid x_2 > \delta\}, \quad \Gamma_-^{(\delta)} = \{x_2 \mid x_2 < -\delta\}. \quad (3.24)$$

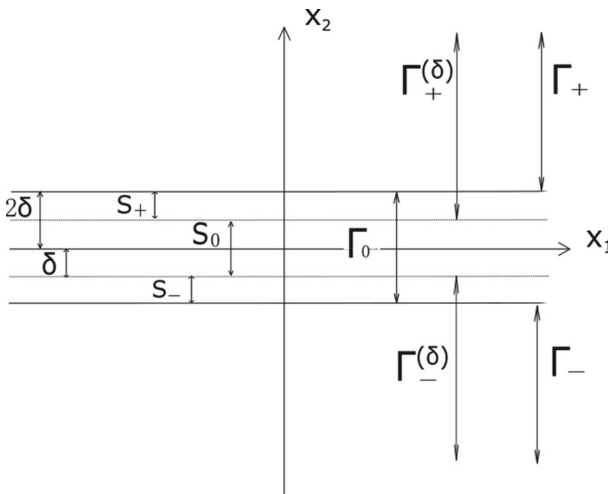


Fig. 1. Regions $\Gamma_+, \Gamma_-, \Gamma_0$

In the sequel, we will further decompose Γ_0 into subdomains

$$S_+ = \{x_2 \mid \delta < x_2 \leq 2\delta\}, \quad S_- = \{x_2 \mid -\delta > x_2 \geq -2\delta\}, \quad S_0 = \{x_2 \mid |x_2| \leq \delta\}. \tag{3.25}$$

Estimating $K\mathcal{I}_{11} + \mathcal{I}_{12}$ from below. We fix K as

$$K := \max \left\{ \int_{\Gamma_+^{(\delta)}} n dx + \int_{\Gamma_0} n dx, \int_{\Gamma_-^{(\delta)}} n dx + \int_{\Gamma_0} n dx \right\}, \tag{3.26}$$

and claim that K is admissible. Indeed, lemma 3.1 with our choice of R^2 assumed large enough so that (3.3) holds implies that the total mass inside Γ_0 does not exceed

$$\int_{\Gamma_0} n dx = \int_{\{x_2 \mid |x_2| \leq 2\delta\}} n dx \leq \frac{(1 + \eta)^2}{2R^2} M \leq 8\pi - \frac{M}{2}. \tag{3.27}$$

Therefore, the Γ_0 strip is the 'cell clear strip'; moreover, since the upper-half plane mass equals $\frac{M}{2}$, the admissibility of K follows, $K \leq \frac{M}{2} + \frac{(1 + \eta)^2}{2R^2} M < 8\pi$.

The motivation for this choice of K comes from the straightforward bound

$$\begin{aligned} & \int_{\Gamma_+^{(\delta)}} n dx \int_{\Gamma_+^{(\delta)}} n \log^+ n dx \\ & + \int_{\Gamma_-^{(\delta)}} n dx \int_{\Gamma_-^{(\delta)}} n \log^+ n dx + \int_{\Gamma_0} n dx \int_{\Gamma_0} n \log^+ n dx \\ & \leq \int_{\Gamma_+^{(\delta)}} n dx \int_{\mathbb{R}_+^2} n \log^+ n dx + \int_{\Gamma_-^{(\delta)}} n dx \int_{(\mathbb{R}_+^2)^c} n \log^+ n dx \\ & + \int_{\Gamma_0} n dx \int_{\mathbb{R}^2} n \log^+ n dx \\ & \leq \max \left\{ \int_{\Gamma_+^{(\delta)}} n dx + \int_{\Gamma_0} n dx, \int_{\Gamma_-^{(\delta)}} n dx + \int_{\Gamma_0} n dx \right\} \cdot \int_{\mathbb{R}^2} n \log^+ n dx =: K\mathcal{I}_{11}. \end{aligned} \tag{3.28}$$

Indeed, we proceed along the lines of [8], appealing to the Logarithmic Hardy–Littlewood–Sobolev inequality in the three regions $\Gamma_+^{(\delta)}$, $\Gamma_-^{(\delta)}$, Γ_0 , obtaining

$$\begin{aligned} & \int_{\Gamma_+^{(\delta)}} n(x) dx \int_{\Gamma_+^{(\delta)}} n \log^+ n dx + 2 \iint_{\Gamma_+^{(\delta)} \times \Gamma_+^{(\delta)}} n(x)n(y) \log |x - y| dx dy \geq C, \\ & \int_{\Gamma_-^{(\delta)}} n(x) dx \int_{\Gamma_-^{(\delta)}} n \log^+ n dx + 2 \iint_{\Gamma_-^{(\delta)} \times \Gamma_-^{(\delta)}} n(x)n(y) \log |x - y| dx dy \geq C, \\ & \int_{\Gamma_0} n(x) dx \int_{\Gamma_0} n \log^+ n dx + 2 \iint_{\Gamma_0 \times \Gamma_0} n(x)n(y) \log |x - y| dx dy \geq C. \end{aligned}$$

We now sum these three inequalities. By (3.28), the sum of their first three terms does not exceed $K\mathcal{I}_{11}$; bookkeeping the overlap of the three domains $\Gamma_+^{(\delta)} \times \Gamma_+^{(\delta)}$, $\Gamma_-^{(\delta)} \times$

$\Gamma_-^{(\delta)}$ and $\Gamma_0 \times \Gamma_0$, consult Fig. 3 we find the sum of the second three terms is \mathcal{I}_{12} modulo the correction $\mathcal{I}_{121} - \mathcal{I}_{122}$ below,

$$\begin{aligned} K\mathcal{I}_{11} + \mathcal{I}_{12} &\geq -C + 4 \iint_{((\Gamma_+^{(\delta)})^c \times \Gamma_+) \cup (\Gamma_- \times (S_+ \cup S_0))} n(x)n(y) \log|x - y| dx dy \\ &\quad - 2 \iint_{(S_+ \times S_+) \cup (S_- \times S_-)} n(x)n(y) \log|x - y| dx dy \tag{3.29} \\ &=: -C + \mathcal{I}_{121} - \mathcal{I}_{122}. \end{aligned}$$

Next applying the fact that $|x - y| \geq \delta$ for all (x, y) in the integral domain of \mathcal{I}_{121} , we estimate the \mathcal{I}_{122} and \mathcal{I}_{122} ,

$$\begin{aligned} \mathcal{I}_{121} &\geq 4M^2 \log \delta, \\ \mathcal{I}_{122} &\leq 2 \iint_{(S_+ \times S_+) \cup (S_- \times S_-)} n(x)n(y) \log^+ |x - y| dx dy \\ &\leq C \iint_{(S_+ \times S_+) \cup (S_- \times S_-)} n(x)n(y)(1 + |x|^2 + |y|^2) dx dy \\ &\leq C(M^2 + 2M \int n(x)|x|^2 dx). \tag{3.30} \end{aligned}$$

Combining (3.30) with (3.29) yields $K\mathcal{I}_{11} + \mathcal{I}_{12} \geq 4M^2 \log \delta - C(1 + M^2 + M \int n|x|^2 dx)$.

Estimating \mathcal{I}_{13} from below. We recall the well-known upper bound on the negative part of the entropy, [7, 8], stating that for f positive function,

$$\begin{aligned} \int_{\mathbb{R}^2} f \log^- f dx &\leq \frac{1}{2} \int_{\mathbb{R}^2} |x|^2 f dx + \log(2\pi) \int_{\mathbb{R}^2} f dx + \frac{1}{e} \\ &\leq C \left(1 + M + \int n|x|^2 dx \right). \tag{3.31} \end{aligned}$$

Combining (3.31) with our previous lower-bound of $K\mathcal{I}_{11} + \mathcal{I}_{12}$ yields

$$\begin{aligned} \mathcal{I}_1(K) &= K\mathcal{I}_{11} + \mathcal{I}_{12} + K\mathcal{I}_{13} \\ &\geq 4M^2 \log \delta - C(M + K) \left(1 + M + \int n|x|^2 dx \right). \tag{3.32} \end{aligned}$$

It remains to upper-bound the \mathcal{I}_2 term in (3.22). Since $\left| \int Hn dx \right| \leq A \int n|x|^2 dx$, it is therefore suffices to show that the second moment of n is bounded for any finite time. Indeed, the time evolution of the second moment can be estimated as follows:

$$\frac{d}{dt} \int n|x|^2 dx \leq 4AM + 4A \int Hn(x) dx \leq 4AM + \frac{A^2}{2} \int n|x|^2 dx,$$

and Gronwall inequality yields the finite bound $|\mathcal{I}_2| \leq \int n(\cdot, t)|x|^2 dx \leq C(A, t) < \infty$.

Finally, equipped with the lower bound on $\mathcal{I}_1(K)$ with $K < 8\pi$ and with the upper bound on \mathcal{I}_2 we revisit (3.23) to conclude that there exists a constant $C = C(M, A, T) < \infty$ such that the entropy $S[n(\cdot, t)]$ is uniformly bounded (independent of ϵ) for any finite time interval $t \in [0, T]$,

$$S[n](t) \leq \frac{1}{(1 - \frac{K}{8\pi})} \left(E[n_0] + C - \frac{1}{2\pi} M^2 \log \delta \right), \quad \forall t \in [0, T]. \tag{3.33}$$

Now by the Propositions 2.1, 2.2, we have that the free energy solution exists on any time interval $[0, T]$, $\forall T < \infty$.

3.4. Step 3—Proof of the Main Theorem

In the proof of Theorem 3.1, we see that the cell population is separated by a ‘cell clear zone’ near the x_1 axis. Since total mass in the ‘‘cell clear zone’’ is small, we can heuristically treat the total cell population as a union of two subgroups with subcritical mass ($< 8\pi$). However, since we lack sufficiently good control over the total number of cells near the x_1 axis, we cannot use this idea to prove the optimal result as stated in Theorem 1.1. The idea of proving Theorem 1.1 is that instead of considering the total cell population as the union of two subgroups separated by one fixed ‘cell clear zone’, we treat it as the union of three subgroups with subcritical mass, namely, the cells in the upper half plane, the lower half plane and the neighborhood of the x_1 axis, respectively. These three subgroups of cells are separated by two ‘cell clear zones’ varying in time.

The main difficulty in the proof is setting up the three new regions such that:

1. mass inside each region is smaller than 8π ;
2. the total mass of cells near their boundaries is well-controlled.

Once the construction is completed, the remaining steps will be similar to step 2.

Proof of Theorem 1.1. We start by constructing the three regions. First we note that the Lemma 3.1 implies that there exists $\delta > 0$ such that the following estimate is satisfied for a fixed $R > 1$ and η chosen small enough:

$$\int_{|x_2| \leq 2\delta} n dx \leq \frac{(1 + \eta)^2}{R^2} \int_{\mathbb{R}^2} n dx \leq \frac{1}{2} M, \quad \forall t > 0. \tag{3.34}$$

Now the region $L = \{(x_1, x_2) \mid |x_2| \leq 2\delta\}$ have total mass less than $\frac{1}{2} M = M_+ < 8\pi$ for all time.

Secondly, we subdivide the region L into J pieces:

$$L = \bigcup_1^J L^i, \quad L^i := \left\{ (x_1, x_2) \mid \frac{2\delta}{J}(i) > |x_2| \geq \frac{2\delta}{J}(i - 1) \right\}.$$

Here $J = J(M) \geq 10$, to be determined later, depends on M . By the pigeon hole principle, there is at least three strips L^i such that

$$\int_{L^i} n(x) dx \leq \frac{2}{J} M_+.$$

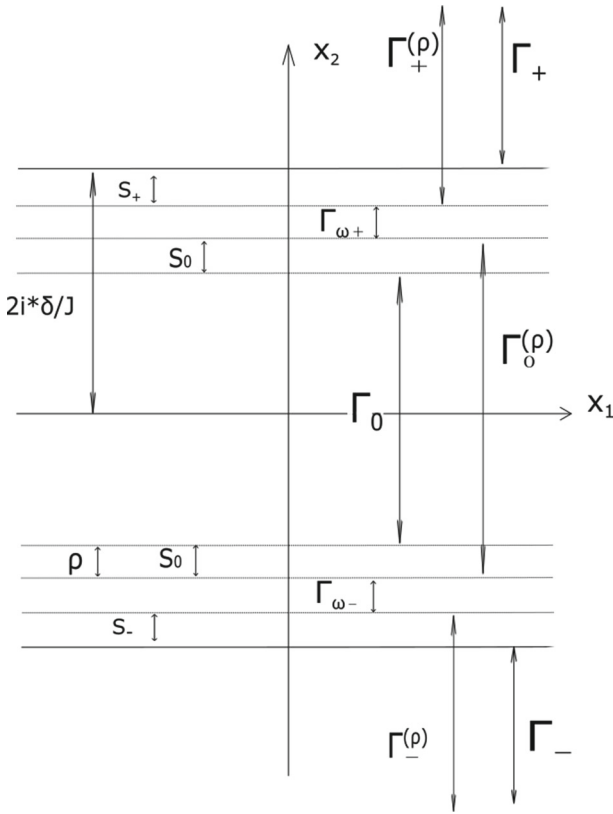


Fig. 2. Regions Γ_+ , Γ_- , Γ_0 and Γ_ω in the proof of the main theorem

Suppose there are only two strips with mass smaller than $\frac{2}{J}M_+$, then total mass in L will be bigger than $(J - 2)\frac{2}{J}M_+ > M_+$, which is a contradiction. Now we pick from these three strips the one which is neither L^1 nor L^J . As a result, this strip L^i does not touch the x_1 axis nor the boundary of L . We denote this i by i^* . The L^{i^*} is the ‘cell clear zone’. Notice that here $i^* = i^*(n, t)$ depends on time.

Finally, we use this i^* to define the regions depicted in Fig. 2. First we define the three regions, each of which has total mass smaller than 8π :

$$\Gamma_+ = \left\{ x_2 \geq \frac{2\delta}{J}i^* \right\}, \quad \Gamma_- = \left\{ x_2 \leq -\frac{2\delta}{J}i^* \right\}, \quad \Gamma_0 = \left\{ |x_2| \leq \frac{2\delta}{J}(i^* - 1) \right\}.$$

Next we set

$$\rho = \frac{2\delta}{3J},$$

and define the ρ neighborhood of the above three regions as follows:

$$\Gamma_+^{(\rho)} = \left\{ x_2 > \frac{2\delta}{J} \left(i^* - \frac{1}{3} \right) \right\}, \quad \Gamma_-^{(\rho)} = \left\{ x_2 < -\frac{2\delta}{J} \left(i^* - \frac{1}{3} \right) \right\},$$

$$\Gamma_0^{(\rho)} = \left\{ |x_2| < \frac{2\delta}{J} \left(i^* - \frac{2}{3} \right) \right\}. \tag{3.35}$$

Now we define the complement Γ_ω of the above three regions $\Gamma_i^{(\rho)}, i = +, -, 0$:

$$\Gamma_\omega = \left\{ \frac{2\delta}{J} \left(i^* - \frac{2}{3} \right) \leq |x_2| \leq \frac{2\delta}{J} \left(i^* - \frac{1}{3} \right) \right\} = \Gamma_{\omega+} \cup \Gamma_{\omega-}, \quad \Gamma_{\omega\pm} = \Gamma_\omega \cap \mathbb{R}_\pm^2.$$

Now we define the complement $\Gamma_\omega^{(\rho)}$ of $\cup_{i \in \{\pm, 0\}} \Gamma_i$ and decompose it into subdomains:

$$\begin{aligned} \Gamma_\omega^{(\rho)} &= (\cup_{i \in \{\pm, 0\}} \Gamma_i)^c = (\Gamma_{\omega+}^{(\rho)}) \cup (\Gamma_{\omega-}^{(\rho)}), \\ \Gamma_{\omega\pm}^{(\rho)} &= \Gamma_\omega^{(\rho)} \cap \mathbb{R}_\pm^2, \\ \Gamma_\omega^{(\rho)} &= \Gamma_\omega \cup S_+ \cup S_- \cup S_0, \\ S_+ &= \Gamma_+^{(\rho)} \setminus \Gamma_+, \quad S_- = \Gamma_-^{(\rho)} \setminus \Gamma_-, \quad S_0 = \Gamma_0^{(\rho)} \setminus \Gamma_0. \end{aligned}$$

Remark 3.4. It is important to notice that the regions we are constructing are changing with respect to the given time t . Therefore, by doing the argument below, we can only show that the entropy is bounded at time t , but since t is an arbitrary finite time, we have the bound on entropy for $\forall t \in [0, T], \forall T < \infty$.

We start estimating the entropy. By the free energy dissipation, we obtain

$$\begin{aligned} E[n_0] &\geq \left(1 - \frac{K}{8\pi} \right) \int_{\mathbb{R}^2} n \log n dx \\ &\quad + \frac{1}{8\pi} \left(K \int_{\mathbb{R}^2} n \log^+ n dx + 2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} n(x)n(y) \log |x - y| dx dy \right) \\ &\quad - \left(\frac{K}{8\pi} \int_{\mathbb{R}^2} n \log^- n dx + \int_{\mathbb{R}^2} H n dx \right) \\ &=: \left(1 - \frac{K}{8\pi} \right) S[n(T)] + \mathcal{I}_1(K) - \mathcal{I}_2(K). \end{aligned} \tag{3.36}$$

To derive entropy bound, we need to estimate \mathcal{I}_1 from below for $K < 8\pi$ and estimate \mathcal{I}_2 from above. We start by estimating \mathcal{I}_1 . Combining the definition of L^{i^*} and (3.34) yields

$$\int_{\Gamma_+^{(\rho)}} n dx \leq M_+ = \frac{1}{2} M < 8\pi, \tag{3.37}$$

$$\int_{\Gamma_-^{(\rho)}} n dx \leq M_+ = \frac{1}{2} M < 8\pi, \tag{3.38}$$

$$\int_{\Gamma_0^{(\rho)}} n dx \leq M_+ = \frac{1}{2} M < 8\pi, \tag{3.39}$$

$$\int_{\Gamma_\omega^{(\rho)}} n dx \leq \int_{L^{i^*}} n dx \leq \frac{2}{J} (M_+). \tag{3.40}$$

Now by the log-Hardy–Littlewood–Sobolev inequality (3.21), we have that

$$\begin{aligned} & \int_{\Gamma_i^{(\rho)}} n(x) dx \int_{\Gamma_i^{(\rho)}} n(x) \log^+ n(x) dx \\ & + 2 \iint_{\Gamma_i^{(\rho)} \times \Gamma_i^{(\rho)}} n(x)n(y) \log |x - y| dx dy \geq -C, \quad i = +, -, 0, \\ & \int_{\Gamma_{\omega^\pm}^{(\rho)}} n(x) dx \int_{\Gamma_{\omega^\pm}^{(\rho)}} n(x) \log^+ n(x) dx \\ & + 2 \iint_{\Gamma_{\omega^\pm}^{(\rho)} \times \Gamma_{\omega^\pm}^{(\rho)}} n(x)n(y) \log |x - y| dx dy \geq -C. \end{aligned}$$

Same as in Section 3.3, we recall the definition of \mathcal{I}_1 (3.36), and use the estimates above to reconstruct the entropy and the potential on the whole \mathbb{R}^2 as follows:

$$\begin{aligned} -C & \leq \left(K \int_{\mathbb{R}^2} n(x) \log^+ n(x) dx + 2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} n(x)n(y) \log |x - y| dx dy \right) \\ & - 2 \iint_{\Omega} n(x)n(y) \log |x - y| dx dy \\ & + 2 \iint_{(S_+ \times S_+) \cup (S_- \times S_-) \cup (S_0^+ \times S_0^+) \cup (S_0^- \times S_0^-)} n(x)n(y) \log |x - y| dx dy \\ & =: 8\pi \mathcal{I}_1 - \mathcal{I}_3 + \mathcal{I}_4. \end{aligned} \tag{3.41}$$

The region Ω and the integral domain of \mathcal{I}_4 are indicated in Fig. 3.³ The K in (3.41) can be estimated using (3.40) as follows:

$$K := M_+ + \int_{\Gamma_\omega^{(\rho)}} n dx \leq \left(1 + \frac{2}{J} \right) M_+. \tag{3.42}$$

By the assumption $M_+ < 8\pi$, we can make J big such that $K < 8\pi$. This is where we choose the $J = J(M)$.

Applying the fact that $|x - y| \geq \frac{2\delta}{3J}, \forall (x, y) \in \Omega$, the \mathcal{I}_3 and \mathcal{I}_4 terms in (3.41) can be estimated as follows:

$$\begin{aligned} \mathcal{I}_3 & \geq CM^2 \log \frac{2\delta}{3J}, \\ \mathcal{I}_4 & \leq 2 \iint_{(S_+ \times S_+) \cup (S_- \times S_-) \cup (S_0^+ \times S_0^+) \cup (S_0^- \times S_0^-)} n(x)n(y) \log^+ |x - y| dx dy \\ & \leq C \iint_{(S_+ \times S_+) \cup (S_- \times S_-) \cup (S_0^+ \times S_0^+) \cup (S_0^- \times S_0^-)} n(x)n(y) (1 + |x|^2 + |y|^2) dx dy \end{aligned}$$

³ Region Ω is the union of the following nine regions, ((1) – (9)):

- 1) $\Gamma_+ \times (\Gamma_+^{(\rho)})^c$, 2) $S_+ \times (\Gamma_+ \cup \Gamma_{\omega^+}^{(\rho)})^c$, 3) $\Gamma_{\omega^+} \times (\Gamma_{\omega^+}^{(\rho)})^c$,
- 4) $S_0^+ \times (\Gamma_0^{(\rho)} \cup \Gamma_{\omega^+}^{(\rho)})^c$, 5) $\Gamma_0 \times (\Gamma_0^{(\rho)})^c$, 6) $S_0^- \times (\Gamma_0^{(\rho)} \cup \Gamma_{\omega^-}^{(\rho)})^c$,
- 7) $\Gamma_{\omega^-} \times (\Gamma_{\omega^-}^{(\rho)})^c$, 8) $S_- \times (\Gamma_-^{(\rho)} \cup \Gamma_{\omega^-}^{(\rho)})^c$, 9) $\Gamma_- \times (\Gamma_-^{(\rho)})^c$.

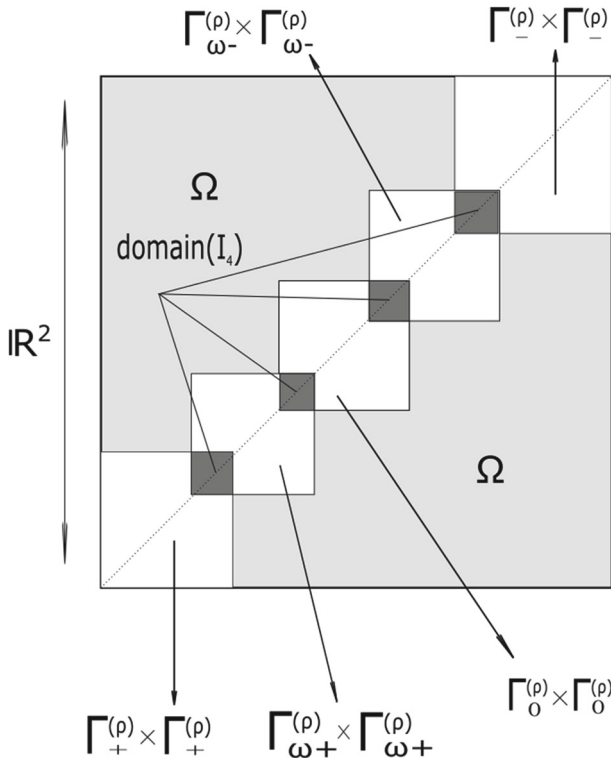


Fig. 3. Region Ω in $\mathbb{R}^2 \times \mathbb{R}^2$

$$\leq C(M^2 + M \int n|x|^2 dx). \tag{3.43}$$

Combining (3.41), (3.42) and (3.43) yields

$$\mathcal{I}_1 \geq CM^2 \log \frac{2\delta}{3J} - C(1 + M^2 + M \int n|x|^2 dx). \tag{3.44}$$

We estimate the \mathcal{I}_2 term in (3.36) using (3.31) and the second-moment bound of n as in the proof of (3.33):

$$\mathcal{I}_2(t) \leq C(M, A, T) < \infty, \quad \forall t \leq T. \tag{3.45}$$

Combining (3.36), (3.44), (3.45) and the second moment bound of n , we obtain that

$$S[n](T) \leq \frac{1}{(1 - \frac{K}{8\pi})} \left(E[n_0] + C(M, A, T) - CM^2 \log \frac{2\delta}{3J} \right) < \infty, \quad \forall T < \infty. \tag{3.46}$$

Once the entropy is bounded for any finite time, the existence is guaranteed by Propositions 2.1 and 2.2.

Appendix A.

In the appendix, we prove the two local existence theorems stated in section 2.2. The proof follows along the same line, as the analysis in [8].

Appendix A.1. Proof of Proposition 2.1

Proof. For any fixed positive ϵ , following the argument as in section 2.5 of [7], we obtain the global solution in $L^2([0, T], H^1) \cap C([0, T], L^2)$ for the regularized Keller–Segel system with advection (2.3).

The goal is to use the Aubin–Lions Lemma to show that the solutions $\{n^\epsilon\}_{\epsilon \geq 0}$ is precompact in certain topology. As in the paper [8], we divide the proof into steps.

Step 1. A priori estimates on n^ϵ and c^ϵ . In this step, we derive several estimates which we will need later.

First we estimate the second moment

$$V := \int_{\mathbb{R}^2} n^\epsilon |x|^2 dx.$$

The time evolution of V can be estimated using the regularised equation (2.3) and the fact that the gradient of the regularized kernel K^ϵ is bounded $|\nabla K^\epsilon(z)| \leq \frac{1}{2\pi|z|}$:

$$\begin{aligned} \frac{d}{dt} V &= 4M + \iint_{\mathbb{R}^2 \times \mathbb{R}^2} n^\epsilon(x, t) n^\epsilon(y, t) (x - y) \nabla K^\epsilon(x - y) dx dy \\ &\quad + 2 \int x \cdot \mathbf{b} n^\epsilon(x) dx \\ &\leq 4M + 2C \int n^\epsilon |x|^2 dx. \end{aligned}$$

By Gronwall, we have that

$$V(t) \leq \frac{4M}{2C} e^{2CT} + V_0 \leq C_V(T) < \infty, \quad \forall 0 \leq t \leq T, \tag{A.1}$$

from which we obtain that $(1 + |x|^2)n^\epsilon \in L^\infty([0, T], L^1)$ uniformly in ϵ .

Next, we estimate $|n^\epsilon \log n^\epsilon|_{L^\infty([0, T]; L^1)}$ and $|\int n^\epsilon c^\epsilon dx|_{L_t^\infty}$. Combining the assumption of the proposition 2.1 and (3.31) yields

$$\int |n^\epsilon \log n^\epsilon| dx \leq \int n^\epsilon (\log n^\epsilon + |x|^2) dx + 2 \log(2\pi) M + \frac{2}{e}. \tag{A.2}$$

Therefore, we proved that $n^\epsilon \log n^\epsilon \in L^\infty([0, T], L^1)$ uniformly in ϵ . Recalling the boundedness of the second moment (A.1) and the representation of c^ϵ as $c^\epsilon = K^\epsilon * n^\epsilon$, we deduce the following estimate using the Young’s inequality $ab \leq e^{a-1} + b \ln b$ for $\forall a, b \geq 1$:

$$|c^\epsilon|(x) = \left| \int K^\epsilon(x - y) n^\epsilon(y) dy \right| \leq C(M, V, |n^\epsilon \log n^\epsilon|_{L^\infty([0, T], L^1)})$$

$$+ C(M) \log(1 + |x|), \tag{A.3}$$

uniformly in ϵ . Combining this with the mass conservation of n^ϵ and the second moment control (A.1), we deduce that

$$\left| \int n^\epsilon c^\epsilon dx \right|_{L^\infty([0,T])} \leq C, \tag{A.4}$$

where C is independent of ϵ .

Next we derive the main a priori estimate, namely, the $L^2([0, T] \times \mathbb{R}^2)$ estimate of $\sqrt{n^\epsilon} \nabla c^\epsilon$. First we calculate the time evolution of $\int n^\epsilon c^\epsilon dx$:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int n^\epsilon c^\epsilon dx &= \int n_t^\epsilon c^\epsilon dx \\ &= \int (\Delta n^\epsilon - \nabla \cdot (\nabla c^\epsilon n^\epsilon) - b \cdot \nabla n^\epsilon) c^\epsilon dx \\ &= \int n^\epsilon \Delta c^\epsilon dx + \int n^\epsilon |\nabla c^\epsilon|^2 dx + \int \mathbf{b} n^\epsilon \cdot \nabla c^\epsilon dx. \end{aligned}$$

Integrating this in time yields

$$\begin{aligned} &\int_0^T \int n^\epsilon |\nabla c^\epsilon|^2 dx dt \\ &= \frac{1}{2} \int n^\epsilon c^\epsilon dx(T) - \frac{1}{2} \int n^\epsilon c^\epsilon dx(0) \\ &\quad - \int_0^T \int n^\epsilon \Delta c^\epsilon dx dt - \int_0^T \int n^\epsilon \mathbf{b} \cdot \nabla c^\epsilon dx dt. \end{aligned} \tag{A.5}$$

Now we estimate the right hand side of (A.5). We see from (A.4) that the first two terms on the right hand side is bounded. Next we estimate the third term on the right hand side of (A.5), which requires information derived from the entropy bound. By the property that $\nabla \cdot \mathbf{b} = 0$, we formally calculate the time evolution of $S[n](t)$ as

$$\frac{d}{dt} S[n](t) = -4 \int |\nabla \sqrt{n}|^2 dx + \int n^2(t) dx. \tag{A.6}$$

The interested reader is referred to [7] for more details. We need to estimate the second term in (A.6). Before doing this, note that, for $K > 1$,

$$\int_{n \geq K} n dx \leq \frac{1}{\log(K)} \int n_+ \log n dx \leq \frac{C}{\log(K)} =: \eta(K) \tag{A.7}$$

can be made arbitrarily small. Now we can use the Gagliardo-Nirenberg-Sobolev inequality together with (A.7) to estimate the second term in (A.6) as follows:

$$\begin{aligned} \int n^2 dx &\leq MK + \int_{n \geq K} n^2 dx \leq MK + \left(\int_{n \geq K} n dx \right)^{1/2} |n|_3^{3/2} \\ &\leq MK + \eta(K)^{1/2} CM^{1/2} |\nabla \sqrt{n}|_2^2. \end{aligned}$$

Combining this with (A.6) and (A.7), we have

$$\frac{d}{dt}S[n](t) = -(4 - \eta(K)^{1/2}CM^{1/2}) \int |\nabla\sqrt{n}|^2 dx + MK. \tag{A.8}$$

The factor $-(4 - \eta(K)^{1/2}CM^{1/2})$ can be made non-positive for K large enough and therefore we have that

$$\int_0^T \int |\nabla\sqrt{n}|^2 dx dt \leq \frac{S[n](0) - S[n](T) + MKT}{(4 - 2\eta(K)^{1/2}M^{1/2}C)}.$$

It follows that $\nabla\sqrt{n}$ is bounded in $L^2([0, T] \times \mathbb{R}^2)$. The derivation for $|\nabla\sqrt{n^\epsilon}|_{L^2([0, T]; L^2)} \leq C$ is similar but more technical, and the interested readers are referred to [7] for more details. As a consequence of the $L^2([0, T] \times \mathbb{R}^2)$ estimate on $\nabla\sqrt{n^\epsilon}$ and of the computation

$$\frac{d}{dt}S[n^\epsilon](t) = -4 \int |\nabla\sqrt{n^\epsilon}|^2 dx + \int n^\epsilon (-\Delta c^\epsilon) dx,$$

we have the estimate

$$\int_0^T \int n^\epsilon (-\Delta c^\epsilon) dx dt \leq C. \tag{A.9}$$

This completes the treatment of the third term on the right hand side of (A.5). Next we estimate the last term $\int_0^T \int \mathbf{b}n^\epsilon \cdot \nabla c^\epsilon dx dt$ in (A.5). First we calculate the time evolution of $\int Gn^\epsilon dx$ as follows:

$$\begin{aligned} \frac{d}{dt} \int Gn^\epsilon dx &= \int G \nabla \cdot (\nabla n^\epsilon - \nabla c^\epsilon n^\epsilon - \mathbf{b}n^\epsilon) dx \\ &= \int \Delta Gn^\epsilon dx + \int \nabla G \cdot \nabla c^\epsilon n^\epsilon dx + \int |\mathbf{b}|^2 n^\epsilon dx \\ &= \int \mathbf{b} \cdot \nabla c^\epsilon n^\epsilon dx + \int |\mathbf{b}|^2 n^\epsilon dx. \end{aligned}$$

Now, integrating in time, we obtain

$$\begin{aligned} \left| \int_0^T \int \mathbf{b} \cdot \nabla c^\epsilon n^\epsilon dx \right| &\leq \left| \int Gn^\epsilon dx(0) \right| + \left| \int Gn^\epsilon dx(T) \right| \\ &\quad + \left| \int_0^T \int |\mathbf{b}|^2 n^\epsilon dx dt \right|. \end{aligned} \tag{A.10}$$

From the assumption $|\mathbf{b}| \lesssim |x|$, we have that the right hand side of (A.10) can be bounded in terms of the second moment V :

$$\left| \int Gn^\epsilon dx \right| + \left| \int |\mathbf{b}|^2 n^\epsilon dx \right| \leq C \int n^\epsilon |x|^2 dx. \tag{A.11}$$

Since the second moment V is bounded (A.1), we have that

$$\left| \int_0^T \int \mathbf{b} \cdot \nabla c^\epsilon n^\epsilon dx \right| \leq C. \tag{A.12}$$

Applying estimates (A.4), (A.9) and (A.12) to (A.5), we obtain

$$\int_0^T \int n^\epsilon |\nabla c^\epsilon|^2 dx dt \leq C < \infty. \tag{A.13}$$

This concludes our first step.

Step 2- Passing to the limit. As in [8], the following Aubin–Lions compactness lemma is applied:

Lemma A.1. (Aubin–Lions lemma)[8] *Take $T > 0$ and $1 < p < \infty$. Assume that $(f_n)_{n \in \mathbb{N}}$ is a bounded sequence of functions in $L^p([0, T]; H)$ where H is a Banach space. If $(f_n)_n$ is also bounded in $L^p([0, T]; V)$ where V is compactly imbedded in H and $\partial f_n / \partial t \in L^p([0, T]; W)$ uniformly with respect to $n \in \mathbb{N}$ where H is imbedded in W , then $(f_n)_{n \in \mathbb{N}}$ is relatively compact in $L^p([0, T]; H)$.*

Our goal now is to find the appropriate spaces V, H, W for n^ϵ . We subdivide the proof into steps, each step determines one space in the lemma.

Space H: Bound on $|n^\epsilon|_{L^2([0, T], L^2)}$: We can estimate the $|n^\epsilon|_2^2$ by applying the following decomposition trick:

$$n^\epsilon = (n^\epsilon - K)_+ + \min \{n^\epsilon, K\}. \tag{A.14}$$

The second part in (A.14) is bounded in $L^p, 1 \leq p \leq \infty$. The first part is bounded in L^1 by

$$|(n^\epsilon - K)_+|_1 \leq \frac{|n^\epsilon \log n^\epsilon|_1}{\log K} = \eta(K), \tag{A.15}$$

which can be made arbitrary small. Now we estimate the time evolution of $\int (n^\epsilon - K)_+^p dx, 1 < p < \infty$ as follows:

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \int (n^\epsilon - K)_+^p dx \\ &= \int (n^\epsilon - K)_+^{p-1} (\Delta n^\epsilon - \nabla \cdot (\nabla c^\epsilon n^\epsilon) - \mathbf{b} \cdot \nabla n^\epsilon) dx \\ &= -\frac{4(p-1)}{p^2} \int |\nabla (n^\epsilon - K)_+^{p/2}|^2 dx - \int (n^\epsilon - K)_+^{p-1} \nabla \cdot (\nabla c^\epsilon n^\epsilon) dx \\ & \quad - \int (n^\epsilon - K)_+^{p-1} \mathbf{b} \cdot \nabla (n^\epsilon - K)_+ dx \\ &\leq -\frac{4(p-1)}{p^2} \int |\nabla (n^\epsilon - K)_+^{p/2}|^2 dx - \frac{p-1}{p} \int (n^\epsilon - K)_+^p \Delta c^\epsilon dx \\ & \quad - K \int (n^\epsilon - K)_+^{p-1} \Delta c^\epsilon dx \\ &=: -\frac{4(p-1)}{p^2} \int |\nabla (n^\epsilon - K)_+^{p/2}|^2 dx + \mathcal{I}_1 + \mathcal{I}_2. \end{aligned} \tag{A.16}$$

Note that since $\nabla \cdot \mathbf{b} \equiv 0$, the term involving \mathbf{b} vanishes. For the T_1 term in (A.16), using the facts that $\Delta c^\epsilon = \Delta K^\epsilon * n^\epsilon$ and $|\Delta K^\epsilon|_1$ is uniformly bounded, we can estimate it as follows:

$$\begin{aligned} \mathcal{I}_1 &\leq \int (n^\epsilon - K)_+^p |\Delta K^\epsilon| * (n^\epsilon - K)_+ dx + K \int (n^\epsilon - K)_+^p dx |\Delta K^\epsilon|_1 \\ &\leq |(n^\epsilon - K)_+|_{p+1}^p \|\Delta K^\epsilon\| * (n^\epsilon - K)_+|_{p+1} + CK \int (n^\epsilon - K)_+^p dx \\ &\leq C|(n^\epsilon - K)_+|_{p+1}^{p+1} + CK|(n^\epsilon - K)_+|_p^p. \end{aligned} \tag{A.17}$$

Similarly, we can estimate the T_2 term in (A.16) as follows:

$$\mathcal{I}_2 \leq CK|(n^\epsilon - K)_+|_p^p + CK^2|(n^\epsilon - K)_+|_{p-1}^{p-1}. \tag{A.18}$$

Combining (A.16), (A.17) and (A.18), we obtain

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \int (n^\epsilon - K)_+^p dx &\leq -\frac{4(p-1)}{p^2} \int |\nabla (n^\epsilon - K)_+^{p/2}|^2 dx \\ &\quad + C|(n^\epsilon - K)_+|_{p+1}^{p+1} + CK|(n^\epsilon - K)_+|_p^p + CK^2|(n^\epsilon - K)_+|_{p-1}^{p-1}. \end{aligned} \tag{A.19}$$

For the highest order term $C|(n^\epsilon - K)_+|_{p+1}^{p+1}$ in (A.19), we use the following Gagliardo-Nirenberg-Sobolev inequality:

$$\int_{\mathbb{R}^2} f^{p+1} dx \leq C \int_{\mathbb{R}^2} |\nabla (f^{p/2})|^2 dx \int_{\mathbb{R}^2} f dx, \quad f \geq 0, \tag{A.20}$$

together with (A.15), to estimate it as follows:

$$\begin{aligned} |(n^\epsilon - K)_+|_{p+1}^{p+1} &\leq C|\nabla \left((n^\epsilon - K)_+^{p/2} \right)|_2^2 |(n^\epsilon - K)_+|_1 \\ &\leq C\eta(K)|\nabla \left((n^\epsilon - K)_+^{p/2} \right)|_2^2. \end{aligned} \tag{A.21}$$

We can take K big such that it is absorbed by the negative dissipation term in (A.19). Now applying Hölder's inequality, Young's inequality and the Gronwall inequality to (A.19), we have that

$$|(n^\epsilon - K)_+|_{L^p}(t) \leq C(T) < \infty, \quad t \in [0, T], \quad p \in (1, \infty).$$

Applying a standard argument, see e.g. [7] proof of Proposition 3.3, we obtain the estimate

$$|n^\epsilon|_{L^\infty([0, T]; L^p)} \leq C(T), \quad p \in (1, \infty). \tag{A.22}$$

In particular, we set $p = 2$ and obtain that

$$|n^\epsilon|_{L^2([0, T]; L^2)} \leq C(T). \tag{A.23}$$

We conclude this step by setting $H := L^2(\mathbb{R}^2)$.

Space V: Bound on $|\nabla n^\epsilon|_{L^2([0,T] \times \mathbb{R}^2)}$: First we calculate the time evolution of the quantity $|n^\epsilon|_2^2$:

$$\begin{aligned} \frac{d}{dt} \int |n^\epsilon|^2 dx &= -2 \int |\nabla n^\epsilon|^2 dx + 2 \int \nabla n^\epsilon \cdot \nabla c^\epsilon n^\epsilon dx - \int \nabla \cdot (\mathbf{b}(n^\epsilon)^2) dx \\ &= -2 \int |\nabla n^\epsilon|^2 dx + 2 \int \nabla n^\epsilon \cdot \nabla c^\epsilon n^\epsilon dx. \end{aligned}$$

Now, integrating in time, we obtain the estimate

$$\begin{aligned} &\int |n^\epsilon|^2 dx(T) - \int |n^\epsilon|^2 dx(0) \\ &= -2 \int_0^T \int |\nabla n^\epsilon|^2 dx dt + 2 \int_0^T \int n^\epsilon (\nabla n^\epsilon \cdot \nabla c^\epsilon) dx dt \\ &\leq -2 \int_0^T \int |\nabla n^\epsilon|^2 dx dt \\ &\quad + 2 \left(\int_0^T \int |n^\epsilon \nabla c^\epsilon|^2 dx dt \right)^{1/2} \left(\int_0^T \int |\nabla n^\epsilon|^2 dx dt \right)^{1/2}. \end{aligned} \tag{A.24}$$

The terms on the left hand side of (A.24) are bounded due to (A.22). For the last term on the right hand side, we can estimate it as follows. The Hardy–Littlewood–Sobolev inequality yields

$$|\nabla c^\epsilon|_4 \leq C |n^\epsilon|_{4/3}, \tag{A.25}$$

which implies

$$|n^\epsilon \nabla c^\epsilon|_2 \leq |n^\epsilon|_4 |\nabla c^\epsilon|_4 \leq C |n^\epsilon|_4 |n^\epsilon|_{4/3}.$$

Combining this and the L^p bound (A.22) yields the boundedness of $n^\epsilon \nabla c^\epsilon$ in $L^\infty([0, T], L^2)$. Applying this fact and (A.22) in (A.24) and set $X = (\int_0^T \int |\nabla n^\epsilon|^2 dx dt)^{1/2}$, we obtain

$$X^2 - 2|n^\epsilon \nabla c^\epsilon|_{L^2([0,T] \times \mathbb{R}^2)} X \leq \int |n^\epsilon|^2 dx(T) - \int |n^\epsilon|^2 dx(0) \leq C.$$

As a result,

$$|\nabla n^\epsilon|_{L^2([0,T] \times \mathbb{R}^2)} \leq C. \tag{A.26}$$

Same as in [8], we set $V := H^1(\mathbb{R}^2) \cap \{n ||x|n^2 \in L^1\}$, which is shown to be compactly imbedded in H there. Thanks to the bound (A.23), (A.26) and the (A.1), we have that $n^\epsilon \in L^2([0, T]; V)$.

Space W: Bound for the $\partial_t n^\epsilon$: In order to estimate the $L^2([0, T]; H^{-1})$ norm of the function $\partial_t n^\epsilon$, we first need to get a bound on the fourth moment of the solution $V_4 := \int n(x_1^4 + x_2^4) dx$. The time evolution of V_4 can be formally estimated as follows:

$$\begin{aligned}
 \frac{d}{dt} V_4 &= \int (\Delta n - \nabla \cdot (\nabla c n) - \mathbf{b} \nabla n)(x_1^4 + x_2^4) dx \\
 &= 12 \int n |x|^2 dx \\
 &\quad - \frac{1}{4\pi} \iint \frac{4(x_1^3 - y_1^3)(x_1 - y_1) + 4(x_2^3 - y_2^3)(x_2 - y_2)}{|x - y|^2} n(x)n(y) dx dy \\
 &\quad + \int \mathbf{b} n \cdot (4x_1^3, 4x_2^3) dx \\
 &\leq C(M) \int n |x|^2 dx + C \int n(x_1^4 + x_2^4) dx. \tag{A.27}
 \end{aligned}$$

Combining the second moment estimate (A.1) and the Gronwall inequality, we have that

$$\int n(x, t) |x|^4 dx \leq C, \quad t \in [0, T]. \tag{A.28}$$

One can adapt this calculation to the regularized solution n^ϵ without much difficulty. We leave the details to the interested reader.

Now we can estimate the $L^2([0, T]; H^{-1})$ norm of the $\partial_t n^\epsilon$. Combining the L^p bound on n^ϵ (A.22), the bound on ∇c (A.25) and the fourth moment control (A.28) and testing the equation (2.3) with $f \in L^2([0, T], H^1(\mathbb{R}^2))$, we have

$$\begin{aligned}
 (\partial_t n^\epsilon, f)_{L^2([0, T] \times \mathbb{R}^2)} &\leq |\nabla n^\epsilon|_{L^2([0, T]; L^2)} |f|_{L^2([0, T]; H^1)} \\
 &\quad + |\nabla c^\epsilon n^\epsilon|_{L^2([0, T]; L^2)} |f|_{L^2([0, T]; H^1)} \\
 &\quad + |\mathbf{b} n^\epsilon|_{L^2([0, T]; L^2)} |f|_{L^2([0, T]; H^1)} \\
 &\leq |\nabla n^\epsilon|_{L^2([0, T]; L^2)} |f|_{L^2([0, T]; H^1)} \\
 &\quad + T^{1/2} |\nabla c^\epsilon|_{L^\infty([0, T]; L^4)} |n^\epsilon|_{L^\infty([0, T]; L^4)} |f|_{L^2([0, T]; H^1)} \\
 &\quad + C T^{1/2} \sup_t V_4^{1/4} |n^\epsilon|_{L^\infty([0, T]; L^3)}^{3/4} |f|_{L^2([0, T]; H^1)} \\
 &\leq C |f|_{L^2([0, T]; H^1)}.
 \end{aligned}$$

As a result, we have that $\partial_t n^\epsilon$ is uniformly bounded in $L^2([0, T]; H^{-1})$.

Combining the results from all the steps above and the Aubin–Lions lemma, we have that $(n^\epsilon)_\epsilon$ is precompact in $L^2([0, T]; L^2)$. We denote n as the limit of one converging subsequence $(n^{\epsilon_k})_{\epsilon_k}$. Moreover, combining (A.25) and the $L^{4/3}([0, T] \times \mathbb{R}^2)$ bound on n^ϵ which can be derived from (A.22), we have that $n^{\epsilon_k} \nabla c^{\epsilon_k}$ converge to $n \nabla c$ in distribution sense.

Step 3- Free energy estimates. By the convexity of the functional $n \rightarrow \int_{\mathbb{R}^2} |\nabla \sqrt{n}|^2 dx$, the fact that $n^{\epsilon_k} \nabla c^{\epsilon_k}$ converge to $n \nabla c$ in distribution sense and weak semi-continuity, we have

$$\begin{aligned}
 \iint_{[0, T] \times \mathbb{R}^2} |\nabla \sqrt{n}|^2 dx dt &\leq \liminf_{k \rightarrow \infty} \iint_{[0, T] \times \mathbb{R}^2} |\nabla \sqrt{n^{\epsilon_k}}|^2 dx dt, \\
 \iint_{[0, T] \times \mathbb{R}^2} n |\nabla c|^2 dx dt &\leq \liminf_{k \rightarrow \infty} \iint_{[0, T] \times \mathbb{R}^2} n^{\epsilon_k} |\nabla c^{\epsilon_k}|^2 dx dt.
 \end{aligned}$$

Moreover, it can be checked that $S[n^\epsilon](t) \rightarrow S[n](t)$ for almost every t , whose proof is similar to the one used in [7] Lemma 4.6.

Next we show the free energy estimate (1.5) using the strong convergence of $\{n^{\epsilon_k}\}$ in $L^2([0, T] \times \mathbb{R}^2)$. The key is to show the following entropy dissipation term is lower semi-continuous for the sequence (n^{ϵ_k}) :

$$\begin{aligned} & \int_0^T \int n^{\epsilon_k} |\nabla \log n^{\epsilon_k} - \nabla c^{\epsilon_k} - \mathbf{b}|^2 dx dt \\ &= 4 \iint_{[0, T] \times \mathbb{R}^2} |\nabla \sqrt{n^{\epsilon_k}}|^2 dx dt + \iint_{[0, T] \times \mathbb{R}^2} n^{\epsilon_k} |\nabla c^{\epsilon_k}|^2 dx dt \\ & \quad + \iint_{[0, T] \times \mathbb{R}^2} n^{\epsilon_k} |\mathbf{b}|^2 dx dt \\ & \quad - 2 \iint_{[0, T] \times \mathbb{R}^2} (n^{\epsilon_k})^2 dx dt - 2 \iint_{[0, T] \times \mathbb{R}^2} n^{\epsilon_k} \mathbf{b} \cdot \nabla c^{\epsilon_k} dx dt \\ &=: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5. \end{aligned} \tag{A.29}$$

For the sake of simplicity, later we use n^ϵ to denote n^{ϵ_k} .

First, we estimate the \mathcal{I}_3 term in (A.29). By the Fatou Lemma, we have the inequality

$$\iint_{[0, T] \times \mathbb{R}^2} n |\mathbf{b}|^2 dx dt \leq \liminf_{\epsilon_k \rightarrow 0} \iint_{[0, T] \times \mathbb{R}^2} n^{\epsilon_k} |\mathbf{b}|^2 dx dt. \tag{A.30}$$

This finishes the treatment of \mathcal{I}_3 .

Next, we show that the term \mathcal{I}_5 in (A.29) actually converges as $\epsilon_k \rightarrow 0$. We decompose the difference between \mathcal{I}_5 and its formal limit into two parts:

$$\begin{aligned} & \left| \int_0^T \int_{\mathbb{R}^2} \mathbf{b} \cdot \nabla c n - \mathbf{b} \cdot \nabla c^\epsilon n^\epsilon dx dt \right| \\ & \leq \int_0^T \int_{\mathbb{R}^2} |\nabla c \cdot \mathbf{b} n - \nabla c^\epsilon \cdot \mathbf{b} n + n \mathbf{b} \cdot \nabla c^\epsilon - \mathbf{b} \cdot \nabla c^\epsilon n^\epsilon| dx dt \\ & \leq \int_0^T \int_{\mathbb{R}^2} |\nabla c - \nabla c^\epsilon| \cdot |\mathbf{b} n| dx dt + \int_0^T \int_{\mathbb{R}^2} |\mathbf{b} n - \mathbf{b} n^\epsilon| \cdot |\nabla c^\epsilon| dx dt \\ & =: \mathcal{I}_{51} + \mathcal{I}_{52}. \end{aligned} \tag{A.31}$$

For the first term \mathcal{I}_{51} in (A.31), applying Hölder and Hardy–Littlewood–Sobolev inequality, we can estimate it as follows:

$$\begin{aligned} \mathcal{I}_{51} & \leq \left(\int_0^T \int_{\mathbb{R}^2} |\nabla(c - c^\epsilon)|_4^4 dx dt \right)^{1/4} \left(\int_0^T \int_{\mathbb{R}^2} |b \sqrt{n}|^2 dx dt \right)^{1/2} \\ & \quad \left(\int_0^T \int_{\mathbb{R}^2} \sqrt{n}^4 dx dt \right)^{1/4} \\ & \leq C \left(\int_0^T \int_{\mathbb{R}^2} |n - n^\epsilon|_{4/3}^4 dx dt \right)^{1/4} \left(\int_0^T \int_{\mathbb{R}^2} |x|^2 n dx dt \right)^{1/2} \end{aligned}$$

$$\begin{aligned}
 & \left(\int_0^T \int_{\mathbb{R}^2} n^2 dx dt \right)^{1/4} \\
 \leq & C \left(\int_0^T |n - n^\epsilon|_2^2 |n - n^\epsilon|_1^2 dt \right)^{1/4} \left(\int_0^T \int_{\mathbb{R}^2} |x|^2 n dx dt \right)^{1/2} \\
 & \left(\int_0^T \int_{\mathbb{R}^2} n^2 dx dt \right)^{1/4} \\
 \leq & C \left(\int_0^T |n - n^\epsilon|_2^2 dt \right)^{1/4} \sup_t |n - n^\epsilon|_1^{1/2} \left(\int_0^T \int_{\mathbb{R}^2} n |x|^2 dx dt \right)^{1/2} \\
 & \left(\int_0^T \int_{\mathbb{R}^2} n^2 dx dt \right)^{1/4}.
 \end{aligned}$$

From the $L^2([0, T] \times \mathbb{R}^2)$ strong convergence of $(n^\epsilon)_\epsilon$, we have that the first factor goes to zero. Other factors are bounded due to (A.1), Fatou's Lemma, L^1 bound on n^ϵ , n and $n \in L^2([0, T] \times \mathbb{R}^2)$. As a result, T_{51} converges to zero. Next we estimate the \mathcal{I}_{52} term in (A.31). Applying the fact that $|\sqrt{|c|} - \sqrt{|a|}| \leq \sqrt{|c - a|}$, the Hölder and Hardy–Littlewood–Sobolev inequality, we have

$$\begin{aligned}
 \mathcal{I}_{52} &= \int_0^T \int_{\mathbb{R}^2} |\mathbf{b}(\sqrt{n^2} - \sqrt{n^{\epsilon 2}})| \cdot |\nabla c^\epsilon| dx dt \\
 &\leq \int_0^T \int_{\mathbb{R}^2} |\mathbf{b}\sqrt{n}(\sqrt{n} - \sqrt{n^\epsilon})| \cdot |\nabla c^\epsilon| dx dt \\
 &\quad + \int_0^T \int_{\mathbb{R}^2} |\mathbf{b}\sqrt{n^\epsilon}(\sqrt{n} - \sqrt{n^\epsilon})| \cdot |\nabla c^\epsilon| dx dt \\
 &\leq |\nabla c^\epsilon|_{L^4([0, T] \times \mathbb{R}^2)} |b\sqrt{n}|_{L^2([0, T] \times \mathbb{R}^2)} \left(\int_0^T \int_{\mathbb{R}^2} \sqrt{|n - n^\epsilon|}^4 dx dt \right)^{1/4} \\
 &\quad + |\nabla c^\epsilon|_{L^4([0, T] \times \mathbb{R}^2)} |b\sqrt{n^\epsilon}|_{L^2([0, T] \times \mathbb{R}^2)} \left(\int_0^T \int_{\mathbb{R}^2} \sqrt{|n - n^\epsilon|}^4 dx dt \right)^{1/4} \\
 &\leq C \sup_t |n^\epsilon|_{L^1(\mathbb{R}^2)}^{1/2} |n^\epsilon|_{L^2([0, T] \times \mathbb{R}^2)}^{1/2} \\
 &\quad \left(\int_0^T \int_{\mathbb{R}^2} (n^\epsilon + n) |x|^2 dx dt \right)^{1/2} |n - n^\epsilon|_{L^2([0, T] \times \mathbb{R}^2)}^{1/2}.
 \end{aligned}$$

By the same reasoning as in T_{51} , we have that this term goes to zero. This finishes the treatment for the term.

Now for the $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_4$ terms in (A.29), we can handle them in the same way as in [7]. Combining all the estimates above we have that

$$\int_0^T \int n |\nabla \log n - \nabla c - \mathbf{b}|^2 dx dt \leq \liminf_{\epsilon_k \rightarrow 0} \int_0^T \int n^{\epsilon_k} |\nabla \log n^{\epsilon_k} - \nabla c^{\epsilon_k} - \mathbf{b}|^2 dx dt. \tag{A.32}$$

The remaining part of the proof is the same as the one in [7]. \square

Appendix A.2. Proof of Proposition 2.2

The proof of the proposition follows along the lines of [8].

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