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Global regularity of the 4D restricted Euler equations

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To Katepalli Sreenivasan on his 60th birthday, with friendship and admiration

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ABSTRACT

We are concerned with the critical threshold phenomena in the restricted Euler (RE) equations. Using the spectral and trace dynamics we identify the critical thresholds for the 3D and 4D restricted Euler equations. It is well known that the 3D RE solutions blow up. Projected on the 3-sphere, the set of initial eigenvalues which give rise to bounded stable solutions is reduced to a single point, which confirms that the 3D RE blowup is generic. In contrast, we identify a surprisingly *rich* set of the initial spectrum on the 4-sphere which yields global smooth solutions; thus, 4D regularity is generic.

1. Restricted Euler equations and spectral dynamics

We are concerned with the questions of global regularity vs. finite-time breakdown of Eulerian flows governed by

$$\partial_t u + u\cdot\nabla_x u = F, \quad x\in\mathbb{R}^n,\ t>0.$$

Here u is the velocity field, $u := (u_1, u_2, \ldots, u_n)^\top : \mathbb{R}^{1+n} \mapsto \mathbb{R}^n$, and its global behavior is dictated by the different models of the forcing $F = F(u, \nabla u, \ldots)$. For forcing involving viscosity and pressure, we meet the well-known Navier–Stokes (NS) equations,

$$\partial_t u + u\cdot\nabla_x u = \nu\Delta u - \nabla p, \quad x\in\mathbb{R}^n,\ t>0, \tag{1.1}$$

augmented with the incompressibility condition, $\nabla\cdot u = 0$ and subject to prescribed initial conditions $u(x,0) = u_0(x)$. In many applications, $\nu > 0$ is sufficiently small so that one can anticipate the behavior of slightly viscous NS solutions to be described by the Euler equations with $\nu = 0$ in (1.1), at least for flows occupying the whole space so that the important effects of boundary layers can be ignored.

The velocity gradient of the incompressible Euler equations, $M := \nabla_x u$ then solves

$$\partial_t M + u\cdot\nabla_x M + M^2 = -(\nabla\otimes\nabla)p. \tag{1.2}$$

Taking the trace of (1.2) while noting that M is trace-free, $tr\ M = \nabla\cdot u = 0$, one finds that $tr\ M^2 = -\Delta p$ which dictates the pressure as $p = -\Delta^{-1}(tr\ M^2)$. The second term in (1.2) therefore amounts to the $n\times n$ time-dependent matrix

$$(\nabla\otimes\nabla)\Delta^{-1}(tr\ M^2) = R[tr\ M^2].$$

Here $R[w]$ denotes the so-called *Riesz matrix* — an $n\times n$ matrix whose entries, $(R[w])_{jk} := R_jR_k(w)$, involve the Riesz transforms R_j, $R_j = -(-\Delta)^{-1/2}\partial_j$, i.e.,

$$R[w] := \left\{R_jR_k(w)\right\}_{j,k=1}^n, \quad \widehat{R_j(w)}(\xi) = -\mathrm{i}\frac{\xi_j}{|\xi|}\hat{w}(\xi)$$

for $1\le j\le n$.

This furnishes an equivalent, self-contained formulation of Euler equations, expressed in terms of the velocity gradient $M = \nabla_x u$, which is governed by,

$$\partial_t M + u\cdot\nabla_x M + M^2 = R[tr\ M^2], \quad M = \nabla_x u, \tag{1.3}$$

and subject to the trace-free initial data, $M(\cdot,0) = M_0$. Observe that the invariance of incompressibility is already taken into account in (1.3) since $tr\ M^2 = tr\ R[tr\ M^2]$ implies that $(\partial_t + u\cdot\nabla_x)tr\ M = 0$ and hence $tr\ M = tr\ M_0 = 0$.

It is the *global* nature of the Riesz matrix, $R[tr\ M^2]$, which makes the issue of regularity for Euler and NS equations such an intricate question to solve, both analytically and numerically, [1]. Various simplifications to this pressure Hessian, $R[trM^2] = -\nabla\otimes\nabla p$

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were sought, e.g., [2–6]. In this paper we focus our attention on the so-called *restricted Euler equations*, proposed in [7,3] as a *localized* alternative of the full Euler equation (1.3). By the definition of the Riesz matrix, one has

$$R[tr\, M^2] = \nabla \otimes \nabla \Delta^{-1}[tr\, M^2] = \nabla \otimes \nabla \int_{\mathbb{R}^n} K(x-y)(tr\, M^2)(y)\mathrm{d}y,$$

where the kernel $K(\cdot)$ is given by

$$K(x) = \begin{cases} \dfrac{1}{2\pi} ln|x| & n = 2, \\ \dfrac{1}{(2-n)\omega_n |x|^{n-2}} & n > 2, \end{cases}$$

with ω_n denoting the surface area of the unit sphere in n dimensions. A direct computation yields

$$\partial_j \partial_k K * tr\, M^2 = \frac{tr\, M^2}{n}\delta_{jk} + \int_{\mathbb{R}^n} \frac{|x-y|^2\delta_{jk} - n(x_j-y_j)(x_k-y_k)}{\omega_n |x-y|^{n+2}} tr\, M^2(y)\mathrm{d}y. \quad (1.4)$$

Ignoring the singular integrals on the right of (1.4), we are left with the local part of the Riesz matrix $R[tr\, M^2]$, given by $tr\, M^2\, I_{n\times n}/n$. We use this local term to approximate the pressure Hessian in (1.3). The resulting restricted Euler (RE) equations amount to

$$\partial_t M + u \cdot \nabla_x M + M^2 = \frac{1}{n} tr\, M^2 I_{n\times n}. \quad (1.5)$$

This is a matrix Ricatti equation for the $n \times n$ matrix M, which should mimic the dynamics of the velocity gradient, ∇u in the full Euler equations. We observe that as in the full Euler equations, incompressibility is maintained in the restricted model, since $trM^2 = tr[tr\, M^2 I_{n\times n}/n]$ implies that $(\partial_t + u \cdot \nabla_x) tr\, M = 0$ and hence $tr\, M = tr\, M_0 = 0$. The 3D RE (1.5) has attracted great attention since it was first introduced in [7,3] as a local approximation to the full 3D Euler equations. It can be used to understand the local topology of the Euler dynamics and to capture certain statistical features of physical turbulent flows, consult [8,9,3].

What about the global regularity of the RE equation (1.5)? the finite-time breakdown of the 3D restricted model goes back to the original work of Viellefosse [3]. In [10] we have shown that the 3D RE solutions break down at a finite time for *all* initial configurations M_0, except for the special case when M_0 has three real eigenvalues,

$$\lambda_1(0) \le \lambda_2(0) \le \lambda_3(0), \quad \left\{\lambda_j(0) = \lambda_j(M_0)\right\}_{j=1}^3,$$

which are aligned along the ray $(-r, -r, 2r)$, $r \in \mathbb{R}^+$. Thus, the finite-time breakdown of the 3D RE equations is *generic*.

In this paper we shall identify and compare between the restricted Euler equations in 3D and 4D case, respectively. To this end, we consider a bounded, divergence-free, smooth vector field $u : \mathbb{R}^n \times [0, T] \to \mathbb{R}^n$. Let $x = x(\alpha, t)$ denote an orbit associated to the flow by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = u(x, t), \quad 0 < t < T, \quad x(\alpha, 0) = \alpha \in \mathbb{R}^n.$$

Then along this orbit, the velocity gradient tensor of the restricted Euler equation (1.5) satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} M + M^2 = \frac{tr\, M^2}{n} I_{n\times n}, \quad \frac{\mathrm{d}}{\mathrm{d}t} := \partial_t + u \cdot \nabla_x.$$

By the spectral dynamics lemma 3.1 in [10], the corresponding eigenvalues of M satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda_i + \lambda_i^2 = \frac{1}{n}\sum_{j=1}^n \lambda_j^2, \quad i = 1, \dots, n. \quad (1.6)$$

This is a closed system for $\Lambda(t) = (\lambda_1(t), \lambda_2(t), \dots, \lambda_n(t))$, which serves as a simple approximation for the evolution of the velocity gradient field.

For arbitrary $n \ge 3$, we use the spectral dynamics of M in order to show the existence of a large set of initial configurations leading to finite-time breakdown of (1.6), generalizing the previous result of [3]. The finite-time breakdown of the n-dimensional RE equations (and the precise topology of the breakdown) was established in [10] after we identified a set of $[n/2] + 1$ global spectral invariants, interesting for their own sake. Yet, this does not exclude the possible existence of other generic sets of initial data, for which global smooth solutions exist. The distinction between these two sets of initial conditions is identified by the so-called *critical threshold* surfaces in configuration space: finite-time breakdown occurs for super-critical initial data on "one side" of the such critical threshold, while the set of *subcritical* initial data on the "other side" of the threshold yields global smooth solutions.

An interesting question therefore arises, namely, whether there exists a critical threshold for the 4D restricted Euler equation. This remarkable critical threshold phenomena was identified in [11,12] for a class of essentially 1D Euler–Poisson equations, and in [13] for a 1D convolution model for nonlinear conservation laws. The 2D critical threshold phenomena has been recently confirmed for a restricted Euler–Poisson system [14] and a rotating Euler equation [15,16]. In all these cases, we identified large, generic sets of subcritical initial data, which evolve to global smooth solutions. This is in contrast to the generic scenario of finite-time blows up in the 3D RE equations.[2]

In this paper we identify the exact critical thresholds for the 4D restricted Eulerian (RE) equations and we conclude with the surprising result that in the 4D case, the RE equations admit a large, generic set of subcritical initial data which give rise to global smooth solutions.

A summary of our results is outlined below. We say that $\Lambda_0 \in \mathbb{R}^n$ is *subcritical* if there exists a global solution in time of (1.6), subject to initial conditions, $\Lambda(0) = \Lambda_0$. A first observation rests on the obvious symmetries of (1.6).

Lemma 1.1. *If Λ is subcritical then so is $r\Lambda, \forall r > 0$. Moreover, $\Lambda_\sigma = \{\lambda_{\sigma(j)}, \forall \sigma \in \pi_n\}$ is also subcritical.*

For the proof we note that if $\Lambda(t)$ is the global solution corresponding to Λ_0, then $r\Lambda(rt)$, $r > 0$ is the global solution corresponding to $r\Lambda_0$. Also, Eqs. (1.6) remain invariant under arbitrary permutation σ which amounts to reordering, exchanging the λ_j-equation with $\lambda_{\sigma(j)}$-equation. It follows that the set of subcritical initial data consists of rays, and therefore, it is enough to consider the projection of this set on the unit sphere. In fact, we can restrict attention to an orthant of any convex set containing the origin. In this context we have

Theorem 1.1. *Solutions to (1.6) with $n = 3$ remain bounded for all time if and only if the initial data $\Lambda_0 := (\lambda_{10}, \lambda_{20}, \lambda_{30})$ lie in the following set*

$$r\{(-1, -1, 2)_\sigma\}.$$

Restricted to one orthant of the unit sphere, we thus find that the 3D RE equations admit only one subcritical point. In this sense,

[2] We should emphasize that generic subcritical data are *not* limited to a perturbative statement of global existence for initial data in the local neighborhood of certain "preferred configurations". Instead, the precise notion of "generic" subcritical sets, quantified below and the references mentioned above, makes it clear the critical threshold phenomena we seek describes a global scenario in configuration space.

the finite-time breakdown of 3D RE is generic. This result was already obtained in [10] by spectral dynamics analysis. In Section 4 we present an alternative, equivalent argument based on trace dynamics of $tr(M^k)$, $1 \le k \le n$, which paves the way for identifying our 4D critical threshold surface in Section 5.

In contrast to this generic 3D finite-time breakdown, the 4D RE equations admit a large class of global smooth solutions. Our 4D results are summarized below.

Theorem 1.2. *Solutions to (1.6) with $n = 4$ remain bounded for all time if and only if the initial data $\Lambda_0 := (\lambda_{10}, \lambda_{20}, \lambda_{30}, \lambda_{40})$ with $\sum_{j=1}^4 \lambda_{j0} = 0$, up to a permutation, lie in one of the following sets*

{i}*Two pairs of arbitrary complex eigenvalues,*

$$\Lambda_0 \in \{(a + \mathrm{i}b, a - \mathrm{i}b, -a + \mathrm{i}c, -a - \mathrm{i}c), \quad bc \neq 0\}.$$

{ii}*One pair of complex eigenvalue with two equal real eigenvalues*

$$\Lambda_0 \in \{(a + \mathrm{i}b, a - \mathrm{i}b, -a, -a), \quad b \neq 0\}.$$

{iii}*Real eigenvalues,*

$$\Lambda_0 \in \{(a + b, a - b, -a, -a), \quad b \in [-2a, 2a], \quad a \ge 0\}.$$

We remark that M is a real matrix of even dimension, therefore its eigenvalues come generically in complex conjugate pairs. If in addition one imposes the traceless condition, we conclude that in fact the form {i} of initial condition is generic.

Expressed in terms of traces $m_k := tr(M^k)$, these initial configurations form a "large" subcritical set which can be realized by its projection on the surface Σ,

$$\Sigma := \{\Lambda \mid 4m_4 - 2m_2^2 - 2m_2 + 3 = 0, \quad m_1 = 0\},$$

$$m_k := \sum_{j=1}^{4} \lambda_j^k.$$

Theorem 1.3. *Solutions to (1.6) with $n = 4$ remain bounded for all time if and only if there exists a $r > 0$ such that the initial data $\Lambda_0 := (\lambda_{10}, \lambda_{20}, \lambda_{30}, \lambda_{40})$ lie in the following set*

$$r\Lambda_0 \in \Sigma \cap \left[\left\{ |m_3| \le \frac{3}{2}(1 - m_2), m_2 \le 1 \right\} \cup \left\{ m_3 = \frac{3}{2}(m_2 - 1), \quad m_2 > 1 \right\} \right].$$

The set stated in Theorem 1.3 is non-trivial; in fact, it contains non-zero neighborhoods.

After this introduction of restricted Euler equations and the associated spectral dynamics, we identify the 4D subcritical initial configurations in terms of eigenvalues in Section 2. An alternative formulation of the spectral dynamics — called trace dynamics is derived in Section 3. Based on the trace dynamics we identify the critical thresholds for 3D case in Section 4 and the 4D model in Section 5. Finally in the Appendix we establish the correspondence between the subcritical eigenvalues and the subcritical set for initial traces.

2. 4D spectral dynamics

Let $\lambda = \lambda_i$ solve the restricted Euler equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda_i + \lambda_i^2 = \frac{1}{4}\sum_{j=1}^{4}\lambda_j^2, \quad i = 1 \cdots 4. \tag{2.1}$$

Two independent global invariants obtained in [10] are

$$(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4) = (\lambda_{10} - \lambda_{20})(\lambda_{30} - \lambda_{40}) \tag{2.2a}$$

and

$$(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4) = (\lambda_{10} - \lambda_{30})(\lambda_{20} - \lambda_{40}). \tag{2.2b}$$

We now prove Theorem 1.2 based on these global invariants. In view of the incompressibility invariant $\sum_{j=1}^4 \lambda_j = 0$, we can express the remaining three spectral degrees of freedom as $\Lambda = (a + b, a - b, -a + c, -a - c)^\top$, where a is real, $a \in \mathbb{R}$, and b or c are either real $b, c \in \mathbb{R}$ or purely imaginary, $b, c \in \mathrm{i}\mathbb{R}$. The two global invariants (2.2) now read

$$bc = b_0c_0 \tag{2.3a}$$

and

$$4a^2 - b^2 - c^2 = 4a_0^2 - b_0^2 - c_0^2. \tag{2.3b}$$

The spectral dynamics (2.1) amounts to the 3×3 closed system,

$$\frac{\mathrm{d}}{\mathrm{d}t}a = -\frac{1}{2}b^2 + \frac{1}{2}c^2, \tag{2.4a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}b = -2ab, \tag{2.4b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}c = 2ac, \tag{2.4c}$$

subject to initial data (a_0, b_0, c_0). Observe that both $b = 0$ and $c = 0$ are global invariants, thus the only equilibrium points of (2.4) when $b_0c_0 \neq 0$ lie along the curves $(0, b^*, c^*)$, $b^* = \pm c^*$. From (2.4b) and (2.4c) it is clear that if either b_0 or c_0 are purely imaginary then they remain so for all time. Thus, we need to discuss three cases in order.

{i} Two pairs of complex eigenvalues, $a_0 \pm \mathrm{i}b_0$ and $-a_0 \pm \mathrm{i}c_0$ with $b_0c_0 \neq 0$. Setting $(a, b, c) \mapsto (a, \mathrm{i}b, \mathrm{i}c)$ in (2.3b) we obtain the global invariant

$$4a^2 + b^2 + c^2 = 4a_0^2 + b_0^2 + c_0^2.$$

In this case, *all* trajectories remain bounded for all time.

{ii} One pair of complex eigenvalues, $a_0 \pm \mathrm{i}b_0$, $b_0 \neq 0$ and $-a_0 \pm c_0$. Setting $(a, b, c) \mapsto (a, \mathrm{i}b, c)$ in (2.3b), then the global invariant (2.3b) becomes

$$4a^2 + b^2 - c^2 = 4a_0^2 + b_0^2 - c_0^2.$$

We distinguish between two cases. If $c_0 = 0$, then by (2.4c) $c(t) \equiv 0$, and the reduced global invariant, $4a^2 + b^2 = 4a_0^2 + b_0^2$, implies that both a and b remain bounded for all $t > 0$. If $c_0 \neq 0$, then the Eq. (2.4a) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}a = \frac{1}{2}\left(b^2 + \frac{(b_0c_0)^2}{b^2}\right),$$

this shows that no finite equilibrium point of the system (2.4a)–(2.4c) is stable, which excludes the possibility of a globally bounded solution when $b_0c_0 \neq 0$.

{iii} Two real eigenvalues, $a_0 \pm b_0$ and $-a_0 \pm c_0$. Again, we distinguish between two cases. Assume that two initial eigenvalues coincide, say $c_0 = 0$ (if $b_0 = 0$, we end up with a similar scenario which amounts to a permutation of the $c_0 = 0$ case). Then $c(t) \equiv 0$ and the remaining (a, b) satisfy the reduced 2×2 system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -b^2/2 \\ -2ab \end{pmatrix},$$

with the corresponding global invariant $4a^2 - b^2 = 4a_0^2 - b_0^2$. Now, since $\frac{\mathrm{d}}{\mathrm{d}t}a = -b^2/2 \le 0$, it follows that $a > 0$ is decreasing while b must approach the stable equilibrium points $(a^* > 0, 0)$ along the positive a-axis, as $\frac{\mathrm{d}}{\mathrm{d}t}b = -2ab$ has the opposite sign of b. Thus, trajectories remain bounded in the invariant sector $|b_0| \le 2a_0$.

Finally, assume no pair of initial eigenvalues coincide, $b_0c_0 \neq 0$. Then, since the global invariants (2.3a) and (2.3b) are not compact, the only possible bounded solutions are those converging to the equilibrium points $(0, \pm c^*, c^*)$. But when substituted into both (2.3a) and (2.3b), this implies that

$4a_0^2 = (b_0 \pm c_0)^2$,

which is satisfied only if at least one pair of initial eigenvalues coincide, i.e. $a_0 \pm b_0 = -a_0 \pm c_0$. We conclude that for real eigenvalues, only those of the form $(a_0 + b_0, a_0 - b_0, -a_0, -a_0)$ with $|b_0| \le 2a_0$, $a_0 \ge 0$ lead to global bounded solution.

3. Trace dynamics

This section is devoted to an alternative formulation of the spectral dynamics in terms of real quantities $m_k := \sum_{j=1}^n \lambda_j^k$, $k = 1, \dots, n$, where $\lambda = \lambda_i$ solves the restricted Euler equation

$$\frac{d}{dt}\lambda_i + \lambda_i^2 = \frac{1}{n}\sum_{j=1}^n \lambda_j^2, \quad i = 1 \cdots n. \tag{3.1}$$

This is motivated by the trace dynamics originally studied in [3] for $n = 3$. The use of trace dynamics enables us to obtain an explicit description of the critical threshold surface for initial configurations.

Here we seek an extension for the general n-dimensional setting, which is summarized in the following

Lemma 3.1 ([10]). *Consider the n-dimensional restricted Euler system (3.1) subject to the incompressibility condition $m_1 := \sum_{j=1}^n \lambda_j = 0$. Then the traces m_k for $k = 2, \dots, n$ satisfy a closed dynamical system, see (3.2)–(3.6) below, which governs the local topology of the restricted flow.*

Proof. Based on the spectral dynamics the evolution equation for each eigenvalue λ_i can be written as

$$\frac{d}{dt}\lambda_i + \lambda_i^2 = \frac{1}{n}m_2, \quad i = 1 \cdots n.$$

By multiplying $k\lambda_i^{k-1}$ and summation over i we obtain

$$\frac{d}{dt}m_k + km_{k+1} = \frac{k}{n}m_2 m_{k-1}, \quad k = 2 \cdots n.$$

Note that $m_1 = 0$ we have

$$\frac{d}{dt}m_2 + 2m_3 = 0, \tag{3.2}$$

$$\frac{d}{dt}m_3 + 3m_4 = \frac{3}{n}m_2^2, \tag{3.3}$$

$\cdots$

$$\frac{d}{dt}m_n + nm_{n+1} = m_{n-1}m_2. \tag{3.4}$$

To close the system, it remains to express m_{n+1} in terms of $(m_1, \dots, m_n)$. To this end we utilize the characteristic polynomial

$$\lambda_j^n + q_1\lambda_j^{n-1} + \cdots q_{n-1}\lambda_j + q_n = 0, \tag{3.5}$$

expressed in terms of the characteristic coefficients

$$q_1 = -m_1 = 0, \quad q_2 = -\frac{1}{2}m_2, \quad q_3 = -m_3/3,$$
$$q_4 = -m_4/4 + m_2^2/8, \quad \cdots.$$

Note that the q's can be expressed in terms of $(m_1, \dots, m_n)$. Using (3.5) one may reduce m_{n+1} in (3.4) to lower-order products. In fact, $\sum_{j=1}^n(\lambda_j \times (3.5)_j)$ gives

$$m_{n+1} + q_2 m_{n-1} + \cdots + q_{n-1}m_2 = 0. \tag{3.6}$$

Substitution into (3.4) yields the closed system we sought for. □

We demonstrate the above procedure by considering the two examples of 3D and 4D critical thresholds.

4. 3D critical thresholds: finite-time blowup

This section is devoted to the study of the 3D critical thresholds, see [3,8]. In the 3D case one has

$$q_1 = 0, \quad q_2 = -\frac{1}{2}m_2, \quad q_3 = \prod_{j=1}^n \lambda_j = -\frac{1}{3}m_3,$$

hence

$$\lambda_i^3 - \frac{1}{2}m_2\lambda_i - \frac{1}{3}m_3 = 0, \quad i = 1, 2, 3.$$

Multiplying by λ_i and taking the summation over i we find

$$m_4 = \frac{1}{2}m_2^2.$$

Thus a closed system is obtained,

$$\frac{d}{dt}m_2 + 2m_3 = 0, \tag{4.1}$$

$$\frac{d}{dt}m_3 + \frac{1}{2}m_2^2 = 0. \tag{4.2}$$

From (4.1) and (4.2) it follows that

$$\frac{d}{dt}[6m_3^2 - m_2^3] = 6m_3\frac{d}{dt}m_3 - 3m_2^2\frac{d}{dt}m_2 = 0,$$

which yields a global invariant

$$6m_3^2 - m_2^3 = \text{Const}.$$

We consider the phase plane (m_2, m_3), except for the separatrix $6m_3^2 = m_2^3$, all other solutions would not approach the origin. The phase plane is divided into two parts by this separatrix. The nonlinearity ensures that trajectories which do not pass the origin must lead to infinity at finite time. In fact for initial data from the region $\{(m_2, m_3), m_2 > \sqrt[3]{6}\, m_3^{2/3}\}$, the corresponding trajectories will remain in this region since the system (4.1) and (4.2) is autonomous. Therefore (4.2) leads to

$$\frac{d}{dt}m_3 < -\frac{1}{2}\sqrt[3]{36}\, m_3^{4/3}. \tag{4.3}$$

Since $\frac{d}{dt}m_3 = -\frac{1}{2}m_2^2$, $m_3(t)$ is always decreasing in time. Even for positive $m_3(0)$, there exists a finite-time T^* such that $m_3(T*) < 0$. The integration of (4.3) over $[T^*, t)$ gives

$$m_3(t) < \left[\frac{3}{2}\sqrt[3]{36}(t - T^*) + m_3(T^*)^{-1/3}\right]^{-3}.$$

This shows that $m_3(t) \to -\infty$ when t approaches a time before

$$T^* + \frac{2}{3\sqrt[3]{36}}(-m_3(T^*))^{-1/3}.$$

Finite-time breakdown can be similarly justified for initial data lying in the region $\{(m_2, m_3), m_2 < \sqrt[3]{6}\, m_3^{2/3}\}$. These facts enable us to conclude the following

Theorem 4.1. *Consider the system (4.1) and (4.2) with initial data $(m_2(0), m_3(0))$. The global bounded solution exists if and only if the initial data lie on the curve*

$$\left\{(m_2, m_3) \mid m_3 = \frac{1}{\sqrt{6}}m_2^{3/2}\right\}.$$

We now turn to interpret this condition in terms of the eigenvalues. Set $\Lambda = (\lambda_1, \lambda_2, \lambda_3)$, the above critical stable set can be written as

$$\Omega = \left\{\Lambda \mid \sum_{k=1}^3 \lambda_k^3 = \frac{1}{\sqrt{6}}\left(\sum_{k=1}^3 \lambda_k^2\right)^{3/2}, \quad \sum_{k=1}^3 \lambda_k = 0\right\}.$$

The homogeneity of the above constraint in terms of eigenvalues implies that if $\Lambda \in \Omega$, then $r\Lambda \in \Omega\ \forall r > 0$.

Without loss of generality we consider the restriction of Ω onto a ball $\sum_{k=1}^{3} \lambda_k^2 = r^2$, denoted by $\Omega(r)$. There are two cases to be considered:

The initial eigenvalues contain complex components, say $\Lambda_0 = (a - b\mathrm{i}, a + b\mathrm{i}, c)$ for real $a, b, c \in \mathbb{R}$. The restricted set $\Omega(\sqrt{6})$ is determined by

$$c + 2a = 0, \quad 2a^2 - 2b^2 + c^2 = 6,$$
$$2a(a^2 - 3b^2) + c^3 = r^3/\sqrt{6} = 6.$$

Eliminating c we have

$$6a^2 - 2b^2 = 6, \quad -6a(a^2 + b^2) = 6$$
$$\Rightarrow 4a^3 - 3a + 1 = (a+1)(2a-1)^2 = 0,$$

which has real roots $a \in \{-1, 0.5, 0.5\}$, from which no real $b \neq 0$ can be found.

The only possible scenario is the real eigenvalue $\Lambda_0 = (a, b, c) \in \mathbb{R}^3$. Restriction again on $\Omega(\sqrt{6})$ we have

$$a + b + c = 0, \quad a^2 + b^2 + c^2 = 6, \quad a^3 + b^3 + c^3 = r^3/\sqrt{6} = 6.$$

Eliminating a, b we have $c^3 - 3c - 2 = 0$ with real roots $c \in \{2, -1, -1\}$. The symmetric property implies that a, b also lie in the set $\{2, -1, -1\}$. In short one has

$$\Omega(\sqrt{6}) = \{\Lambda | (-1, -1, 2), (-1, 2, -1), (2, -1, -1)\}.$$

This when combined with the above scaling property leads to the result stated in Theorem 1.1.

5. 4D critical thresholds: global regularity

In the 4D case one has

$$q_1 = 0, \quad q_2 = -\frac{1}{2}m_2, \quad q_3 = -\frac{1}{3}m_3, \quad q_4 = -\frac{m_4}{4} + \frac{m_2^2}{8}.$$

Hence

$$\lambda_i^4 - \frac{1}{2}m_2\lambda_i^2 - \frac{1}{3}m_3\lambda_i - \frac{m_4}{4} + \frac{m_2^2}{8} = 0, \quad i = 1 \cdots 4.$$

Multiplying by λ_i and taking the summation we obtain

$$m_5 = \frac{1}{2}m_2m_3 + \frac{1}{3}m_3m_2 = \frac{5}{6}m_2m_3.$$

Therefore the resulting closed system becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}m_2 = -2m_3, \tag{5.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}m_3 = \frac{3}{4}m_2^2 - 3m_4, \tag{5.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}m_4 = -\frac{7}{3}m_3m_2. \tag{5.3}$$

From (5.1) and (5.3) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m_4 - \frac{7}{12}m_2^2\right) = 0,$$

which gives a global invariant

$$m_4 - \frac{7}{12}m_2^2 = m_{40} - \frac{7}{12}m_{20}^2. \tag{5.4}$$

Substitution of this into (5.2) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}m_3 = -m_2^2 - 3\left(m_4 - \frac{7}{12}m_2^2\right).$$

In order to ensure global bounded solution (excluding globally decreasing m_3) it is necessary to consider trajectories for which

$$m_4(t) - \frac{7}{12}m_2^2(t) = -\frac{l^2}{3}, \quad l > 0. \tag{5.5}$$

We thus have a closed system for (m_2, m_3)

$$\frac{\mathrm{d}}{\mathrm{d}t}m_2 = -2m_3, \quad \frac{\mathrm{d}}{\mathrm{d}t}m_3 = -m_2^2 + l^2 \tag{5.6}$$

with a moving parameter l determined by (5.5) with $t = 0$. This system has two critical points $(-l, 0)$ and $(l, 0)$; it is easy to verify that as equilibrium points of system (5.6), $(-l, 0)$ is a spiral and $(l, 0)$ is a saddle for the corresponding linearized system.

This structure suggests that part of separatrix' of this system may serve as the critical threshold. Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(3m_3^2 - m_2^3 + 3l^2m_2\right) = 6m_3\frac{\mathrm{d}}{\mathrm{d}t}m_3 - 3m_2^2\frac{\mathrm{d}}{\mathrm{d}t}m_2 + 3l^2\frac{\mathrm{d}}{\mathrm{d}t}m_2 = 0.$$

Thus the 2nd global invariant when passing $(m_2, m_3) = (l, 0)$ becomes

$$3m_3^2 - m_2^3 + 3l^2m_2 = 2l^3,$$

yielding two separatrixes

$$3m_3^2 = m_2^3 - 3l^2m_2 + 2l^3 = (m_2 + 2l)(m_2 - l)^2. \tag{5.7}$$

We note in passing that the two invariants (5.4) and (5.7) are in fact the same spectral invariants we had before in (2.3), which are now reformulated in terms of the traces m_2 and m_3. Thus, for example, the straightforward identity

$$12m_4 - 7m_2^2 \equiv -4(4a^2 - b^2 - c^2)^2 - 48(bc)^2,$$

reveals the relation between the trace-based invariant (5.4) and the spectral invariant (2.3b).

In the phase plane (m_2, m_3), this consists of a closed curve for $-2l \leq m_2 \leq l$ and two open branches for $m_2 > l$. The phase plane analysis suggests that the global bounded solution exists if and only if the initial data satisfy (5.5) and

$$(m_2(0), m_3(0)) \in \Gamma_l,$$

where

$$\Gamma_l := \left\{(m_2, m_3) \mid |m_3| \leq \frac{l - m_2}{\sqrt{3}}\sqrt{m_2 + 2l}, \quad -2l \leq m_2 \leq l\right\}$$
$$\cup \left\{m_3 = \frac{m_2 - l}{\sqrt{3}}\sqrt{m_2 + 2l}, \quad m_2 > l\right\}$$

and the moving parameter l is determined by (5.5). Also we can show that if initial data do not belong to Γ_l then the solution becomes unbounded in finite time.

Fig. 1 depicts trajectories for system (5.6) in (m_2, m_3) plane, which contain the boundary of the non-trivial set $\Gamma \equiv \Gamma_l$ with (5.5).

From above analysis it follows that the solutions remain bounded for all time if and only if the initial data Λ_0 lies in the following set

$$\Lambda_0 \in \cup_{l>0}\left\{S_l \mid (m_2, m_3) \in \Gamma_l\right\}, \tag{5.8}$$

where,

$$S_l := \left\{\Lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \mid m_4 - \frac{7}{12}m_2^2 = -\frac{l^2}{3}\right\},$$

and $\Gamma_l := \Gamma_{1l} \cup \Gamma_{2l} \cup \Gamma_{3l}$ where

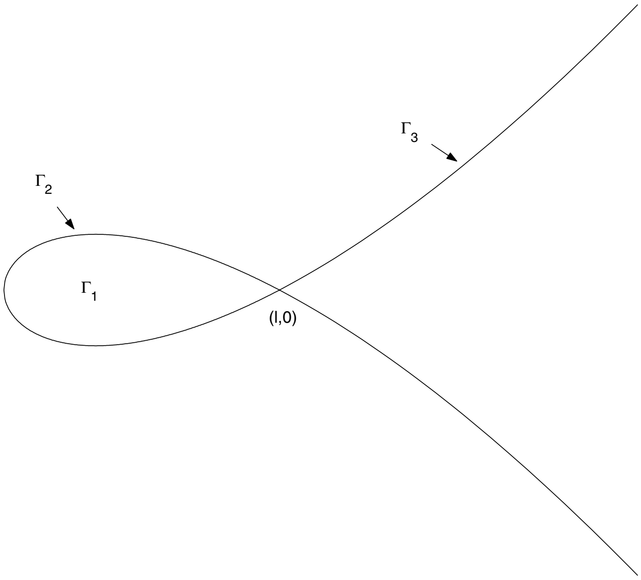

Fig. 1. The domain Γ_ℓ of subcritical configurations in $m_2 - m_3$ plane which lead to global 4D solutions.

$$\Gamma_{1l} := \left\{(m_2, m_3) \mid |m_3| < \frac{l - m_2}{\sqrt{3}}\sqrt{m_2 + 2l}, -2l \le m_2 < l\right\},$$

$$\Gamma_{2l} := \left\{(m_2, m_3) \mid |m_3| = \frac{l - m_2}{\sqrt{3}}\sqrt{m_2 + 2l}, -2l \le m_2 < l\right\},$$

$$\Gamma_{3l} := \left\{(m_2, m_3) \mid m_3 = \frac{m_2 - l}{\sqrt{3}}\sqrt{m_2 + 2l}, m_2 \ge l\right\}.$$

We now turn to Theorem 1.3. Let $r > 0$ be a moving parameter, we restrict attention to the parameterized surface $m_2 + 2l = 3r^2$. Clearly the constraint $m_2 \ge -2l$ is ensured for any real r. For any $r > 0$, the set S restricted on this surface is represented as

$$4m_4 - 2m_2^2 - 2m_2r^2 + 3r^4 = 0.$$

This is a parabolic cylinder in the space (m_2, m_3, m_4). Applying the scaling property stated in Lemma 1.1, we may set $r = 1$, and denote the set S with constraint $m_2 + 2l = 3$ and $m_1 = 0$ as

$$\Sigma := \{\Lambda \mid 4m_4 - 2m_2^2 - 2m_2 + 3 = 0, \quad m_1 = 0\},$$

$$m_k := \sum_{j=1}^{4} \lambda_j^k.$$

The first half of the set $\Gamma_l|_\Sigma$ is supported where $-2l \le m_2 \le l$, together with $m_2 + 2l = 3$, i.e., $l = \frac{3-m_2}{2}$, leading to $m_2 \le 1$.

In this case, the restriction

$$|m_3| \le l - m_2 = \frac{3 - m_2}{2} - m_2 = \frac{3}{2}(1 - m_2),$$

yields

$$\Omega_1 = \left\{(m_2, m_3) \mid |m_3| \le \frac{3}{2}(1 - m_2), \quad m_2 \le 1\right\}.$$

For reals Λ, $m_2 \ge 0$; the fact of no lower bound for m_2 suggests that any complex eigenvalue with zero divergence may well lie in $\{\Sigma, (m_2, m_3) \in \Omega_1\}$.

The second $\Gamma_l|_\Sigma$-constraint, supported on $m_3 = m_2 - l$ requiring $m_2 > l = \frac{3-m_2}{2}$, i.e., $m_2 > 1$ leading to

$$\Omega_2 = \left\{(m_2, m_3) \mid m_3 = \frac{3}{2}(m_2 - 1), \quad m_2 > 1\right\}.$$

The above set $\{\Sigma \mid (m_2, m_3) \in [\Omega_1 \cup \Omega_2]\}$ is 'fat'. Note the 3D case is similar to the special case $l = 0$ which restricts to a large subcritical set.

6. Concluding remarks

This work studies the restricted Euler (RE) equations for the velocity gradient matrix in four spatial dimensions. Our study led to the surprising conclusion that for a generic set of initial conditions, the 4D RE solutions remain bounded in time. This is in sharp contrast to the results in three spatial dimensions, where the solutions are bounded in time only when the initial conditions belong to a co-dimension 2 set. It is in this sense that the energy transfer in the 4D RE equations is found to be better than the 3D case. We mention in this context the two recent works of [17,18] on 4D incompressible Navier–Stokes equations, which also suggest a more efficient energy transfer in 4D than in 3D case.

Our investigation of the stable manifold in the 4D case is based on two different sets of variables: (i) eigenvalues of the velocity gradient matrix; (ii) traces of the velocity gradient matrix. The spectral formulation was introduced by us in [10] as a versatile tool for studying critical thresholds in Eulerian dynamics. It is simple and easy to analyze. The trace formulation was introduced by Vieillefosse [3] in his study of the 3D restricted Euler. It yields explicit expressions for critical thresholds in terms of the traces, m_j's.

How does the restricted Euler dynamics relate to real flows? it is clear that the finite-time breakdown of the 3D RE model does not necessarily bear on the full, non-restricted Euler equations. The question whether one can infer global regularity of solutions for the full, non-restricted 4D Euler equations is left open. We hope that our global existence result of the 4D RE equations will help to shed light on this question. In doing so, the 4D setup could clarify differences in the global regularity behavior of the full Euler equations, depending on the *dimension* of the underlying space.

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Appendix

Finally we turn to interpretation of eigenvalues presented in Theorem 1.2 in terms of the subcritical sets in Theorem 1.3, or the equivalent set (5.8).

{i} Two pairs of complex eigenvalues. The eigenvalues must be $\lambda_1 = a + bi$, $\lambda_2 = a - bi$, $\lambda_3 = -a + ci$, $\lambda_4 = -a - ci$, where $a, b, c \in \mathbb{R}$ and $b \ne 0$, $c \ne 0$. A direct calculation gives

$$m_2 = 4a^2 - 2b^2 - 2c^2,$$
$$m_3 = 6a(c^2 - b^2),$$
$$m_4 = 4a^4 + 2b^4 + 2c^4 - 12a^2(b^2 + c^2),$$
$$m_2^2 - l^2 = 3\left(m_4 - \frac{1}{4}m_2^2\right) = 3(b^2 - c^2)^2 - 24a^2(b^2 + c^2),$$
$$l^2 = (4a^2 + b^2 + c^2)^2 + 12b^2c^2.$$

It follows that $-2l \le m_2 \le l$ and $l > 4a^2 + b^2 + c^2$. Then

$$\frac{l - m_2}{\sqrt{3}}\sqrt{m_2 + 2l} > \frac{3(b^2 + c^2)}{\sqrt{3}}\sqrt{12a^2} = 6|a|(b^2 + c^2) > |m_3|.$$

Thus we know $\{S_l \mid (m_2, m_3) \in \Gamma_l\} = \{S_l \mid (m_2, m_3) \in \Gamma_{1l}\}$.

{ii} One pair of complex eigenvalues and two real eigenvalues. The four eigenvalues must be $\lambda_1 = a + bi$, $\lambda_2 = a - bi$, $\lambda_3 = -a + c$, $\lambda_4 = -a - c$, where $a, b, c \in \mathbb{R}$ and $b \ne 0$. Changing c in Case II to $-ci$ we immediately obtain

$$m_2 = 4a^2 - 2b^2 + 2c^2,$$
$$m_3 = -6a(b^2 + c^2),$$
$$m_4 = 4a^4 + 2b^4 + 2c^4 - 12a^2(b^2 - c^2),$$
$$m_2^2 - l^2 = 3\left(m_4 - \frac{1}{4}m_2^2\right) = 3(b^2 + c^2)^2 - 24a^2(b^2 - c^2),$$
$$l^2 = (4a^2 + b^2 - c^2)^2 - 12b^2c^2.$$

Suppose $-2l \le m_2 \le l$ for $l > 0$, we distinguish two cases:

(1) If $4a^2 + b^2 - c^2 \ge 0$, then $l \le 4a^2 + b^2 - c^2$, and

$$\frac{l - m_2}{\sqrt{3}}\sqrt{m_2 + 2l} \le \frac{3(b^2 - c^2)}{\sqrt{3}}\sqrt{12a^2} = 6|a(b^2 - c^2)| \le |m_3|.$$

It becomes an equality if and only if $c = 0$.

(2) If $4a^2 + b^2 - c^2 < 0$, then $l < c^2 - 4a^2 - b^2$, and $l - m_2 = b^2 - c^2 - 8a^2 < 0$. It is a contradiction with $m_2 < l$. For $m_2 > l$, the constraint $m_3 = \frac{m_2 - l}{\sqrt{3}}\sqrt{m_2 + 2l}$ leads to the relation (A.1), i.e.,

$$p = \left(3m_3^2 - m_2^3 - 9m_2\left(m_4 - \frac{7}{12}m_2^2\right)\right)^2 + 108\left(m_4 - \frac{7}{12}m_2^2\right)^3 = 0.$$

Calculation shows that $p = 432b^2c^2(b^2 + (2a - c)^2)^2(b^2 + (2a + c)^2)^2$. So $p = 0$ if and only if $c = 0$. Thus we know that $\{S_l \mid (m_2, m_3) \in \Gamma_l\} = \{S_l \mid (m_2, m_3) \in \Gamma_{2l}\}$, and two real eigenvalues must be equal.

{iii} all the eigenvalues are real. Suppose the four real eigenvalues are a, b, c and $-(a + b + c)$, $a, b, c \in \mathbb{R}$, then $m_2 \ge 0$. From the set S in (5.8) it follows

$$\frac{1}{3}(m_2^2 - l^2) = m_4 - \frac{1}{4}m_2^2 \ge 0,$$

here we have used the inequality $(\alpha + \beta + \gamma + \delta)^2 \le 4(\alpha^2 + \beta^2 + \gamma^2 + \delta^2)$. These together lead to $m_2 \ge l$. Thus if all the eigenvalues are real, then $\{S_l \mid (m_2, m_3) \in \Gamma_l\} = \{S_l \mid (m_2, m_3) \in \Gamma_{3l}\}$.

Because of the homogeneousness, we can assume the four real eigenvalues are $1 + s$, $-1 + w$, -1 and $1 - s - w$ (if $\Lambda_0 \ne 0$). Let us do the following calculation.

From $m_3 = \frac{m_2 - l}{\sqrt{3}}\sqrt{m_2 + 2l}$ it follows that

$$3m_3^2 = (m_2 - l)^2(m_2 + 2l) = m_2^3 - 3m_2l^2 + 2l^3,$$
$$\Rightarrow [3m_3^2 - m_2^3 + 3m_2l^2]^2 = 4(l^2)^3.$$

Using $l^2 = -3(m_4 - \frac{7}{12}m_2^2)$ we have

$$p := \left(3m_3^2 - m_2^3 - 9m_2\left(m_4 - \frac{7}{12}m_2^2\right)\right)^2 + 108\left(m_4 - \frac{7}{12}m_2^2\right)^3 = 0. \quad (A.1)$$

Calculation shows that

$$p = -27(s + 2)^2w^2(s - w + 2)^2(2s + w)^2 \times (s + 2w - 2)^2(s + w - 2)^2.$$

So $p = 0$ if and only if $s = -2$ or $w = 0$ or $s - w + 2 = 0$ or $2s + w = 0$ or $s + 2w - 2 = 0$ or $s + w - 2 = 0$. They are all the same if we consider the homogeneousness and permutation. Now we know that the four eigenvalues must be in the form $r(1 + s, -1, -1, 1 - s)$. We claim that the range for s is $[-2, 2]$.

(i) For $-2 \le s \le 2$, it is easy to check that $\Lambda \in \{S_l \mid (m_2, m_3) \in \Gamma_{3l}\}$.

(ii) For $s > 2$ or $s < -2$, we calculate l, m_2, m_3 to obtain

$$l = s^2 - 4, \quad m_2 = 4 + 2s^2, \quad m_3 = 6s^2.$$

So

$$\frac{m_2 - l}{\sqrt{3}}\sqrt{m_2 + 2l} = \frac{s^2 + 8}{\sqrt{3}}\sqrt{4s^2 - 4} = \frac{2(s^2 + 8)}{\sqrt{3}}\sqrt{s^2 - 1}.$$

Hence

$$\left[\frac{m_2 - l}{\sqrt{3}}\sqrt{m_2 + 2l}\right]^2 - m_3^2 = \frac{3}{4}(s^2 - 4)^3 > 0.$$

We now can conclude that if all the eigenvalues are real, then $\{S_l \mid (m_2, m_3) \in \Gamma_l\} = \{S_l \mid (m_2, m_3) \in \Gamma_{3l}\}$, and Λ_0 must be in the form $r(1 + s, 1 - s - 1, -1,)$ (plus arbitrary permutation), where $-2 < s < 2$ and $r > 0$.

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