Hyperbolic Systems with Different Time Scales

EITAN TADMOR
California Institute of Technology

1. Introduction

We consider the linear hyperbolic system

$$\text{(1.1a)}\qquad \begin{aligned}\frac{\partial u}{\partial t} &= P\left(x, t, \frac{\partial}{\partial x}\right)u + F, && t > 0,\\ u(x, 0) &= f(x), && t = 0,\end{aligned}$$

in a bounded domain Ω with smooth boundary Γ on which linear relations between u components

$$\text{(1.1b)}\qquad Du_{|\Gamma} = 0, \qquad t \geqq 0,$$

are given as boundary conditions. Here $u = (u^{(1)}, \cdots, u^{(n)})'$, $F = (F^{(1)}, \cdots, F^{(n)})'$, $f = (f^{(1)}, \cdots, f^{(n)})'$ are n-dimensional vector functions which depend smoothly on $x = (x_1, \cdots, x_s)$, t and

$$\text{(1.1c)}\qquad P\left(x, t, \frac{\partial}{\partial x}\right) = \varepsilon^{-1}P_0\left(x, t, \frac{\partial}{\partial x}\right) + P_1\left(x, t, \frac{\partial}{\partial x}\right), \qquad 0 < \varepsilon \ll 1,$$

where the n-dimensional coefficient matrices of

$$\text{(1.1d)}\qquad \begin{aligned}P_0 &= \frac{1}{2}\sum_{j=1}^{s}\left[\tilde{A}_j(x, t)\frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_j}(\tilde{A}_j(x, t)\cdot)\right] + \tilde{B}(x, t),\\ P_1 &= \frac{1}{2}\sum_{j=1}^{s}\left[\tilde{\tilde{A}}_j(x, t)\frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_j}(\tilde{\tilde{A}}_j(x, t)\cdot)\right] + \tilde{\tilde{B}}(x, t),\end{aligned}$$

$$\text{(1.1e)}\qquad \begin{aligned}\tilde{A}_j &= \tilde{A}_j^*, & \tilde{\tilde{A}}_j &= \tilde{\tilde{A}}_j^*, & j = 1, 2, \cdots, s,\\ \tilde{B} &= -\tilde{B}^*, & \tilde{\tilde{B}} &= -\tilde{\tilde{B}}^*,\end{aligned}$$

are of order $O(1)$ and smoothly depend on x, t.

Let $v \equiv v(x)$ be a vector function and denote by

$$\|v\|_{\xi,\eta} = \sum_{|\zeta| \leq \xi}\left\|\frac{\partial^{|\zeta|}v}{\partial x_1^{\zeta_1}\cdots\partial x_s^{\zeta_s}}\right\|_\eta,^{1}$$

$$\zeta = (\zeta_1, \cdots, \zeta_s) \in \mathbb{N}^s, \quad |\zeta| = \sum_{j=1}^{s}\zeta_j,$$

[1] If v, w are vector functions, the $\|v\|_\eta^\eta = \int_\Omega |v|^\eta\,dx$ denotes the usual L_η-norm. In particular, $\|v\|_2^2 \equiv \|v\|^2 = (v, v)$, where $(v, w) = \int_\Omega w^* v\,dx$ denotes the usual L_2-inner product.

Communications on Pure and Applied Mathematics, Vol. XXXV, 839–866 (1982)

CCC 0010-3640/82/060839-28$03.80

the usual Sobolev norm. Then we are interested in showing that, under appropriate assumptions, the solution of (1.1) satisfies the

ENERGY ESTIMATE. In any finite time interval $0 \leqq t \leqq T$, there exists a constant $K_0 = K_0(T)$ independent of ε^{-1}, $u(x, 0)$, $F(x, t)$ and their derivatives, such that the solution of (1.1) satisfies the estimate

$$\text{(E)} \qquad \left\| \frac{\partial^\nu u(x,t)}{\partial t^\nu} \right\|_{\mu,2} \leqq K_0 \left[\sup_{0 \leqq \tau \leqq t} \left\| \frac{\partial^\nu F(x,\tau)}{\partial \tau^\nu} \right\|_{\mu,2} + \left\| \frac{\partial^\nu u(x, t=0)}{\partial t^\nu} \right\|_{\mu,2} \right].$$

Thus estimate (E) guarantees that for the solution of system (1.1) to remain bounded independently of the fast time scale of order $O(\varepsilon^{-1})$, one has to prepare the initial data in such a way that $\partial^\nu u(x,t)/\partial t^\nu|_{t=0}$, $\nu = 0, 1, \cdots$, are of order $O(1)$. (The variation of the initial space derivatives $\partial^{|\mu|} u(x,t)/\partial x^\mu|_{t=0}$ is independent of the fast scale since we have assumed that the initial data $f(x)$ are sufficiently smooth.)

We postulate

ASSUMPTION 1.1. *The symbol* $\hat{P}_0(i\omega) \equiv i \sum_{j=1}^s \omega_j \tilde{A}_j + \tilde{B}$ *has a fixed rank which is independent of* $\omega = (\omega_1, \cdots, \omega_s)$, x, t.

In the appendix we prove that for Assumption 1.1 to hold, the assumed fixed rank of the symbol $\hat{P}_0(i\omega)$ has to be an even integer, which throughout the paper is denoted by

$$\text{(1.1f)} \qquad \operatorname{rank}\left[i \sum_{j=1}^s \omega_j \tilde{A}_j + \tilde{B} \right] = 2p, \quad |\omega| \leqq \infty, \quad x \in \Omega, \quad t \geqq 0.$$

ASSUMPTION 1.2. *The operator P is half-bounded, i.e., there exists a constant* α *independent of* ε^{-1}, *such that, for all smooth functions w satisfying the boundary conditions* (1.1b), *the estimate*

$$\text{(1.2)} \qquad \mathcal{R}e\,(w, Pw) \leqq \alpha \|w\|^2$$

holds.

Remark. We can make the growth factor α in (1.2) as a negative as we need, by introducing into (1.1) a new decaying variable $u_*(x,t) = e^{-\beta t} u(x,t)$, which satisfies system (1.1) with $P \to P_* = P - \beta$, while the growth factor α is replaced by $\alpha_* = \alpha - \beta$.

Using integration by parts, Assumption 1.2 implies that the homogeneous system associated with (1.1),

$$\text{(1.3)} \qquad \frac{\partial v}{\partial t} = Pv, \qquad x \in \Omega, \qquad Dv_{|\Gamma} = 0,$$

is uniformly well posed, i.e., the energy estimate

$$\|v(x,t)\| \leqq e^{\alpha(t-t_0)}\|v(x,t_0)\|, \qquad t \geqq t_0 \geqq 0, \tag{1.4}$$

holds.

By Duhamel's principle, we can estimate $\|u(x,t)\|$ in terms of the initial and inhomogeneous terms, $\|u(x,t=0)\|$, $\|F(x,t)\|$, thus proving estimate (E) for $\nu = 0$. Now, differentiating system (1.1) with respect to either its time or space arguments in the one-dimensional case, one obtains (the differentiated u is denoted by $\dot{u}$)

$$\frac{\partial \dot{u}}{\partial t} = P\dot{u} + \varepsilon^{-1}\dot{P}_0 u + \dot{P}_1 u + \dot{F}, \qquad x \in \Omega = [0, 1],$$

$$D\dot{u} + \dot{D}u = 0, \qquad x \in \Gamma = \{0, 1\}.$$

Then, assuming $\tilde{A}_j$, $\tilde{B}$, D are constant matrices independent of x, t, we can apply Duhamel's principle once more using the boundedness of $\|u(x,t)\|$ and see that estimate (E) holds also for $\nu = 1$. Repeating the process we obtain estimate (E) for the higher derivatives. (See [2], Section 3, for the multi-dimensional case where the boundary conditions are, in particular, periodic.) The above process breaks down however in the presence of variable coefficients, since direct application of Duhamel's principle yields estimates depending on ε^{-1}, which is reflected by the nonvanishing term $\varepsilon^{-1}\dot{P}_0 u$.

In this paper we generalize the theory developed by H. O. Kreiss in [2] for estimating the solution of system (1.1) with variable coefficients in the special case $p = 1$, and prove the energy estimate (E) for the general case $p \geqq 1$. We assume that the reader is familiar with [2].

We start by considering the one-dimensional problem deriving, in Section 2, a normal representation for the symbol $\hat{P}_0(i\omega)$, where we distinguish between three normal forms of that representation. In fact, the normal representation derived is the key for proving the one-dimensional energy estimate as carried out in Sections 3, 4, and 5. We also show in Section 2 that in the three different cases, the $2p$ fast characteristic velocities of the system split into p pairs of velocities traveling in opposite directions.

In Sections 3 and 4 we prove estimate (E) for the first and second normal forms of $\hat{P}_0(i\omega)$, respectively. These are the appropriate block generalizations of those studied in [2], Sections 5 and 6. The energy estimate for the third normal form, which actually consists of $p-1$ different subcases (and evidently cannot exist in the special case $p = 1$), is proved in Section 5, where we employ combined techniques previously used in studying the first two normal forms.

The study of the multi-dimensional problem can be carried out similarily, by first deriving a normal form of the multi-dimensional symbol $\hat{P}_0(i\omega)$, and then employing it to obtain an energy estimate. The detailed analysis of that case will be published in a forthcoming paper.

2. A Normal Form for Problems in One Space Dimension

We consider the system

$$\text{(2.1a)}\qquad \frac{\partial u}{\partial t}=\left(\frac{1}{\varepsilon}P_0+P_1\right)u+F, \qquad t>0,$$
$$u(x,0)=f(x), \qquad t=0,$$

in one space dimension $0\leqq x\leqq 1$, together with boundary conditions

$$\text{(2.1b)}\qquad D_0u(0,t)=D_1u(1,t)=0, \qquad t\geqq 0,$$

which are linear relations between u-components, expressing the dependence of the incoming characteristic variables on the outgoing ones. Here

$$\text{(2.1c)}\qquad P_0u=\tfrac{1}{2}[\tilde{A}u_x+(\tilde{A}u)_x]+\tilde{B}u, \quad P_1u=\tfrac{1}{2}[\tilde{\tilde{A}}u_x+(\tilde{\tilde{A}}u)_x]+\tilde{\tilde{B}}u,$$

where, by our Assumption 1.1 with $\omega=\infty$, the symmetric matrix $\tilde{A}$ has exactly $2p$ eigenvalues $\kappa\neq 0$.[2] Therefore, without restricting generality, $\tilde{A}$ and $\tilde{\tilde{A}}$ may be expressed in the form

$$\text{(2.1d)}\qquad \tilde{A}=\begin{pmatrix}A^{II} & 0\\ 0 & 0\end{pmatrix}, \quad A^{II} \text{ nonsingular}, \quad \tilde{\tilde{A}}=\begin{pmatrix}0 & 0\\ 0 & A^{IIII}\end{pmatrix},$$

where A^{II}, A^{IIII} denote the upper left and lower right blocks of order $2p$ and $n-2p$, respectively—a conventional notation which will be used throughout the paper.

In order to prove that the energy estimate (E) holds, we need a special form of the symbol $\hat{P}_0(i\omega)$. In our next theorem we discuss this in a somewhat generalized formulation which is going to be used later on.

THEOREM 2.1 (Compare [2], Lemma 4.1). *Consider the pencil $\omega\mathcal{A}+\mathcal{B}$, where*

$$\mathcal{A}=\begin{pmatrix}A^{II} & 0\\ 0 & 0\end{pmatrix}$$

is a real symmetric matrix, $\mathcal{B}$ is a Hermitian matrix whose elements are either in $\mathbb{R}$ or $i\mathbb{R}$ (i.e., are either real or purely imaginary), and assume that, for all real ω, $|\omega|\leqq\infty$, it has rank $2p$. Then, there exists an ω-independent orthogonal matrix

$$\text{(2.2)}\qquad U=\begin{pmatrix}U^{II} & 0\\ 0 & U^{IIII}\end{pmatrix}$$

[2] The example

$$\tilde{A}=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \qquad \tilde{B}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

shows that indeed, it is necessary to consider the limit case $\omega=\infty$ for $\tilde{A}$ to have rank $2p$.

such that

$$\text{(2.3a)}\quad \omega U^*\mathcal{A}U + U^*\mathcal{B}U = \omega\begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{12}^* & A_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} & B_{13} & 0 \\ B_{12}^* & B_{22} & 0 & 0 \\ B_{13}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with

$$\text{(2.3b)}\qquad \operatorname{rank}(B_{13}) = m, \qquad \operatorname{rank}(\omega A_{22} + B_{22}) = 2(p-m),$$

where A_{11}, B_{11}, B_{13} and A_{22}, B_{22} are square blocks of order m and $2p-m$, respectively.

Proof: The eigenspace connected with $\kappa = 0$ consists of vectors $x = (x^I, x^{II})'$ which are determined by the system

$$\text{(2.4)}\qquad \begin{aligned} \omega A^{II}x^I + B^{II}x^I + B^{III}x^{II} &= 0, \\ B^{III*}x^I + B^{IIII}x^{II} &= 0. \end{aligned}$$

For $\omega = \infty$ the above system which, by the nonsingularity of A^{II}, becomes

$$x^I = 0, \qquad B^{IIII}x^{II} = 0,$$

satisfies our assumption of having rank $2p$ only if $B^{IIII} \equiv 0$. Then, the weakly coupled system (2.4) has the assumed rank $2p$ only if $\operatorname{rank}(B^{III}) = m$ with $0 \leqq m \leqq p$.

We can construct orthogonal transformations U^{II}, U^{IIII} such that

$$U^{II*}B^{III}U^{IIII} = \begin{pmatrix} B_{13} & 0 \\ 0 & 0 \end{pmatrix},$$

where B_{13} is a nonsingular block of order m (for example, one may take $\begin{pmatrix} B_{13} & 0 \\ 0 & 0 \end{pmatrix}$ to be the singular value decomposition of B^{III}). Applying the orthogonal transformation $U = \begin{pmatrix} U^{II} & 0 \\ 0 & U^{IIII} \end{pmatrix}$, we obtain the desired form (2.3a), while system (2.4) which, for

$$A^{II} = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix}, \qquad B^{II} = \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix}$$

and

$$x^I = (x^{[1]}, x^{[2]})', \qquad x^{II} = (x^{[3]}, x^{[4]})',$$

partitioned correspondingly, becomes

$$\begin{aligned} (\omega A_{11} + B_{11})x^{[1]} + (\omega A_{12} + B_{12})x^{[2]} + B_{13}x^{[3]} &= 0, \\ (\omega A_{12}^* + B_{12}^*)x^{[1]} + (\omega A_{22} + B_{22})x^{[2]} \qquad &= 0, \\ + B_{13}^*x^{[1]} \qquad\qquad &= 0, \end{aligned}$$

has the assumed rank $2p$ only if rank (B_{13}) + rank $(\omega A_{22}+B_{22})$ + rank $(B_{13}^*) = 2p$, i.e., rank $(\omega A_{22}+B_{22}) = 2(p-m)$. This completes the proof of the theorem.

The construction of the othogonal transformation

$$U=\begin{pmatrix} U^{II} & 0 \\ 0 & U^{IIII} \end{pmatrix}$$

in (2.2) is such that

$$U^{II*}B^{III}U^{IIII}=\begin{pmatrix} B_{13} & 0 \\ 0 & 0 \end{pmatrix},\qquad \text{rank } (B^{III}) = \text{rank } (B_{13}). \tag{2.5}$$

We postulate

ASSUMPTION 2.1 ([2], Assumption 4.2). *The matrix B^{III} has fixed rank independent of x, t.*

Then, employing Theorem 2.1 for $\omega\tilde{A}+i\tilde{B}$, we can construct an orthogonal matrix

$$U=\begin{pmatrix} U^{II} & 0 \\ 0 & U^{IIII} \end{pmatrix}$$

smoothly depending on x, t which transforms $\omega\tilde{A}+i\tilde{B}$ to the normal representation (2.3). Introducing a new variable

$$u\to\begin{pmatrix} U^{II} & 0 \\ 0 & U^{IIII} \end{pmatrix}u$$

into system (2.1), we find that, corresponding to rank $(B^{III}) = m$, $0 \leqq m \leqq p$, the operator P_0 has one of $p+1$ possible normal representations which we split up into the following three normal forms:

The *first normal form* (with $m=0$), where

$$P_0u=\frac{1}{2}\left[\begin{pmatrix} A^{II} & 0 \\ 0 & 0 \end{pmatrix}\frac{\partial u}{\partial x}+\frac{\partial}{\partial x}\left(\begin{pmatrix} A^{II} & 0 \\ 0 & 0 \end{pmatrix}u\right)\right]+\begin{pmatrix} B^{II} & 0 \\ 0 & 0 \end{pmatrix}u, \tag{2.6a}$$

with $2p$-order blocks satisfying

$$|i\omega A^{II}+B^{II}|\neq 0,\qquad \text{for all real } \omega,\qquad |\omega|\leqq\infty; \tag{2.6b}$$

the *second normal form* (with $m=p$), where

$$\begin{aligned} P_0u=\frac{1}{2}&\left[\begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{12}^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\frac{\partial u}{\partial x}+\frac{\partial}{\partial x}\left(\begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{12}^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}u\right)\right] \\ &+\begin{pmatrix} B_{11} & B_{12} & B_{13} & 0 \\ -B_{12}^* & 0 & 0 & 0 \\ -B_{13}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}u, \end{aligned} \tag{2.7a}$$

with all non-zero blocks being of order p and

(2.7b) $$|A_{12}|\cdot|B_{13}|\neq 0;$$

the *third normal form* (with $1\leqq m\leqq p-1$, $p>1$), where

$$P_0u=\frac{1}{2}\left[\begin{pmatrix}A_{11} & A_{12} & 0\\ A_{12}^* & A_{22} & 0\\ 0 & 0 & 0\end{pmatrix}\frac{\partial u}{\partial x}+\frac{\partial}{\partial x}\left(\begin{pmatrix}A_{11} & A_{12} & 0\\ A_{12}^* & A_{22} & 0\\ 0 & 0 & 0\end{pmatrix}u\right)\right]$$
$$+\begin{pmatrix}B_{11} & B_{12} & B_{13} & 0\\ -B_{12}^* & B_{22} & 0 & 0\\ -B_{13}^* & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}u,$$

A_{11}, B_{11}, B_{13} and A_{22}, B_{22} being square blocks of order m and $2p-m$, respectively, and

$$\operatorname{rank}(i\omega A_{22}+B_{22})=2(p-m).$$

Upon applying the orthogonal transformation $U=\operatorname{diag}[I_m;U_{22};I_{n-2p}]$, where U_{22} diagonalizes A_{22}, we find that the third normal representation of the operator P_0 takes the form

(2.8a)
$$P_0u=\frac{1}{2}\left[\begin{pmatrix}A_{11} & A_{12,1} & A_{12,2} & 0\\ A_{12,1}^* & \begin{pmatrix}A_{22,1}\\ 0\end{pmatrix} & \begin{pmatrix}0\\ 0\end{pmatrix} & 0\\ A_{12,2}^* & & & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\frac{\partial u}{\partial x}+\frac{\partial}{\partial x}\left(\begin{pmatrix}A_{11} & A_{12,1} & A_{12,2} & 0\\ A_{12,1}^* & \begin{pmatrix}A_{22,1}\\ 0\end{pmatrix} & \begin{pmatrix}0\\ 0\end{pmatrix} & 0\\ A_{12,2}^* & & & 0\\ 0 & 0 & 0 & 0\end{pmatrix}u\right)\right]$$
$$+\begin{pmatrix}B_{11} & B_{12,1} & B_{12,2} & B_{13} & 0\\ -B_{12,1}^* & \multicolumn{2}{c}{\multirow{2}{*}{(B_{22})}} & 0 & 0\\ -B_{12,2}^* & & & 0 & 0\\ -B_{13}^* & 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}}\\ 0 & 0 & 0 & & \end{pmatrix}u,$$

$A_{12,2}$, B_{13}, being m-th order nonsingular blocks and

(2.8b) $$\operatorname{rank}\left[i\omega\begin{pmatrix}A_{22,1} & 0\\ 0 & 0\end{pmatrix}+B_{22}\right]=2(p-m),\quad\text{for all real }\omega,\quad |\omega|\leqq\infty.^3$$

Remarks. (i) We note that the first and second normal forms may be considered as limit cases of the third one, with $m=0$ and $m=p$, respectively.

[3] Since $\operatorname{rank}(i\omega A_{22}+B_{22})=2(p-m)$ we have $\operatorname{rank}(A_{22,1})\leqq 2(p-m)$ and clearly $\operatorname{rank}(A_{12,2})\leqq m$. Equating $\operatorname{rank}(A^{II})=\operatorname{rank}(A_{12,2})+\operatorname{rank}(A_{22,1})+\operatorname{rank}(A_{12,2}^*)=2p$, we find that equality must take place in the first two weak inequalities. Hence $A_{12,2}$ is nonsingular and (2.8b) holds also for $\omega=\infty$.

(ii) By Assumption 1.1, the dimension of system (2.1), n, is not less than $2p$ and therefore the first normal form (2.6) can always exist. In the second and third normal forms (the former is considered as the limit case $m=p$ of the latter), the variation of rank $(B^{III}) = \text{rank}\,(B_{13}) = m$, $1 \leqq m \leqq p$, is further restricted by

$$1 \leqq m \leqq \min(p, n-2p), \qquad n > 2p. \tag{2.9a}$$

In particular, the second normal form can exist only if the dimension n satisfies

$$n \geqq 3p. \tag{2.9b}$$

For A^{II} we have in all three normal cases

$$(-1)^p |A^{II}| > 0. \tag{2.10}$$

Indeed, in the first normal form, (2.10) follows by expanding

$$0 \neq |i\omega A^{II} + B^{II}| \equiv (-1)^p \omega^{2p} |A^{II}| + \cdots + |B^{II}|, \qquad |\omega| \leqq \infty,$$

and noting that $|B^{II}| > 0$ as a determinant of a nonsingular antisymmetric matrix. In the second normal form, (2.10) follows by multiplying

$$A^{II} = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & 0 \end{pmatrix}$$

by the $2p$-dimensional matrix $J_{2p} = \text{antidiag}\,(1; 1; \cdots; 1)$,

$$|J_{2p}|\,|A^{II}| = |J_p A_{12}^*|\,|J_p A_{12}| = |A_{12}|^2,$$

and noting that $|J_{2p}| = (-1)^p$. Finally, in the third normal form, (2.10) follows by multiplying $J_{2p} A^{II}$,

$$|J_{2p}|\,|A^{II}| = |J_m A_{12,2}^*|\,|J_{2p-2m} A_{22,1}|\,|J_m A_{12,2}| = (-1)^{p-m} |A_{22,1}|\,|A_{12,2}|^2,$$

where by induction $(-1)^{p-m}|A_{22,1}| > 0$.

From (2.10) it follows that in the particular case $p=1$ (which is, for example, the case of Euler equations with the sound speed representing the fast scale) we have $p=1$ pairs of characteristic velocities traveling in opposite directions. In our next lemma we show that this result holds also in the general case $p \geqq 1$.

LEMMA 2.1. *The $2p$ fast characteristic velocities of system* (2.1) *consist of p positive velocities and p negative ones.*

Proof: In the first normal form (2.6), we consider the characteristic polynomial

$$Q(r, \omega) \equiv |rI_{2p} - (\omega A^{II} + iB^{II})| \tag{2.11}$$

whose r-roots are real, and whose number of signed roots is independent of ω since, by (2.6b), $Q(r=0,\omega)\neq 0$. Taking $\omega=0$ we find that the $2p$ roots of $Q(r,\omega=0)$ are the eigenvalues of iB^{II} (B^{II} antisymmetric) which split into p pairs, each of which consists of two real eigenvalues with different signs. Letting $\omega\to\infty$, we see that the eigenvalues of A^{II} are split similarly.

In the second normal form (2.7), we note that, by (2.7b), A_{12} is nonsingular and hence $A_{12}A_{12}^*$ is positive definite. It follows that there exists a nonsingular matrix U_{11}, which diagonalizes (under congruence) both $A_{12}A_{12}^*>0$ and A_{11} into I_p and some diagonal matrix, say Λ, respectively. Let

$$U^{II}=\begin{pmatrix} U_{11} & 0 \\ 0 & I_p \end{pmatrix};$$

then the number of signed eigenvalues κ of A^{II} is determined by

$$(2.12)\quad |\kappa I_{2p}-U^{II}A^{II}U^{II*}|=\left|\begin{pmatrix} \kappa I_p-\Lambda & -U_{11}A_{12} \\ -A_{12}^*U_{11}^* & \kappa I_p \end{pmatrix}\right|=|\kappa^2 I_p-\kappa\Lambda-I_p|,$$

i.e., $\kappa_{j+}\kappa_{j-}=-1$.

In the third normal form (2.8) we use induction on the rank $2p$. For $p=1$ the result follows from (2.10). (In fact, in the special case $p=1$, only the first two normal forms which have already been discussed above can exist.) By the induction assumption, (2.8b) implies that $A_{22,1}$ has $p-m$ positive eigenvalues and $p-m$ negative ones and hence is congruently similar to

$$U_{22,1}A_{22,1}U_{22,1}^*=\begin{pmatrix} I_{p-m} & 0 \\ 0 & -I_{p-m} \end{pmatrix}.$$

Let V denote the nonsingular (and in fact unitary) matrix

$$2^{-1/2}\begin{pmatrix} I_{p-m} & -I_{p-m} \\ I_{p-m} & I_{p-m} \end{pmatrix};$$

then upon applying $U^{II}=\operatorname{diag}[I_m;\,VU_{22,1};\,I_m]$ we find that A^{II} is congruently similar to (and therefore has the same number of signed eigenvalues as) the block upper antidiagonal matrix

$$(2.13)\qquad U^{II}A^{II}U^{II*}=\begin{pmatrix} A_{11} & A_{12,1} & A_{12,2} \\ A_{12,1}^* & \begin{pmatrix} 0 & I \\ I & \end{pmatrix} & 0 \\ A_{12,2}^* & 0 & 0 \end{pmatrix}.$$

We recall that every (nonsingular block) upper antidiagonal symmetric matrix of order $2p$ has p positive eigenvalues and p negative ones.[4] This gives us the desired result in the third normal case as well.

Remark. While carrying out the proof of Lemma 2.1 we have used the fact that $\hat{P}_0(i\omega = 0) = \tilde{B}$ is nonsingular. If in Assumption 1.1 one excludes the case $\omega = 0$ (for example, if $\tilde{B} \equiv 0$), one can no longer split the fast velocities obtained earlier. Indeed, in case only the principle part of system (2.1) is considered, the eigenvalues of the first normal form can take arbitrary signs. In the second and third normal forms, there are exactly $2m$ eigenvalues, $1 \leqq m \leqq p$, (where, as usual, $m = p$ is related to the second form) split into m pairs of different signs; the remaining $2(p-m)$ eigenvalues may have arbitrary signs.

Finally, we close this section by recalling that P_1 always has the form

$$P_1 u = \frac{1}{2}\left[\begin{pmatrix} 0 & 0 \\ 0 & A^{IIII} \end{pmatrix}\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\left(\begin{pmatrix} 0 & 0 \\ 0 & A^{IIII} \end{pmatrix} u\right)\right] + \tilde{\tilde{B}} u. \tag{2.14}$$

Concerning A^{II} we make

ASSUMPTION 2.2 ([2], Assumption 5.2′). *There exist constants $K, \delta > 0$, such that*

$$\|A^{IIII^{-1}}\| \leqq K \quad \text{for} \quad 0 \leqq x \leqq \delta, \quad 1-\delta \leqq x \leqq 1. \tag{2.15}$$

We note that introducing a new variable $u_*(x, t) = \psi(x) u(x, t)$ into system (2.1), where $\psi(x) \in C_0^\infty$ is the cut-off function,

$$\psi(x) = \begin{cases} 1 & \text{for } \delta \leqq x \leqq 1-\delta, \\ 0 & \text{for } x \leqq \delta/2, x \geqq 1-\delta/2, \end{cases}$$

we can assume, without restricting generality, that A^{IIII} is nonsingular in the interior domain as well (for details see [2], Section 5).

3. One-Dimensional Problem of the First Normal Form

We consider the system (2.1) with P_0 of the first normal form (2.6), i.e., for $u = (u^I, u^{II})'$ we have the system of equations

$$\begin{aligned} u_t^I &= \frac{1}{\varepsilon} P^{II} u^I + \tilde{\tilde{B}}^{III} u^{II} + F^I, \\ u_t^{II} &= P^{IIII} u^{II} - \tilde{\tilde{B}}^{III*} u^I + F^{II}, \end{aligned} \tag{3.1a}$$

[4] This result was in fact derived in the above discussion of the second normal form. Indeed, in our special case (2.13), one may take

$$A_{12} \text{ as } \begin{pmatrix} * & A_{12.2} \\ I & 0 \end{pmatrix}, \qquad A_{11} \text{ as } \begin{pmatrix} A_{11} & * \\ * & 0 \end{pmatrix}$$

(where $*$ stands for the appropriate rectangular matrices) and then employ the congruent similarity introduced in (2.12).

where

$$(3.1b)\qquad \begin{aligned} P^{II}u^{I} &= \tfrac{1}{2}[A^{II}u_x^{I} + (A^{II}u^{I})_x] + \tilde{B}^{II}u^{I}, && \|A^{II^{-1}}\| \leq \text{const.} \\ P^{IIII}u^{II} &= \tfrac{1}{2}[A^{IIII}u_x^{II} + (A^{IIII}u^{II})_x] + \tilde{\tilde{B}}^{IIII}u^{II}, && \|A^{IIII^{-1}}\| \leq \text{const.} \end{aligned}$$

($\tilde{B}^{II}$—given by $B^{II} + \varepsilon \tilde{\tilde{B}}^{II}$—is smooth and of order $O(1)$ like all the other blocks).

Without loss of generality we may assume that the matrices A^{II}, A^{IIII} are diagonal at $x = 0, 1$; i.e.,

$$(3.2)\quad A^{II}_{|x=0,1} = \begin{pmatrix} \Lambda_\alpha & 0 \\ 0 & -\Lambda_\beta \end{pmatrix}_{|x=0,1}, \quad A^{IIII}_{|x=0,1} = \begin{pmatrix} \Gamma_\alpha & 0 \\ 0 & -\Gamma_\beta \end{pmatrix}, \quad \Lambda_j, \Gamma_j > 0, \quad j = \alpha, \beta.$$

Then the left and right boundary conditions (2.1b) with the corresponding partitioning $u^{I} = (u^{I}_\alpha, u^{I}_\beta)'$, $u^{II} = (u^{II}_\alpha, u^{II}_\beta)'$ can be written, respectively, as

$$(3.3a)\qquad \begin{pmatrix} u^{I}_\beta(0,t) \\ u^{II}_\beta(0,t) \end{pmatrix} = \begin{pmatrix} L_{II} & L_{III} \\ L_{III} & L_{III} \end{pmatrix} \begin{pmatrix} u^{I}_\alpha(0,t) \\ u^{II}_\alpha(0,t) \end{pmatrix},$$

and

$$(3.3b)\qquad \begin{pmatrix} u^{I}_\alpha(1,t) \\ u^{II}_\alpha(1,t) \end{pmatrix} = \begin{pmatrix} R_{II} & R_{III} \\ R_{III} & R_{IIII} \end{pmatrix} \begin{pmatrix} u^{I}_\beta(1,t) \\ u^{II}_\beta(1,t) \end{pmatrix},$$

expressing the dependence of the incoming characteristic variables on the outgoing ones. Here L_{ij}, R_{ij}, $i, j = I, II$, are in general rectangular blocks (in fact, in our special case, Lemma 2.1 implies that they are all square blocks of order p), and for simplicity only, are assumed to be independent of t. Otherwise the additional inhomogeneous boundary terms, generated in the time differentiated system which we intend to estimate, can be eliminated by subtracting the appropriately constructed vector function[5] and replacing the inhomogeneous term F by some other smooth vector F_* (for details see [2], Lemma 5.1).

By Assumption 1.2, the operator

$$P = \begin{pmatrix} \dfrac{1}{\varepsilon} P^{II} & \tilde{\tilde{B}}^{III} \\ -\tilde{\tilde{B}}^{III*} & P^{IIII} \end{pmatrix}$$

is half-bounded; i.e., for all $u = (u^{I}, u^{II})'$ satisfying the boundary conditions (3.3), we have

$$(3.4)\quad \mathcal{R}e\,(Pu, u) = \mathcal{R}e \left[\frac{1}{\varepsilon}(P^{II}u^{I}, u^{I}) + (P^{IIII}u^{II}, u^{II}) \right] \leq \alpha(\|u^{I}\|^2 + \|u^{II}\|^2),$$

with some constant α *independent of* ε. Hence, by considering $u = (u^{I}, u^{II})'$ where

[5] With the help of Lemma 3.1 below.

$u^I(x,t)$ satisfies

$$D_{II}u^I=0:\ \begin{aligned} u^I_\beta(0,t)&=L_{II}u^I_\alpha(0,t),\\ u^I_\alpha(1,t)&=R_{II}u^I_\beta(1,t), \end{aligned} \tag{3.5}$$

and $u^{II}(x,t)=(u^{II}_\alpha(x,t),u^{II}_\beta(x,t))'$ smoothly connecting $u^{II}(0,t)=(0,L_{III}u_\alpha(0,t))'$ with $u^{II}(1,t)=(R_{III}u^I_\beta(1,t),0)'$, it follows that

$$\mathcal{R}e\,(P^{II}u^I,u^I)\leqq 0 \quad \text{for} \quad u^I \text{ satisfying } D_{II}u^I=0. \tag{3.6}$$

That is, P^{II} is strictly half-bounded (there is no energy growth in time). Integrating (3.6) by parts and taking into account (3.5) we obtain

$$\begin{aligned} \mathcal{R}e\,(P^{II}u^I,u^I)&=u^{I*}(x,t)A^{II}(x,t)u^I(x,t)|_{x=0}^{x=1}\\ &=u^{I*}_\beta(1)[R^*_{II}\Lambda_\alpha(1,t)R_{II}-\Lambda_\beta(1,t)]u^I_\beta(1)\\ &\quad+u^{I*}_\alpha(0)[L_{II}\Lambda_\beta(0,t)L_{II}-\Lambda_\alpha(0,t)]u^I_\alpha(0)\leqq 0 \end{aligned} \tag{3.7}$$

with arbitrary $u^I_\alpha(0)\equiv u^I_\alpha(0,t)$, $u^I_\beta(1)\equiv u^I_\beta(1,t)$. Hence Assumption 1.2 implies (3.6), and from (3.7) one derives the standard inequalities

$$L^*_{II}\Lambda_\beta(0,t)L_{II}-\Lambda_\alpha(0,t)\leqq 0, \tag{3.8a}$$

$$R^*_{II}\Lambda_\alpha(1,t)R_{II}-\Lambda_\beta(1,t)\leqq 0. \tag{3.8b}$$

Thus the boundary values are reflected in such a way that no energy enters the interior domain through the boundaries.

Now if we slightly strengthen the weak inequalities (3.8) by requiring the

DISSIPATIVITY CONDITION. *At least one of the strict inequalities*

$$L^*_{II}\Lambda_\beta(0,t)L_{II}-\Lambda_\alpha(0,t)<0, \quad R^*_{II}\Lambda_\alpha(1,t)R_{II}-\Lambda_\beta(1,t)<0, \tag{3.9}$$

holds.

I.e., requiring the boundary conditions to be dissipative rather than only energy conserving, we are able to prove

LEMMA 3.1 ([2], Assumption 5.2). *For every smooth F^I, the two-point boundary value problem*

$$P^{II}v^I=F^I, \quad D_{II}v^I=g \tag{3.10a}$$

has a unique solution, satisfying

$$\|v^I\|+\|v^I_x\|\leqq K(\|F^I\|+|g|). \tag{3.10b}$$

Proof: Assume $\lambda=0$ is an eigenvalue of (3.10a) with corresponding eigensolution $\phi\neq 0$ satisfying

$$P^{II}\phi=0, \qquad D_{II}\phi=0. \tag{3.11}$$

Multiplying (3.11) by ϕ and integrating by parts we obtain

$$0=\mathcal{R}e\,(P^{II}\phi,\phi)=\phi_\beta^*[R_{II}^*\Lambda_\alpha R_{II}-\Lambda_\beta]\phi_{\beta|x=1}+\phi_\alpha^*[L_{II}^*\Lambda_\beta L_{II}-\Lambda_\alpha]\phi_{\alpha|x=0}.$$

By the Dissipative Condition (3.9), it follows that either $\phi_\alpha(x)_{|x=0}=0$ or $\phi_\beta(x)_{|x=1}=0$ and therefore in view of the boundary conditions (3.11), either $\phi(x)_{|x=0}=0$ or $\phi(x)_{|x=1}=0$. Hence $\phi\equiv 0$, contradicting the assumption that $\lambda=0$ is an eigenvalue, and (3.10) follows.

By Assumption 1.2, P is half-bounded. Hence the energy estimate (1.4) is satisfied, and, by Lemma 3.1, system (3.10a) is uniquely solvable. Thus, both Assumptions 5.1 and 5.2 of [2] hold, implying

THEOREM 3.1 ([2], Theorem 5.1). *The system* (2.1) *with* P_0 *of the first normal form* (2.6), *satisfies the energy estimate* (E).

Remark. Alternatively one can prove Theorem 3.1 by replacing the Dissipative Condition (3.9) by the somewhat milder assumption (3.10)—see [2], Section 5. The Dissipative Condition was introduced here, however, to indicate that (3.9) followed by (3.10) actually places a very weak additional limitation on our system, both theoretically and practically. Indeed, the boundary coefficients L_{II}, R_{II} which, by Assumption 1.2, are *weakly* restricted by (3.8), are to be further restricted by the similar *strict* inequalities (3.9).

Satisfying the energy estimate (E), it follows from Theorem 3.1 that the solution of (2.1) will remain bounded independently of ε^{-1} if its initial conditions are chosen so that the initial time derivatives are bounded. We refer to [2], Theorem 5.2, in discussing the procedure to construct such initial conditions. (Note that if, in particular, $n=2p$, then for $\partial^j u/\partial t^j|_{t=0}$, $j=0,1,\cdots,\nu$, to be bounded, $\|u(t=0)\|$ must be, as can be expected, of order $O(\varepsilon^\nu)$, since no slow scale variables are present in the system.)

4. One-Dimensional Problem of the Second Normal Form

We consider the system (2.1) with P_0 of the second normal form (2.7), i.e.,

$$P_0u=\frac{1}{2}\left[\begin{pmatrix}A_{11}&A_{12}&0\\A_{12}^*&0&0\\0&0&0\end{pmatrix}\frac{\partial u}{\partial x}+\frac{\partial}{\partial x}\left(\begin{pmatrix}A_{11}&A_{12}&0\\A_{12}^*&0&0\\0&0&0\end{pmatrix}u\right)\right]$$
$$+\begin{pmatrix}B_{11}&B_{12}&B_{13}&0\\-B_{12}^*&0&0&0\\-B_{13}^*&0&0&0\\0&0&0&0\end{pmatrix}u, \tag{4.1}$$

where A_{12}, B_{13} are nonsingular matrices of order p.

Denote by $u^{[1]}$ the first p components of u,

$$u^{[1]}=(u^{(1)},\cdots,u^{(p)})'. \tag{4.2}$$

Then, starting with row $p+1$, the next $2p$ equations of system (2.1) take the form

$$A^*_{12}u^{[1]}_x+R_1u^{[1]}=\varepsilon\psi_1\{u,u_t,F\}, \tag{4.3a}$$

$$-B^*_{13}u^{[1]}=\varepsilon[S_1u_x+\psi_2\{u,u_t,F\}]. \tag{4.3b}$$

Here, $\psi_j\{\cdot,\cdot,\cdot\}$ are bounded linear vector functions of their arguments (a property which we shall denote by curly brackets) and S_2 is a $p\times n$ rectangular matrix consisting of the first p rows of $[0_{(n-2p)\times p};A^{IIII}_{(n-2p)\times(n-2p)}]$ (by (2.9b), they exist).

Following [2], Section 6, the way to derive the energy estimate (E) in the second normal form is to show first that $u^{[1]}$—by satisfying the overdetermined system (4.3)—must be of order $O(\varepsilon)$. We shall therefore prove

LEMMA 4.1 ([2], Lemma 6.2). *There exists a constant $K>0$, such that for $u^{[1]}$ satisfying* (4.3) *we have*

$$\|u^{[1]}_x\|+\|u^{[1]}\|\leqq\varepsilon K(\|u\|+\|u_t\|+\|F\|). \tag{4.4}$$

Proof: We give a somewhat simpler version of the proof of Lemma 6.1 in [2]. By Assumption 2.1 the nonsingular $-B^*_{13}$ has a bounded inverse for $0\leqq x\leqq 1$, and therefore (4.3b) can be rewritten as

$$u^{[1]}=\varepsilon[(S_2u)_x+\chi_2\{u,u_t,F\}],\quad \|S_2\|\leqq\text{const}. \tag{4.5}$$

Let

$$v^{[1]}=S_2u+\int_0^x\chi_2(\xi)\,d\xi,\qquad \chi_2(\xi)\equiv\chi_2\{u(\xi,t),u_t(\xi,t),F(\xi,t)\}. \tag{4.6a}$$

Then (4.5) becomes

$$u^{[1]}=\varepsilon v^{[1]}_x. \tag{4.6b}$$

By inserting this into (4.3a), which is first divided by the nonsingular A^*_{12}, we obtain

$$v^{[1]}_{xx}+R_2v^{[1]}_x=\chi_1\{u,u_t,F\},\quad \|R_2\|\leqq\text{const}. \tag{4.7}$$

Now, for sufficiently small $\eta>0$ and smooth w we have a Sobolev inequality:

$$\|w_x\|\leqq[\eta^2\|w_{xx}\|^2+\eta^{-2}\|w\|^2]^{1/2}\leqq\eta\|w_{xx}\|+\eta^{-1}\|w\|. \tag{4.8}$$

Then, by (4.7), (4.8) we estimate

$$\|v^{[1]}_{xx}\|\leqq\text{const.}\,(\|v^{[1]}_x\|+\|\chi_1\|)\leqq\text{const.}\,(\eta\|v^{[1]}_{xx}\|+\eta^{-1}\|v^{[1]}\|+\|\chi_1\|). \tag{4.9}$$

By choosing η small enough, (4.9) implies

$$\|v_{xx}^{[1]}\| \leqq \text{const.}\,(\|v^{[1]}\| + \|\chi_1\|) \tag{4.10}$$

which, using (4.6a) to estimate $v^{[1]}$ in terms of u and χ_2, gives us

$$\varepsilon^{-1}\|u_x^{[1]}\| = \|v_{xx}^{[1]}\| \leqq \text{const.}\,(\|u\| + \|u_t\| + \|F\|). \tag{4.11}$$

Finally, by (4.8) we can estimate $\|v_x^{[1]}\|$ in terms of $\|v_{xx}^{[1]}\|$ and $\|v^{[1]}\|$, and using (4.10), (4.6a) we obtain

$$\varepsilon^{-1}\|u^{[1]}\| = \|v_x^{[1]}\| \leqq \text{const.}\,(\|u\| + \|\chi_2\| + \|\chi_1\|) \leqq \text{const.}\,(\|u\| + \|u_t\| + \|F\|), \tag{4.12}$$

which, together with (4.11), yields (4.4).

With the help of Lemma 4.1 we can now prove

THEOREM 4.1 ([2], Theorem 6.1). *The system* (2.1), *with* P_0 *of the second normal form* (2.7), *satisfies the energy estimate* (E).

Proof: The proof proceeds as in Theorem 6.1 of [2]. We first want to estimate the first time derivative u_t, by considering the differentiated system (2.1) with respect to t:

$$\dot{u}_t = \frac{1}{\varepsilon}P_0\dot{u} + P_1\dot{u} + \dot{F} + \frac{1}{\varepsilon}\dot{P}_0 u + \dot{P}_1 u, \qquad \cdot \equiv \frac{\partial}{\partial t}. \tag{4.13}$$

The last $n-2p$ equations of (2.1), after eliminating the space derivative by using Assumption 2.2 about the nonsingularity of A^{IIII}, give us

$$\|u_x^{II}\| \leqq \text{const.}\,(\varepsilon^{-1}\|u^{[1]}\| + \|u\| + \|u_t\| + \|F\|). \tag{4.14}$$

Hence, by Lemma 4.1,

$$\|\dot{P}_1 u\| \leqq \text{const.}\,(\|u_x^{II}\| + \|u\|) \leqq \text{const.}\,(\|u\| + \|u_t\| + \|F\|). \tag{4.15}$$

Moreover, denote

$$u^{[2]} = (u^{(p+1)}, \cdots, u^{(2p)})', \qquad u^{[3]} = (u^{(2p+1)}, \cdots, u^{(3p)})' \tag{4.16}$$

(see (4.2)). Then using the first p equations of (2.1) and the nonsingularity of A_{12}, we have $\varepsilon^{-1}u_x^{[2]} = \phi\{u, u_t, F, \varepsilon^{-1}u_x^{[1]}, \varepsilon^{-1}(u^{[1]}, u^{[2]}, u^{[3]})'\}$. Thus we can rewrite

$$\frac{1}{\varepsilon}\dot{P}_0 u = \frac{1}{\varepsilon}Cu + \Phi\{u, u_t, F, \varepsilon^{-1}u_x^{[1]}\}, \tag{4.17a}$$

where

$$C = \begin{pmatrix} 0 & C_{12} & C_{13} & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{4.17b}$$

and, by Lemma 4.1,

(4.17c) $$\|\Phi\{u, u_t, F, \varepsilon^{-1}u_x^{[1]}\}\| \leqq \text{const.} (\|u\| + \|u_t\| + \|F\|).$$

Having obtained (4.15), (4.17) we now multiply the differentiated equation (4.13) by u_t and integrate over time; then the half-boundedness Assumption 1.2 implies[6]

(4.18a) $$\|u_t(x,t)\| \Big|_{t=0}^{t=T} \leqq (\alpha + \text{const.}) \int_0^T \|u_t\| \, dt + \varepsilon^{-1}\left[|(u, Cu)| \Big|_{t=0}^{t=T} + \left| \int_0^T (u, (Cu)_t) \, dt \right| \right] + K,$$

where the constant K is bounded by

(4.18b) $$K \leqq \text{const.} \left(\int_0^T \|u\|^2 + \|\dot{F}\|^2 + \|F\|^2 \, dt \right).$$

Now, we observe that $\varepsilon^{-1}(u, Cu)$ and $\varepsilon^{-1}(u, (Cu)_t)$ depend linearly on $\varepsilon^{-1}u^{[1]}$ and hence by Lemma 4.1 are bounded. Then, by (4.18) with $\alpha + \text{const.} \leqq 0$ (which, as noted in Section 1, is no restriction), we can estimate $\|u_t\|$ in terms of $\|u_t(x, t=0)\|$, $\|\dot{F}\|$, $\|F\|$ and $\|u\|$, where by Duhamel's principle $\|u\|$ in turn can be estimated in terms of $\|u(x, t=0)\|$ and $\|F\|$. Thus we arrive at an energy estimate for u_t. Using the differential equation (2.1), we can now also estimate u_x, thus proving estimate (E) for $\nu = 1$. Differentiating (2.1) repeatedly with respect to t gives us the energy estimate (E) for higher derivatives as well. This proves the theorem.

We shall now discuss how to prepare the initial data with bounded time derivatives and hence, by Theorem 4.1, guarantee a bounded solution for later times. Our system (2.1) with P_0, P_1 given, respectively, by (4.1), (2.14) takes the form (for simplicity assume the matrix coefficients to be constant)

(4.19a) $$\varepsilon u_t^{[1]} = A_{11}u_x^{[1]} + A_{12}u_x^{[2]} + B_{11}u^{[1]} + B_{12}u^{[2]} + B_{13}u^{[3]} + \varepsilon \tilde{B}^{II[1]}u^{I} + \varepsilon \tilde{B}^{III[1]}u^{II} + \varepsilon F^{[1]},$$

(4.19b) $$\varepsilon u_t^{[2]} = A_{12}^{*}u_x^{[1]} - B_{12}^{*}u^{[1]} + \varepsilon \tilde{B}^{II[2]}u^{I} + \varepsilon \tilde{B}^{III[2]}u^{II} + \varepsilon F^{[2]},$$

(4.19c) $$u_t^{II} = -\varepsilon^{-1}B_{13}^{*}u^{[1]} + A^{IIII}u_x^{II} - \tilde{B}^{III*}u^{I} + \tilde{B}^{IIII}u^{II} + F^{II},$$

where the superscript $[j]$, $j = 1, 2, 3$, denotes appropriate partitioning corresponding to (4.2), (4.16).

[6] For simplicity only we assume that the boundary conditions (2.1b) are independent t, so by (1.2), $\mathcal{R}e\,(\dot{u}, P_0\dot{u}) \leqq \alpha\|\dot{u}\|^2$. Otherwise, the extra inhomogeneous boundary terms $\dot{D}_\eta u(\eta, t)$, $\eta = 0, 1$, can be eliminated by subtracting an appropriately constructed vector function with corresponding update of F.

Let V^{II} be unitary and diagonalize A^{II}. Then by Lemma 2.1 exactly p variables of $V^{II}u^{I}$, $u^{I}=(u^{[1]}, u^{[2]})'$, are inflow variables and the remaining p are outflow ones. In view of Lemma 4.1, $u^{[1\varepsilon]}\equiv\varepsilon^{-1}u^{[1]}$ should be of order $O(1)$; thus naturally the boundary conditions should connect $u^{[1\varepsilon]}$ and $u^{[2]}$ with u^{II}. Hence, the fast scale variables of the system are determined at the boundaries by

(4.20a) $$u^{[1\varepsilon]}(0,t)=L_{[1\varepsilon],2}u^{[2]}(0,t)+L_{[1\varepsilon],II}u^{II}(0,t),$$

(4.20b) $$u^{[1\varepsilon]}(1,t)=R_{[1\varepsilon],2}u^{[2]}(1,t)+R_{[1\varepsilon],II}u^{II}(1,t).$$

For the slow scale variables, u^{II}, we have (see (3.3))

(4.20c) $$u^{II}_{\beta}(0,t)=L_{II,2}u^{[2]}(0,t)+L_{IIII}u^{II}_{\alpha}(0,t),$$

(4.20d) $$u^{II}_{\alpha}(1,t)=R_{II,2}u^{[2]}(1,t)+R_{IIII}u^{II}_{\beta}(1,t).$$

We want to assure that the first time derivative, u_t, is bounded. By Lemma 4.1, therefore,

(4.21a) $$u^{[1\varepsilon]}\equiv\varepsilon^{-1}u^{[1]}=O(1)$$

with $u^{[1\varepsilon]}_x$ also of order $O(1)$. Having obtained (4.21a), the boundedness of $u^{[2]}_t$, u^{II}_t follows from (4.19b, c). For $u^{[1]}_t$ to be bounded we also need

(4.21b) $$A_{12}u^{[2]}_x+B_{12}u^{[2]}+B_{13}u^{[3]}=O(\varepsilon),$$

by (4.19a). To satisfy (4.21), determine u^{II} [7] and the right-hand side of (4.21b). Next, by the boundary conditions (4.20c, d) at most p components of $u^{[2]}$ are determined at either $x=0$ or $x=1$,[7] and the remaining components, if any, are chosen at one point. Thus $u^{[2]}$ is uniquely determined by (4.21b). Finally, we smoothly define $u^{[1\varepsilon]}$ between its boundary values $x=0$ and $x=1$, given, respectively, by (4.20a) and (4.20b).[7]

Higher derivatives can be handled similarly repeating differentiation of (4.19) with respect to t. In particular, let $\varepsilon\to 0$. Then $(u^{[1\varepsilon]}, u^{[2]}, u^{II})'$ converge to the solution of the reduced system

(4.22a) $$A_{12}w^{[2]}_x+B_{12}w^{[2]}+B_{13}w^{[3]}=0,$$

(4.22b) $$w^{[2]}_t=A^*_{12}w^{[1]}_x-B^*_{12}w^{[1]}+\tilde{B}^{II[2]}w^{I}+\tilde{B}^{III[2]}w^{II}+F^{[2]},$$

(4.22c) $$w^{II}_t=-B^*_{13}w^{[1]}+A^{IIII}w^{II}_x-\tilde{B}^{III[2]*}w^{[2]}+\tilde{B}^{IIII}w^{II}+F^{II},$$

and we can derive asymptotic expansions, for details see [2], Section 6.

[7] For simplicity assume u^{II} has at most (and therefore exactly) p components, $u^{II}\equiv u^{[3]}$, or otherwise handle $u^{II}\equiv(u^{[3]}, u^{[4]})'$ separately. Also assume $\tilde{B}^{III[1]}=0$ so that u^{II} and $u^{[1]}$ are not coupled through (4.19c). The case $\tilde{B}^{III[1]}\neq 0$ can be handled by expanding in terms of the solution of the reduced system (4.22) given below.

5. One-Dimensional Problem of the Third Normal Form

We consider the system (2.1) with P_0 of the third normal from (2.8), i.e.,

$$P_0 u = \frac{1}{2}\left[\begin{pmatrix} A_{11} & A_{12,1} & A_{12,2} & 0 \\ A^*_{12,1} & \begin{pmatrix} A_{22,1} & 0 \\ 0 & 0 \end{pmatrix} & & 0 \\ A^*_{12,2} & & & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\left(\begin{pmatrix} A_{11} & A_{12,1} & A_{12,2} & 0 \\ A^*_{12,1} & \begin{pmatrix} A_{22,1} & 0 \\ 0 & 0 \end{pmatrix} & & 0 \\ A^*_{12,2} & & & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}u\right)\right]$$

$$\text{(5.1a)} \qquad + \begin{pmatrix} B_{11} & B_{12,1} & B_{12,2} & B_{13} & 0 \\ -B^*_{12,1} & \multicolumn{2}{c}{(B_{22})} & 0 & 0 \\ -B^*_{12,2} & & & 0 & 0 \\ -B^*_{13} & 0 & 0 & \multicolumn{2}{c}{0} \\ 0 & 0 & 0 & & \end{pmatrix}u$$

with $A_{12,2}$, B_{13} nonsingular blocks of order m, $0<m<p$, and

$$\text{(5.1b)} \qquad \operatorname{rank}\left[i\omega\begin{pmatrix} A_{22,1} & 0 \\ 0 & 0 \end{pmatrix} + B_{22}\right] = 2(p-m).$$

By (5.1b) it follows (applying the same argument of letting $\omega \to \infty$ as in Theorem 2.1) that the lower right block in the corresponding partitioning of B_{22} must vanish. Thus

$$\text{(5.1c)} \qquad B_{22} = \begin{pmatrix} B_{22,1} & B_{22,2} \\ -B^*_{22,2} & 0 \end{pmatrix}.$$

Furthermore, the side condition (5.1b) suggests applying Theorem 2.1 once more in order to obtain the normal form of the main subsymbol

$$i\omega\begin{pmatrix} A_{22,1} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} B_{22,1} & B_{22,2} \\ -B^*_{22,2} & 0 \end{pmatrix}$$

and hence simplifying the normal form of the overall operator P_0 given in (5.1). Indeed, in our next lemma we show that by doing this repeatedly, we can take $B_{22,2}$ in (5.1c) to be zero as well, namely, the main subsymbol

$$i\omega\begin{pmatrix} A_{22,1} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} B_{22,1} & 0 \\ 0 & 0 \end{pmatrix}$$

is then of the first normal form.

LEMMA 5.1. *Consider the operator P_0 of the third normal form given by* (5.1). *Then there exists an orthogonal matrix*

$$\text{(5.2)} \qquad \mathbf{U} = \begin{pmatrix} \mathbf{U}^{II} & 0 \\ 0 & I \end{pmatrix}$$

such that

(5.3a) $$\mathbf{U}^*P_0\mathbf{U}u=\tfrac{1}{2}[\mathbf{A}u_x+(\mathbf{A}u)_x]+\mathbf{B}u,$$

where **A**, **B** *are given, respectively, by*

(5.3b)
$$\mathbf{A}=\begin{pmatrix}[\mathbf{A}_{11}] & [\mathbf{A}_{12,1}] & [\mathbf{A}_{12,2}] & 0\\ [\mathbf{A}_{12,1}^*] & [\mathbf{A}_{22,1}] & & \\ [\mathbf{A}_{12,2}^*] & & 0 & \\ 0 & & & \end{pmatrix},$$
$$\mathbf{B}=\begin{pmatrix}[\mathbf{B}_{11}] & [\mathbf{B}_{12,1}] & [\mathbf{B}_{12,2}] & [\mathbf{B}_{13}] & 0\\ [-\mathbf{B}_{12,1}^*] & [\mathbf{B}_{22,1}] & & & \\ [-\mathbf{B}_{12,2}^*] & & & & \\ [-\mathbf{B}_{13}^*] & & 0 & & \\ 0 & & & & \end{pmatrix}.$$

Here, the **m***-dimensional* $\mathbf{A}_{12,2}$ ($\mathbf{A}_{12,2}$ *is nonsingular*) *has a* (*block*) *upper antidiagonal form and* $\mathbf{B}_{12,2}$ *has a* (*block*) *upper antidiagonal Hessenberg form given, respectively, by*

(5.3c)
$$\mathbf{A}_{12,2}=\begin{pmatrix} * & \cdots & * & \cdots & [\mathbf{A}^{(1,1)}]\\ \vdots & & & \iddots & \\ * & & [\mathbf{A}^{(j,j)}] & & \\ \vdots & & & & 0\\ [\mathbf{A}^{(q,q)}] & \iddots & & & \end{pmatrix},$$
$$\mathbf{B}_{12,2}=\begin{pmatrix} * & \cdots & * & \cdots & \cdots & [\mathbf{B}^{(1,1)}]\\ \vdots & & & & [\mathbf{B}^{(2,2)}] & [\mathbf{B}^{(2,1)}]\\ * & & & \iddots & & \\ \vdots & & [\mathbf{B}^{(j,j)}] & [\mathbf{B}^{(j,j-1)}] & \iddots & \\ \vdots & \iddots & & & & 0\\ [\mathbf{B}^{(q,q)}] & [\mathbf{B}^{(q,q-1)}] & \iddots & & & \end{pmatrix},$$

where $\mathbf{A}^{(j,j)}$, $\mathbf{B}^{(j,j)}$, $\mathbf{B}^{(j,j-1)}$ *are* m_j*-dimensional blocks,* $\mathbf{A}^{(j,j)}$, $\mathbf{B}^{(j,j-1)}$ *and the* **m***-dimensional block* $\mathbf{B}_{13}$ *are nonsingular, with*

(5.3d) $$\mathbf{m}=\sum_{j=1}^{q}m_j\leqq p,\quad 0<m_q\leqq\cdots<m_2<m_1=m<p,$$

and

(5.3e) $$\operatorname{rank}(i\omega\mathbf{A}_{22,1}+\mathbf{B}_{22,1})=2(p-\mathbf{m}).$$

Proof: By Theorem 2.1, there exists an orthogonal $V^{(1)}$ which transforms the subsymbol

$$i\omega\begin{pmatrix} A_{22,1} & 0 \\ 0 & 0 \end{pmatrix} + B_{22}$$

into one of three possible normal forms.

In the case of the first normal form (see (2.6)) (5.3) is obtained with $q=1$ identifying $\mathbf{A}_{ij,k}$, $\mathbf{B}_{ij,k}$ and $\mathbf{m}=m_1$ in (5.3) with $A_{ij,k}$, $B_{ij,k}$ and m in (5.1), respectively.

In the case of the second normal form (see (2.7)), which by (2.9b) may exist only if $2m \geqq p$, (5.3) is obtained with $q=2$ and $m_1 = m \geqq m_2 = p-m$ (hence here $\mathbf{m} = m_1 + m_2 = p$ and $\mathbf{A}_{22,1} \equiv \mathbf{B}_{22,1} \equiv 0$).

In the case of the third normal form (see (2.8)), with some typical order of the corner blocks say m_*, where, by (2.9a), $0 < m_* < \min(p-m, m)$, a structure like (5.3) is obtained with $q=2$ and $m_1 = m > m_2 = m_*$, $\mathbf{m} = m_1 + m_2 < p$, consisting of a new subsymbol of rank $2(p-\mathbf{m})$ for which the normal characterization step described above can be applied once more.

Thus, after at most p such steps the result follows with

$$\mathbf{U}^{II} = \prod_j \begin{pmatrix} I & 0 \\ 0 & V^{(j)} \end{pmatrix}.$$

Remarks. (i) According to the iterative constructive proof of Lemma 5.1, it is clear that the final form (5.3) is obtained when the process is terminated in one of two ways: either with the subsymbol of the the first normal form where the weak inequalities in (5.3d) are *both* strict, or, with the subsymbol of the second normal form further reduced to be included in the corner blocks so that *both* equalities in (5.3d) hold. The subsymbol then vanishes.

(ii) As remarked in Section 2, the first and second normal forms may be considered as limit cases of the third normal form with $m=0$ and $m=p$, respectively. Thus, the form (5.3) with the side condition (5.3d) replaced by the modified condition

$$\mathbf{m} = \sum_{j=1}^{q} m_j, \qquad 0 \leqq m_q \leqq \cdots < m_2 < m_1 \leqq p, \tag{5.4}$$

to include also these two limit cases, gives us the most general form of the symbol $\hat{P}_0(i\omega)$.

Having the *explicit* representation of the third normal form as given by (5.3), we now may proceed to obtain the desired energy estimate (E). We do so by first using the special structure of the $\mathbf{m}$-dimensional corner blocks $\mathbf{A}_{12,2}$, $\mathbf{B}_{12,2}$ in (5.3c) to show that the first $\mathbf{m}$ components of the solution

$$\mathbf{u}^{[1]} \equiv (u^{(1)}, \cdots, u^{(\mathbf{m})})' \tag{5.5}$$

must be of order $O(\varepsilon)$ (compare—in the second normal form—Lemma 4.1).

LEMMA 5.2. *There exists a constant $K>0$, such that*

(5.6) $$\|u_x^{[1]}\|+\|\mathbf{u}^{[1]}\|\leqq \varepsilon K(\|u\|+\|u_t\|+\|F\|).$$

Proof: The row equations of (2.1), associated with the (non-zero) lower-left corner blocks of **A** and **B** in (5.3), are given, respectively, by

(5.7a) $$\mathbf{A}_{12,2}^*\mathbf{u}_x^{[1]}+R_1\mathbf{u}^{[1]}=\varepsilon\Psi\{u, u_t, F\},\qquad R_1\equiv \mathbf{A}_{12,2x}^*-\tfrac{1}{2}\mathbf{B}_{12,2}^*,$$

(5.7b) $$\mathbf{B}_{13}^*u^{[1]}=\varepsilon[S_1u_x+\Psi_1\{u, u_t, F\}],\qquad u^{[1]}\equiv(u^{(1)},\cdots,u^{(m_1)})'.$$

We introduce the compatible partitioning

(5.8a) $$\mathbf{u}^{[1]}=(u^{[1]},\cdots,u^{[q]})',$$

where

(5.8b) $$u^{[j]}\equiv(u^{(m_{j-1}+1)},\cdots,u^{(m_j)})',\quad j=1,2,\cdots,q,\qquad m_0\equiv$$

To simplify notations let us also denote $\mathbf{B}^{(1,0)}\equiv\mathbf{B}_{13}$. Equations (5.7b), (5.7a) then take the corresponding partitioned form

(5.9a) $$\mathbf{B}^{(1,0)*}u^{[1]}=\varepsilon[S_1u_x+\Psi_1\{u, u_t, F\}],\qquad j=1,$$

(5.9b) $$\mathbf{B}^{(j,j-1)*}u^{[j]}=\varepsilon\Psi_j\{u, u_t, F\}+\sum_{k=1}^{j-1}(R_{jk}u^{[k]}+S_{jk}u_x^{[k]}),$$
$$S_{j,j-1}=A^{(j,j)*},\qquad j=2,3,\cdots,q.$$

Trying to follow the proof of Lemma 4.1 we note that only $u^{[1]}$—the first component of $\mathbf{u}^{[1]}$—admits the overdetermined equation (5.9a) in order to give us an ε-order estimate. However, making use of the strong coupling between successive components of $\mathbf{u}^{[1]}$—as expressed by the nonsingularity of $\mathbf{B}^{(j,j-1)*}$, $\mathbf{A}^{(j,j)*}$ appearing in (5.9b)—we are able to show that each one of these components, $u^{[j]}$, can be estimated in terms of the former ones; Hence the ε-order estimate is also valid for the remaining $u^{[j]}$, $j=2,3,\cdots,q$. The detailed proof along these lines is given below.

Starting with (5.9a), the terms $u^{[k]}$, $k=1,\cdots,j-1$, appearing in the right-hand side of (5.9b) can be replaced by their explicit representation obtained from (5.9b) with $k=2,3,\cdots,j$, respectively. Thus $u^{[j]}$ can be expressed as a bounded[8] linear combination of $O(\varepsilon)$ terms and $u_x^{[k]}]_{k=1}^{j-1}$, namely

(5.10) $$u^{[j]}\equiv u^{[j]}\{\varepsilon u_x, \varepsilon\Psi_k]_{k=1}^{k=j}, u_x^{[k]}]_{k=1}^{k=j-1}\}.$$

Let us denote

(5.11) $$u^{[1,j]}\equiv(u^{[1]},\cdots,u^{[j]})',\qquad j=1,2,\cdots,q,$$

where in particular (see (5.8a)), $u^{[1,q]}\equiv\mathbf{u}^{[1]}$. Equations (5.7a)—after division by

[8] Using the nonsingulairty of $\mathbf{B}^{(j,j-1)*}$.

the nonsingular $\mathbf{A}^*_{12,2}$—and (5.10) can be rewritten, respectively, in the concise form

$$u_x^{[1,q]}+\mathbf{R}_q u^{[1,q]}=\varepsilon\mathbf{\Psi}_{1,q}\{u, u_t, F\}, \tag{5.12a}$$

$$u^{[1,j]}=\varepsilon[\mathbf{S}_j u_x+\mathbf{\Psi}_{2,j}\{u, u_t, F\}]+\mathbf{T}_j u_x^{[1,j-1]}, \qquad j=1,2,\cdots,q. \tag{5.12b}$$

(We understand that, for the case $j=1$, $u^{[1,0]}\equiv 0$, and (5.12b) is reduced to (5.9a).)

Equations (5.12a), and (5.12b) with $j=q$, give us for $u^{[1,q]}$ an overdetermined system similar to (4.3)—the one discussed in Lemma 4.1. Repeating the proof of Lemma 4.1 in our modified case (5.12) where there are additional $O(1)$ spatial derivatives in the right-hand side of (5.12b) (or alternatively use Lemma 4.1 directly, rewriting first (5.12b) with the help of $\mathbf{S}_{*q}\equiv\mathbf{S}_q+\varepsilon^{-1}\mathbf{T}_q$), we obtain

$$\|u_x^{[1,q]}\|+\|u^{[1,q]}\|\leqq \varepsilon K_0\|u\|+\|u_t\|+\|F\|+K_{q-1}\|u^{[1,q-1]}\|.$$

Thus $u_x^{[q]}$, $u^{[q]}$ are bounded in terms of the first $q-1$ components $u^{[k]}$, $k=1,2,\cdots,q-1$, which enable us to reduce (5.12a) to

$$u_x^{[1,q-1]}+\mathbf{R}_{q-1}u^{[1,q-1]}=\varepsilon\mathbf{\Psi}_{1,q-1}\{u, u_t, F\}. \tag{5.13a}$$

Using (5.12b) for $j=q-1$,

$$u^{[1,q-1]}=\varepsilon[\mathbf{S}_{q-1}u_x+\mathbf{\Psi}_{2,q-1}\{u, u_t, F\}]+\mathbf{T}_{q-1}u_x^{[1,q-2]}, \tag{5.13b}$$

we have a reduced overdetermined system satisfied by $u^{[1,q-1]}$ for which Lemma 4.1 can be applied once more.

After q such steps, where the estimate follows in the j-th step,

$$\|u_x^{[1,j]}\|+\|u^{[1,j]}\|\leqq K_0(\|u\|+\|u_t\|+\|F\|)+K_{j-1}\|u^{[1,j-1]}\| \tag{5.14}$$

is used to obtain a reduced system for $u^{[1,j-1]}$; we finally arrive at

$$u_x^{[1,1]}+\mathbf{R}_1 u^{[1,1]}=\varepsilon\mathbf{\Psi}_{1,1}\{u, u_t, F\}, \tag{5.15a}$$

$$u^{[1,1]}=\varepsilon[\mathbf{S}_1 u_x+\mathbf{\Psi}_{2,1}\{u, u_t, F\}]. \tag{5.15b}$$

By Lemma 4.1, $u^{[1]}\equiv u^{[1,1]}$ satisfies the estimate (4.4), and in view of (5.14) the same holds for $u^{[1,j]}$, $j=2,3,\cdots,q$. In particular, for $j=q$, $u^{[1,q]}\equiv\mathbf{u}^{[1]}$ satisfies the desired estimate (5.6) which completes the proof.

We continue by considering the $2(p-\mathbf{m})$-dimensional subsymbol $i\omega\mathbf{A}_{22,1}+\mathbf{B}_{22,1}$ associated with

$$P_{22,1}\equiv\frac{1}{2}\left[\mathbf{A}_{22,1}\frac{\partial}{\partial x}+\frac{\partial}{\partial x}(\mathbf{A}_{22,1}\cdot)\right]+\mathbf{B}_{22,1}. \tag{5.16}$$

(We exclude the trivial case whre $\mathbf{m}=p$—see Remark (i) above.) Let

$$\mathbf{u}^{[2]}\equiv(u^{(\mathbf{m}+1)},\cdots,u^{(2p-\mathbf{m})})', \quad \mathbf{u}^{[3]}\equiv(u^{(2p-\mathbf{m}+1)},\cdots,u^{(2p)})' \tag{5.17}$$

which together with (5.5) give us the partitioning which corresponds to that of the third normal form of $\mathbf{A}$ in (5.3b).

By Assumption 1.2 the operator P is half-bounded, namely, for all $\mathbf{u}=(\mathbf{u}^{[1]}, \mathbf{u}^{[2]}, \mathbf{u}^{[3]}, \mathbf{u}^{II})'$ satisfying the boundary conditions (2.1b) we have

$$\mathcal{R}e\,(P\mathbf{u}, \mathbf{u}) \equiv \frac{1}{\varepsilon}\mathbf{u}^{[2]*}\mathbf{A}_{22,1}\mathbf{u}^{[2]}|_{x=0}^{x=1} + \frac{1}{\varepsilon}\mathbf{u}^{[1]}\mathbf{A}_{11}\mathbf{u}^{[1]}|_{x=0}^{x=1}$$

$$(5.18)\qquad + \frac{1}{\varepsilon}[\mathbf{u}^{[1]*}\mathbf{A}_{12,1}\mathbf{u}^{[2]} + \mathbf{u}^{[1]*}\mathbf{A}_{12,2}\mathbf{u}^{[3]}]|_{x=0}^{x=1} + \mathcal{R}e\,(P^{IIII}\mathbf{u}^{II}, \mathbf{u}^{II})$$

$$\leqq \alpha[\|\mathbf{u}^{[1]}\|^2 + \|\mathbf{u}^{[2]}\|^2 + \|\mathbf{u}^{[3]}\|^2 + \|\mathbf{u}^{II}\|^2],$$

with some ε-independent constant α. By Lemma 5.2 only the first of the four-term summation in (5.18) is of order $O(\varepsilon^{-1})$. Hence for ε sufficiently small it follows that

$$(5.19a)\qquad \mathcal{R}e\,(P_{22,1}\mathbf{u}^{[2]}, \mathbf{u}^{[2]}) \equiv \mathbf{u}^{[2]*}\mathbf{A}_{22,1}\mathbf{u}^{[2]}|_{x=0}^{x=1} \leqq 0,$$

for $\mathbf{u}^{[2]}$ with appropriate partitioning $\mathbf{u}^{[2]} = (\mathbf{u}_\alpha^{(2)}, \mathbf{u}_\beta^{(2)})$ satisfying

$$(5.19b)\qquad D_{22,1}\mathbf{u}^{[2]} = 0:\ \begin{matrix} \mathbf{u}_\beta^{[2]}(0,t) = L_{2,2}\mathbf{u}_\alpha^{[2]}(0,t) \\ \mathbf{u}_\alpha^{[2]}(1,t) = R_{2,2}\mathbf{u}_\beta^{[2]}(1,t). \end{matrix}$$

Without loss of generality, assuming that at the boundaries $\mathbf{A}_{22,1}$ takes the diagonal form

$$(5.20)\qquad \mathbf{A}_{22,1|x=0,1} = \begin{pmatrix} \Lambda_{22,1\alpha} & \\ & -\Lambda_{22,1\beta} \end{pmatrix}_{|x=0,1}, \qquad \Lambda_{22,1j} > 0, \qquad j = \alpha, \beta,$$

(5.19) is equivalent to (see (3.8))

$$(5.21a)\qquad L_{2,2}^*\Lambda_{22,1\beta}(0,t)L_{2,2} - \Lambda_{22,1\alpha}(0,t) \leqq 0,$$

$$(5.21b)\qquad R_{2,2}^*\Lambda_{22,1\alpha}(1,t)R_{2,2} - \Lambda_{22,1\beta}(1,t) \leqq 0.$$

Now, we slightly strengthen (5.21), requiring (compare (3.9) in the first normal form):

Dissipativity Condition. *At least one of the strict inequalities*

$$(5.22)\qquad L_{2,2}^*\Lambda_{22,1\beta}(0,t)L_{2,2} - \Lambda_{22,1\alpha} < 0, \qquad R_{2,2}^*\Lambda_{22,1\alpha}(1,t)R_{2,2} - \Lambda_{22,1\beta}(1,t) < 0$$

holds.

Assuming the boundary conditions (5.19b) to be maximal dissipative, (5.22), Lemma 3.1 implies

Lemma 5.3. *For every smooth* $\mathbf{F}^{[2]}$, *the two-point boundary value problem*

$$(5.23a)\qquad P_{22,1}\mathbf{v}^{[2]} = \mathbf{F}^{[2]}, \qquad D_{22,1}\mathbf{v}^{[2]} = g$$

has a unique solution, satifying

$$\|\mathbf{v}^{[2]}\|+\|\mathbf{v}_x^{[2]}\|\leqq K(\|\mathbf{F}^{[2]}\|+|g|). \tag{5.23b}$$

According to Lemma 5.1, the third normal form (5.3) may be considered as a composition of two different parts. The first is the subsymbol (5.3e) of the first normal form for which Lemma 5.3 holds; the energy estimate for this form was analyzed in Section 3. The second consists of the corner blocks (5.3c–d) having a structure like the second normal form for which Lemma 5.2 holds; the energy estimate for this form was analyzed in Section 4. By employing the techniques used in Sections 3 and 4, we are finally able to complete the proof for the case of the composed third normal form (5.3).

THEOREM 5.1. *The system* (2.1) *with* P_0 *of the third normal form* (5.3), *satisfies the energy estimate* (E).

Proof: Differentiating the system (2.1) with respect to t we obtain

$$\dot{\mathbf{u}}_t=\frac{1}{\varepsilon}P_0\dot{\mathbf{u}}+P_1\dot{\mathbf{u}}+\dot{F}+\frac{1}{\varepsilon}\dot{P}_0\mathbf{u}+\dot{P}_1\mathbf{u}, \qquad \dot{}\equiv\frac{\partial}{\partial t}. \tag{5.24}$$

We estimate the term $(1/\varepsilon)\dot{P}_0\mathbf{u}$ in two steps. First we consider $(1/\varepsilon)\dot{P}_{22,1}\mathbf{u}$, where we proceed as in Theorem 3.1.[9] By Lemma 5.3 the boundary value problem

$$P_{22,1}\mathbf{v}_*^{[2]}=-\dot{P}_{22,1}\mathbf{u}^{[2]}, \qquad D_{22,1}\mathbf{v}_*^{[2]}=0, \tag{5.25a}$$

has a unique bounded solution with

$$\|\mathbf{v}_*^{[2]}\|+\|\mathbf{v}_{*x}^{[2]}\|\leqq K(\|\mathbf{u}_x^{[2]}\|+\|\mathbf{u}^{[2]}\|)\leqq \text{const.}\ (\|\mathbf{u}\|+\|\mathbf{u}_t\|). \tag{5.25b}$$

Replacing $\mathbf{u}^{[2]}$ by $\mathbf{u}^{[2]}-\mathbf{v}_*^{[2]}$ introduces an additional bounded inhomogeneous vector g_{**} at the boundaries, which is eliminated by subtracting the solution of

$$P_{22,1}\mathbf{v}_{**}^{[2]}=0, \qquad D_{22,1}\mathbf{v}_{**}^{[2]}=g_{**}, \qquad |g_{**}|\leqq \text{const.}\ (\|\mathbf{u}\|+\|\mathbf{u}_t\|). \tag{5.26a}$$

By Lemma 5.3, (5.26a) is indeed uniquely solvable with

$$\|\mathbf{v}_{**}^{[2]}\|+\|\mathbf{v}_{**x}^{[2]}\|\leqq K|g_{**}|\leqq \text{const.}\ (\|\mathbf{u}\|+\|\mathbf{u}_t\|). \tag{5.26b}$$

Now, in terms of the new variable $\mathbf{w}=(\mathbf{w}^{[1]},\mathbf{w}^{[2]},\mathbf{w}^{[3]},\mathbf{w}^{II})'$,

$$\mathbf{w}^j=\mathbf{u}^j, \qquad j=[1],[3],II, \qquad \mathbf{w}^{[2]}=\mathbf{u}^{[2]}-(\mathbf{v}_*^{[2]}+\mathbf{v}_{**}^{[2]}), \tag{5.27}$$

the differentiated system (5.24) takes the form

$$\dot{\mathbf{w}}_t=\frac{1}{\varepsilon}P_0\dot{\mathbf{w}}+P_1\dot{\mathbf{w}}+\dot{F}_*+\frac{1}{\varepsilon}[\dot{P}_0-\dot{P}_{22,1}]\mathbf{w}+\dot{P}_1\mathbf{w} \tag{5.28a}$$

[9] See [2], Theorem 5.1.

with

$$\|\dot{F}_*\| \leqq \text{const.}\,(\|\mathbf{u}\| + \|\mathbf{u}_t\| + \|F\| + \|\dot{F}\|), \tag{5.28b}$$

complemented by the homogeneous boundary conditions (2.1b).

Next we consider $(1/\varepsilon)[\dot{P}_0 - \dot{P}_{22,1}]\mathbf{w}$, where we proceed as in Theorem 4.1. We can now write

$$\frac{1}{\varepsilon}[\dot{P}_0 - \dot{P}_{22,1}]\mathbf{w} = \frac{1}{\varepsilon}\mathbf{C}\mathbf{w} + \Phi\{\mathbf{w}, \mathbf{w}_t, F, \varepsilon^{-1}\mathbf{w}_x^{[1]}\}, \tag{5.29a}$$

where

$$\mathbf{C} = \begin{pmatrix} 0 & \mathbf{C}_{12} & \mathbf{C}_{13} & \mathbf{C}_{14} & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{5.29b}$$

By Lemma 5.2, (5.25b) and (5.26b),

$$\|\Phi\{\mathbf{w}, \mathbf{w}_t, F, \varepsilon^{-1}\mathbf{w}_x^{[1]}\}\| \leqq \text{const.}\,(\|\mathbf{u}\| + \|\mathbf{u}_t\| + \|F\| + \|\dot{F}\|). \tag{5.29c}$$

Similarly, for the term $\dot{P}_1\mathbf{w}$ we have

$$\|\dot{P}_1\mathbf{w}\| \leqq \text{const.}\,(\|\mathbf{u}\| + \|\mathbf{u}_t\| + \|F\| + \|\dot{F}\|). \tag{5.30}$$

Integrating (5.24) over time, noting that the terms $\varepsilon^{-1}(\mathbf{w}, \mathbf{C}\mathbf{w})$, $\varepsilon^{-1}(\mathbf{w}, (\mathbf{C}\mathbf{w})_t)$ are bounded and taking into account (5.25b), (5.26b) and (5.29c) we finally conclude that the energy estimate (E) holds for $\nu = 1$. Higher derivatives can be handled in like manner.

Let us now discuss how bounded initial conditions should be determined so that by Theorem 5.1 the solution will remain bounded for $t > 0$. For simplicity assume the matrix coefficients to be constant, neglect the slow scale low-order term $\tilde{B} = 0$ and let u^{II} consist of only m scalar components. The system (2.1) with P_0 given in (5.3) then takes the form

$$\begin{aligned} \varepsilon\mathbf{u}_t^{[1]} = \mathbf{A}_{11}\mathbf{u}_x^{[1]} + \mathbf{A}_{12,1}\mathbf{u}_x^{[2]} + \mathbf{A}_{12,2}\mathbf{u}_x^{[3]} + \mathbf{B}_{11}\mathbf{u}^{[1]} \\ + \mathbf{B}_{12,1}\mathbf{u}^{[2]} + \mathbf{B}_{12,2}\mathbf{u}^{[3]} + \mathbf{B}_{13}\mathbf{u}^{II} + \varepsilon F^{[1]}, \end{aligned} \tag{5.31a}$$

$$\varepsilon\mathbf{u}_t^{[2]} = \mathbf{A}_{12,1}^*\mathbf{u}_x^{[1]} - \mathbf{B}_{12,1}^*\mathbf{u}^{[1]} + P_{22,1}\mathbf{u}^{[2]} + \varepsilon F^{[2]}, \tag{5.31b}$$

$$\varepsilon\mathbf{u}_t^{[3]} = \mathbf{A}_{12,2}^*\mathbf{u}_x^{[1]} - \mathbf{B}_{12,2}^*\mathbf{u}^{[1]} + \varepsilon F^{[3]}, \tag{5.31c}$$

$$\mathbf{u}_t^{II} = -\varepsilon^{-1}\mathbf{B}_{13}^*\mathbf{u}^{[1]} + \mathbf{A}^{IIII}\mathbf{u}_x^{II} + F^{II}. \tag{5.31d}$$

By Lemma 5.2, $\mathbf{u}^{[1\varepsilon]} \equiv \varepsilon^{-1}\mathbf{u}^{[1]}$ is of order $O(1)$; thus the complementing boundary conditions (2.1b) take the corresponding partitioned form (see (5.19b))

$$\begin{bmatrix} \mathbf{u}^{[1\varepsilon]} \\ \mathbf{u}_\beta^{[2]} \end{bmatrix} = \begin{bmatrix} L_{[1\varepsilon],2} & L_{[1\varepsilon],3} \\ L_{2,2} & L_{2,3} \end{bmatrix} \begin{bmatrix} \mathbf{u}_\alpha^{[2]} \\ \mathbf{u}^{[3]} \end{bmatrix} + \begin{bmatrix} L_{[1\varepsilon],II} \\ L_{2,II} \end{bmatrix} \mathbf{u}^{II}, \qquad x = 0, \tag{5.32a}$$

$$\begin{bmatrix} \mathbf{u}^{[1\varepsilon]} \\ \mathbf{u}_\alpha^{[2]} \end{bmatrix} = \begin{bmatrix} R_{[1\varepsilon],2} & R_{[1\varepsilon],3} \\ R_{2,2} & R_{2,3} \end{bmatrix} \begin{bmatrix} \mathbf{u}_\beta^{[2]} \\ \mathbf{u}^{[3]} \end{bmatrix} + \begin{bmatrix} R_{[1\varepsilon],II} \\ R_{2,II} \end{bmatrix} \mathbf{u}^{II}, \qquad x = 1, \tag{5.32b}$$

and with corresponding partitioning of $A^{II\,II}$ at the boundaries (see (3.2)),

(5.32c) $$\mathbf{u}^{II}_{\beta}(0,t)=L_{II,2}\mathbf{u}^{[2]}(0,t)+L_{II,3}\mathbf{u}^{[3]}(0,t)+L_{II\,II}\mathbf{u}^{II}_{\alpha}(0,t),$$

(5.32d) $$\mathbf{u}^{II}_{\alpha}(1,t)=R_{II,2}\mathbf{u}^{[2]}(1,t)+R_{II,3}\mathbf{u}^{[3]}(1,t)+R_{II\,II}\mathbf{u}^{II}_{\beta}(1,t).$$

To assure that $u_{t|t=0}$ is bounded, we must have, by Lemma 5.2,

(5.33a) $$\mathbf{u}^{[1\varepsilon]}\equiv\varepsilon^{-1}\mathbf{u}^{[1]}=O(1)$$

with $\mathbf{u}^{[1\varepsilon]}_x$ also of order $O(1)$. Using (5.32a), the boundedness of $\mathbf{u}^{[3]}_t$, $\mathbf{u}^{II}_t$ follows from (5.31c–d). Also it follows from (5.31b) that one must have

(5.33b) $$P_{22,1}\mathbf{u}^{[2]}=O(\varepsilon),$$

in order for $\mathbf{u}^{[2]}_t$ to be bounded and, similarly, for $\mathbf{u}^{[1]}_t$ to be bounded, (5.31a) implies

(5.33c) $$\mathbf{A}_{12,2}\mathbf{u}^{[3]}+\mathbf{B}_{12,1}\mathbf{u}^{[3]}_x+\mathbf{A}_{12,1}\mathbf{u}^{[2]}+\mathbf{B}_{12,1}\mathbf{u}^{[2]}+\mathbf{B}_{13}\mathbf{u}^{II}=O(\varepsilon).$$

To satisfy (5.33) determine $\mathbf{u}^{II}$, the right-hand side of (5.33b), (5.33c), and specify $\mathbf{u}^{[3]}$ at the boundaries arbitrarily. Then by (5.32c–d), $\mathbf{u}^{[2]}$ is also determined at the boundaries and, according to Lemma 5.2, $\mathbf{u}^{[2]}$ is uniquely determined by (5.33b) everywhere. Then $\mathbf{u}^{[3]}$ is found as the solution for (5.33c), and finally $\mathbf{u}^{[1\varepsilon]}$ is obtained by smoothly connecting its given boundary values in (5.32a), (5.32b). Higher derivatives can be handled similarly and, in particular, letting $\varepsilon\to 0$, we find that $(\mathbf{u}^{[1\varepsilon]},\mathbf{u}^{[2]},\mathbf{u}^{[3]},\mathbf{u}^{II})'$ converges to the solution of the reduced system

(5.34a) $$\mathbf{A}_{12,2}\mathbf{w}^{[3]}_x+\mathbf{B}_{12,2}\mathbf{w}^{[3]}+\mathbf{A}_{12,1}\mathbf{w}^{[2]}_x+\mathbf{B}_{12,1}\mathbf{w}^{[2]}+\mathbf{B}_{13}\mathbf{w}^{II}=0,$$

(5.34b) $$P_{22,1}\mathbf{w}^{[2]}=0,$$

(5.34c) $$\mathbf{w}^{[3]}_t=\mathbf{A}^*_{12,2}\mathbf{w}^{[1]}_x-\mathbf{B}_{12,2}\mathbf{w}^{[1]}+F^{[3]},$$

(5.34d) $$\mathbf{w}^{II}_t=-\mathbf{B}^*_{13}\mathbf{w}^{[1]}+\mathbf{A}^{II\,II}\mathbf{w}^{II}_x+F^{II}.$$

Appendix

THEOREM. *Assume the symbol* $\hat{P}_0(i\omega)=i\sum_{j=1}^{s}\omega_j\tilde{A}_j+\tilde{B}$ *has a fixed rank independent of* ω, x, t (*Assumption* 1.1). *Then* $\hat{P}_0(i\omega)$ *has an even rank.*

Proof: For simplicity let us consider the one-dimensional case ($s=1$) assuming

(A.1) $$\operatorname{rank}[i\omega\tilde{A}+\tilde{B}]=2p+1, \qquad \text{integer } p\geqq 0.$$

Clearly, (A.1) cannot hold for $p=0$. The general case then follows by induction.

Without restricting generality, let $\tilde{A}$ be partitioned into

(A.2) $$\tilde{A}=\begin{pmatrix}\tilde{A}^{II} & 0\\ 0 & 0\end{pmatrix},$$

where $\tilde{A}^{II}$ is a $(2p+1)$-dimensional nonsingular block. As in the proof of Theorem 2.1 it follows by considering $\omega \to \infty$ that $\tilde{B}$ has the corresponding partitioning

$$\tilde{B} = \begin{pmatrix} \tilde{B}^{II} & \tilde{B}^{III} \\ -\tilde{B}^{III*} & 0 \end{pmatrix} \tag{A.3a}$$

with

$$\operatorname{rank}(\tilde{B}^{III}) = m, \qquad 0 \leqq 2m \leqq 2p+1. \tag{A.3b}$$

The case $m=0$ is impossible since the $(2p+1)$-dimensional *antisymmetric* block $\tilde{B}^{II}$ is singular. Hence rank $P_0(i\omega=0) = \operatorname{rank}(\tilde{B}^{II}) < 2p+1$ contradicting (A.1). Thus

$$\operatorname{rank}(\tilde{B}^{III}) = m, \qquad 0 < m < p. \tag{A.3c}$$

Now, let

$$\tilde{U}^{II*}\tilde{B}^{III}\tilde{U}^{IIII} = \begin{pmatrix} \tilde{B}_{13} & 0 \\ 0 & 0 \end{pmatrix} \tag{A.4}$$

be the singular value decomposition of $\tilde{B}^{III}$, and B_{13} an m-dimensional nonsingular block. Also form the corresponding partitioning

$$\tilde{A}^{II} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{12}^* & \tilde{A}_{22} \end{pmatrix}, \qquad \tilde{B}^{II} = \begin{pmatrix} \tilde{B}_{11} & \tilde{B}_{12} \\ -\tilde{B}_{12}^* & \tilde{B}_{22} \end{pmatrix}.$$

Then upon employing the orthogonal transformation

$$\tilde{U} = \begin{pmatrix} \tilde{U}^{II} & 0 \\ 0 & \tilde{U}^{IIII} \end{pmatrix}$$

we find that the kernel of $P_0(i\omega)$ consists of $x = (x^{[1]}, x^{[2]}, x^{[3]}, x^{[4]})'$ satisfying

$$\begin{aligned} (i\omega\tilde{A}_{11} + \tilde{B}_{11})\tilde{x}^{[1]} + (i\omega\tilde{A}_{12} + \tilde{B}_{12})\tilde{x}^{[2]} + \tilde{B}_{13}x^{[3]} &= 0, \\ (i\omega\tilde{A}_{12}^* - \tilde{B}_{12}^*)\tilde{x}^{[1]} + (i\omega\tilde{A}_{22} + \tilde{B}_{22})\tilde{x}^{[2]} &= 0, \\ -\tilde{B}_{13}^*\tilde{x}^{[1]} &= 0. \end{aligned}$$

The system has the assumed rank $2p+1$ only if

$$\operatorname{rank}(\tilde{B}_{13}) + \operatorname{rank}(i\omega\tilde{A}_{22} + \tilde{B}_{22}) + \operatorname{rank}(\tilde{B}_{13}^*) = 2p+1,$$

i.e., the rank $(i\omega\tilde{A}_{22} + \tilde{B}_{22}) = 2(p-m)+1$, which by (A.3c) contradicts the induction assumption.

Acknowledgment. I would like to thank Professor H.-O. Kreiss for suggesting the problem and for his encouragement during the preparation of this paper.

The material presented here was discussed by the author in a lecture given at the Courant Institute. He is now affiliated with ICASE, NASA Langley Research Center.

Added in proofs: To shed a different (and simpler) light on Lemma 2.1 and the theorem in the Appendix, we note that the continuous dependence of the signature of an Hermitian matrix on its rank implies under Assumption 1.1, signature $[i\hat{P}_0(i\omega)]$ = signature $[i\hat{P}_0(i\omega)|_{\omega=0} = i\tilde{B}] = 0$ by the antisymmetry of $\tilde{B}$; hence the rank of $\hat{P}_0(i\omega)$ is necessarily even, $2p$, consisting of p pairs of eigenvalues of opposite signs.

Bibliography

[1] Gantmacher, F. R., *The Theory of Matrices*, Chelsea Publishing Company, New York, 1960, Vol. II.

[2] Kreiss, H.-O., *Problems with different time scales for partial differential equations*, Comm. Pure, Appl. Math., 33, 1980, pp. 399–439.

Received October, 1981.