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BURGERS' EQUATION WITH VANISHING HYPER-VISCOSITY*

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Abstract. We prove that bounded solutions of the vanishing hyper-viscosity equation, $u_t + f(u)_x + (-1)^s\varepsilon\partial_x^{2s}u = 0$ converge to the entropy solution of the corresponding convex conservation law $u_t + f(u)_x = 0$, $f'' > 0$. The hyper-viscosity case, $s > 1$, lacks the monotonicity which underlines the Krushkov BV theory in the viscous case $s = 1$. Instead we show how to adapt the Tartar-Murat compensated compactness theory together with a weaker entropy dissipation bound to conclude the convergence of the vanishing hyper-viscosity.

Key words. Conservation law, hyper-viscosity, entropy dissipation, compensated compactness

AMS subject classifications. 35L65, 35B30, 65N35.

1. Convergence with vanishing hyper-viscosity. Consider the convex conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0, \quad f'' > 0, \tag{1.1}$$

subject to initial conditions, $u(x, 0) = u_0$. We are concerned with the convergence of its hyper-viscosity regularization of order $s \geq 1$

$$\frac{\partial u^\varepsilon}{\partial t} + \frac{\partial}{\partial x} f(u^\varepsilon(x,t)) = (-1)^{s+1}\varepsilon \frac{\partial^{2s}}{\partial x^{2s}} u^\varepsilon(x,t). \tag{1.2}$$

The viscous case corresponding to $s = 1$ is well understood within the classical Krushkov theory, which is built on the monotonicity of the associated solution operator, e.g., [Daf00, §VI]. The prototype is Burgers' equation governed by the quadratic flux $f(u) = u^2/2$. The hyper-viscosity case for $s > 1$, however, lacks monotonicity and the Krushkov BV theory seems out of reach. Instead we show how to adapt the Tartar-Murat compensated compactness theory, [Tar75, Mur78] in the present non-monotone framework. A similar approach originated with [Sch82] for the vanishing diffusion-dispersion problem where the RHS of (1.2) is replaced by $\varepsilon u^\varepsilon_{xx} + \delta_\varepsilon u^\varepsilon_{xxx}$. In the particular borderline case, $\delta_\varepsilon \sim \varepsilon^2$, limit solutions may in fact violate Krushkov entropy condition, [KL02]. Otherwise, entropy solution limits are recovered by compensated arguments as long as diffusion dominates, $\delta_\varepsilon \ll \varepsilon^2$, [Sch82, KL02]. We should point out that in the present context, *hyper*-viscosity with $s > 1$ yields a weaker entropy dissipation bound than in the viscosity dominated case $s = 1$, consult (1.6) below. We show that this hyper-viscosity entropy dissipation estimate will suffice.

To begin with, we rescale the hyper-viscosity amplitude $\varepsilon \equiv \varepsilon_N = N^{-(2s-1)}$. Denote $u_N \equiv u^{\varepsilon_N}$, then (1.2) reads

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$$\frac{\partial u_N}{\partial t} + \frac{\partial}{\partial x} f(u_N(x,t)) = \frac{(-1)^{s+1}}{N^{2s-1}} \frac{\partial^{2s}}{\partial x^{2s}} u_N(x,t) =: \mathcal{I}(u_N). \tag{1.3}$$

The rescaling in (1.3) is made such that u_N has a smallest scale of order $1/N$, in the sense of satisfying , consult Lemma 1.2 below,

$$\|\partial_x^p u_N(x,t)\|_{L^2([0,T],L^\infty_{loc}(x))} \leq Const. N^p \cdot \|u_N(x,t)\|_{L^2([0,T],L^\infty_{loc}(x))}, \qquad p < s.$$

This estimate is motivated by the fact that u_N is closely related to its N-term Fourier projection, $u_N \sim P_N u_N$. Indeed, the approach taken here follows closely our discussion on the spectral hyper-viscosity method introduced in [Tad93], consult (2.1) below, which directly governs the approximate N-projection $u_N \sim P_N u$. As in [Tad93], we restrict attention to the periodic case.

We begin with the behavior of the quadratic entropy of the hyper-viscosity solution, $U(u_N) = \frac{1}{2}u_N^2$. Multiplication of (1.3) by u_N implies

$$\frac{1}{2}\frac{\partial}{\partial t} u_N^2 + \frac{\partial}{\partial x}\int^{u_N} \xi f'(\xi) d\xi = \frac{(-1)^{s+1}}{N^{2s-1}} u_N \frac{\partial^{2s}}{\partial x^{2s}} u_N =: \mathcal{II}(u_N). \tag{1.4}$$

The expression on the right (1.4) represents the quadratic entropy dissipation + production of the hyper-viscosity solution. Successive "differentiation by parts" enables us to rewrite this expression as

$$\mathcal{II}(u_N) \equiv \frac{1}{N^{2s-1}} \sum_{\substack{p+q=2s-1 \\ q \geq s}} (-1)^{s+p+1} \frac{\partial}{\partial x}\left[\frac{\partial^p u_N}{\partial x^p}\frac{\partial^q u_N}{\partial x^q}\right] - \frac{1}{N^{2s-1}}\left(\frac{\partial^s u_N}{\partial x^s}\right)^2$$

$$:= \mathcal{II}_1(u_N) + \mathcal{II}_2(u_N). \tag{1.5}$$

and spatial integration leads to the following (compare [Tad93, Lemma 4.1])

LEMMA 1.1. *[Entropy dissipation estimate]. The hyper-viscosity solution u_N satisfies the following apriori estimate*

$$\|u_N(\cdot,T)\|^2_{L^2} + \frac{1}{N^{2s-1}}\|\partial_x^s u_N\|^2_{L^2(x,[0,T])} \leq \|u_N(\cdot,0)\|^2_{L^2(x)} \leq K_0^2. \tag{1.6}$$

Here and below K_0 stands for an N-independent L^2-bound depending solely on the initial energy, $K_0 \geq \|u_N(\cdot,0)\|_{L^2}$.

Next, we decompose u_N into low and high modes, $u_N = u_N^I + u_N^{II}$,

$$u_N(x,t) = \sum_{|k|\leq N} \widehat{u}_N(k,t)e^{ikx} + \sum_{|k|>N} \widehat{u}_N(k,t)e^{ikx} =: u_N^I(x,t) + u_N^{II}(x,t). \tag{1.7}$$

Observe that the entropy dissipation bound for u_N^I in (1.6), $N\sum_{|k|\leq N}(|k|/N)^{2s}|\widehat{u}(k,t)|^2_{L^2[0,T]} \leq K_0^2$, is considerably weaker in the hyper-viscosity case, $s > 1$, than in the standard viscosity regularization with $s = 1$. In the latter case, (1.6) amounts to the H^1-bound $\|\partial_x u_N\|_{L^2(x,[0,T])} \leq K_0\sqrt{N}$. Nevertheless, interpolation of (1.6) in the general hyper-viscosity case, $s > 1$, still enables us to control the L^2-growth of $\partial_x u_N$,

$$\|\partial_x u_N\|_{L^2(x,[0,T])} \leq Const. \|\partial_x^s u_N\|^{\frac{1}{s}}_{L^2(x,[0,T])} \|u_N\|^{1-\frac{1}{s}}_{L^2(x,[0,T])} \leq Const_T N^{1-\frac{1}{2s}}. \tag{1.8}$$

To proceed we prepare three estimates. We begin with the higher modes in u_N^{II}. Here we utilize the entropy dissipation estimate (1.6), to find that for $p < s$

$$\left\|\partial_x^p u_N^{II}(x,t)\right\|^2_{L^2([0,T],L^\infty(x))} \leq \int_{t=0}^T \Big(\sum_{|k|>N} |k|^p \cdot |\hat{u}(k,t)|\Big)^2 dt \leq$$

$$\leq \Big(\int_{t=0}^T \sum_{|k|>N} |k|^{2s} \cdot |\hat{u}(k,t)|^2 dt\Big) \cdot \sum_{|k|>N} \frac{1}{|k|^{2(s-p)}} \leq K_0^2 N^{2p}, \quad p < s. \tag{1.9}$$

In particular $\|u_N^{II}\|_{L^2([0,T],L^\infty(x)} \leq K_0$ and we conclude $1/N$ is the smallest scale in the hyper-viscosity solution u_N is in the sense that

Lemma 1.2. *There exists a constant such that $\forall p < s$ the following holds*

$$\left\|\partial_x^p u_N(x,t)\right\|_{L^2([0,T],L^\infty(x))} \leq Const.N^p \cdot \Big[K_0 + K_\infty\Big], \quad K_\infty(T) := \|u_N(x,t)\|_{L^2([0,T],L^\infty(x))}. \tag{1.10}$$

To verify (1.10) we first note that the lower modes grouped in u_N^I form an N-degree trigonometric polynomial for which Bernstein's inequality applies, $\|\partial_x^p u_N^I(x,t)\|_{L^\infty(x)} \leq Const.N^p \cdot \|u_N^I(x,t)\|_{L^\infty(x)}$. This together with (1.9) yield

$$\begin{aligned}
\left\|\partial_x^p u_N(x,t)\right\|^2_{L^2([0,T],L^\infty(x))} &= \int_{t=0}^T \|\partial_x^p u_N^I(x,t)\|^2_{L^\infty(x)} dt + \int_{t=0}^T \|\partial_x^p u_N^{II}(x,t)\|^2_{L^\infty(x)} dt \\
&\leq Const.N^{2p} \int_{t=0}^T \|u_N^I(x,t)\|^2_{L^\infty(x)} dt + K_0^2 N^{2p} \\
&\leq 2Const.N^{2p} \Big[\int_{t=0}^T \|u_N(x,t)\|^2_{L^\infty(x)} dt + \int_{t=0}^T \|u_N^{II}(x,t)\|^2_{L^\infty(x)} dt\Big] + K_0^2 N^{2p} \\
&\leq Const.N^{2p} \Big[K_\infty^2(T) + K_0^2\Big]. \quad \blacksquare
\end{aligned} \tag{1.11}$$

Next, we treat the higher derivatives, $\partial_x^q u_N$ with $s \leq q \leq 2s$. The hyper-viscosity equation (1.3) relates the highest $2s$-derivative to the first-order ones,

$$\|\partial_x^{2s} u_N\| \leq N^{2s-1} \Big[\|\partial_t u_N\| + f_\infty \|\partial_x u_N\|\Big], \quad f_\infty := \sup_{x,[0,T]} |f'(u_N)| \tag{1.12}$$

Spatial integration of (1.3) against $\partial_t u_N$ yields

$$\|\partial_t u_N\|^2_{L^2(x)} + (\partial_t u_N, f'(u_N)\partial_x u_N)_{L^2(x)} = -\frac{1}{2N^{2s-1}} \frac{d}{dt} \|\partial_x^s u_N\|^2_{L^2(x)},$$

and by temporal integration we can bound the time derivative, $\|\partial_t u_N\|_{L^2(x,[0,T])}$ in terms of the spatial one $\|\partial_x u_N\|_{L^2(x,[0,T])}$,

$$\|\partial_t u_N\|^2_{L^2(x,[0,T])} \leq \frac{1}{2}\|\partial_t u_N\|^2_{L^2(x,[0,T])} + \frac{1}{2} f_\infty^2 \|\partial_x u_N\|^2_{L^2(x,[0,T])} + \frac{1}{2N^{2s-1}} \|\partial_x^s u_N(\cdot,0)\|^2_{L^2}.$$

Inserting this into (1.12) we find, in view of (1.8),

$$\|\partial_x^{2s}u_N\|_{L^2(x,[0,T])} \leq Const.N^{2s-1}\Big[2f_\infty\|\partial_x u_N\|_{L^2(x,[0,T])} + \frac{1}{2N^{2s-1}}\|\partial_x^s u_N(\cdot,0)\|^2_{L^2}\Big] \leq$$

$$\leq Const N^{2s-1}\Big[f_\infty N^{1-\frac{1}{2s}} + C_0^2\Big] \leq Const_\infty N^{2s-\frac{1}{2s}}. \tag{1.13}$$

Here and below $Const_\infty$ denote different constants depending on $\|u_N\|_{L^\infty(x,[0,T])}$ and C_0 is a bound on initial smoothness,

$$N^{-(s-\frac{1}{4s})}\|\partial_x^s u_N(\cdot,0)\|_{L^2} \leq C_0 < \infty, \tag{1.14}$$

measuring a minimal amount of H^s smoothness of the initial data which prevents the formation of an initial layer. In particular, (1.14) allows growing initial oscillations as long as $\|\partial_x^s u_N(\cdot,0)\|_{L^2} \sim N^{(s-\frac{1}{4s})}$. We summarize by stating

Lemma 1.3. *There exists a positive constant $\delta = \delta(s) \sim 1/s$ such that $\forall q,\ s \leq q < 2s$ the following holds*

$$\Big\|\partial_x^q u_N(x,t)\Big\|_{L^2(x,[0,T])} \leq Const_\infty \cdot N^{q-\delta}, \quad s \leq q < 2s. \tag{1.15}$$

To verify (1.15) we interpolate (1.13) and (1.6) to conclude (with $\theta := \frac{q}{s} - 1$)

$$\|\partial_x^q u_N(x,t)\|_{L^2(x,[0,T])} \leq Const\|\partial_x^{2s}u_N(x,t)\|^\theta_{L^2(x,[0,T])} \times \|\partial_x^s u_N(x,t)\|^{1-\theta}_{L^2(x,[0,T])} \leq$$

$$\leq Const_\infty N^{(2s-\frac{1}{2s})\theta} \times N^{(s-\frac{1}{2})(1-\theta)} \leq Const_\infty N^{(s+\frac{1}{2}-\frac{1}{2s})\frac{q}{s}+(\frac{1}{2s}-1)} =$$

$$= Const_\infty N^{q-\delta_q}, \qquad \delta_q := \frac{q}{s}\Big(\frac{1}{2s} - \frac{1}{2}\Big) + 1 - \frac{1}{2s}, \tag{1.16}$$

and (1.15) follows with $\delta(s) = \delta_{2s-1} = (2s-1)/2s^2 \sim 1/s$ (and in fact (1.15) is verified for $q = 2s$ with the slightly smaller $\delta(s) = 1/2s$). ■

Finally, lemma 1.2 and its L^2 version in (1.15) yield,

$$\Big\|\Big[\frac{\partial^p u_N}{\partial x^p}\frac{\partial^q u_N}{\partial x^q}\Big]\Big\|_{L^1([0,T],L^2_{loc}(x))} \leq \|\partial_x^p u_N\|_{L^2([0,T],L^\infty(x))} \times \|\partial_x^q u_N\|_{L^2_{loc}(x,t)} \leq$$

$$\leq Const \cdot N^p\big[K_\infty(T) + K_0\big] \times Const_\infty N^{q-\delta}$$

$$\leq Const_\infty N^{p+q-\delta}, \quad p < s \leq q < 2s. \tag{1.17}$$

Equipped with the small scale upperbounds in (1.10), (1.15) and (1.17) we now turn to the main result, stating

Theorem 1.4. *[Convergence]. Consider the hyper-viscosity solution* (1.2) *subject to L^2-bounded initial data, $\|u^\varepsilon(\cdot,0)\|_{L^2} \leq K_0$ so that (1.14) holds and assume $u^\varepsilon(\cdot,t)$ remains uniformly bounded, $\|u^\varepsilon(\cdot,t)\|_{L^\infty(x,[0,T])} < \infty$. Then u^ε converges to the unique entropy solution of the convex conservation law (1.1).*

Remark. (on L^∞-stability.) The L^∞-stability with 2nd order viscosity, $s = 1$, can be deduced by L^p-iterations, monotonicity or entropy decay arguments. The issue of an L^∞ bound for vanishing hyper-viscosity of order $s > 1$ remains an open question.

Proof. We seek H^{-1}-stability in the sense that both the local error on the right hand-side of (1.3), $\mathcal{I}(u_N)$, and the *quadratic* entropy dissipation + production on the right of (1.4), $\mathcal{II}(u_N)$, belong to a compact subset of $H^{-1}_{loc}(x,t)$. By compensated compactness arguments, this will suffice to deduce the L^p_{loc}-strong, $p < \infty$ convergence of u_N to a weak solution of the convex law (1.1).
Consider the first expression, $\mathcal{I}(u_N)$ on the right of (1.3). The inequality (1.6) with $q = 2s - 1$ implies that $\mathcal{I}(u_N)$ tends to zero in $H^{-1}_{loc}(x,t)$, for

$$\left\|\mathcal{I}(u_N) \equiv \frac{(-1)^{s+1}}{N^{2s-1}}\partial_x^{2s}u_N\right\|_{L^2([0,T],H^{-1}_{loc}(x))} \leq \text{Const.}\frac{1}{\sqrt{N}}\|u_N\|_{L^2_{loc}(x,t)} \leq \frac{K_0}{\sqrt{N}} \to 0. \tag{1.18}$$

We now turn to the entropy dissipation term $\mathcal{II}(u_N)$ in (1.5): its first half tends to zero in $H^{-1}_{loc}(x,t)$, for by (1.17) we have $\forall p + q = 2s - 1$,

$$\left\|\mathcal{II}_1(u_N)\right\|_{L^1([0,T],H^{-1}_{loc}(x))} \leq \frac{1}{N^{2s-1}}\sum_{\substack{p+q=2s-1\\ q\geq s}}\left\|\frac{\partial}{\partial x}\left[\frac{\partial^p u_N}{\partial x^p}\frac{\partial^q u_N}{\partial x^q}\right]\right\|_{L^1([0,T],H^{-1}_{loc}(x))} \leq$$

$$\leq \text{Const}_\infty \cdot \frac{1}{N^{2s-1}}\sum_{\substack{p+q=2s-1\\ q\geq s}} N^{p+q-\delta} \leq \frac{C_s}{N^\delta} \to 0, \quad C_s \sim s. \tag{1.19}$$

The second half of $\mathcal{II}$ in (1.5), $-\frac{1}{N^{2s-1}}\left(\frac{\partial^s u_N}{\partial x^s}\right)^2$, is bounded in $L^1_{loc}(x,t)$, consult Lemma 1.1 and hence by Murat's Lemma [Mur78], belongs to a compact subset of $H^{-1}_{loc}(x,t)$

$$\mathcal{II}_2(u_N) \underset{H^{-1}_{loc}(x,t)}{\longrightarrow} \leq 0. \tag{1.20}$$

We conclude that the entropy dissipation of the hyper-viscosity solution — for both linear and quadratic entropies, belongs to a compact subset of $H^{-1}_{loc}(x,t)$. It follows that the hyper-viscosity solution u_N converges strongly (in $L^p_{loc}, \forall p < \infty$) to a weak solution of (1.1). Indeed, since these entropy dissipation terms tend either to zero or to a negative measure, the convergence to the unique entropy solution follows. ■

2. Related models.

2.1. Spectral hyper-viscosity (SV) method. We consider the spectral viscosity method (SV)

$$\frac{\partial u_N}{\partial t} + \partial_x[P_N f(u_N)] = -N\sum_{|k|\leq N}\sigma\Big(\frac{|k|}{N}\Big)\widehat{u}_k(t)e^{ikx}. \tag{2.1}$$

where $\sigma(\xi)$ is a symmetric low pass filter satisfying

$$\sigma(\xi) \geq \Big(|\xi|^{2s} - \frac{1}{N}\Big)_+.$$

The SV method was introduced in [Tad89] for $s = 1$ and the convergence of its hyper-viscosity version in [Tad93] is the forerunner of the present approach; consult [MOT93, Ma98, GMT01] for non-periodic extensions. The (hyper-)SV method allows for an increasing order of parabolicity as long as $C_s N^{-\delta} \longrightarrow 0$ holds in (1.19), i.e., (recall $\delta(s) \sim 1/s$), we require $s^s << N$. In particular, for $s \sim (\log N)^\mu, \mu < 1$ for example, one is led to a low pass filter, $\sigma(\xi) = (\xi^{2s} - 1/N)_+$ which allows for an increasing portion of the spectrum to stay viscous free, i.e., spectral viscosity is introduced only at modes with wavenumbers $|k| \geq Const.N(\log N)^{-\mu/2}$ while retaining high-order of accuracy at first half of the viscous-free spectrum

$$\frac{\partial u_N}{\partial t} + \frac{\partial}{\partial x}[P_N f(u_N)] = -N \sum_{m_N \leq |k| \leq N} \Big(\frac{|k|}{N}\Big)^s \widehat{u}_k(t) e^{ikx}, \quad m_N \sim Const.N(\log N)^{-\mu/2}.$$

Unlike the regular viscosity case, the solution operator associated with (1.2) with $s > 1$ is *not* monotone — here there are "spurious" oscillations, on top of the Gibbs' oscillations due to the Fourier projection in (2.1). The convergence statement of the hyper-SV method (2.1) in [Tad93] and its analogous statement in theorem 1.4 show that oscillations of either type do not cause instability. Moreover, these oscillations contain, in some weak sense, highly accurate information on the exact entropy solution; this could be revealed by post-processing the spectral (hyper)-viscosity approximation, e.g. [GT85, MOT93, TT02].

2.2. Convergence with vanishing Kuramoto-Sivashinsky viscosity. We are concerned with the convergence of its vanishing viscosity regularization which is modeled after the 4th-order Kuramoto-Sivashinsky (KS) dissipation

$$\frac{\partial u^\varepsilon}{\partial t} + \frac{\partial}{\partial x} f(u^\varepsilon(x,t)) = -\varepsilon \frac{\partial^2}{\partial x^2} u^\varepsilon(x,t) - \varepsilon^3 \frac{\partial^4}{\partial x^4} u^\varepsilon(x,t). \tag{2.2}$$

Denote $u_N = u^{\varepsilon_N}$ corresponding to KS viscosity amplitude of order $\varepsilon \equiv 1/N$, then (2.2) reads

$$\frac{\partial u_N}{\partial t} + \frac{\partial}{\partial x} f(u_N(x,t)) = -\frac{1}{N}\frac{\partial^2 u_N}{\partial x^2} - \frac{1}{N^3}\frac{\partial^4 u_N}{\partial x^4} =: \mathcal{I}(u_N). \tag{2.3}$$

The rescaling made in (2.3) is such that u_N has a smallest scale of order $1/N$, in the sense that

$$\|\partial_x^p u_N(x,t)\|_{L^2([0,T],L^\infty_{loc}(x))} \leq Const.N^p \cdot \|u_N(x,t)\|_{L^2([0,T],L^\infty_{loc}(x))}, \quad p < 2. \tag{2.4}$$

As before, (2.4) is deduced by separating small and large scale of order $\sim N$ and using the following quadratic entropy dissipation estimate

$$\frac{1}{2}\frac{\partial}{\partial t} u_N^2 + \frac{\partial}{\partial x}\int^{u_N} \xi f'(\xi) d\xi = -\frac{1}{N} u_N \frac{\partial^2 u_N}{\partial x^2} - \frac{1}{N^3} u_N \frac{\partial^4 u_N}{\partial x^4} := \mathcal{II}(u_N). \tag{2.5}$$

The expression on the right (2.5) represents the quadratic entropy dissipation + production of the KS-viscosity solution. Successive "differentiation by parts" enables us to rewrite this expression as

$$\begin{aligned}\mathcal{II}(u_N) \equiv\ & \partial_x \Big[-\frac{1}{N^3}\Big(u_N \frac{\partial^3 u_N}{\partial x^3} - \frac{\partial u_N}{\partial x}\frac{\partial^2 u_N}{\partial x^2}\Big) - \frac{1}{N} u_N \frac{\partial u_N}{\partial x}\Big] + \Big[\frac{1}{N}\Big(\frac{\partial u_N}{\partial x}\Big)^2 - \frac{1}{N^3}\Big(\frac{\partial^2 u_N}{\partial x^2}\Big)^2\Big] \\ & := \mathcal{II}_1(u_N) + \mathcal{II}_2(u_N),\end{aligned}$$

and spatial integration leads to the following

LEMMA 2.1. *[Entropy dissipation estimate]. The KS-viscosity solution u_N satisfies the following apriori estimate*

$$(2.6)\ \|u_N(\cdot,t)\|^2_{L^2} + \frac{1}{N^3}\Big|\Big|\frac{\partial^2 u_N}{\partial x^2}\Big|\Big|^2_{L^2_{loc}(x,t)} \leq \|u_N(\cdot,0)\|^2_{L^2(x)} + \frac{1}{N}\Big|\Big|\frac{\partial u_N}{\partial x}\Big|\Big|^2_{L^2_{loc}(x,t)}.$$

Using this entropy dissipation estimate, one can argue the convergence of the vanishing KS viscosity along the lines of the hyper-viscosity case. The question whether the vanishing KS limit is an entropy solution of (1.2) remains open.

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