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THE WELL-POSEDNESS OF THE KURAMOTO–SIVASHINSKY EQUATION*

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Abstract. The Kuramoto–Sivashinsky equation arises in a variety of applications, among which are modeling reaction-diffusion systems, flame-propagation and viscous flow problems. It is considered here, as a prototype to the larger class of generalized Burgers equations: those consist of quadratic nonlinearity and arbitrary linear parabolic part. We show that such equations are well-posed, thus admitting a unique smooth solution, continuously dependent on its initial data. As an attractive alternative to standard energy methods, existence and stability are derived in this case, by "patching" in the large short time solutions without "loss of derivatives".

Key words. Kuramoto–Sivashinsky equation, fixed point iterations, existence, uniqueness, stability

AMS(MOS) subject classifications. Primary 35Q20; secondary 35K55

1. Introduction. The equation referred to in the title is of the form

$$\frac{\partial\phi}{\partial t}+|\nabla\phi|^2+\Delta\phi+\Delta^2\phi=0.$$

This equation was independently advocated by Kuramoto [2], in connection with reaction-diffusion systems, and by Sivashinsky [4], modeling flame propagation; it also arises in the context of viscous film flow [5] and bifurcating solutions of the Navier–Stokes equations.[1]

In this paper we study the well-posedness question associated with the one-dimensional version of the Kuramoto–Sivashinsky equation (abbreviated hereafter as the K-S equation)

$$\frac{\partial\phi}{\partial t}+\left(\frac{\partial\phi}{\partial x}\right)^2+\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^4\phi}{\partial x^4}=0. \tag{1.1}$$

It is shown that the Cauchy problem connected with (1.1) is well-posed: the K-S equation admits a unique smooth solution, continuously dependent on its initial data. In fact, all the results quoted below equally apply to the more general equation

$$\frac{\partial\phi}{\partial t}+\left(\frac{\partial\phi}{\partial x}\right)^2=P\left(\frac{\partial}{\partial x}\right)\phi=0, \tag{1.2a}$$

with a linear part, strongly parabolic of arbitrary order $\nu>\frac{3}{2}$,

$$\mathrm{Re}\,\hat{P}(i\xi)\geqq \mathrm{Const}\cdot|\xi|^{\nu},\qquad |\xi|\to\infty. \tag{1.2b}$$

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[1] G. Sivashinsky, private communication.

Existence and stability results given here, are obtained by modifying Taylor's recipe, [6, p. 96], for treating the existence question in the special case of Burgers equation, $\hat{P}(i\xi)=\xi^2$. According to that recipe, roughly speaking, dissipation is used to compensate nonlinearity, so that short time solutions can be constructed without running into the familiar phenomenon of "loss of derivatives". Coupled with an L^2-decay estimate, short time solutions are then "patched" together, in the large. A study along these lines is carried out in §2 below, where existence and stability questions are treated in connection with the K-S equation. Existence and uniqueness in this case were previously proved by energy methods, see e.g., Aimar and Penel [1], Nicolaenko and Scheurer [3]. The technical details are avoided in §2: these are postponed to §4, all proved by virtue of a single standard estimate on the *linear* dissipative part of the equation, given in §3.

The above study thus suggests itself, with handling arbitrary linear dissipative parts. In §5 we conclude by quoting the corresponding results to such generalized Burgers equations.

2. Existence and stability. We start by putting the K-S equation in a conservative form: we differentiate (1.1), obtaining that the new decayed variable $u\equiv u(x,t;\ \eta)=e^{-\eta t}\partial\phi/\partial x$, $\eta>0$, satisfies

$$\frac{\partial u}{\partial t}+e^{\eta t}\frac{\partial(u^2)}{\partial x}+\eta u+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^4 u}{\partial x^4}=0; \tag{2.1a}$$

a solution for the initial value problem (2.1a) is sought, $u(t)$, $t\geqq 0$, subject to initial condition

$$u(x,t=0)=f(x). \tag{2.1b}$$

Both the pure Cauchy problem, $-\infty<x<\infty$, and the periodic problem, say $-\pi/2\leqq x\leqq\pi/2$, are discussed. We explicitly treat the first infinite case by means of Fourier expansion; the somewhat simpler periodic case can be likewise handled, using Fourier series instead.

If we let $\hat{P}(i\xi)\equiv\hat{P}(i\xi;\ \eta)=\eta-\xi^2+\xi^4$ denote the symbol associated with the spatial linear part of (2.1a) and let $\hat{Q}(\xi,t)\equiv\hat{Q}(\xi,t;\ \eta)=e^{-t\hat{P}(i\xi;\eta)}$ be its transformed solution operator, then by Duhammel's principle (2.1) admits the following integral representation

$$u(t)=Q(t;\ \eta)*f+\int_0^t e^{\eta\tau}\cdot Q(t-\tau;\ \eta)*\frac{\partial}{\partial x}(u^2(\tau))\,d\tau. \tag{2.2}$$

Abbreviate the right-hand side of (2.2) by $\mathbf{J}_\eta[u;\ f]$; to simplify notation, we will occasionally suppress the explicit dependence on the initial data, thus writing

$$\mathbf{J}_\eta[u]\equiv\mathbf{J}_\eta[u;\ f]=Q(t;\ \eta)*f+\int_0^t e^{\eta\tau}\cdot Q(t-\tau;\ \eta)*\frac{\partial}{\partial x}(u^2(\tau))\,d\tau. \tag{2.3}$$

The question of existence of a solution for (2.1) is now transformed into that of a fixed point solution for $\mathbf{J}_\eta[u]$. Fixing T, $T>0$, we seek a fixed point solution for $\mathbf{J}_\eta[u]$ in $L^\infty([0,T],\ L^2)$, equipped with the standard norm $\|u\|=\sup_{0\leqq t\leqq T}|u(\cdot;\ t)|$.[2] The existence of such a fixed point solution is guaranteed, at least for a short time, as a consequence of

[2] We adopt the notation of single bars to denote spatial norming; for example, $|w|_{H^s}=(\int(1+|\xi|^2)^s|\hat{w}(\xi)|^2\,d\xi)^{1/2}$. Similarly, double bars are reserved to space-time norming; for example, $\|w\|_s=\sup_{0\leqq t\leqq T}|w(\cdot,t)|_{H^s}$. In particular, $|w|=|w|_{H^0}=(\int w^2(x)\,dx)^{1/2}$, $\|w\|=\|w\|_0$.

LEMMA 2.1 (short time contraction). *Given v, w in $L^{\infty}([0,T], L^2)$ and $\mathbf{J}_\eta[\cdot]=\mathbf{J}_\eta[\cdot;f]$ as in* (2.3). *Then, there exists a constant $\eta_0 \geqq 0$, such that for $\eta \geqq \eta_0$ we have,*

$$\|\mathbf{J}_\eta[v]-\mathbf{J}_\eta[w]\| \leqq M(T;\,\eta)\cdot(\|v\|+\|w\|)\cdot\|v-w\|. \tag{2.4a}$$

Here, $M(T;\,\eta)$ is given by,

$$M(T;\,\eta)=2e^{\eta T}\cdot T^{1/8}. \tag{2.4b}$$

By virtue of Lemma 2.1 we find

COROLLARY 2.2 (short time boundedness). *Set $T=T_1$, $T_1>0$, such that*

$$4M(T_1;\,\eta)\cdot|f|<1. \tag{2.5a}$$

Then, for $\eta \geqq \eta_0$ we have,

$$\|\mathbf{J}_\eta^{[n]}[f]\| \leqq 2|f|, \qquad n=0,1,\cdots. \tag{2.5b}$$

Thus, the fixed point iterations, $\mathbf{J}_\eta^{[n]}[f]$ remain inside the origin centered ball of radius $2|f|$. Hence—since by Lemma 2.1 $\mathbf{J}_\eta[\cdot]$ contracts inside that ball, having a Lipschitz constant $4M(T_1;\,\eta)\cdot|f|<1$—the existence of a fixed point solution for $\mathbf{J}_\eta[u]$ follows, at least for a short time interval, $0 \leqq t \leqq T_1$. Furthermore, *the length of that existence interval, T_1, depends on no higher than the initial L^2-norm*. This latter fact plays a central role in the foregoing analysis; in particular, it enables the local solution just constructed, to be continued to a global one, with the help of

LEMMA 2.3 (large time decay). *Let $u(t;\,\eta)\equiv u(x,t;\,\eta)$ be a solution of* (2.1). *Then, there exists a constant $\eta_0 \geqq 0$, such that for $\eta \geqq \eta_0$ we have*

$$|u(t_2;\,\eta)| \leqq e^{-(\eta-\eta_0)(t_2-t_1)}\cdot|u(t_1;\,\eta)|, \qquad 0 \leqq t_1 \leqq t_2 \leqq T. \tag{2.6}$$

Verification of Lemma 2.3 is straightforward: multiplying (2.1a) by $u(x,t;\,\eta)$, integrating by parts while noting the vanishing contribution of the nonlinear term,[3] we find

$$1/2\,\frac{d}{dt}|u(t)|^2=-\eta|u(t)|^2+\left|\frac{\partial u}{\partial x}(t)\right|^2-\left|\frac{\partial^2 u}{\partial x^2}(t)\right|^2;$$

invoking the Parseval relation, the last equality yields

$$1/2\,\frac{d}{dt}|u(t)|^2 \leqq \max_{\xi}(-\hat{P}(i\xi;\,\eta))\cdot|u(t)|^2,$$

and integration finally leads us to (2.6) with $\eta_0=\frac{1}{4}$. We remark that in the periodic case, $-\pi/2 \leqq x \leqq \pi/2$, one can invoke instead Poincaré's inequality,

$$\int_{-\pi/2}^{\pi/2}\left|\frac{\partial u}{\partial x}\right|^2=\int_{-\pi/2}^{\pi/2}\left[\frac{\partial u}{\partial x}-\frac{1}{\pi}\cdot\int_{-\pi/2}^{\pi/2}\frac{\partial u}{\partial x}\right]^2 \leqq \int_{-\pi/2}^{\pi/2}\left|\frac{\partial^2 u}{\partial x^2}\right|^2,$$

leading, in a similar way, to (2.6) with $\eta_0=0$. Observe that in general the exponential growth bound, η_0, may depend on the period.

To conclude the existence of solution in the large, we now fix η, $\eta \geqq \eta_0$, with appropriately chosen η_0 in either the finite or infinite case; then, short time

[3] With the infinite pure Cauchy problem, $u(x,t)$ is required to vanish at $x=\pm\infty$, indeed, $|u(t)|_{H^1}<\infty$ according to Theorem 2.6 below.

solutions—constructed according to Lemma 2.1—can be successively "patched" together, over time intervals which—according to Lemma 2.3—are of a *fixed* (non-shrinking) length T_1. Integrating, we obtain a global solution for the K-S equation, $\phi=\phi(x,t)$; the solution so obtained is—up to integration factor—unique. Thus we finally arrive at

THEOREM 2.4 (existence). *The* K-S *equation* (1.1), *with prescribed initial data* $\phi(t=0)$ *in* H^1, *admits a unique solution*, $\phi=\phi(x,t)$, *which satisfies*,

$$\left|\frac{\partial\phi}{\partial x}(t)\right| \leqq e^{\eta_0 T}\cdot\left|\frac{\partial\phi}{\partial x}(t=0)\right|, \qquad 0\leqq t\leqq T<\infty. \tag{2.7}$$

In fact, $\phi(t)$, $t\geqq 0$, belongs to H^1: a further L^2 estimate needed here, is discussed in §4 below.

The global solution referred to in Theorem 2.4, is constructed by patching together short time solutions, using a single L^2 a priori estimate. Such a patching procedure differs from existence proofs via standard energy methods, e.g., [1], [3], where higher a priori estimates are called for. Instead, we rely here on having a derivative-free Lipschitz contraction factor, so that short time solutions can be constructed, without running into the familiar phenomenon of "loss of derivatives". We note that solving the integrodifferential equation (2.2) by fixed point iterations results in the existence of a solution satisfying the original *differential* equation (2.1), in a *weak* sense. Concerning the existence of such a solution under a stronger topology, one observes that (2.1a) contains two destabilizing sources: the focusing effect ("loss of derivatives") caused by the nonlinear term, and the exponential divergence of the second order dissipative term. It is the balance of these two terms by the fourth-order dissipation, which leads us to the important derivative-free Lipschitz contraction factor in this case. Making a finer study of that balance, we are able to conclude that the solution constructed above is, in fact, smooth enough to be interpreted as a classical one. To this end, we sharpen Lemma 2.1, stating

LEMMA 2.5 (short time contraction). *Given* v, w *in* $L^\infty([0,T],\ H^s)$ $s\geqq 0$, *and* $\mathbf{J}_\eta[\cdot]=\mathbf{J}_\eta[\cdot;\ f]$ *as in* (2.3). *Then, there exists a constant* $\eta_0\geqq 0$, *such that for* $\eta\geqq\eta_0$ *we have*,

$$\left\|\mathbf{J}_\eta[v]-\mathbf{J}_\eta[w]\right\|_{s+2}\leqq 2^s\cdot M(T;\ \eta)\cdot\left(\|v\|_s+\|w\|_s\right)\cdot\|v-w\|_s. \tag{2.8}$$

Thus, each fixed point iteration gives us a smoother *correction*. In particular, setting s to be zero, we find on account of Corollary 2.2 that $\{\mathbf{J}_\eta^{[n]}[f]\}_{n\geqq 0}$ form a Cauchy sequence in the $L^\infty([0,T_1],\ H^2)$—origin centered ball of radius $2|f|$. Hence, the fixed point iterations $\mathbf{J}_\eta^{[n]}[f]$ converge to a unique, short time solution, $u=u(x,t)$ in $L^\infty([0,T_1],\ H^2)$. Thanks to the L^2-decay estimate in Lemma 2.3, such short time solutions can be patched in the large as before, integrated once and yielding

THEOREM 2.6 (existence). *The* K-S *equation* (1.1), *with prescribed initial data* $\phi(t=0)$ *in* H^3, *admits a unique solution*, $\phi=\phi(x,t)$, *which satisfies*,

$$\left|\frac{\partial\phi}{\partial x}(t)\right|_{H^2}\leqq \frac{5}{4}e^{\alpha T}\cdot\left|\frac{\partial\phi}{\partial x}(t=0)\right|_{H^2}, \qquad 0\leqq t\leqq T\leqq\infty. \tag{2.9}$$

Finally, we turn to examine the question of stability: allowing the initial data to vary as well, we have the final extension to the short time contraction lemma, which now reads

LEMMA 2.7 (short time contraction). *Given v, w in $L^\infty([0,T], H^s)$ with $f=v(t=0)$, $g=w(t=0)$ in H^{s+2}. Then, there exists a constant $\eta_0 \geqq 0$ such that for $\eta \geqq \eta_0$ we have,*

$$\text{(2.10)}\qquad \|\mathbf{J}_\eta[v;\, f]-\mathbf{J}_\eta[w;\, g]\|_{s+2} \leqq |f-g|_{H^{s+2}}+2^s\cdot M(t;\,\eta)\cdot\left(\|v\|_s+\|w\|_s\right)\cdot\|v-w\|_s.$$

Now let $v(t)=\mathbf{J}_\eta[v(t);\ v(t=0)]$, $w(t)=\mathbf{J}_\eta[w(t);\ w(t=0)]$ be two different fixed point solutions of (2.1a), whose initial data $f=v(t=0)$ and $g=w(t=0)$ are assumed to be in H^2; according to Theorem 2.6, $u(t)$ and $v(t)$ belong to H^2 later on, $t\geqq 0$, and as a consequence of Lemma 2.7 with $s=0$, we have short time stability

$$|v(t)-w(t)|_{H^2}\leqq\frac{1}{1-M(T_1;\,\eta)\cdot(|f|+|g|)}|v(t=0)-w(t=0)|_{H^2},\qquad 0\leqq t\leqq T_1.$$

Successive application of the last inequality yields the desired stability result, which we state as our final

THEOREM 2.8 (stability). *Let ϕ, ψ be two different solutions of the* K-S *equation* (1.1), *with initial data $\phi(t=0)$, $\psi(t=0)$ lying in H^3. Then, there exist constants C and $\beta\geqq 0$ (both may depend on $|(\partial\phi/\partial x)(t=0)|+|(\partial\psi/\partial x)(t=0)|$), such that the following estimate holds*:

$$\text{(2.11)}\qquad \left|\frac{\partial\phi}{\partial x}(t)-\frac{\partial\psi}{\partial x}(t)\right|_{H^2}\leqq C\cdot e^{\beta t}\cdot\left|\frac{\partial\phi}{\partial x}(t=0)-\frac{\partial\psi}{\partial x}(t=0)\right|_{H^2},\qquad 0\leqq t\leqq T\leqq\infty.$$

3. An estimate on the dissipative kernel. The following classical estimate is in the heart of the matter.

LEMMA 3.1. *Given $\omega(x)$ in W_p^m, $1\leqq p\leqq 2$, and real r, $r\geqq\frac{1}{2}-1/p$. Then, there exist constants, $C=C_{p,r}$ and $\eta_0\geqq 0$, such that for $\eta\geqq\eta_0$ we have,*

$$\text{(3.1)}\qquad |Q(t;\,\eta)*\omega|_{H^{m+r}}\leqq C\cdot e^{-(\eta-\eta_0)t}\cdot t^{-(r-1/2+1/p)/4}\cdot|\omega|_{W_p^m}.$$

Remark. We adopt here the standard notation, W_p^m, to denote the L^p-type Sobolev space of order m, consisting of those functions whose derivatives up to order m belong to L^p. (Although not specifically referred to, a fractional Sobolev space with nonintegral m should be interpreted as a Besov space: to comply with notation, we therefore restrict attention to integral orders, with the understanding that final results can be interpolated into Besov space.)

For completeness, we include here a short calculation verifying (3.1): setting $\mu=p/(2-p)$ and letting μ' be its conjugate, $1/\mu+1/\mu'=1$; then the Hölder inequality yields

$$\text{(3.2)}\qquad |Q(t;\,\eta)*\omega|_{H^{m+r}}\leqq\left[\int_{-\infty}^{\infty}\left(1+|\xi|^2\right)^{\mu r}e^{-2\mu t(\eta-\xi^2+\xi^4)}d\xi\right]^{1/2\mu} \times\left[\int_{-\infty}^{\infty}\left(1+|\xi|^2\right)^{\mu' m}|\hat{\omega}(\xi)|^{2\mu'}d\xi\right]^{1/2\mu'}.$$

Since by the Hausdorff–Young inequality the Fourier transform is of type $(2\mu',\ (2\mu')'=p)$, the second factor on the right of (3.2), $|\hat{\omega}|_{W_{2\mu'}^m}$, does not exceed

$$\text{(3.3)}\qquad \left[\int_{-\infty}^{\infty}\left(1+|\xi|^2\right)^{\mu' m}|\hat{\omega}(\xi)|^{2\mu'}d\xi\right]^{1/2\mu'}\leqq(2\pi)^{1/2-1/p}\cdot|\omega|_{W_p^m}.$$

Next, we split the first factor on the right of (3.2),

$$e^{-\eta t}\left[\int_{-\infty}^{\infty}\left(1+|\xi|^2\right)^{\mu r}e^{-2\mu t(\xi^4-\xi^2)}\,d\xi\right]^{1/2\mu}=e^{-\eta t}\left[\int_{|\xi|\leqq\sqrt{2}}\cdots+\int_{|\xi|>\sqrt{2}}\cdots\right]^{1/2\mu};$$

the first of the two integrals admits a pessimistic bound of

$$\int_{|\xi|\leqq\sqrt{2}}\left(1+|\xi|^2\right)^{\mu r}e^{-2\mu t(\xi^4-\xi^2)}\,d\xi\leqq 2\sqrt{2}\,3^{\mu r}e^{\mu t/2},$$

while the second one is estimated by

$$\int_{|\xi|>\sqrt{2}}\left(1+|\xi|^2\right)^{\mu r}e^{-2\mu t(\xi^4-\xi^2)}\,d\xi$$
$$\leqq 2^{\mu r+1}\int_{\xi=0}^{\infty}\xi^{2\mu r}e^{-\mu t\xi^4}\,d\xi=2^{\mu r-1}\Gamma\left(\frac{2\mu r+1}{4}\right)(\mu t)^{-(2\mu r+1)/4}.$$

Added together, we find that the first factor on the right of (3.2), does not exceed

$$\left[\int_{-\infty}^{\infty}\left(1+|\xi|^2\right)^{\mu r}e^{-2\mu t(\eta-\xi^2+\xi^4)}\,d\xi\right]^{1/2\mu} \tag{3.4}$$
$$\leqq B_{p,r}\cdot e^{-(\eta-\eta_0)t}\cdot t^{-(2\mu r+1)/8\mu},\qquad \eta_0=\frac{1}{4},$$

with Stirling's formula giving us a bound of

$$B_{p,r}=(4\pi e)^{1/2\mu}3^{r/2}\left(r+\frac{1}{p}\right)^{r-1/2+1/p}.$$

Recalling that $(2\mu')'=p$, (3.2), (3.3) and (3.4) yield the required estimate (3.1) with $C_{p,r}=(2\pi)^{1/2-1/p}B_{p,r}$.

Remark 1. In the *infinite* case under consideration, an exponential growth bound, $\eta_0=\frac{1}{4}$, was found. In general, η_0 may depend on the period, in the spirit of an earlier remark; for example, $\eta_0=0$, in the π-periodic case.

Remark 2. For future reference, we quote here the constants $C_{p,r}$ in two special cases: as can be readily verified, $C_{2,0}=1$ (indeed, such an estimate also follows by a straightforward integration by parts, essentially contained in the verification of Lemma 2.3 above); also, by sharpening the above pessimistic bounds, one finds $C_{1,3}<8$.

4. Proof of main results. We first study the operator $\mathbf{J}_\eta[\cdot;\ \cdot]$ introduced in (2.3), whose fixed point solutions are sought. Equipped with Lemma 3.1, we are able to derive the following summary stability estimate

$$\text{(S)}\quad \left|\mathbf{J}_\eta[v(t);\ f]-\mathbf{J}_\eta[w(t);\ g]\right|_{H^{s+2}}\leqq e^{-(\eta-\eta_0)t}\cdot|f-g|_{H^{s+2}}$$
$$+2^{s+1}\cdot e^{\eta t}\cdot t^{1/8}\cdot\sup_{0\leqq\tau\leqq t}|v(\tau)+w(\tau)|_{H^s}\cdot\sup_{0\leqq\tau\leqq t}|v(\tau)-w(\tau)|_{H^s}.$$

To verify (S)—assuming the quantities on the right are finite and $\eta\geqq\eta_0$—we consider the difference

$$\mathbf{J}_\eta[v(t);\ f]-\mathbf{J}_\eta[w(t);\ g]=Q(t;\ \eta)*(f-g)$$
$$+\int_0^t e^{\eta\tau}\cdot Q(t-\tau;\ \eta)*\frac{\partial}{\partial x}\left(v^2(\tau)-w^2(\tau)\right)d\tau,$$

so that after taking norms on both sides we have

$$|\mathbf{J}_\eta[v(t);\, f]-\mathbf{J}_\eta[w(t);\, g]|_{H^{s+2}} \leqq |Q(t;\,\eta)*(f-g)|_{H^{s+2}} + \int_0^t e^{\eta\tau}\cdot\left|Q(t-\tau;\,\eta)*\frac{\partial}{\partial x}(v^2(\tau)-w^2(\tau))\right|_{H^{s+2}} d\tau.$$

Now applying Lemma 3.1 with respect to both terms on the right of the last inequality: the first term with $(r,p,m)=(0,2,s+2)$, and the second one with $(r,p,m)=(3,1,s-1)$; recalling the earlier quoted constants $C_{2,0}=1$ and $C_{1,3}<8$, we find

$$|\mathbf{J}_\eta[v(t);\, f]-\mathbf{J}_\eta[w(t);\, g]|_{H^{s+2}} \leqq e^{-(\eta-\eta_0)t}\cdot|f-g|_{H^{s+2}} + 8\cdot\int_0^t e^{\eta\tau}\cdot e^{-(\eta-\eta_0)(t-\tau)}\cdot(t-\tau)^{-7/8}\cdot\left|\frac{\partial}{\partial x}(v^2(\tau)-w^2(\tau))\right|_{W_1^{s-1}} d\tau.$$

The last integral bounds the interaction between the linear dissipative part of the equation, and the nonlinear differentiated quadratic term; the loss of derivative due to the latter is compensated here by dissipation, weighted with the L^1 topology. In order to return to the usual L^2 setup, we apply the Leibniz rule and Cauchy–Schwarz inequality to find

$$\left|\frac{\partial}{\partial x}(v^2(\tau)-w^2(\tau))\right|_{W_1^{s-1}} \leqq 2^{s+1}\cdot|v(\tau)+w(\tau)|_{H^s}\cdot|v(\tau)-w(\tau)|_{H^s}.$$

Inserted into the last integral and carrying out the integration, we end up with the required estimate (S).

We now turn to prove the results in §2, starting with:

Short time contractions (Lemma 2.1, Lemma 2.5, Lemma 2.7). Taking supremum over both sides of the (S) estimate with varying t, $0\leqq t\leqq T$, and equipped with the notation of

$$M(T;\,\eta)=2e^{\eta T}\cdot T^{1/8}$$

in (2.4b), we find

$$\|\mathbf{J}_\eta[v;\, f]-\mathbf{J}_\eta[w;\, g]\|_{s+2} \leqq |f-g|_{H^{s+2}}+2^s\cdot M(T;\,\eta)\cdot(\|v\|_s+\|w\|_s)\cdot\|v-w\|_s,$$

so that Lemma 2.7 follows. Taking the special case $f=g$ proves Lemma 2.5, and further setting $s=0$, yields Lemma 2.1,

$$\|\mathbf{J}_\eta[v]-\mathbf{J}_\eta[w]\| \leqq \|\mathbf{J}_\eta[v]-\mathbf{J}_\eta[w]\|_2 \leqq M(T;\,\eta)\cdot(\|v\|+\|w\|)\cdot\|v-w\|.$$

(Observe that in the case of Lemma 2.1, where no gain of derivatives is involved, one can in fact improve the contraction factor $M(T;\,\eta)$ to be $\frac{2}{7}e^{\eta T}T^{7/8}$.)

An immediate consequence of Lemma 2.1 is the following:

Short time boundedness (Corollary 2.2). Setting $v=\mathbf{J}_\eta^{[n-1]}[f]$ and $w=0$ in Lemma 2.1, we find

$$\|\mathbf{J}_\eta[\mathbf{J}_\eta^{[n-1]}(f)]\| \leqq \|\mathbf{J}_\eta[v;\, f]-\mathbf{J}_\eta[w=0;\, f]\|+\|\mathbf{J}_\eta[w=0;\, f]\| \leqq M(T;\,\eta)\cdot\|\mathbf{J}_\eta^{[n-1]}[f]\|^2+\|Q(t;\,\eta)*f\|.$$

We now consider a temporal interval of length T_1 such that $4M(T_1;\ \eta)\cdot\lfloor f\rceil<1$: assuming $\|\mathbf{J}_\eta^{[n-1]}[f]\|\leqq 2\lfloor f\rceil$ in that interval, then together with Lemma 3.1 taking $(r,p,m)=(0,2,0)$, we obtain

$$\left\|\mathbf{J}_\eta^{[n]}[f]\right\|\leqq 4M(T_1;\ \eta)|f|\cdot|f|+|f|\leqq 2|f|,$$

and Corollary 2.2 follows by induction.

Owing to the last two results in the small, one may construct fixed point solutions, $u(t)$, as local solutions over time intervals $[T_N,T_{N+1}]$ $N=0,1,2,\cdots$, such that $4M(T_{N+1}-T_N;\ \eta)\cdot|u(T_N)|<1$. Thanks to the large L^2-estimate in Lemma 2.3, the local solutions just constructed can be patched in the large, over *fixed* length time intervals, $T_N=NT_1$, $N=0,1,\cdots$, obtaining

Existence. (Theorem 2.4, Theorem 2.6). Given the initial data $\phi(t=0)$ in H^1, we set $f=(\partial\phi/\partial x)(t=0)$ for the initial value problem (2.1); let $u(t)$, $t\geqq 0$, be its global solution, constructed according to the above recipe. Integrated once, we obtain a solution for the K-S equation, $\phi(x,t)=\int^x u(\xi,t)\,d\xi$, which satisfies—choosing $\eta=\eta_0$ in Lemma 2.3—

$$\left|\frac{\partial\phi}{\partial x}(t)\right|\leqq e^{\eta_0 T}\cdot\left|\frac{\partial\phi}{\partial x}(t=0)\right|,\qquad 0\leqq t\leqq T.$$

This proves Theorem 2.4. In order to show that $u=\partial\phi/\partial x$ possesses a certain degree of smoothness, at least that of the initial data, we appeal to the short time contraction estimate in Lemma 2.5 with $s=0$:

$$\left\|\mathbf{J}_\eta[v]-\mathbf{J}_\eta[w]\right\|_2\leqq M(T;\ \eta)\cdot(\|v\|+\|w\|)\cdot\|v-w\|.$$

Consider first the time interval $[0,T=T_1]$ and let $u=\mathbf{J}_\eta[u]$ the fixed point solution there; choosing $v=u$ and $w=0$, we find

$$\|u\|_2=\left\|\mathbf{J}_\eta[u]\right\|_2\leqq M(T_1;\ \eta)\|u\|^2+\|Q*f\|_2.$$

Using Lemma 2.3 and Lemma 3.1 with $(r,p,m)=(0,2,2)$, we end up with

$$\|u\|_2\leqq M(T_1;\ \eta)|f|^2+|f|_2^2\leqq\frac{5}{4}|f|_2.$$

Successive application of the last inequality over the accumulated patching intervals, implies

$$|u(t;\ \eta)|_{H^2}\leqq\left(\frac{5}{4}\right)^{t/T_1+1}\cdot|f|_{H^2}.$$

Choosing $\eta=\eta_0$, Theorem 2.6 now follows with $\alpha=\eta_0+\ln(\frac{5}{4})$,

$$\left|\frac{\partial\phi}{\partial x}(t)\right|_{H^2}\leqq\frac{5}{4}e^{\alpha T}\cdot\left|\frac{\partial\phi}{\partial x}(t=0)\right|_{H^2},\qquad 0\leqq t\leqq T<\infty.$$

Remark. We note that the above solution $\phi=\phi(x,t)$ lies, in fact, in the same Sobolev space the initial data belong to, H^s, $0\leqq s\leqq 2$. This follows from a complementing L^2-estimate which we now derive: multiplying (1.1) by ϕ and integrating by parts, we find

$$\frac{1}{2}\frac{d}{dt}|\phi(t)|^2\leqq\left|\frac{\partial\phi}{\partial x}(t)\right|^2-\left|\frac{\partial^2\phi}{\partial x^2}(t)\right|^2+|\phi(t)|_{L^\infty}\cdot\left|\frac{\partial\phi}{\partial x}(t)\right|^2.$$

We interpolate in a somewhat nonstandard way, $|\phi|_{L^\infty} \leqq \varepsilon|\phi| + C\cdot\varepsilon^{-1}\cdot|\phi_x|$, so that by appropriately choosing $\varepsilon = \gamma\cdot|(\partial\phi/\partial x)(t)|^{-2}$, the last inequality implies

$$\frac{1}{2}\frac{d}{dt}|\phi(t)|^2 \leqq \gamma\cdot|\phi(t)|^2 + \gamma^{-1}K,$$

with $K = K(|(\partial\phi/\partial x)(t)|)$. Thanks to Lemma 2.3, we can control

$$K\left(\left|\frac{\partial\phi}{\partial x}(t)\right|\right) \leqq K\left(\left|\frac{\partial\phi}{\partial x}(t=0)\right|\right),$$

and L^2-boundedness now follows

$$|\phi(t)| \leqq e^{\gamma t}\cdot\left[|\phi(t=0)| + \gamma^{-1}\cdot K\left(\left|\frac{\partial\phi}{\partial x}(t=0)\right|\right)\right],$$

with arbitrarily small exponential growth factor γ, $\gamma > 0$. Regarding the periodic case, $-\pi/2 \leqq x \leqq \pi/2$, one may subtract the average

$$\bar{\phi}(t) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\phi(x,t)\,dx,$$

so that by invoking Poincaré's inequality for $\phi(t) - \bar{\phi}(t)$ rather than interpolating, we find

$$|\phi(t) - \bar{\phi}(t)| \leqq |\phi(t=0) - \bar{\phi}(t=0)| + K\left(\left|\frac{\partial\phi}{\partial x}(t=0)\right|\right)\cdot t^{1/2}.$$

5. A generalized Burgers equation. The results of the last sections were so organized, in order to emphasize that the only a priori estimate required for the proofs, concerns the linear dissipative part of the equation, see Lemma 3.1. Hence, the following generalization can be easily worked out.

We consider the generalized Burgers equation

$$\frac{\partial u}{\partial t} + \frac{\partial(u^2)}{\partial x} + P\left(\frac{\partial}{\partial x}\right)u = 0 \tag{5.1a}$$

whose linear part, $\partial/\partial t + P(\partial/\partial x)$, is assumed strongly parabolic of order ν,

$$\operatorname{Re}\hat{P}(i\xi) \geqq \text{Const}\cdot|\xi|^\nu, \qquad |\xi| \to \infty. \tag{5.1b}$$

Regarding the corresponding kernel, $\hat{Q}(t;\,\eta) = e^{-t(\eta + \hat{P}(i\xi))}$, we have, in analogy with Lemma 3.1,

$$|Q(t;\,\eta) * \omega|_{H^{m+r}} \leqq C\cdot e^{-(\eta-\eta_0)t}\cdot t^{-(r-1/2+1/p)/\nu}\cdot|\omega|_{W_p^m}. \tag{5.2}$$

In particular, considering $Q(t;\,\eta)$ operating from L^1 to H^{1+s}, it is found to have an operator norm with an *integrable* singularity, $t^{-(s+3/2)/\nu}$, provided $s < \nu - \frac{3}{2}$. Arguments similar to those introduced in §2, then lead us to

THEOREM 5.1. *Let u, v be two different solutions of the generalized Burgers equation* (5.1), *with initial data lying in H^s, $s < \nu - 3/2$. Then, there exist constants, C and $\beta \geqq 0$* (*both may depend on $|u(t=0)| + |v(t=0)|$*), *such that the following estimate holds*:

$$|u(t) - v(t)|_{H^s} \leqq C\cdot e^{\beta t}\cdot|u(t=0) - v(t=0)|_{H^s}. \tag{5.3}$$

We end up noting that the above recipe suggests itself, in studying the all important question regarding the long-time behavior of solutions for (5.1).

Remark. The special case $P(\partial/\partial x)=(-\partial^2/\partial x^2)^{\nu/2}$ can be considered as a one-dimensional degenerate case of the formal d-dimensional Navier–Stokes equations; global regularity in the latter case follows with dissipativity of order $\nu>1+d/2$, (see, e.g., Rose and Sulem, J. de Physique, 39 (1978), pp. 441–484). In either way, one finds $\nu=\frac{3}{2}$ as the critical order of dissipativity which guarantees regularity in the one-dimensional case, $d=1$.

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