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On a new scale of regularity spaces with applications to Euler's equations

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Abstract

We introduce a new ladder of function spaces which is shown to fill in the gap between the weak $L^{p\infty}$ spaces and the larger Morrey spaces, M^p. Our motivation for introducing these new spaces, denoted by $\vee^{pq}$, is to gain more accurate information on (compact) embeddings of Morrey spaces in appropriate Sobolev spaces. It is here that the secondary parameter q (and a further logarithmic refinement parameter α, denoted by $\vee^{pq}(\log\vee)^\alpha$) gives a finer scaling, which allows us to make the subtle distinctions necessary for embedding in spaces with a fixed order of smoothness.

We utilize an H^{-1}-stability criterion which we have recently introduced (Lopes Filho M C, Nussenzveig Lopes H J and Tadmor E 2001 Approximate solution of the incompressible Euler equations with no concentrations *Ann. Insitut H Poincaré* C **17** 371–412), in order to study the strong convergence of approximate Euler solutions. We show how the new refined scale of spaces, $\vee^{pq}(\log\vee)^\alpha$, enables us to approach the borderline cases which separate between H^{-1}-compactness and the phenomena of concentration–cancellation. Expressed in terms of their $\vee^{pq}(\log\vee)^\alpha$ bounds, these borderline cases are shown to be intimately related to uniform bounds of the total (Coulomb) energy and the related vorticity configuration.

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1. Introduction

We introduce a new ladder of function spaces which is shown to fill in the gap between Marcenkwetiz weak-L^p spaces, and the larger Morrey spaces, M^p. The former measure the total mass of measurable fs over *arbitrary sets*; the latter measure the total mass of fs over *arbitrary balls*. The newly introduced scale of spaces, denoted by $\vee^{pq}$, measures a ℓ_q-weighted distribution of the total mass of a measurable f over an arbitrary *collection of disjoint balls*.

Our motivation for introducing these new spaces, denoted by $\vee^{pq}$, is to gain more accurate information on (compact) embeddings of Morrey spaces in appropriate Sobolev spaces. It is here that the secondary parameter q (and a further logarithmic refinement parameter α, denoted by $\vee^{pq}(\log\vee)^{\alpha}$) give a finer scaling, which allows us to make the subtle distinctions necessary for embedding in spaces with a fixed order of smoothness.

In section 2 we prove that the new scale of spaces, $\vee^{pq}$, $q \geqslant p \geqslant 1$, interpolates the gap between $\vee^{pp} \subset L^{p\infty}$ and $\vee^{p\infty} = M^p$. The further logarithmic refinement, $\vee^{pq}(\log\vee)^{\alpha}$ is introduced in order to address the case $p = 1$. In section 3 we study the compact embeddings of $\vee^{pq}(\log\vee)^{\alpha}(\mathbb{R}^N)$. The compactness results are accomplished by a precise characterization of the decay of the wavelet coefficients for $\vee^{pq}(\log\vee)^{\alpha}$-functions. We are particularly interested in H^{-1}-compactness. For $\Omega \subset \mathbb{R}^N$, it is shown that $\vee^{p2}(\log\vee)^{\alpha}(\Omega)$ is H^{-1}-compact if $p > 2N/(N+2)$, or, if $p = 2N/(N+2)$ and $\alpha > \frac{1}{2}$. This should be compared with the compact embedding of Morrey spaces, consult [21, theorem 4.2], which states that $M^p(\Omega) \overset{\text{comp}}{\hookrightarrow} H^{-1}$ for the restricted range of $p > N/2$. Equipped with the new scale of spaces $\vee^{p2}(\log\vee)^{\alpha}(\mathbb{R}^N)$ we are now able to resolve the question of compactness for the gap of ps, $p \leqslant N/2 < 2$. Specifically, we show that the question of $H^{-1}_{\mathrm{loc}}(\mathbb{R}^N)$-compactness is characterized by the borderline cases of $X_2 := \vee^{12}(\log\vee)^{1/2}_c(\mathbb{R}^2)$ in the two-dimensional (2D) case, and $X_3 := \vee^{\frac{6}{5}2}_c(\mathbb{R}^3)$ in the three-dimensional (3D) case.

The new scale of spaces, through the precise characterization of their H^{-1}-compactness, is put into use in section 4, where we discuss the approximate solution of the incompressible Euler equations. Recently, we introduced in [21] a sharp local condition for the lack of concentrations in (and hence the L^2 convergence of) sequences of such approximate solutions. Simply stated, the sequence of associated vorticities is required to be H^{-1}_{loc}-compact, and it is in this context that the $\vee^{pq}$-bounds are shown to play a fundamental role. Indeed, in both the $N = 2$ and the $N = 3$-dimensional cases, we show that the corresponding X_N-bounds on the vorticities are intimately related to the uniform bound on the Coulomb energy of the solutions and their vorticity configurations.

In the two-dimensional case we end up with a rather complete classification which is summarized in the following statement (consult corollary 4.1 and theorem 4.1 below).

Theorem. *Let $\{u^{\varepsilon}(\cdot,t)\}$ be a family of approximate solutions of the 2D Euler equations, and assume that the corresponding sequence of vorticities $\{\omega^{\varepsilon}(\cdot,t)\}$ is uniformly bounded in $\widetilde{\vee}^{12}(\log\widetilde{\vee})^{\alpha}_c(\mathbb{R}^2)$, $\alpha > 0$.*

(a) No concentration. If $\alpha > \frac{1}{2}$, then $\{u^{\varepsilon}(\cdot,t)\}$ is strongly compact in $L^{\infty}([0,T]; L^2_{\mathrm{loc}}(\mathbb{R}^2))$, with a strong L^2-limit, $u(\cdot,t)$, which is a weak solution of the 2D Euler equations.

(b) Concentration–cancellation. If $\alpha \in (0, \frac{1}{2}]$, then $\{u^{\varepsilon}(\cdot,t)\}$ has a L^2-weak limit, $u(\cdot,t)$ which is a finite-energy solution of the 2D Euler equations.

One signed measures, say $\omega^{\varepsilon}(\cdot,0) \geqslant 0$ are shown to be bounded in $\widetilde{\vee}^{12}(\log\widetilde{\vee})^{1/2}_c(\mathbb{R}^2)$ (consult lemma 4.1 below), and thus part (b) of the above theorem is recast as an extended version of Delort's result [14] in the language of $\vee$-spaces.

The new ladder of spaces establishes a direct linkage between questions related to the configuration of the N-dimensional pseudo-energy and regularity in the borderline case X_N. The configuration of three-dimensional vorticity, $\boldsymbol{\omega}^{\varepsilon}(\cdot,t)$ involves local stretching effects and nonlinear energy saturation associated with small sets of increasingly intense vorticity. The following result relates these issues to the uniform bound in the borderline case $X_3 = \vee^{\frac{6}{5}2}$.

Theorem. *Let $\{u^{\varepsilon}(\cdot,t)\}$ be a family of approximate solutions of the 3D Euler equations and assume that the corresponding sequence of compactly supported vorticities, $\{\boldsymbol{\omega}^{\varepsilon}(\cdot,t)\}$ satisfies the local alignment condition (4.8). Then the following holds:*

$$\|\boldsymbol{\omega}^{\varepsilon}(\cdot,t)\|_{\vee^{\frac{6}{5}2}(\Omega)} \leqslant \textit{constant} \qquad \Omega \subset \mathbb{R}^3.$$

We close by noting that the new scale of spaces, $\widetilde{\vee}^{pq}(\Omega)$, is not necessarily comparable with the scale of Morrey spaces, $\widetilde{M}^r(\Omega)$, unless additional information, for example, the packing measure of Ω is provided. Thus, for example, $\widetilde{M}^{3/2}(\Omega)$ is a borderline Morrey space for $H^{-1}_{\mathrm{loc}}(\mathbb{R}^3)$ compactness, which was shown by Giga and Miyakawa [15] to guarantee the existence of the related 3D Navier–Stokes solutions. Compared with the corresponding borderline case $X_3(\Omega) = \widetilde{\vee}^{\frac{6}{5}2}(\Omega)$, we find in corollary 3.1 below that the latter is larger, $\widetilde{M}^{3/2}(\Omega) \subset X_3(\Omega)$, for Ωs with finite packing measure so that $\pi^{h_1}(\Omega) < \infty$.

2. The spaces $\vee^{pq}(\log \vee)^{\alpha}(\Omega)$

Given a domain $\Omega \subset \mathbb{R}^N$, we consider the set $\mathcal{B}(\Omega)$ of all collections of mutually disjoint balls contained in Ω, $\mathcal{B}(\Omega) = \{B_j \mid \cup B_j \subseteq \Omega\}$, balls with sufficiently small radius $B_j = B_{R_j}(x_j)$, $R_j \leqslant R_0 < \frac{1}{2}$.

Definition 2.1. *The space $\vee^{pq}(\Omega)$, $1 \leqslant p \leqslant q \leqslant \infty$, consists of all fs in $L^1_{\mathrm{loc}}(\Omega)$ such that for all collections, $\{B_j\} \subset \mathcal{B}(\Omega)$, the following estimate holds:*

$$\sup_{\{B_j\}\subset\mathcal{B}(\Omega)} \left(\sum_j \left(R_j^{-N/p'} \int_{B_j} |f(x)|\, \mathrm{d}x \right)^q \right)^{1/q} \leqslant \text{constant} \qquad 1 \leqslant p \leqslant q \leqslant \infty. \tag{2.1}$$

The smallest of such constants in (2.1) is the $\vee^{pq}$-norm of f. Thus, if we let $⨍_{B_R(x_0)} |f(x)|\,\mathrm{d}x$ denote the average mass of $|f|$ over the ball $B_R(x_0)$ centred on x_0, and $\bar{f} = (\bar{f}_1, \bar{f}_2, \ldots)$ denote the vector of averages, $\bar{f}_j := ⨍_{x\in B_j} |f(x)|\,\mathrm{d}x$, then

$$\|f\|_{\vee^{pq}(\Omega)} := \sup_{R_j < R_0} \|\{R_j^{N/p} \bar{f}_j\}\|_{\ell^q} \qquad q \geqslant p. \tag{2.2}$$

Occasionally, we shall need a further logarithmic refinement

$$\vee^{pq}(\log \vee)^{\alpha}(\Omega) := \left\{ f \in L^1(\Omega) \mid \sup_{R_j<R_0} \|\{R_j^{N/p} |\log R_j|^{\alpha} \bar{f}_j\}\|_{\ell^q} < \infty \right\} \qquad q > p. \tag{2.3}$$

We abbreviate $\vee^{pq,\alpha} = \vee^{pq}(\log \vee)^{\alpha}$. We shall also need the corresponding extension dealing with bounded measures, $\mu \in \mathcal{BM}(\Omega)$. With $\bar{\mu}_j := |\mu|(B_j)/|B_j|$ we set

$$\widetilde{\vee}^{pq,\alpha} = \left\{ \mu \in \mathcal{BM}(\Omega) \mid \sup_{R_j<R_0} \|\{R_j^{N/p} |\log R_j|^{\alpha} \bar{\mu}_j \right\} \|_{\ell^q} \qquad q \geqslant p.$$

For $\Omega = \mathbb{R}^N$, the space $\vee^{pq,\alpha}_{\mathrm{loc}}$ is defined as the Fréchet space determined by the family $\{\|f\|_{\vee^{pq,\alpha}(B_k(0))}\}_{k\in\mathbb{N}}$.

The norm $\|f\|_{\vee^{pq,\alpha}(\Omega)}$ quantifies the (ir-)regularity of f by measuring a weighted distribution of its singularities, distributed over a 'packing' of Ω by a covering of balls. Clearly, the use of balls in these definitions is not essential, and they can be replaced, for example, by sequences of non-thin cubes, $\{\mathcal{C}_j\}$, for which $(\operatorname{div} \mathcal{C}_j)^N \leqslant \text{constant} \times |\mathcal{C}_j|$. Thus, if a bounded $\Omega \subset \mathbb{R}^N$ is covered by a lattice of disjoint cubes, $\Omega \subset \cup \mathcal{C}_j$, $\mathcal{C}_j(\cdot) = \mathcal{C}(\cdot + j)$, $j \in Z^N$, each of size $|\mathcal{C}_j| = R^N$, then $f \in \vee^{pq,\alpha}(\Omega)$ implies

$$\left(\sum_j \left(\int_{\mathcal{C}_j} |f(x)|\,\mathrm{d}x \right)^q \right)^{1/q} \leqslant R^{N/p'} |\log R|^{-\alpha} \qquad \mathcal{C}_j(\cdot) := \mathcal{C}(\cdot + j) \quad j \in Z^N. \tag{2.4}$$

We note in passing that as we refine the covering, say by a dyadic refinement of the lattice in (2.4), the corresponding $\vee$-sum, $\|\{R_j^{N/p} \bar{f}_j\}\|_{\ell^q}$ need not increase for $q \geqslant p$.

We want to place the scale of new spaces, $\vee^{pq,\alpha}$, in the context other known spaces, and this is carried out in terms of the known Lorentz–Zygmund and Morrey spaces. Here is a brief readers' digest which will enable us to introduce the necessary notation, and we refer the reader to [2] for a detailed description.

Let f^* denote the usual decreasing rearrangement of f. For a bounded $\Omega \subset \mathbb{R}^n$, the space $L^{pq,\alpha}(\Omega) = L^{pq}(\log L)^\alpha(\Omega)$ consists of all measurable functions fs such that

$$\left(\int_{s=0}^{|\Omega|} [s^{1/p}|\log s|^\alpha f^*(s)]^q \, \mathrm{d}s/s\right)^{1/q} < \infty$$

we shall be exclusively concerned with the weak spaces corresponding to $q = \infty$, where

$$\|f\|_{L^{p\infty,\alpha}(\Omega)} = \sup_{s \leqslant |\Omega|} s^{1/p}|\log s|^\alpha f^*(s).$$

For consistency of notation, however, we retain here the secondary index $q = \infty$, and we refer the interested reader to [1, 2], for a detailed study of the logarithmic refinement indexed with $\alpha > 0$.

If we replace f^* with its maximal function, $f^{**} := \frac{1}{s}\int_0^s f^*(r)\,\mathrm{d}r$, we obtain the closely related Lorentz–Zygmund spaces $L^{(pq,\alpha)}(\Omega) = \{f \mid \|s^{1/p}|\log s|^\alpha f^{**}\|_{L^q(\mathrm{d}s/s)} < \infty\}$. The $L^{(pq,\alpha)}$s are rearrangement-invariant spaces which include as special cases both the Lorentz spaces, $L^{(pq)} = L^{(pq,0)}$, and the logarithmic Orlicz spaces $L^{(11,\alpha-1)} = L(\log L)^\alpha(\Omega)$ [1, theorem 11.1], Again, we are exclusively interested here in the case of the secondary index $q = \infty$: using the maximality of $f^{**}(s) = \sup_{|E|=s} ⨍_E |f|$, we find that $L^{(p\infty,\alpha)}$ consists of all fs such that

$$L^{(p\infty,\alpha)}(\Omega) = \left\{ f \mid \int_E |f(y)|\,\mathrm{d}y \leqslant \text{constant} \times |E|^{1/p'}\big|\log|E|\big|^{-\alpha}, \right.$$
$$\left. \forall E \subset \Omega, |E| < E_0 < 1 \right\}. \tag{2.5}$$

We note in passing that $L^{(pq,\alpha)}$ coincide with $L^{pq,\alpha}$ for $p > 1$ [1, corollary 8.2]. For $p = 1$, however, the spaces $L^{(1q,\alpha)}$ (denoted by $\mathcal{L}^{1q}(\log\mathcal{L})^\alpha$ in [1, section 11]) are strictly smaller than the corresponding $L^{1q,\alpha}$. Thus, with $\alpha = 0$, for example, the $L^{(1q)}$s are varying between $L^{(11)} = L(\log L)$ and $L^{(1\infty)} = L^1$.

Finally, if we replace in (2.5) the arbitrary sets Es by balls, we enlarge the Lorentz–Zygmund spaces, arriving at the scale of Morrey spaces,

$$M^{p,\alpha}(\Omega) := \left\{ f \in L^1(\Omega) \mid \int_{B_R(x_0)\subset\Omega} |f(x)|\,\mathrm{d}x \leqslant CR^{N/p'}|\log R|^{-\alpha},\ \forall R \leqslant R_0 < 1 \right\}. \tag{2.6}$$

The case $\alpha = 0$ yields the classical Morrey space M^p, e.g. [15, 16]; the logarithmic refinement of $M^{p,\alpha}$ was recently used in [21], motivated by the corresponding logarithmic refinement in Lorentz–Zygmund spaces. Following [15, 21], we let $\widetilde{M}^{p,\alpha}(\Omega)$ denote the corresponding Morrey scale for bounded measures, $\mu \in \mathcal{BM}(\Omega)$

$$\|\mu\|_{\widetilde{M}^{p,\alpha}} := \sup_{R<R_0<1} \big[R^{-N/p'}|\log R|^\alpha|\mu|(B_R(x))\big].$$

We now turn to discuss the scale of spaces $\vee^{pq}$ for $p \leqslant q \leqslant \infty$. Their definition in (2.1) makes apparent the role of the parameter q as the usual secondary index, so that $\vee^{pq}$ form a 'scale' of intermediate spaces between $\vee^{p\infty}$ and $\vee^{pp}$. Indeed, one can interpolate an $\vee^{pq}$ bound

$$\|f\|_{\vee^{pq}} \leqslant \|f\|_{\vee^{pp}}^{\theta} \cdot \|f\|_{\vee^{p\infty}}^{1-\theta} \qquad \theta = p/q \leqslant 1. \tag{2.7}$$

More precisely, using real interpolation arguments along the lines of, for example, [9, theorem 7.5], one finds $\vee^{pq}$ as an interpolation space of $\vee^{p\infty}$ and $\vee^{pp}$,

$$\vee^{pq} = (\vee^{p\infty}, \vee^{pp})_{\theta,q} \qquad \theta = p/q \leqslant 1.$$

It is therefore enough to consider the two end cases $q = p$ and $q = \infty$. We start with the latter.

Clearly, $\vee^{p\infty,\alpha}$ consists of all $L^1_{\text{loc}}(\Omega)$ fs whose behaviour is determined by their average mass on just *one* ball, i.e. (2.6) holds.

Lemma 2.1. *For $p \geqslant 1$ we have*

$$\vee^{p\infty,\alpha}(\Omega) = M^{p,\alpha}(\Omega) \qquad p \geqslant 1. \tag{2.8}$$

Next we turn to discuss the spaces $\vee^{pp,\alpha}$, which are shown to be in between the Lorentz–Zygmund spaces $L^{pp,\alpha} \equiv L^p(\log L)^\alpha$ (consult [1, corollary 10.2]) and $L^{p\infty,\alpha} \equiv L^{p\infty}(\log L)^\alpha$. The following lemma is in the heart of this matter.

Lemma 2.2. *For $p \geqslant 1$ we have*

$$L^p(\log L)^\alpha(\Omega) \subset \vee^{pp,\alpha}(\Omega) \subset L^{p\infty}(\log L)^\alpha(\Omega) \qquad p \geqslant 1 \quad \alpha \geqslant 0. \tag{2.9}$$

Proof. Consider an arbitrary open measurable set $E \subseteq \Omega$ of size $|E| = t < 1$. To verify the right half of (2.9), we need to estimate the decay rate of $\int_E |f(x)|\,\mathrm{d}x$ as $t \downarrow 0$. To this end, we cover E by the family of interior balls $\{B_{R_x}(x) \subset E\}$. By Vitali's covering lemma, e.g. [28, section 1.6], we can select a subfamily of countably many disjoint balls, $\{B_j = B_{R_j}(x_j) \mid \cup B_j \subset E\}$, which cover at least a fixed fraction of E, namely, the complement of $E_1 := \cup_j B_{R_j}(x_j)$ does not exceed $|E - E_1| \leqslant \theta t$ with $\theta = \left(\frac{4}{5}\right)^N$.

We write

$$\int_E |f(x)|\,\mathrm{d}x = \int_{E-E_1} |f(x)|\,\mathrm{d}x + \sum_j \int_{B_j} |f(x)|\,\mathrm{d}x. \tag{2.10}$$

Assuming that $f \in \vee^{pp,\alpha}(\Omega)$, then the last summation on the right does not exceed

$$\begin{aligned}\sum_j \int_{B_j} |f(x)|\,\mathrm{d}x &\leqslant \left(\sum_j \left(R_j^{-N/p'} |\log R_j|^\alpha \int_{B_j} |f(x)|\,\mathrm{d}x\right)^p\right)^{1/p} \\ &\quad\times \left(\sum_j |\log R_j|^{-\alpha p'} R_j^{Np'/p'}\right)^{1/p'} \\ &\leqslant \text{constant} \times t^{1/p'} |\log t|^{-\alpha} \qquad \text{constant} = N^\alpha \|f\|_{\vee^{pp,\alpha}(\Omega)}.\end{aligned}$$

Next, consider the maximal function $F(t) := \sup_{|E|=t} \int_E |f(x)|\,\mathrm{d}x$. Using the fact that $|E - E_1| \leqslant \theta t$ together with (2.11), then (2.10) yields

$$F(t) \leqslant F(\theta t) + \text{constant} \times t^{1/p'} |\log t|^{-\alpha}. \tag{2.11}$$

Recalling that $F(t)$ is, in fact, the primitive of the decreasing rearrangement f^*, $F(t) = \int_0^t f^*(s)\,\mathrm{d}s$, the desired $\|f\|_{L^{p\infty}}$-bound follows from (2.11)

$$f^*(t) \leqslant \frac{F(t) - F(\theta t)}{(1-\theta)t} \leqslant \text{constant} \times t^{-1/p} |\log t|^\alpha.$$

For the reversed implication on the left of (2.9), we use the Hölder inequality which yields the following straightforward $\vee^{pp}$-bound for L^p functions:

$$\sum_j \left(R_j^{-N/p'} \int_{B_j} |f(x)|\,\mathrm{d}x \right)^p \leqslant \sum_j R_j^{-Np/p'} \int_{B_j} |f(x)|^p\,\mathrm{d}x\, R_j^{Np/p'}$$
$$= \int_{\cup B_j} |f(x)|^p\,\mathrm{d}x \leqslant \|f\|_{L^p}^p \qquad p \geqslant 1 \tag{2.12}$$

and thus, the left-hand side of (2.9) with $\alpha = 0$ follows. For general $\alpha > 0$ we need a logarithmic refinement based on the duality between $L^p(\log L)^\alpha$ and $L^{p'}(\log L)^{-\alpha}$, consult, for example, [2, corollary 8.15], [1, theorem 8.4], yielding

$$\sum_j \left(R_j^{-N/p'} |\log R_j|^\alpha \int_{B_j} |f(x)|\,\mathrm{d}x \right)^p \leqslant \sum_j R_j^{-Np/p'} |\log R_j|^{\alpha p} \|f(x)\|_{L^p(\log L)^\alpha(B_j)}^p$$
$$\times \left(\int_{t=0} \left[(1 + |\log t|)^{-\alpha} 1_{0 \leqslant t \leqslant R_j^N} \right]^{p'} \mathrm{d}t \right)^{p/p'}$$
$$\leqslant \text{constant} \times \sum_j R_j^{-Np/p'} |\log R_j|^{\alpha p} \|f(x)\|_{L^p(\log L)^\alpha(B_j)}^p R_j^{Np/p'} |\log R_j|^{-\alpha p}$$
$$\leqslant \text{constant} \times \|f(x)\|_{L^p(\log L)^\alpha(\cup B_j)}^p \qquad p \geqslant 1 \quad \alpha \geqslant 0. \tag{2.13}$$

Thus, the $\vee^{pp,\alpha}$ bound of f implies that the left-hand side of (2.9) holds. □

Remarks.

1. We note in passing an alternative derivation of (2.9). Setting $F^{(p,\alpha)}(t) := t^{-1/p'} |\log t|^\alpha F(t)$, then (2.11) yields

$$F^{(p,\alpha)}(t) \leqslant \theta^{1/p'} \left| \frac{\log t}{\log(\theta t)} \right|^\alpha F^{(p,\alpha)}(\theta t) + \text{constant}.$$

Successive application of this recursion relation yields

$$F^{(p,\alpha)}(t) \leqslant \sum_k^\infty \theta^{k/p'} \left| \frac{\log t}{\log(\theta^k t)} \right|^\alpha$$
$$= |\log t|^\alpha \sum_k \frac{\theta^{k/p'}}{(k|\log\theta| + |\log t|)^\alpha}$$
$$\leqslant \begin{cases} \text{constant} & p > 1 \\ \text{constant} \times |\log t| & p = 1 \quad \alpha > 1. \end{cases}$$

For $p > 1$, we conclude, as before, $\vee^{pp,\alpha} \subset L^{(p\infty,\alpha)} = L^{p\infty,\alpha}$—the logarithmic refinement corresponding to (2.9). For $p = 1, \alpha > 1$, however, this approach only yields $\vee^{11,\alpha} \subset L^{(1\infty,\alpha-1)}$, whereas the derivation of lemma 2.2 led to a tighter bound in terms of $L^{1\infty,\alpha}$. We note that the space $L^{1\infty,\alpha}$ is indeed smaller (at least for $\alpha > 1$) than the space $L^{(1\infty,\alpha-1)}$ [1, theorem 12.1].

2. The following example, due to DeVore [7], shows that $\vee^{pp}(\mathbb{R}_+)$ lies *strictly* inside $L^{p\infty}(\mathbb{R}_+)$. To this end observe that the averages of the $L^{p\infty}$ function $f(x) = x^{-1/p}$, averaged over the dyadic intervals $I_j = [2^{-j}, 2^{-j+1}]$, are given by $\bar{f}_j = ⨍_{I_j} |f(x)|\,\mathrm{d}x = c_p 2^{j/p}$, and hence $\{2^{-j/p}\bar{f}_j \equiv \text{constant}\} \notin \ell_p$. In fact, this shows that $L^{p\infty} \not\subset \vee^{pq}$, $q < \infty$.
3. For a different kind of inclusion relations in terms of Besov spaces we refer to (3.4) below, asserting that $\vee^{pp'}(\Omega) \subset B_\infty(L^{p'}(\Omega))$ for $p \leqslant 2$.

In summary, we see that the new spaces $\vee^{pq,\alpha}$ offer a new ladder which covers the gap between the weak Lorentz–Zygmund spaces corresponding to $q = p$, and the larger Morrey spaces corresponding to $q = \infty$, namely

$$L^{pp,\alpha} \subset \vee^{pp,\alpha}_{\text{loc}} \subset L^{p\infty,\alpha} \cdots \vee^{pq,\alpha}_{\text{loc}} \cdots \subset \vee^{p\infty,\alpha}_{\text{loc}} = M^{p,\alpha}_{\text{loc}} \qquad p \geqslant 1. \tag{2.14}$$

3. Compact embeddings

Our motivation for introducing the new spaces $\vee^{pq,\alpha}$ was to gain a more accurate information on (compact) embeddings of Morrey spaces in appropriate Sobolev spaces. It is here that the secondary parameters q and α give a finer scaling, which allows us to make the subtle distinctions necessary in embedding in spaces with a fixed order of smoothness. To avoid an excessive number of indices, we begin with a prototype configuration, referring to the specific situation encountered in [21]. The general case will be stated later (in theorem 3.2 below).

According to [21, theorems 4.2 and 4.3], the Morrey spaces $\widetilde{M}^{p,\alpha}(\Omega)$ are precompact in $H^{-1}(\Omega)$ as long as

$$\widetilde{M}^{p,\alpha}(\Omega) \overset{\text{comp}}{\hookrightarrow} H^{-1}(\Omega) \qquad \left(p - \frac{N}{2}\right)_+ + (\alpha - 1)_+ > 0. \tag{3.1}$$

We distinguish between two borderline cases.

- In the two-dimensional case, we find that $\widetilde{M}^{1,\alpha}(\mathbb{R}^2)$ is precompact in $H^{-1}(\mathbb{R}^2)$ for $\alpha > 1$. On the other hand, counterexamples constructed in [11, 22] show that $\widetilde{M}^{1,1/2}(\mathbb{R}^2) \cap \mathcal{BM}^+_c(\mathbb{R}^2)$ is not compactly embedded in $H^{-1}(\mathbb{R}^2)$. Thus, the gap $\frac{1}{2} < \alpha < 1$ remains open with regard to the question of compact embedding of $\widetilde{M}^{1,\alpha}(\mathbb{R}^2)$ in $H^{-1}_{\text{loc}}(\mathbb{R}^2)$.
- The gap is even wider for $p > 1$. Considering the Lebesgue/Lorentz hierarchy (here we ignore the logarithmic subscaling, taking $\alpha = 0$), one finds the critical Lebesgue exponent $(p^*)' = \frac{2N}{N+2}$, so that all $L^{p,\infty}_c(\mathbb{R}^N)$ with $p > \frac{2N}{N+2}$ are compactly embedded in $H^{-1}_{\text{loc}}(\mathbb{R}^N)$. The Morrey hierarchy is different: according to (3.1), Morrey spaces $M^p_c(\mathbb{R}^N)$ are $H^{-1}_{\text{loc}}(\mathbb{R}^N)$-compact for a smaller range of exponents with $p > N/2$. Though the Morrey spaces are bigger than the corresponding weak-L^p, $L^{p,\infty} \subset M^p$, they both admit the same scaling. Thus, for example, with $N = 3$ we are left with the open question with regard to the 'correct' scaling exponent within the intermediate gap $\frac{6}{5} < p < \frac{3}{2}$, which will suffice for compact embedding in $H^{-1}_{\text{loc}}(\mathbb{R}^3)$.

Equipped with the new scale of *intermediate* spaces $\vee^{pq,\alpha}$, we are able to address the question of compactness for the above gaps, by sharpening (3.1) as follows.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let $\{f^\varepsilon\} \subset C^\infty_c(\Omega)$ be a bounded sequence in $\vee^{p2,\alpha}(\Omega)$. If either:*

(a) $p > 2N/(N+2)$*, or,*

(b) $p = 2N/(N+2)$ *and* $\alpha > \frac{1}{2}$

then $\{f^\varepsilon\}$ *is precompact in* $H^{-1}_{\text{loc}}(\mathbb{R}^N)$.

Proof. We assume that Ω is included within the N-box, $\mathcal{C}_0 = [-2^{k_0}, 2^{k_0}]^N$. We will consider an orthonormal wavelet basis for $L^2(\Omega)$, $\{\psi_{jk}\}$. This basis may be built using a (finite) wavelet set, $\Psi = \{\psi\}$, supported in $\mathcal{C}_0$, which we will require to belong to $H^1(\mathbb{R}^N)$ (consult [6, section 10.1], [10, section 3.6] or [23] for a brief overview). Specifically, the wavelet basis consists of

$$\psi_{jk}(x) := 2^{kN/2}\psi(2^k x - j) \qquad k \in Z_0^+ := Z^+ - k_0 \quad j \in Z^N \quad \psi \in \Psi$$

which are supported in the dyadic cubes $\mathcal{C}_{jk} := 2^{-k}(\mathcal{C}_0 + j)$; of course, $\text{div}(\mathcal{C}_{jk}) \sim R_k = 2^{-k}$ for all js, and we consider the wavelet expansion of each f^ε:

$$f^\varepsilon = \sum_{\psi\in\Psi}\sum_{k\in Z_0^+}\sum_{j\in Z^N}\hat{f}^\varepsilon_{jk}\psi_{jk} \qquad \hat{f}^\varepsilon_{jk} = \int_{\mathcal{C}_{jk}} f^\varepsilon \psi_{jk}\,\mathrm{d}x.$$

The ψ_{jk}s are H^{-1}-orthogonal, each of which does not exceed $\|\psi_{jk}\|^2_{H^{-1}} \leqslant \min\left\{2^{-2k}\int|\widehat{\psi}(\eta)|^2/|\eta|^2, 1\right\}$, and hence

$$\|f^\varepsilon\|^2_{H^{-1}} = \sum_{\psi\in\Psi}\sum_{(j,k)\in(Z^N, Z_0^+)} |\hat{f}^\varepsilon_{jk}|^2\|\psi_{jk}\|^2_{H^{-1}} \leqslant \text{constant} \times \sum_{k\in Z_0^+} 2^{-2k}\sum_{j\in Z^N}|\hat{f}^\varepsilon_{jk}|^2.$$

Next we observe that $\cup_j \mathcal{C}_{jk}$ is a covering of disjoint cubes, each of volume of $R_k^N = 2^{-kN}$. Hence, the application of (2.4) for $f^\varepsilon \in \vee^{p2,\alpha}$ (with $R = R_k = 2^{-k}$) yields

$$\begin{aligned}\sum_{j\in Z^N}|\hat{f}^\varepsilon_{jk}|^2 &\leqslant 2^{kN}\sum_{j\in Z^N}\left(\int_{\mathcal{C}_{jk}}|f^\varepsilon(x)|\,\mathrm{d}x\right)^2 \\ &\leqslant \text{constant} \times 2^{kN}\|f^\varepsilon\|^2_{\vee^{p2,\alpha}}2^{-2kN/p'}|1+k_+|^{-2\alpha}. \end{aligned} \tag{3.2}$$

It follows that the f^εs are bounded in H^{-1}. Indeed, using (3.2) we find the upper bound

$$\|f^\varepsilon\|^2_{H^{-1}} \leqslant \text{constant} \times \sum_{k\in Z_0^+} 2^{-2k}\,2^{kN}2^{-2kN/p'}|1+k_+|^{-2\alpha}$$

which shows that f^ε are H^{-1}-bounded if either (a) or (b) holds. Moreover, we have H^{-1}-compactness of $\{f^\varepsilon\}$ in view of the *uniform* summability

$$\left\|\sum_{k>K}\sum_{j\in Z^N}\hat{f}^\varepsilon_{jk}\psi_{jk}\right\|^2_{H^{-1}} \leqslant \text{constant} \times \sum_{k>K} 2^{k(N-2N/p'-2)}|1+k_+|^{-2\alpha}$$

$$\overset{K\to\infty}{\longrightarrow} 0 \quad \text{uniformly in } \varepsilon.$$

The uniform high-frequency decay (in H^{-1}) converts weak compactness in H^{-1} into a strong one. $\square$

Remarks.

1. The compact embedding stated in theorem 3.1 is extended to more general families of measures. Arguing along [21, theorem 4.3] we find

$$\widetilde{\bigtriangledown}^{p2,\alpha}(\Omega) \overset{\text{comp}}{\hookrightarrow} H^{-1}(\Omega) \qquad \left(p - \frac{2N}{N+2}\right)_+ + \left(\alpha - \frac{1}{2}\right)_+ > 0.$$

2. The scale of space $\vee^{pq,\alpha}$ enables one to make the (compact) embeddings precise in more general Besov spaces, $B^s_\eta(L^r(\Omega))$ spaces (measuring s-order of smoothness in $L^r_{\text{loc}}(\mathbb{R}^N)$ with secondary index η). The latter is characterized by a bounded wavelet expansion based on a scaled basis of pre-wavelets $\psi_{jk}(x) = 2^{kN/r}\psi(2^k x - j)$. Assume ψ has a certain order of smoothness, say, s_0, then [8, 9]

$$\|f\|^\eta_{B^s_\eta(L^r(\mathbb{R}^N))} \sim \sum_{k\in Z} 2^{ks\eta}\left(\sum_{j\in Z^N} |\hat{f}_{jk}|^r\right)^{\eta/r} \qquad -\infty < s < s_0 \quad 1 < r < \infty.$$

Arguing as before we arrive at

Theorem 3.2. *For a bounded* $\Omega \subset \mathbb{R}^N$ *we have*

$$\vee^{pq,\alpha}(\Omega) \overset{\text{comp}}{\hookrightarrow} B^s_\eta(L^q(\Omega)) \qquad \begin{cases} \dfrac{1}{p} < \dfrac{1}{q'} - \dfrac{s}{N} & \alpha \geqslant 0 \\ \dfrac{1}{p} = \dfrac{1}{q'} - \dfrac{s}{N} & \alpha > 1/\eta. \end{cases} \tag{3.3}$$

The case $(\eta, q, s) = (2, 2, -1)$ corresponds to theorem 3.1. The limiting case $(\eta, q, s) = (\infty, p', 0)$ yields the (non-compact) embedding

$$\vee^{pp'}(\Omega) \subset B_\infty(L^{p'}(\Omega)). \tag{3.4}$$

Equipped with the scale of spaces $V^{p2,\alpha}$ of theorem 3.1, we return to examine the gap mentioned earlier concerning H^{-1} compactness. For the full gap of ps, $p \in [\frac{N+2}{2N}, \frac{N}{2}]$, to be H^{-1}-compact requires $N/2 < 2$, where we are left with precisely the two relevant cases of two- and three-dimensional problems. We distinguish between the two borderline cases.

- In the two-dimensional case we find that

$$\widetilde{\vee}^{12,\alpha}_c(\mathbb{R}^2) \overset{\text{comp}}{\hookrightarrow} H^{-1}_{\text{loc}}(\mathbb{R}^2) \qquad \alpha > \tfrac{1}{2}. \tag{3.5}$$

We recall that for $\alpha > 1$, $\widetilde{M}^{1,\alpha}_{\text{loc}}$ is $H^{-1}_{\text{loc}}(\mathbb{R}^2)$-compact, while $\widetilde{M}^{1,1/2}(\mathbb{R}^2) \cap \mathcal{BM}^+_c(\mathbb{R}^2)$ is not. Using (3.5), we are now able to address the open issue of compact embedding of $\widetilde{M}^{1,\alpha}_{\text{loc}} = \widetilde{\vee}^{1\infty,\alpha}$ in the gap $\frac{1}{2} \leqslant \alpha \leqslant 1$. We conclude that half of this gap, quantified in terms of $\widetilde{\vee}^{1q,\alpha}(\mathbb{R}^2)$, $\alpha > \frac{1}{2}$ with the secondary index $1 \leqslant q \leqslant 2$ is $H^{-1}_{\text{loc}}(\mathbb{R}^2)$-compact, and, as we shall see below, the conclusion (3.5) is sharp in the sense that $H^{-1}_{\text{loc}}(\mathbb{R}^2)$-compactness is lost for $\widetilde{\vee}^{1q,\frac{1}{2}}(\mathbb{R}^2)$ with the secondary index in the other half, $2 \leqslant q \leqslant \infty$. In particular, we identify as a borderline case for H^{-1}-compactness, the space $\widetilde{\vee}^{12}(\log\widetilde{\vee})^{1/2}(\mathbb{R}^2)$ which consists of all measures such that

$$\widetilde{\vee}^{12}(\log\widetilde{\vee})^{1/2}(\Omega) = \left\{\mu \mid \sup_{\{B_j\}\subset\mathcal{B}(\Omega)} \sum_j |\log R_j|(|\mu|(B_j))^2 \leqslant \text{constant}\right\} \qquad \Omega \subset \mathbb{R}^2. \tag{3.6}$$

- In the three-dimensional case we find

$$\widetilde{\vee}^{p2}_c(\mathbb{R}^3) \overset{\text{comp}}{\hookrightarrow} H^{-1}_{\text{loc}}(\mathbb{R}^3) \qquad p > \tfrac{6}{5}. \tag{3.7}$$

We recall the different scales of H^{-1}-compactness: for Moerry spaces, $\widetilde{M}^p_{\text{loc}}(\mathbb{R}^3) \overset{\text{comp}}{\hookrightarrow} H^{-1}$ for $p > \frac{3}{2}$, while for Lorentz spaces, $L^{p\infty}_{\text{loc}}(\mathbb{R}^3) \overset{\text{comp}}{\hookrightarrow} H^{-1}$ for $p > \frac{6}{5}$. Using our new scale of spaces, we can now address the issue of H^{-1}-compactness of $\widetilde{M}^p_{\text{loc}} = \widetilde{\vee}^{p\infty}$ in the gap

$\frac{3}{2} > p > \frac{6}{5}$. We conclude that for $p > \frac{6}{5}$, half of this gap, quantified in terms of $\widetilde{\nabla}^{pq}_{\mathrm{loc}}(\mathbb{R}^3)$ with $1 \leqslant q \leqslant 2$, is H^{-1}-compact. In particular, we realize that as in the case of Lorentz scale, $p = \frac{6}{5}$ is the 'correct' critical index for $H^{-1}(\mathbb{R}^3)$-compactness, and we identify as a borderline case the space $\widetilde{\nabla}^{\frac{6}{5}2}(\mathbb{R}^3)$ which consists of all μs such that

$$\widetilde{\nabla}^{\frac{6}{5}2}(\Omega) = \left\{ \mu \mid \sup_{\{B_j\} \subset \mathcal{B}(\Omega)} \sum_j \frac{1}{R_j} (|\mu|(B_j))^2 \leqslant \text{constant} \right\} \qquad \Omega \subset \mathbb{R}^3. \tag{3.8}$$

It is instructive at this point to compare the regularity statement of $\widetilde{\nabla}^{\frac{6}{5}2}(\mathbb{R}^3)$ versus the regularity of the 3D borderline case in the Morrey scale, $\widetilde{M}^{3/2}(\mathbb{R}^3)$. The latter consists of those μs whose total mass over arbitrary balls decays at least linearly with the radius,

$$\widetilde{M}^{3/2}(\Omega) = \left\{ \mu \mid |\mu|(B_R) \leqslant \text{constant} \times R \right\} \qquad \Omega \subset \mathbb{R}^3.$$

The $\widetilde{\nabla}^{\frac{6}{5}2}(\mathbb{R}^3)(\Omega)$-bound in (3.8) allows for a slower decay of the total mass—up to order one-half for a single ball, yet this slower rate should take into account a *collection* of disjoint balls. In general, therefore, the two different bounds are not comparable unless additional information regarding the *asymptotic behaviour* of covering balls in (3.8) is provided. For example, an $\widetilde{M}^{3/2}(\Omega)$ bound of μ yields

$$\|\mu\|^2_{\widetilde{\nabla}^{\frac{6}{5}2}(\Omega)} \leqslant \sup_{R_j \leqslant R_0} \sum_j \frac{1}{R_j} (|\mu|(B_j))^2 \leqslant \sum_j R_j \, \|\mu\|^2_{\widetilde{M}^{3/2}(\Omega)} \tag{3.9}$$

and hence, if Ω has a finite *packing measure*, $\pi^{h_1}(\Omega)$, so that it can be packed by covering balls with the finite sum of diameters, we conclude

Corollary 3.1. *Assume that $\Omega \subset \mathbb{R}^3$ has a finite packing measure, $\pi^{h_1}(\Omega) < \infty$, $h_1(t) = t$. Then*

$$\widetilde{M}^{3/2}(\Omega) \subset \widetilde{\nabla}^{\frac{6}{5}2}(\Omega).$$

4. Approximate solutions of Euler's equations

We are concerned with flows of an incompressible ideal fluid modelled by the Euler equations

$$\begin{aligned} &u_t + u \cdot \nabla u = -\nabla p \\ &\operatorname{div} u = 0 \\ &\text{initial and boundary data} \end{aligned} \tag{4.1}$$

where $u := (u_1, \ldots, u_N)$ and p are the velocity and pressure of the flow. One way to address the question of existence of (weak) solutions for (4.1) is by producing a family of *approximate solutions*, $\{u^\varepsilon(\cdot, t)\}$ and justifying the passage to the limit, say $\varepsilon \downarrow 0$. We recall the definition of *approximate solutions* over any fixed time interval $[0, T]$. We seek a family of incompressible velocity fields, $\{u^\varepsilon\}$, $\operatorname{div} u^\varepsilon = 0$, uniformly bounded in $L^\infty([0, T]; L^2_{\mathrm{loc}}(\mathbb{R}^N)) \cap Lip((0, T); H^{-L}_{\mathrm{loc}}(\mathbb{R}^N))$ such that they satisfy the approximate consistency with (4.1). Namely, for any test vector field $\Phi \in C_c^\infty([0, T) \times \mathbb{R}^N)$ with $\operatorname{div} \Phi = 0$ we have

$$\int_0^T \int_{\mathbb{R}^N} \Phi_t \cdot u^\varepsilon + (D\Phi\, u^\varepsilon) \cdot u^\varepsilon \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^N} \Phi(x, 0) \cdot u^\varepsilon(x, 0)\,\mathrm{d}x \longrightarrow 0 \qquad \text{as} \quad \varepsilon \to 0. \tag{4.2}$$

The uniform bound in $L^\infty([0,T]; L^2_{\mathrm{loc}}(\mathbb{R}^N))$ states the uniform bound on the kinetic energy. In the generic case, these weak formulations hold in some negative Sobolev space tested against vector fields in $H^s_c([0,T)\times\mathbb{R}^N)$. Together with the L^2-energy bound, it follows that u^ε has the *Lip* regularity with a uniform bound in $Lip((0,T); H^{-L}_{\mathrm{loc}}(\mathbb{R}^N))$ for some $L = L(s,N) > 1$, e.g. [11, 18]. This (weak) regularity in time enables us to define the manner in which u^ε assumes prescribed initial data.

The L^2-energy bound implies that we can extract a weak-* converging subsequence, $\{u^{\varepsilon_k}\} \rightharpoonup u$ in $L^\infty([0,T]; L^2_{\mathrm{loc}}(\mathbb{R}^N))$, and thus we are facing one of two possibilities. Either there is *strong* L^2-convergence $u^{\varepsilon_k}(\cdot,t) \to u(\cdot,t)$ in $L^1[0,T]$, so that by passing to the limit (in both the linear and quadratic terms) in (4.2), $u(\cdot,t)$ is found to be a weak solution of (4.1). The other possibility is lack of strong convergence, $u^\varepsilon \rightharpoonup u$. In this case the L^2 energy concentrates on a subset $E \subset \Omega\times[0,T]$ characterized by a positive *reduced defect measure* introduced in $\theta(E) > 0$ [12],

$$\theta(E) := \limsup \int_E |u(x,t) - u^\varepsilon(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{4.3}$$

Outside this concentration set $\limsup_{\varepsilon\to 0}\int_{E^c}|u^\varepsilon - u|^2\,\mathrm{d}x\,\mathrm{d}t = 0$. Greengard and Thomann [17] have shown that the concentration set E has Hausdorff dimension $H(E) \geqslant 1$. Upper bounds on the 2D Hausdoff dimension $H(E)$ can be found in [26].

The phenomena of energy concentration does not exclude the possibility of convergence to a weak solution. DiPerna and Majda initiated in [11–13] the study of the *concentration–cancellation* phenomena, where subtle cancellation justifies the passage to the limit $u_i^{\varepsilon_k}u_j^{\varepsilon_k} \rightharpoonup u_i u_j$, $i \neq j$, so that despite the concentration of energy, the weak-$*$ $\lim u^{\varepsilon_k} = u$ is found to be a weak solution of (4.1).

It is physically relevant to classify many approximate flows into one of the two scenarios outlined above according to the behaviour of their vorticity fields, $\boldsymbol{\omega}^\varepsilon(\cdot,t) := \nabla\times u^\varepsilon(\cdot,t)$. A sharp criterion for strong L^2-convergence was introduced in the recent work [21]. The so-called H^{-1}*-stability* criterion requires the associated vorticity field $\boldsymbol{\omega}^\varepsilon(\cdot,t)$ to form a precompact subset in $C((0,T), H^{-1}_{\mathrm{loc}}(\mathbb{R}^N))$. The main result [21, theorem 1.1] states that an H^{-1}-stable family of approximate solutions, $\{u^\varepsilon\}$, admits a subsequence which is strongly convergent to a weak solution in $L^\infty([0,T], L^2_{\mathrm{loc}}(\mathbb{R}^N))$.

We will utilize the H^{-1} stability criterion to study the strong convergence of approximate Euler solutions. In particular, our new refined scale of spaces, $\vee^{pq,\alpha}(\mathbb{R}^N)$, will enable us to 'approach' the borderline cases which separate the phenomena of concentration–cancellation. We distinguish between two- and three-dimensional flows.

4.1. The 2D Euler equations

Incompressible flows in two space dimensions become considerably simpler (than the $N > 2$ case), since the 2D vorticity equation is reduced to the scalar transport equation

$$\omega_t + u\cdot\nabla\omega = 0. \tag{4.4}$$

It is governed by a divergence-free velocity field, u, which is recovered by the Biot–Savart law $u = K * \omega$ with $K(\xi) := \xi^\perp/(2\pi|\xi|^2)$. It follows that any *rearrangement invariant* space, X, is a regularity space for the vorticity equation (4.4), so that X-regularity of $\omega^\varepsilon(\cdot,t)$ is retained in time. Thus, consider a specific example family of approximate Euler solutions, $\{u^\varepsilon(\cdot,t)\}$, associated with the mollified initial data, $u_0^\varepsilon = K_\varepsilon * \omega_0$, where K_ε denotes the mollified kernel $K_\varepsilon := \eta_\varepsilon * K$. It follows (consult [21, corollary 2.2] for the precise details) that if the initial vorticity, ω_0, belongs to such a rearrangement-invariant space X, $X_{\mathrm{loc}} \overset{\mathrm{comp}}{\hookrightarrow} H^{-1}_{\mathrm{loc}}(\mathbb{R}^2)$, then

H^{-1}-stability is retained for later times, and hence $\{u^\varepsilon\}$ has a strong limit, $u(\cdot,t)$, which is a weak solution associated with the initial velocity $u_0 = K * \omega_0 \in X$ without concentrations.

The 2D rearrangement-invariant examples of H^{-1}-compactness revisited in [21] (generalizing [3, 20, 24]), include

(a) Orlicz spaces, $L(\log L)^\alpha_c(\mathbb{R}^2)$, $\alpha > \frac{1}{2}$; and the slightly larger
(b) Lorentz spaces $L^{(1,q)}_c(\mathbb{R}^2)$, $q < 2$.

We also mention the borderline cases which are not compactly embedded in $H^{-1}_{\text{loc}}(\mathbb{R}^2)$,

(c) $X = L(\log L)^{1/2}_c(\mathbb{R}^2)$ and $X = L^{(1,2)}_c(\mathbb{R}^2)$.

Despite the lack of compactness in these borderline cases, it was shown in [21, theorems 2.2 and 2.4] that special X-sequences of approximate vorticities corresponding to mollified initial data in these borderline cases, $\omega_0^\varepsilon = \eta_\varepsilon * \omega_0$, $\omega_0 \in X$, are, in fact, $H^{-1}_{\text{loc}}(\mathbb{R}^2)$-compact.

The 2D problem beyond rearrangement-invariant spaces was studied in [21, section 3] in terms of Morrey spaces, $M^{1,\alpha}_c(\mathbb{R}^2)$, which are compactly embedded in $H^{-1}_{\text{loc}}(\mathbb{R}^2)$ for $\alpha > 1$. The study of Morrey spaces in this context was motivated by the DiPerna–Majda conjecture on the concentration–cancellation phenomenon of *one-signed* vorticities. Majada [22], has shown how the Morrey regularity in $\widetilde{M}^{1,1/2}_c(\mathbb{R}^2)$ of such one-signed vorticities plays a fundamental role in his simplified proof of the concentration–cancellation argument of Delort [14]. The new ladder of spaces, $\vee^{1q,\alpha}(\mathbb{R}^2)$, provides us with more precise information on the regularity of one-signed measures which could not be classified in terms of the missing gap in the ladder of Morrey spaces, $M^{1,\alpha}(\mathbb{R}^2)$, $\frac{1}{2} < \alpha < 1$.

We begin with an immediate consequence of our main theorem 3.1 regarding approximate vorticities, $\omega^\varepsilon(\cdot,t) \in X_\alpha := \widetilde{V}^{(12,\alpha)}_c(\mathbb{R}^2)$. Taking into account the definition of approximate solutions, we have that $\{\omega^\varepsilon\}$ are uniformly bounded,

$$\{\omega^\varepsilon\} \hookrightarrow Lip((0,T), H^{-L-1}_{\text{loc}}(\mathbb{R}^2)) \cap L^\infty((0,T), X_\alpha))$$

where according to (3.5), $X_\alpha \overset{\text{comp}}{\hookrightarrow} H^{-1}_{\text{loc}}(\mathbb{R}^2) \overset{\text{comp}}{\hookrightarrow} H^{-L-1}$. It follows that $\{\omega^\varepsilon\} \overset{\text{comp}}{\hookrightarrow} C((0,T), H^{-1}_{\text{loc}}(\mathbb{R}^2))$ and by our H^{-1}-stability result [21], we conclude

Corollary 4.1. *Let $\{u^\varepsilon\}$ be a family of approximate solutions of the 2D Euler equations (4.1), and assume that the corresponding sequence of vorticities $\{\omega^\varepsilon\}$ is uniformly bounded in $L^\infty([0,T]; \widetilde{\vee}^{(12,\alpha)}_c(\mathbb{R}^2))$, with $\alpha > \frac{1}{2}$. Then $\{u^\varepsilon\}$ is strongly compact in $L^\infty([0,T]; L^2_{\text{loc}}(\mathbb{R}^2))$, and has a strong limit, $u(\cdot,t)$, which is a weak solution with no concentrations.*

Seeking a strategy for obtaining *a priori* $\vee^{12,\alpha}$-bounds of the type required in the last corollary, we are led to the following.

Question. *Consider a sequence of approximate vorticities, $\omega^\varepsilon(\cdot,t)$, corresponding to mollified initial data, $\omega_0^\varepsilon = \eta_\varepsilon * \omega_0$ with $\omega_0 \in \widetilde{\vee}^{12,\alpha}_c(\mathbb{R}^2)$. Does the sequence $\{\omega^\varepsilon(\cdot,t)\}$ remain in $\widetilde{\vee}^{12,\alpha}_c(\mathbb{R}^2)$ for $t > 0$?*

Though the general question remains open, we offer one possible strategy for obtaining *a priori* $\vee$-type bounds in the special case of one-signed vorticities. To this end, we let $H(\omega)$ denote the 'pseudo-energy'

$$H(\omega) := -\frac{1}{2\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \log|x-y|\omega(x)\,\omega(y)\,\mathrm{d}x\,\mathrm{d}y$$

noting that it is an invariant quantity associated with smooth vorticities,

$$H(\omega(t)) = H(\omega_0).$$

Indeed, expressed in terms of the streamfunction, $\psi = \frac{1}{2}\pi \log|x| * \omega$, the velocity is given by $u = \nabla^{\perp}\psi$, and the energy associated with the 2D flow reads

$$\int_{B_R(0)} |u|^2 \,\mathrm{d}x = \int_{B_R(0)} \nabla^{\perp}\psi \cdot \nabla^{\perp}\psi \,\mathrm{d}x = -\int_{B_R(0)} \omega\psi \,\mathrm{d}x + \int_{\partial B_R(0)} \nabla^{\perp}\psi \cdot \boldsymbol{t}\psi \,\mathrm{d}s$$

and hence, assuming a far-field behaviour which is invariant in time (there is no far-field decay of this boundary term), we conclude that, in fact, $H(\omega(\cdot,t))$ measures the invariance of the total energy $\int |u(\cdot,t)|^2 \,\mathrm{d}x$.

Equipped with the invariance of pseudo-energy we now turn to consider the $\widetilde{\vee}^{12,\alpha}$-bound of one-signed vorticities.

Lemma 4.1. *Let $\{u^{\varepsilon}\}$ be a family of approximate solutions of the 2D Euler equations with one signed measured vorticities, $\{\omega_0^{\varepsilon} \in \mathcal{BM}_c^+\}$. Then*

$$\|\omega^{\varepsilon}(\cdot,t)\|_{\widetilde{\vee}^{12}(\log\widetilde{\vee})_c^{1/2}(\mathbb{R}^2)} \leqslant \textit{constant}. \tag{4.5}$$

Proof. We consider an arbitrary collection of disjoint balls, $\{B_j\}_j$ with sufficiently small radii, $B_j = B_{R_j}(x_j)$, $R_j < \frac{1}{2}$. We partition the energy between its self-induced part, H_{si}, and the interaction energy, H_{ie} [4]

$$\begin{aligned} H(\omega^{\varepsilon}(\cdot,t)) &= -\frac{1}{2\pi}\sum_k \iint_{B_{R_j}\times B_{R_j}} \log|x-y| \,\mathrm{d}\omega^{\varepsilon}(x,t)\,\mathrm{d}\omega^{\varepsilon}(y,t) \\ &\quad -\frac{1}{2\pi}\sum_{j\neq k} \iint_{B_{R_j}\times B_{R_k}} \log|x-y| \,\mathrm{d}\omega^{\varepsilon}(x,t)\,\mathrm{d}\omega^{\varepsilon}(y,t) \\ &=: H_{si}(\omega(t)) + H_{ie}(\omega(t)). \end{aligned}$$

First, we note a lower bound on the interaction energy either when $\omega^{\varepsilon}(\cdot,t)$ remains compactly supported, say in $B_{R_t}(0)$, so that $\log|x-y| \leqslant (\log|2R_t|)_+$, or, following [22], using the fact that $(\log|x-y|)_+ \leqslant 2(|x|^2+|y|^2)$. In either case we find $H_{ie}(\omega^{\varepsilon}(\cdot,t))$ to be bounded from below; for example, in the second case we find

$$\begin{aligned} -H_{ie}(\omega^{\varepsilon}(\cdot,t)) &\leqslant \frac{1}{\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2} (|x|^2+|y|^2)\,\mathrm{d}\omega^{\varepsilon}(x,t)\,\mathrm{d}\omega^{\varepsilon}(y,t) \\ &\leqslant \frac{2}{\pi} I_0(\omega^{\varepsilon}(\cdot,t)\, I_2(\omega^{\varepsilon}(\cdot,t) \leqslant \text{constant}_0. \end{aligned}$$

The last uniform bound follows from the fact that the first two moments, $I_0(\omega^{\varepsilon}(\cdot,t)$ and $I_2(\omega^{\varepsilon}(\cdot,t)$, are global invariants of 2D flows (or at least bounded quantities for approximate flows).

Second, we note that the $\widetilde{\vee}^{12}(\log\widetilde{\vee})^{1/2}$-bound of ω^{ε} is a *lower bound* for the self-induced energy: indeed, in view of the positivity of ω^{ε},

$$\begin{aligned} \frac{1}{2\pi}\sum_j |\log 2R_j| \left(\int_{B_j} |\,\mathrm{d}\omega^{\varepsilon}(x,t)|\right)^2 &\leqslant -\frac{1}{2\pi}\sum_j \iint_{B_{R_j}\times B_{R_j}} \log|x-y|\,\mathrm{d}\omega^{\varepsilon}(x)\,\mathrm{d}\omega^{\varepsilon}(y) \\ &= H_{si}(\omega^{\varepsilon}(\cdot,t)). \end{aligned}$$

The $\vee$-bound (4.5) follows from the last two estimates,

$$\begin{aligned} \frac{1}{2\pi}\|\omega^{\varepsilon}(\cdot,t)\|^2_{\widetilde{\vee}^{12}(\log\widetilde{\vee})^{1/2}(\mathbb{R}^2)} &\leqslant H_{si}(\omega^{\varepsilon}(\cdot,t)) = H(\omega^{\varepsilon}(\cdot,t)) - H_{ie}(\omega^{\varepsilon}(\cdot,t)) \\ &\leqslant H(\omega_0) + \text{constant}_0. \end{aligned}$$

□

According to theorem 3.1, $\widetilde{\triangledown}^{12,\alpha}(\mathbb{R}^2)$ are compactly embedded in $H^{-1}_{\rm loc}(\mathbb{R}^2)$ for $\alpha > \frac{1}{2}$, and by the main stability result of [21], therefore, no concentration phenomenon occurs in this range when $\|\omega^\varepsilon(\cdot,t)\|_{\widetilde{\triangledown}^{12,\alpha}(\mathbb{R}^2)} \leqslant$ constant $\times\ \alpha > \frac{1}{2}$. In particular, the $\widetilde{\triangledown}^{12}(\log\widetilde{\triangledown})^{1/2}$ regularity of one-signed measures is, in fact, a borderline case and, analogous with our previous discussion of the borderline cases of Orlicz and Lorentz cases, we raise the following.

Question. *Consider a sequence of approximate vorticities, corresponding to mollified initial data, $\omega_0^\varepsilon = \eta_\varepsilon * \omega_0$, $\omega_0 \in \widetilde{\triangledown}^{12}(\log\widetilde{\triangledown})_c^{1/2}(\mathbb{R}^2)$. Does the sequence $\{\omega^\varepsilon(\cdot,t)\}$ remain compact in $H^{-1}_{\rm loc}(\mathbb{R}^2)$ for $t > 0$?*

An affirmative answer to this question implies that for 2D initial vorticities with one-signed (in fact, more general) measures, one can construct a solution by a limiting argument which avoids the phenomena of concentration (consult [21, lemma 2.3] regarding the issue of temporal continuity).

As we noted before in the context of the borderline cases $X = L(\log L)_c^{1/2}(\mathbb{R}^2)$ and $X = L_c^{(1,2)}(\mathbb{R}^2)$, they both lack $H^{-1}_{\rm loc}(\mathbb{R}^2)$-compactness, hence only special X-sequences are expected to form compact subsets in $H^{-1}_{\rm loc}(\mathbb{R}^2)$, e.g. approximate vorticities corresponding to mollified initial data. Similarly, we note that only the special $\vee^{12}(\log\vee)^{1/2}(\mathbb{R}^2)$-sequence can be expected to form H^{-1}-compact sequences. The following counterexample due to DiPerna and Majda [13, proposition 3.1], demonstrates a family of steady vorticities, $\{\omega^\varepsilon\}$, which are positive and hence uniformly bounded in $\widetilde{\triangledown}^{12}(\log\widetilde{\triangledown})^{1/2}$, yet it lacks H^{-1}-compactness. To this end, pick a non-negative $C_0^\infty(0,1)$ radial vorticity, $\omega(r)$, and consider its dilations

$$\omega^\varepsilon(x) := \frac{1}{\varepsilon^2\sqrt{|\log\varepsilon|}}\omega\left(\frac{|x|}{\varepsilon}\right) \qquad \Gamma(r) := \int_0^r s\omega(s) < \infty.$$

A straightforward computation shows that the induced velocity field satisfies the steady Euler equations

$$u^\varepsilon(x) = \frac{1}{\varepsilon\sqrt{|\log\varepsilon|}}u\left(\frac{x}{\varepsilon}\right) \qquad u(x) = \frac{x^\perp}{|x|^2}\Gamma(|x|)$$

with finite kinetic energy, and for which $u_i^\varepsilon(x)u_j^\varepsilon(x) \rightharpoonup \pi\Gamma^2(\infty)\delta(x)\delta_{ij}$.

The lack of $H^{-1}_{\rm loc}(\mathbb{R}^2)$-compactness for *general* sequences in the borderline case $X = \widetilde{\triangledown}^{12}(\log\widetilde{\triangledown})_c^{1/2}(\mathbb{R}^2)$ indicates the possibility of energy concentration, and in this context we show that if energy concentration does take place then the $\widetilde{\triangledown}^{12}(\log\widetilde{\triangledown})^{1/2}$ bound is sufficient to guarantee the concentration–cancellation phenomena. The following is a generalization of Delort's result [14].

Theorem 4.1. *Let $\{u^\varepsilon(\cdot,t)\}$ be a family of approximate solutions of the 2D Euler equations (4.1), and assume that the corresponding sequence of vorticities $\{\omega^\varepsilon\}$ is uniformly bounded in $L^\infty([0,T];\widetilde{\triangledown}^{12}(\log\widetilde{\triangledown})_c^{\alpha}(\mathbb{R}^2))$, $\alpha > 0$. Then the L^2-weak limit, $u^\varepsilon \rightharpoonup u(\cdot,t)$ is a finite-energy solution of the 2D Euler equation (4.1).*

Proof. A weak formulation of the 2D Euler's equations (4.4)

$$\omega_t + K*\omega\cdot\nabla\omega = 0$$

reads, consult [27],

$$\int_0^\infty\int_{\mathbb{R}^2}\phi_t\omega^\varepsilon(x,t)\,{\rm d}x\,{\rm d}t + \int_0^\infty\int_{\mathbb{R}^2\times\mathbb{R}^2}H_\phi(x,y,t)\,\omega^\varepsilon(x,t)\,\omega^\varepsilon(y,t)\,{\rm d}x\,{\rm d}y\,{\rm d}t$$
$$+\int_{\mathbb{R}^2}\phi(x,0)\,\omega_0^\varepsilon(x)\,{\rm d}x = 0 \qquad \forall\phi\in C_c^\infty([0,\infty)\times\mathbb{R}^2)$$

where the kernel $H_\phi(x, y, t)$ is given by

$$H_\phi(x, y, t) := \frac{\nabla\phi(x,t) - \nabla\phi(y,t)}{4\pi|x-y|}\,\frac{(x-y)^\perp}{|x-y|}.$$

By a density argument we may restrict our attention to a test function of the form $\phi(x,t) = \psi(t)\,\varphi(x)$. We let $\rho(|x|) \in C_0^\infty(0,2)$ be a positive cut-off function with $\rho(|x|) \equiv 1$ for $|x| \leqslant 1$. The main issue is passage to a limit in the quadratic term (corresponding to the mixed term *weak*-$*\lim u_1^\varepsilon u_2^\varepsilon$), which is decomposed in the by-now standard fashion, consult [14], [22, section 2], [27], ...

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}^2\times\mathbb{R}^2} H_\phi(x,y,t)\,\omega^\varepsilon(x,t)\,\omega^\varepsilon(y,t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\
&\qquad = \int_0^\infty\int_{\mathbb{R}^2\times\mathbb{R}^2} \psi(t)\left(1-\rho\left(\frac{|x-y|}{\delta}\right)\right)H_\varphi(x,y)\,\omega^\varepsilon(x,t)\,\omega^\varepsilon(y,t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\
&\qquad\quad + \int_0^\infty\int_{\mathbb{R}^2\times\mathbb{R}^2} \psi(t)\rho\left(\frac{|x-y|}{\delta}\right)H_\varphi(x,y)\,\omega^\varepsilon(x,t)\,\omega^\varepsilon(y,t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\
&\qquad =: I_\delta(\omega^\varepsilon) + J_\delta(\omega^\varepsilon).
\end{aligned}$$

By Delort's lemma [14, proposition 1.2.3], $\psi(t)\big(1-\rho\big(\frac{|x-y|}{\delta}\big)\big)H_\varphi(x,y)$ is a 'nice' kernel such that

$$\lim_{\varepsilon\downarrow 0} I_\delta(\omega^\varepsilon) = \int_0^\infty\int_{\mathbb{R}^2\times\mathbb{R}^2}\psi(t)\left(1-\rho\left(\frac{|x-y|}{\delta}\right)\right)H_\varphi(x,y)\,\mathrm{d}\omega(x,t)\,\mathrm{d}\omega(y,t)\,\mathrm{d}t.$$

It remains to estimate the behaviour of $J_\delta(\omega^\varepsilon)$ which is supported near the singularity along the diagonal $x = y$, and it is here that the $\widetilde{\triangledown}^{12,\alpha}$-bound plays an essential role. To this end, we cover $\mathbb{R}^2$ with a net of $2\delta\times 2\delta$ cubes, $\mathcal{C}_j = 2\delta\mathcal{C}(\cdot + 2\delta j)$, $j\in Z^2$, with $\mathcal{C}$ denoting the 2D unit cube. Decomposing the integration in $J_\delta(\omega^\varepsilon)$ over $\mathbb{R}^2 = \cup_j\mathcal{C}_j$, we find

$$\begin{aligned}
J_\delta(\omega^\varepsilon) &= \int_0^\infty \psi(t)\sum_{j,k}\int_{(x,y)\in(\mathcal{C}_j\times\mathcal{C}_k)}\rho\left(\frac{|x-y|}{\delta}\right)H_\varphi(x,y)\,\omega^\varepsilon(x,t)\,\omega^\varepsilon(y,t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\
&\leqslant C_\varphi\int_0^\infty|\psi(t)|\sum_{j,k}\int_{\substack{(x,y)\in(\mathcal{C}_j\times\mathcal{C}_k)\\ |x-y|\leqslant 2\delta}}|\omega^\varepsilon(x,t)|\,|\omega^\varepsilon(y,t)|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \qquad C_\varphi := \|H_\varphi\|_{L^\infty}.
\end{aligned}$$

For each cell $\mathcal{C}_j$, only its immediate neighbouring cells, $\mathcal{C}_k$, $|k-j|_\infty\leqslant 1$, participate in the summation on the right of (4.6), so that $|x-y|\leqslant 2\delta$ whenever $(x,y)\in(\mathcal{C}_j,\mathcal{C}_k)$. For each j (respectively, k) there are precisely nine such neighbouring cells (including the cell $k=j$ itself) which contribute to the self-induced energy. Since ρH_φ is bounded along the diagonal we find, in view of (2.4)

$$\begin{aligned}
J_\delta(\omega^\varepsilon) &\leqslant C_\varphi\int_0^\infty|\psi(t)|\Bigg[\sum_{|j-k|\leqslant 1}\frac{1}{2}\left(\int_{x\in\mathcal{C}_j}|\omega^\varepsilon(x,t)|\,\mathrm{d}x\right)^2\\
&\qquad + \sum_{|j-k|\leqslant 1}\frac{1}{2}\left(\int_{x\in\mathcal{C}_k}|\omega^\varepsilon(y,t)|\,\mathrm{d}y\right)^2\Bigg]\,\mathrm{d}t\\
&\leqslant 9C_\varphi\int_0^\infty|\psi(t)|\left(\int_{x\in\mathcal{C}_j}|\omega^\varepsilon(x,t)|\,\mathrm{d}x\right)^2\mathrm{d}t\\
&\leqslant 9C_\varphi\|\psi\|_{L^\infty}\,\|\omega^\varepsilon(x,t)\|^2_{L^1_{\mathrm{loc}}(\mathbb{R}_t;\widetilde{\triangledown}^{12}(\log\widetilde{\triangledown})^\alpha)}|\log\rho|^{-2\alpha} \qquad C_\varphi = \|H_\varphi\|_{L^\infty}.
\end{aligned}$$

It follows that $J_\delta(\omega^\varepsilon)$ tends to zero uniformly in ε, $|J_\delta(\omega^\varepsilon)| \leqslant \text{constant} \times |\log\rho|^{-2\alpha} \underset{\rho\to 0}{\to} 0$, and we conclude that $H_\phi(x,y,t)\,\omega^\varepsilon(x,t)\,\omega^\varepsilon(y,t) \rightharpoonup H_\phi(x,y,t)\,\mathrm{d}\omega(x,t)\,\mathrm{d}\omega(y,t)$. □

4.2. The 3D Euler equations

According to the compact embedding (3.1), a family of 3D vorticities, $\{\boldsymbol{\omega}^\varepsilon(\cdot,t)\}$, which is uniformly bounded in $L^\infty([0,T];\widetilde{M}^p_{\mathrm{loc}}(\mathbb{R}^3))$ with $p>\frac{3}{2}$, induces a velocity field with the L^2-strong limit, $u(\cdot,t)$, which is a global weak solution of the 3D Euler equations (consult [21, theorem 4.5]). Note that unlike the 2D problem, however, Morrey space estimates do not have the physical interpretation as circulation decay estimates. And moreover, there is no known strategy of obtaining *a priori* estimates on the $\widetilde{M}^p$-size of the vorticity, $\|\boldsymbol{\omega}(\cdot,t)\|_{\widetilde{M}^p(\mathbb{R}^3)}$, in time. We want to show that our new scale of spaces offers a better tool to handle the issue of compactness in terms of physically relevant invariant quantities.

As in the 2D case, we begin with the following.

Corollary 4.2. *Let $\{u^\varepsilon\}$ be a family of approximate solutions of the three-dimensional Euler equations (4.1), and assume that the corresponding sequence of vorticities $\{\boldsymbol{\omega}^\varepsilon\}$ is uniformly bounded in $L^\infty([0,T];(\widetilde{\vee}^{p2}(\mathbb{R}^3))$, with $p>\frac{6}{5}$. Then, $\{u^\varepsilon\}$ is strongly compact in $L^\infty([0,T];L^2_{\mathrm{loc}}(\mathbb{R}^3))$, and hence it has a strong limit, $u(\cdot,t)$, which is a weak solution with no concentrations.*

There is no known strategy to obtain *a priori* $\vee^{p2}(\mathbb{R}^3)$-bounds on the vorticity, and there is no *a priori* reason to expect that they are invariants of 3D flows. There is one notable exception, however, which is linked precisely to the borderline case of $\vee^{p2}(\mathbb{R}^3)$ with $p=\frac{6}{5}$. We explore this exceptional case below. First, we recall the one special 3D invariant which is the pseudo-energy (the Coulomb energy)

$$H(\boldsymbol{\omega}(x,t)) := \frac{1}{8\pi}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\langle\boldsymbol{\omega}(x,t),\boldsymbol{\omega}(y,t)\rangle}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y = H(\boldsymbol{\omega}_0).$$

Next, we cover space with a 3D lattice, $\mathbb{R}^3=\cup_j\mathcal{C}_j$, and as before, we partition the energy, $H(\boldsymbol{\omega}(x,t)) = H_{si}(\boldsymbol{\omega}(x,t)) + H_{ie}(\boldsymbol{\omega}(x,t))$, into its self-induced, short-range part, $H_{si}(\boldsymbol{\omega}(x,t))$, and long-range interaction energy, $H_{ie}(\boldsymbol{\omega}(x,t))$, namely

$$H_{si}(\boldsymbol{\omega}(x,t)) = \frac{1}{8\pi}\sum_j\iint_{\mathcal{C}_j\times\mathcal{C}_j}\frac{\langle\boldsymbol{\omega}(x,t),\boldsymbol{\omega}(y,t)\rangle}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y$$

$$H_{ie}(\boldsymbol{\omega}(x,t)) = \frac{1}{8\pi}\sum_{j\neq k}\iint_{\mathcal{C}_j\times\mathcal{C}_k}\frac{\langle\boldsymbol{\omega}(x,t),\boldsymbol{\omega}(y,t)\rangle}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y.$$

To proceed, we make two claims regarding *lower bounds* of the two portions of the pseudo-energy, similar to the 2D configuration (but much harder to prove).

(a) A lower bound on the interaction energy

$$H_{ie}(\boldsymbol{\omega}^\varepsilon(x,t)) = \frac{1}{8\pi}\sum_{j\neq k}\iint_{\mathcal{C}_j\times\mathcal{C}_k}\frac{\langle\boldsymbol{\omega}^\varepsilon(x,t),\boldsymbol{\omega}^\varepsilon(y,t)\rangle}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y \geqslant -\text{constant}_{ie}. \tag{4.6}$$

(b) For sufficiently small cubes, $\mathcal{C}_j$,

$$\boldsymbol{\omega}^\varepsilon(x,t) \sim ⨍_{\mathcal{C}_j}\boldsymbol{\omega}^\varepsilon(y,t)\,\mathrm{d}y \qquad x\in\mathcal{C}_j \tag{4.7}$$

which leads to a lower bound of the self-induced energy

$$H_{si}(\omega^\varepsilon(\cdot,t)) = \frac{1}{8\pi}\sum_j \iint_{\mathcal{C}_j\times\mathcal{C}_j} \frac{\langle \boldsymbol{\omega}(x,t), \boldsymbol{\omega}(y,t)\rangle}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y$$

$$\geqslant \frac{1}{8\pi}\sum_j \frac{1}{2R_j}\left(\int_{\mathcal{C}_j} |\boldsymbol{\omega}^\varepsilon(x,t)|\,\mathrm{d}x\right)^2.$$

The last two estimates yield the $V^{\frac{6}{5}2}(\mathbb{R}^3)$-bound on the vorticity, consult (3.8),

$$\frac{1}{16\pi}\sum_j \frac{1}{R_j}\left(\int_{\mathcal{C}_j} |\boldsymbol{\omega}^\varepsilon(x,t)|\,\mathrm{d}x\right)^2 \leqslant H_{si}(\boldsymbol{\omega}^\varepsilon(\cdot,t)) = H(\boldsymbol{\omega}^\varepsilon(\cdot,t)) - H_{ie}(\boldsymbol{\omega}^\varepsilon(\cdot,t))$$

$$\leqslant H_0 + \text{constant}_{ie} \qquad H_0 = H(\boldsymbol{\omega}_0^\varepsilon).$$

The new ladder of spaces is establishing a direct linkage between questions related to the global configuration of the 3D pseudo-energy and the borderline case of $\vee^{\frac{6}{5}2}(\mathbb{R}^3)$-regularity. Similar to the 2D framework we are now led to inquire about the $H^{-1}(\mathbb{R}^3)$ compactness of this borderline case.

Question. *Consider a sequence of approximate vorticities, $\boldsymbol{\omega}^\varepsilon(\cdot,t) \in L^\infty([0,T], \vee^{\frac{6}{5}2}(\mathbb{R}^3))$. What are the possible configurations of the pseudo-energy so that the sequence $\{\boldsymbol{\omega}^\varepsilon(\cdot,t)\}$, is compact in $L^\infty([0,T], H^{-1}_{\text{loc}}(\mathbb{R}^3))$?*

We note that an answer to this question maps a possible strategy of constructing solutions to the 3D Euler equation for large time. The estimates claimed in (4.6) and (4.7) demonstrated this issue. For a detailed discussion on the configurations of the self-induced energy the interaction energy and the relation to vortex stretching we refer to Chorin [4, chapter 5].

We conclude this section by pointing out one such strategy which leads to the desired $\vee^{\frac{6}{5}2}$-bound in the 3D case. To this end we let

$$\boldsymbol{\xi}(x,t) := \frac{\boldsymbol{\omega}^\varepsilon(x,t)}{|\boldsymbol{\omega}^\varepsilon(x,t)|} \qquad \boldsymbol{\omega}^\varepsilon(x,t) \neq 0$$

denote the direction of the vorticity $\boldsymbol{\omega}^\varepsilon$. The stretching effect of $\boldsymbol{\omega}^\varepsilon$ is controlled by the difference $|\boldsymbol{\xi}^\varepsilon(x,t) - \boldsymbol{\xi}^\varepsilon(y,t)|$ and we make

Assumption 4.1. *There exist constants $\delta > 0$ and $\theta = \theta_\delta < 1$, such that whenever $|\boldsymbol{\omega}^\varepsilon(x,t)|, |\boldsymbol{\omega}^\varepsilon(y,t)| > K_0$, there holds*

$$|\boldsymbol{\xi}^\varepsilon(x,t) - \boldsymbol{\xi}^\varepsilon(y,t)| \leqslant \sqrt{2}\theta \qquad \forall |x-y| \leqslant \delta. \tag{4.8}$$

Squaring (4.8) yields $2 - 2\langle \boldsymbol{\xi}^\varepsilon(x,t), \boldsymbol{\xi}^\varepsilon(y,t)\rangle = |\boldsymbol{\xi}^\varepsilon(x,t) - \boldsymbol{\xi}^\varepsilon(y,t)|^2 \leqslant 2\theta^2$, and hence, whenever $|\boldsymbol{\omega}^\varepsilon(x,t)|, |\boldsymbol{\omega}^\varepsilon(y,t)| > K_0$, we have

$$\langle \boldsymbol{\omega}^\varepsilon(x,t), \boldsymbol{\omega}^\varepsilon(y,t)\rangle \geqslant (1-\theta^2)|\boldsymbol{\omega}^\varepsilon(x,t)||\boldsymbol{\omega}^\varepsilon(y,t)| \qquad |x-y| \leqslant \delta. \tag{4.9}$$

Thus, under assumption (4.8) there is a local alignment of the *direction of the vorticity*, $\boldsymbol{\xi}^\varepsilon(\cdot,t)$, whenever its magnitude, $|\boldsymbol{\omega}^\varepsilon(\cdot,t)|$, becomes too large. Assumption 4.1 is inspired by Constantin and Fefferman [5], who proved the existence of 3D Navier–Stokes solutions under the short-range alignment assumption

$$|\boldsymbol{\xi}^\varepsilon(x,t) - \boldsymbol{\xi}^\varepsilon(y,t)| \leqslant \frac{|x-y|}{\delta} \qquad |x-y| \leqslant \delta \qquad |\boldsymbol{\omega}^\varepsilon(x,t)|, |\boldsymbol{\omega}^\varepsilon(y,t)| > K_0.$$

Equipped with the alignment assumption 4.1 we prove that $\boldsymbol{\omega}^\varepsilon(\cdot,t)$ remains uniformly bounded in the borderline space $X_3 = \vee_c^{\frac{6}{5}2}(\mathbb{R}^3)$

Theorem 4.2. *Let* $\{u^\varepsilon(\cdot,t)\}$ *be a family of approximate solutions of the 3D Euler equations (4.1). Assume that the corresponding sequence of vorticities,* $\{\boldsymbol{\omega}^\varepsilon(\cdot,t)\}$*, is compactly supported and satisfies the local alignment condition (4.8). Then the following holds:*

$$\|\boldsymbol{\omega}^\varepsilon(\cdot,t)\|_{\vee^{\frac{6}{5}2}(\Omega)} \leqslant \text{constant}_T \qquad \Omega \subset \mathbb{R}^3 \quad t \leqslant T. \tag{4.10}$$

Remark. The requirement of $\boldsymbol{\omega}^\varepsilon(\cdot,t)$ having compact support is made for simplicity and could be replaced by a weaker requirement of fast enough decay at infinity.

Proof. We begin by partitioning the total energy between its short-range, self-induced part, and its long-range interaction energy, $H(\boldsymbol{\omega}^\varepsilon(\cdot,t)) = H_{si}(\boldsymbol{\omega}^\varepsilon(\cdot,t)) + H_{ie}(\boldsymbol{\omega}^\varepsilon(\cdot,t))$. The partition is taken at a scale level δ dictated by the alignment assumption in (4.8),

$$H_{si}(\boldsymbol{\omega}^\varepsilon(\cdot,t)) := \frac{1}{8\pi}\iint_{\mathbb{R}^3\times\mathbb{R}^3} 1_{|x-y|\leqslant\delta} \frac{\langle\boldsymbol{\omega}^\varepsilon(x,t),\boldsymbol{\omega}^\varepsilon(y,t)\rangle}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y$$

$$H_{ie}(\boldsymbol{\omega}^\varepsilon(\cdot,t)) := \frac{1}{8\pi}\iint_{\mathbb{R}^3\times\mathbb{R}^3} 1_{|x-y|\geqslant\delta} \frac{\langle\boldsymbol{\omega}^\varepsilon(x,t),\boldsymbol{\omega}^\varepsilon(y,t)\rangle}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y.$$

In the 3D case we have the advantage that the interaction energy is lower-bounded (by $H(\boldsymbol{\omega}^\varepsilon(\cdot,t)) = -H_0$), or equivalently, that $H_{si}(\boldsymbol{\omega}^\varepsilon) \leqslant 2H_0$. Indeed, computing the 3D Fourier transform, $\eta(\xi) := \mathcal{F}(|x|^{-1}1_{|x|<\delta}) = |\xi|^{-2}(1-\cos(|\xi|\delta))$, yields for the weighted L^2_η norm,

$$\begin{aligned} H_{si}(\boldsymbol{\omega}^\varepsilon(\cdot,t)) &= \frac{1}{8\pi}\|\hat{\boldsymbol{\omega}}^\varepsilon(\xi,t)\|^2_{L^2_{\eta(\xi)}} \leqslant \frac{2}{8\pi}\int_{\xi\in\mathbb{R}^3}\frac{|\hat{\boldsymbol{\omega}}^\varepsilon(\xi,t)|^2}{|\xi|^2}\,\mathrm{d}\xi \\ &= 2H(\boldsymbol{\omega}^\varepsilon(\cdot,t)) \leqslant 2H_0. \end{aligned} \tag{4.11}$$

Next, we split the vorticity between its bounded and unbounded parts at 'height' K_0

$$\boldsymbol{\omega}^\varepsilon(x,t) = \boldsymbol{\omega}^\varepsilon(x,t)1_{\Omega\cap\{x\ |\ |\omega^\varepsilon(x,t)|\leqslant K\}} + \boldsymbol{\omega}^\varepsilon(x,t)1_{\Omega\cap\{x\ |\ \omega^\varepsilon(x,t)|<K_0\}} =: \boldsymbol{\omega}^\varepsilon_- + \boldsymbol{\omega}^\varepsilon_+$$

and we show that the bounded part of the vorticity, $\boldsymbol{\omega}^\varepsilon_-(\cdot,t)$, has a finite contribution to the self-induced energy. We start by expanding

$$H_{si}(\boldsymbol{\omega}^\varepsilon(\cdot,t)) \equiv H_{si}(\boldsymbol{\omega}^\varepsilon_+(\cdot,t)) + H_{si}(\boldsymbol{\omega}^\varepsilon_-(\cdot,t)) + 2H_{si}(\boldsymbol{\omega}^\varepsilon_-(\cdot,t),\boldsymbol{\omega}^\varepsilon_+(\cdot,t))$$

with the third term on the right denoting the bilinear positive form (positivity follows along the lines of (4.11) or consult [19, theorem 9.8])

$$H_{si}(\boldsymbol{f},\boldsymbol{g}) := \frac{1}{8\pi}\iint_{|x-y|\leqslant\delta}\frac{\langle\boldsymbol{f}(x),\boldsymbol{g}(y)\rangle}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y.$$

The Cauchy–Schwartz inequality then yields

$$2|H_{si}(\boldsymbol{\omega}^\varepsilon_-(\cdot,t),\boldsymbol{\omega}^\varepsilon_+(\cdot,t))| \leqslant \tfrac{1}{2}H_{si}(\boldsymbol{\omega}^\varepsilon_+(\cdot,t)) + 8H_{si}(\boldsymbol{\omega}^\varepsilon_-(\cdot,t))$$

and in view of (4.11) we end up with the upper bound

$$\begin{aligned} \tfrac{1}{2}H_{si}(\boldsymbol{\omega}^\varepsilon_+(\cdot,t)) &\leqslant H_{si}(\boldsymbol{\omega}^\varepsilon(\cdot,t)) + 7H_{si}(\boldsymbol{\omega}^\varepsilon_-(\cdot,t)) \\ &\leqslant 2H_0 + 7\times\text{constant}_{K_0} \qquad H_0 = H(\boldsymbol{\omega}^\varepsilon(\cdot,0)). \end{aligned} \tag{4.12}$$

Here we have used the fact that $\boldsymbol{\omega}^\varepsilon(\cdot,t)$ are compactly supported so that

$$H_{si}(\boldsymbol{\omega}^\varepsilon_-(\cdot,t)) \leqslant \frac{K_0^2}{8\pi}\int_{x\in\operatorname{supp}\boldsymbol{\omega}^\varepsilon(\cdot,t)}\int_{y\in B_\rho(x)}\frac{1}{|x-y|}\,\mathrm{d}y\,\mathrm{d}x \leqslant \text{constant}_{K_0}$$

with, say, $\text{constant}_{K_0} \sim K_0^2 |\,\text{div}(\text{supp}\,\boldsymbol{\omega}^\varepsilon(\cdot,t))|^3\delta^2$. Of course, one can relax the requirement of compact support, asking a fast enough decay of $\boldsymbol{\omega}^\varepsilon(x,t)$, $|x| \to \infty$.

Given a collection of disjoint balls, $\{B_j = B_{R_j}(x_j)\}$, we claim the short-range part of the energy controls the $\vee$-size of $\boldsymbol{\omega}_+^\varepsilon$, when measured over all balls with radii $R_j \leqslant R_0 < \delta/4$; indeed, in view of (4.9), our alignment assumption implies

$$H_{si}(\boldsymbol{\omega}_+^\varepsilon(\cdot,t)) \geqslant \frac{1}{8\pi}\sum\iint_{(x,y)\in B_j\times B_j} \frac{(1-\theta^2)|\boldsymbol{\omega}_+^\varepsilon(x,t)|\cdot|\boldsymbol{\omega}_+^\varepsilon(y,t)|}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y$$
$$\geqslant \frac{1-\theta^2}{16\pi}\sum_j \frac{1}{R_j}\left(\int_{B_j}|\boldsymbol{\omega}_+^\varepsilon(x,t)|\,\mathrm{d}x\right)^2$$

and by varying over all collections of such balls we find a lower bound for the self-induced part of the energy in terms of its $\vee^{\frac{6}{5}2}$-norm

$$H_{si}(\boldsymbol{\omega}_+^\varepsilon(\cdot,t)) \geqslant \frac{1-\theta^2}{16\pi}\|\boldsymbol{\omega}_+^\varepsilon\|^2_{\vee^{\frac{6}{5}2}(\Omega)}.$$

Using this estimate together with (4.12), the asserted $\vee^{\frac{6}{5}2}$-bound follows:

$$\|\boldsymbol{\omega}_+^\varepsilon\|^2_{\vee^{\frac{6}{5}2}(\Omega)} \leqslant \frac{32\pi}{1-\theta^2}(2H_0 + 7\times \text{constant}_{K_0}). \qquad \square$$

The $\vee^{\frac{6}{5}2}_{\text{loc}}(\mathbb{R}^3)$-bound derived in theorem 4.2 implies that $\{\boldsymbol{\omega}^\varepsilon(\cdot,t)\}$ is uniformly bounded in $H^{-1}_{\text{loc}}(\mathbb{R}^3)$. This, in turn, can be strengthened into H^{-1}_{loc}-compactness, for example, as long as the velocity field remains uniformly $L^{p>2}_{\text{loc}}$-bounded. The proof is essentially an application of Murat's lemma [25]. Arguing along the lines of [21, theorem 4.6] we conclude

Corollary 4.3. *Let $\{u^\varepsilon(\cdot,t)\}$ be a family of approximate solutions of the 3D Euler equations (4.1) such that $\{u^\varepsilon\}$ is uniformly bounded in $L^\infty((0,T), L^p(\mathbb{R}^3))$ with $p > 2$, and assume that the compactly supported vorticities, $\{\boldsymbol{\omega}^\varepsilon(\cdot,t)\}$ satisfy the local alignment condition (4.8). Then $\{u^\varepsilon\}$ is strongly compact in $L^\infty((0,T), L^2_{\text{loc}}(\mathbb{R}^2))$, and hence it has a strong limit, $u(\cdot,t)$, which is a weak solution of (4.1).*

We close by noting that there is no known strategy to guarantee the $L^p_{\text{loc}}(\mathbb{R}^3)$-bound on the velocity for $p > 2$.

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Note added in proof. If we let $Pf(x) := \sum_j ⨍_{B_j} |f|\cdot\chi_{B_j}(x)$ denote the Haar projection of f subject to the partition $\{B_j\}$, then a straightforward computation shows $\|f\|_{V^{pp}} = \sup\{\|Pf\|_{L^p(\Omega)} \mid \{B_j\} \subset \mathcal{B}(\Omega)\}$, and by a density argument therefore, $V^{pp} \subset L^p$. It follows that V^{pq} forms the scale of interpolation spaces between $V^{pp} = L^p$ and M^p.

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