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COMPENSATED COMPACTNESS FOR 2D CONSERVATION LAWS

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Abstract. We introduce a new framework for studying two-dimensional conservation laws by compensated compactness arguments. Our main result deals with 2D conservation laws which are *nonlinear* in the sense that their velocity fields are a.e. not co-linear. We prove that if u^ε is a family of uniformly bounded approximate solutions of such equations with H^{-1}-compact entropy production and with (a minimal amount of) uniform time regularity, then (a subsequence of) u^ε convergences strongly to a weak solution. We note that no translation invariance in space — and in particular, no spatial regularity of $u(\cdot, t)$ is required. Our new approach avoids the use of a large family of entropies; by a judicious choice of entropies, we show that only *two* entropy production bounds will suffice. We demonstrate these convergence results in the context of vanishing viscosity, kinetic BGK and finite volume approximations. Finally, the intimate connection between our 2D compensated compactness arguments and the notion of multi-dimensional nonlinearity based on kinetic formulation is clarified.

Keywords: Conservation laws; entropy bounds; compensated compactness; kinetic formulation.

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1. Introduction and Statement of Main Results

Currently, there are four main approaches to study the existence of solutions for quasilinear hyperbolic conservation laws. First was the standard tool of compactness based on *a priori* BV bounds. Then, from the mid-eighties through the mid-nineties, the other three approaches of compensated compactness, measure valued solutions and kinetic formulations were developed, all of which appeal to *a priori* entropy production bounds.

Compactness arguments based on BV bounds were proven as the most effective tool for studying *general* one-dimensional systems of conservation laws. The long line of results in this direction is stretched from Glimm's celebrated result [12] to the recent general existence result of Bianchini and Bressan [1]. Applications to finite difference approximations — from Glimm's scheme to high-resolution scalar schemes, e.g. [13, 31] and the references therein, are primary numerical examples for the success of the compactness approach. The approach is limited, however, to essentially one-dimensional systems. Moreover, the multidimensional BV-based scalar existence theory of Krushkov [14], hinges in an essential manner on the translation invariance of the underlying solution operator. An alternative approach is offered by the compensated compactness theory developed by Tartar [33, 34] and Murat [18, 20]. Here, the hard-to-get BV estimates are replaced with L^2-type entropy production bounds (L^2-type for quadratic entropies and likewise, for general strictly convex entropies). The example of spectral approximations is in order; rather than using BV bounds which are difficult to realize in the dual spectral space, the convergence of the spectral viscosity approximation introduced in [29, 30] is achieved by adapting L^2-type entropy bounds. A similar situation is encountered with hyperviscosity limits, where lack of monotonicity excludes simple derivation of *a priori* BV-bounds. Instead, convergence (of uniformly bounded solutions) follows by compensated compactness arguments, e.g. [32]. So far, existence results based on compensated compactness arguments were restricted to one-dimensional conservation laws. DiPerna's theory of measure valued solutions for nonlinear conversation laws is a third approach to construct entropy solutions. This approach applies to multidimensional problems by appealing to *all* entropies associated with the nonlinear conservation laws. The examples of finite volume schemes on irregular multidimensional grid is in order. Lack of translation invariance excludes BV bounds (even in the L^1-contractive 1D case!); instead, convergence follows from entropy consistency, see [7, 11] for example. This argument depends in an essential manner on having a large family of entropies and hence its applications are so far restricted to scalar equations. The fourth and last approach for studying the existence (and regularity) of solutions, introduced in [16], is based on application of averaging lemmas for the underlying kinetic formulations. Here, the example of convergence for FV scheme [36] is in order. These kinetic arguments apply to scalar as well as systems which admit a kinetic formulation and are not necessarily restricted to one space dimension. Our discussion of the analytical methods available for studying multidimensional conservation laws is by no means inclusive and we mention in passing the examples of compensated compactness and regularity in Hardy spaces, [8], or the geometrical optics studies, e.g. [5], ... We refer to [4] for an extensive bibliography.

Our purpose in this paper is three fold. First, we present a framework for implementing compensated compactness arguments in two space dimensions, thus extending the current framework beyond the 1D applications. Second, our new approach avoids the use of a large (one-parameter) family of entropies; in Sec. 3 we show that by a judicious choice of entropies, only *two* entropy production bounds will suffice, in analogy to the one-entropy in the 1D case discussed in Sec. 2. Finally, a third aspect is to highlight the role of *nonlinearity* in excluding oscillations in the 2D case. Specifically, our main result in Theorem 3.1 below deals with 2D conservation laws $u_t + f_1(u)_{x_1} + f_2(u)_{x_2} = 0$, which are *nonlinear* in the sense that their velocity field (f_1, f_2) is a.e. not co-linear, consult (3.11) or (3.15) below. Let u^ε is a family of uniformly bounded approximate solutions with H^{-1}-compact entropy production (here r^ε are the corresponding residuals, $r^\varepsilon := u^\varepsilon_t + f_1(u^\varepsilon)_{x_1} + f_2(u^\varepsilon)_{x_2}$)

$$\{\eta'(u^\varepsilon)r^\varepsilon\} \in L^p([0,T];X) \quad \text{with } X \hookrightarrow H^{-1}_{\text{loc}}(\mathbb{R}^2_x) \text{ and } p \geq 1, \quad \text{for } \eta = f_1, f_2.$$

Assuming the time regularity bound, $\partial_t u^\varepsilon \in L^q_{\text{loc}}(\mathbb{R}_t; \mathcal{M}(\mathbb{R}^2_x))$, $q > 1$, then (a subsequence of) u^ε convergences strongly to a weak solution. We note in passing that no translation invariance in space — and in particular, no spatial regularity of $u(\cdot, t)$ is required beyond the necessary uniform bound. In this context we clarify, in Sec. 4 below, the intimate connection with the notion of multi-dimensional nonlinearity introduced in [16] and we bring closer the relation between our 2D compensated compactness arguments and the multi-dimensional kinetic arguments.

2. Strong Convergence — A Single Entropy Suffices in the 1D Case

We consider the scalar conservation law

$$\partial_t u + \partial_x f(u) = 0, \tag{2.1}$$

subject to initial conditions, $u(x,0) = u_0$. The entropy solution of (2.1) could be realized by the vanishing viscosity limit, $u = s\lim u^\varepsilon$ where u^ε satisfies the viscosity equation

$$\partial_t u^\varepsilon + \partial_x f(u^\varepsilon) = \varepsilon \partial_{xx} u^\varepsilon. \tag{2.2}$$

In the usual approach of compensated compactness developed by Tartar [33, 34] and Murat [18], the H^{-1}-compact entropy production for the whole family of Krushkov entropies is sought, in order to conclude the strong convergence $u^\varepsilon \to u$,

$$\partial_t \big[|u^\varepsilon - c|\big] + \partial_x \big[\text{sgn}(u^\varepsilon - c)(f(u^\varepsilon) - f(c))\big] \hookrightarrow H^{-1}_{\text{loc}}(\mathbb{R}_x, \mathbb{R}_t), \tag{2.3}$$

Here one makes use of two *a priori* estimates:

$\{\mathcal{A}1\}$ A uniform bound, $\|u^\varepsilon\|_{L^\infty_{\text{loc}}(\mathbb{R}_x, \mathbb{R}_t)} \leq \text{Const.}$
$\{\mathcal{A}2\}$ An entropy production bound $\sqrt{\varepsilon}\|\partial_x u^\varepsilon\|_{L^2_{\text{loc}}(\mathbb{R}_x, \mathbb{R}_t)} \leq \text{Const.}$

Granted these two *a priori* bounds, Tartar [35] and independently Chen and Lu (consult [4, Theorem 2.7]), have shown that the H^{-1}-compact entropy production of a *single* entropy is sufficient to enforce strong convergence to a weak solution of (2.1). A similar "single entropy"-approach was initiated by Rascle for 1D *systems*, consult [26, 27]. Since the above references are not readily available, the three-step convergence argument is sketched below. First, the viscous term on the right of (2.2) is clearly H^{-1}-compact (vanishing of order $\sqrt{\varepsilon}$). For the second step we integrate (2.2) against $f'(u)$ obtaining, with $F'(w) := (f'(w))^2$,

$$\partial_t f(u^\varepsilon) + \partial_x F(u^\varepsilon) = \varepsilon f'(u^\varepsilon)\partial_{xx}u^\varepsilon \equiv \varepsilon\partial_{xx}f(u^\varepsilon) - \varepsilon f''(u^\varepsilon)(u^\varepsilon_x)^2 =: I_\varepsilon + II_\varepsilon. \quad (2.4)$$

The *a priori* bounds $\{\mathcal{A}1\}$, $\{\mathcal{A}2\}$, imply that the first term on the right is $H^{-1}_{\text{loc}}(\mathbb{R}_x, \mathbb{R}_t)$-compact,

$$\|I_\varepsilon\|_{H^{-1}_{\text{loc}}(\mathbb{R}_x,\mathbb{R}_t)} \le \sqrt{\varepsilon}\|f'(u^\varepsilon)\|_{L^\infty} \times \sqrt{\varepsilon}\|\partial_x u^\varepsilon\|_{L^2_{\text{loc}}(\mathbb{R}_x,\mathbb{R}_t)} \le \text{Const.}\sqrt{\varepsilon} \to 0;$$

the second term is L^1-bounded

$$\|II_\varepsilon\|_{L^1_{\text{loc}}(\mathbb{R}_x,\mathbb{R}_t)} \le \|f''(u^\varepsilon)\|_{L^\infty} \times \varepsilon\|\partial_x u^\varepsilon\|^2_{L^2_{\text{loc}}(\mathbb{R}_x,\mathbb{R}_t)} \le \text{Const.}$$

and hence by standard embedding, it is compact in $W^{-1}_{\text{loc}}(L^r(\mathbb{R}_x, \mathbb{R}_t))$ for $r < 2$. We now argue along the lines of Murat [19]. The sum on the right of the (2.4) is $W^{-1}_{\text{loc}}(L^r)$-compact while by $\{\mathcal{A}1\}$ the gradient on the left side is bounded in $W^{-1}_{\text{loc}}(L^\infty)$, and hence by interpolation, these terms are compactly embedded in H^{-1}_{loc}. Thus, the right-hand sides of both (2.2) and (2.4) are H^{-1}_{loc}-compact. In fact, this H^{-1}_{loc}-compactness remains valid if we replace $u^\varepsilon, f(u^\varepsilon)$ and $F(u^\varepsilon)$ on the left-hand sides (2.2) and (2.4) with $u^\varepsilon - \bar{u}, f(u^\varepsilon) - f(\bar{u})$ and $F(u^\varepsilon) - F(\bar{u})$, respectively, where $\bar{u} := \mathsf{wlim}\, u^\varepsilon$. In the third step, we consider the expression

$$D(w) := (w - \bar{u}) \times (F(w) - F(\bar{u})) - (f(w) - f(\bar{u}))^2.$$

Granted the above H^{-1}_{loc}-compactness, we can now invoke the the div-curl lemma which states that by extracting subsequences if necessary, the weak-* limit of $D(u^\varepsilon)$ is given by

$$\begin{aligned}\mathsf{wlim}\, D(u^\varepsilon) &= \mathsf{wlim}(u^\varepsilon - \bar{u}) \times \mathsf{wlim}(F(u^\varepsilon) - F(\bar{u})) - \big(\mathsf{wlim}(f(u^\varepsilon) - f(\bar{u}))\big)^2 \\ &= -(\bar{f} - f(\bar{u}))^2 \le 0, \quad \bar{f} := \mathsf{wlim}\, f(u^\varepsilon). \end{aligned} \quad (2.5)$$

But on the other hand, recalling F as the primitive of $(f')^2$ implies that $D(\cdot)$ is non-negative, for by Cauchy–Schwarz inequality

$$\begin{aligned}(f(w) - f(\bar{u}))^2 &= \left(\int_{\bar{u}}^{w} f'(v)dv\right)^2 \le (w - \bar{u})\int_{\bar{u}}^{w}(f'(v))^2 dv \\ &= (w - \bar{u}) \times (F(w) - F(\bar{u})).\end{aligned}$$

Therefore, the weak limit of $D(u^\varepsilon)$ is nonnegative, which is reconciled with (2.5) when the desired convergence of the approximate flux holds, namely, $\mathsf{wlim}\, f(u^\varepsilon) = \bar{f} = f(\bar{u})$. Passing to the limit in (2.2) we conclude that $\bar{u}$ is a weak solution, $\partial_t \bar{u} + \partial_x f(\bar{u}) = 0$.

Is $\bar{u}$ *the* entropy solution? In general, the convergence $\text{wlim}\, u^\varepsilon = \bar{u}$ need not be a strong limit and the $\bar{u}$ limit need not be the entropy solution, but more can be said provided additional information on the *nonlinearity* of f is available. In the convex case, for example, $\bar{f} = f(\bar{u})$ implies strong convergence of $u^\varepsilon \to \bar{u}$ and a single entropy inequality implies $\bar{u}$ is the entropy solution of (2.1), ([23] or [15]). The statement of strong convergence can be extended to any interval of nonlinearity of f, either by the arguments of [34], or by using the above Cauchy–Schwarz inequality as in [25, 28]. Indeed, our arguments above show the a.e. strong convergence of (a subsequence of) $D(u^\varepsilon) \to 0$. Therefore, if we quantify the nonlinearity of f, assuming that

$$f(u) \text{ is not affine on any nontrivial interval} \tag{2.6}$$

we conclude that (a subsequence) $u^\varepsilon \to \bar{u}$ and that $\bar{u}$ is the unique entropy solution, consult [6].

3. Strong Convergence — Two Entropies Suffice in the 2D Case

In this section we turn our attention to the two-dimensional case. Here we introduce a proper notion of multidimensional nonlinearity and relate it to the strong convergence of approximate solutions. Our reasoning is based on compensated compactness arguments and as in the one-dimensional case, these arguments do not involve *a priori* spatial BV estimates.

3.1. *Compensated compactness in 2D conservation laws*

We begin with the prototype viscous approximation. Let u^ε be solution of the 2D viscous conservation law

$$\partial_t u^\varepsilon + \partial_{x_1} f_1(u^\varepsilon) + \partial_{x_2} f_2(u^\varepsilon) = \varepsilon \Delta u^\varepsilon, \tag{3.1}$$

subject to $u^\varepsilon(x,0) = u_0^\varepsilon(x)$. As before, we utilize two *a priori* bounds associated with (3.1),

$\{\mathcal{A}1\}$ A uniform bound, $\|u^\varepsilon\|_{L^\infty_{\text{loc}}(\mathbb{R}^2_x,\mathbb{R}_t)} \le \text{Const.}$, and
$\{\mathcal{A}2\}$ An entropy production bound, $\sqrt{\varepsilon}\|\nabla_x u^\varepsilon\|_{L^2_{\text{loc}}(\mathbb{R}^2_x,\mathbb{R}_t)} \le \text{Const.}$;

we add a third type of a time regularity bound,

$\{\mathcal{A}3\}$ $\|\partial_t u^\varepsilon\|_{L^q_{\text{loc}}(\mathbb{R}_t;\mathcal{M}(\mathbb{R}^2_x))} \le \text{Const.},$ with $q > 1$.

Since the solution operator associated with (3.1) is L^1-contractive, $\|\partial_t u^\varepsilon(\cdot,t)\|_{\mathcal{M}(\mathbb{R}^2_x)} \le \|\partial_t u^\varepsilon(\cdot,0)\|_{\mathcal{M}(\mathbb{R}^2_x)}$ and $\{\mathcal{A}3\}$ with $q=\infty$ holds for sufficiently regular initial data, say

$$\|u_0^\varepsilon\|_{BV} + \varepsilon\|\nabla_x u_0^\varepsilon\|_{BV} \le \text{Const.} \tag{3.2}$$

We note that the $\{\mathcal{A}3\}$-bound hinges on the translation invariance *in time*. In typical cases, this requires BV bounded initial data (and in fact, BV bounded initial total flux so that $\|f_1(u_0^\varepsilon)_{x_1} + f_2(u_0^\varepsilon)_{x_2}\|_{\mathcal{M}(\mathbb{R}_x^2)} <$ Const. will do), but otherwise it is independent of *a priori* spatial BV bound $\|u^\varepsilon(\cdot,t)\|_{BV} <$ Const.

We begin by multiplying (3.1) against ${f_1}'(u^\varepsilon)$ and ${f_2}'(u^\varepsilon)$, obtaining

$$({f_1}'(u^\varepsilon))^2\partial_{x_1}u^\varepsilon + ({f_1}'(u^\varepsilon){f_2}'(u^\varepsilon))\partial_{x_2}u^\varepsilon = {f_1}'(u^\varepsilon)\varepsilon\Delta u^\varepsilon - \partial_t {f_1}' \tag{3.3}$$

$$({f_1}'(u^\varepsilon){f_2}'(u^\varepsilon))\partial_{x_1}u^\varepsilon + ({f_2}'(u^\varepsilon))^2\partial_{x_2}u^\varepsilon = {f_2}'(u^\varepsilon)\varepsilon\Delta u^\varepsilon - \partial_t {f_2}'. \tag{3.4}$$

The entropy production bound ($\mathcal{A}2$) implies for arbitrary $\phi \in C^2$, that the product $\phi'(u^\varepsilon)\varepsilon\Delta u^\varepsilon$ can be decomposed as the sum of two terms, $\sqrt{\varepsilon}\nabla \cdot (\phi'(u^\varepsilon)\nabla u^\varepsilon)$ in $L^2([0,T],\mathcal{X})$ with "$\mathcal{X} = \sqrt{\varepsilon}H^{-1}_{\text{loc}}(\mathbb{R}_x^2)$" which is H^{-1}-compact, and $-\sqrt{\varepsilon}\phi''(u^\varepsilon)|\nabla_x u^\varepsilon|^2$ in $L^1_{\text{loc}}(\mathbb{R}_t,\mathcal{M}(\mathbb{R}_x^2))$; likewise, the bounds assumed in $\{\mathcal{A}1\}$, $\{\mathcal{A}3\}$ imply that $\partial_t\phi(u^\varepsilon)$ is $L^1_{\text{loc}}(\mathbb{R}_t,\mathcal{M}(\mathbb{R}_x^2))$-bounded. Thus, if we let F_{11}, F_{22} and F_{12} denote the (indefinite) primitives of $({f_1}'(u))^2, ({f_2}'(u))^2$ and ${f_1}'(u){f_2}'(u)$, respectively, then (3.3), (3.4) tell us that each of the gradients,

$$\partial_{x_1}F_{11}(u^\varepsilon) + \partial_{x_2}F_{12}(u^\varepsilon) \quad \text{and} \quad \partial_{x_1}F_{12}(u^\varepsilon) + \partial_{x_2}F_{22}(u^\varepsilon)$$

is the sum of two terms — one is bounded in $L^2([0,T];\ \mathcal{X} \hookrightarrow H^{-1}_{\text{loc}}(\mathbb{R}_x^2))$ and the other in $L^1_{\text{loc}}(\mathbb{R}_t,\mathcal{M}(\mathbb{R}_x^2))$. In addition, by $\{\mathcal{A}1\}$ these gradients are bounded in $L^\infty([0,T];\ W^{-1}_{\text{loc}}(L^\infty(\Omega)))$. Moreover, the time regularity bound $\{\mathcal{A}3\}$ implies that the gradients in (3.1) are bounded in $C^\lambda([0,T]; W^{-1}_{\text{loc}}(L^1(\mathbb{R}_x^2)))$ with $\lambda = 1/q'$ and therefore, recalling that $q > 1$, that their $W^{-1}_{\text{loc}}(L^1)$-norms are equi-continuous in time. We can now invoke the following "time-dependent" version of Murat lemma [19], consult Lemma 6.1 below for the precise statement, which states that equi-continuity and the L^∞-bounds together with the $L^2([0,T],\mathcal{X}) + L^1(\mathbb{R}_t,\mathcal{M})$ decomposition yield $H^{-1}_{\text{loc}}(\mathbb{R}_x^2)$-compactness,

$$\partial_{x_1}F_{11}(u^\varepsilon) + \partial_{x_2}F_{12}(u^\varepsilon) \hookrightarrow L^\infty([0,T]; H^{-1}_{\text{loc}}(\mathbb{R}_x^2)), \tag{3.5}$$

$$\partial_{x_1}F_{12}(u^\varepsilon) + \partial_{x_2}F_{22}(u^\varepsilon) \hookrightarrow L^\infty([0,T]; H^{-1}_{\text{loc}}(\mathbb{R}_x^2)). \tag{3.6}$$

The div-curl lemma implies that the weak limits, $\bar{F}_{ij} := \text{wlim}\, F_{ij}(u^\varepsilon(\cdot,t))$, satisfy

$$\text{wlim}\big[F_{11}(u^\varepsilon)F_{22}(u^\varepsilon) - F_{12}^2(u^\varepsilon)\big] = \overline{F_{11}} \cdot \overline{F_{22}} - \overline{F_{12}}^2, \tag{3.7}$$

or equivalently,

$$\text{wlim}\big[(F_{11} - \overline{F_{11}})(F_{22} - \overline{F_{22}}) - (F_{12} - \overline{F_{12}})^2\big] = 0, \quad F_{ij} = F_{ij}(u^\varepsilon). \tag{3.8}$$

To proceed, we consider the nonnegative form

$$D(w) := (F_{11}(w) - F_{11}(c))(F_{22}(w) - F_{22}(c)) - (F_{12}(w) - F_{12}(c))^2$$

where $c = c(x,t)$ denotes an arbitrary fixed state, independent of u^ε, which is yet to be determined. Cauchy–Schwarz inequality shows that $D(w)$ is indeed nonnegative,

$$(F_{12}(w) - F_{12}(c))^2 = \left(\int_c^w {f_1}'(v){f_2}'(v)dv \right)^2 \le \int_c^w ({f_1}'(v))^2 dv \int_c^w ({f_2}'(v))^2 dv$$
$$= (F_{11}(w) - F_{11}(c))(F_{22}(w) - F_{22}(c)). \tag{3.9}$$

Using (3.8) we conclude

$$\begin{aligned}\mathsf{wlim}\, D(u^\varepsilon) &= \mathsf{wlim}\big[(F_{11}(u^\varepsilon) - F_{11}(c))(F_{22}(u^\varepsilon) - F_{22}(c)) - (F_{12}(u^\varepsilon) - F_{12}(c))^2\big]\\ &= \mathsf{wlim}\big[(F_{11}(u^\varepsilon) - \overline{F_{11}}) + (\overline{F_{11}} - F_{11}(c))\big]\\ &\quad \cdot \big[(F_{22}(u^\varepsilon) - \overline{F_{22}}) + (\overline{F_{22}} - F_{22}(c))\big] - \big[(F_{12}(u^\varepsilon) - \overline{F_{12}})^2\\ &\quad + 2(F_{12}(u^\varepsilon) - \overline{F_{12}})(\overline{F_{12}} - F_{12}(c)) + (\overline{F_{12}} - F_{12}(c))^2\big]\\ &= \big[(\overline{F_{11}} - F_{11}(c))(\overline{F_{22}} - F_{22}(c)) - (\overline{F_{12}} - F_{12}(c))^2\big].\end{aligned} \tag{3.10}$$

We now choose $c = c(x,t)$ such that $\int^c ({f_1}'(v))^2 dv = \overline{F_{11}}$; such $c(x,t)$ certainly exists since $0 \le \overline{F_{11}} \le \int_{u_{\min}}^{u_{\max}} ({f_1}'(v))^2 dv$. With this choice of c we find $F_{11}(c) - \overline{F_{11}} = 0$ and (3.9),(3.10) tell us that $0 \le D(u^\varepsilon) \rightharpoonup 0$. Since $D(u^\varepsilon)$ is bounded then $\mathsf{wlim}\, D^2(u^\varepsilon) = \mathsf{wlim}\, D(u^\varepsilon) = 0$ and hence $D(u^\varepsilon)$ converges strongly, $s\lim D(u^\varepsilon) = 0$.

In fact, more is true. We first note that $D(w)$ has a minimum at $u = c$ for by (3.9)

$$\begin{aligned}D'(w) &= ({f_1}'(w))^2(F_{22}(w) - F_{22}(c)) + ({f_2}'(w))^2(F_{11}(w) - F_{11}(c))\\ &\quad - 2{f_1}'(w){f_2}'(w)(F_{12}(w) - F_{12}(c)) \begin{cases} \ge 0, & w > c\\ \le 0, & w < c\end{cases}.\end{aligned}$$

Next, we assume that ${f_1}'$ and ${f_2}'$ are *linearly independent* in the sense that their linear combinations $s(\xi, v) := \xi_1 {f_1}'(v) + \xi_2 {f_2}'(v)$ do not identically vanish, i.e.

$$\forall |\xi| = 1: \quad s(\xi, \cdot) \not\equiv 0 \;\text{ on any nontrivial interval.} \tag{3.11}$$

Then, the Cauchy–Schwarz inequality (3.9) is strict, which in turn implies that $D(c) = 0$ is in fact a *strict* minimum, $D(w) > D(c), \forall\, w \ne c$. The strong convergence $D(u^\varepsilon) \to D(c) = 0$ then implies that (a subsequence of) u^ε converges strongly, $u^\varepsilon(\cdot,t) \to c(\cdot,t) = \bar{u}(\cdot,t)$ and the diagonal procedure coupled with equi-continuity in time, $\|u^\varepsilon(\cdot,t\|_{\mathcal{M}(\mathbb{R}^2_x)} \in C^{1/q'}([0,T])$ imply strong convergence in space-time. We summarize by stating

Theorem 3.1. *Consider the 2D scalar conservation law*

$$\partial_t u + \partial_{x_1} f_1(u) + \partial_{x_2} f_2(u) = 0 \tag{3.12}$$

and assume it is nonlinear in the sense that (3.11) *holds. Let u^ε be a family of uniformly bounded approximate solutions of* (3.12),

$$\partial_t u^\varepsilon + \partial_{x_1} f_1(u^\varepsilon) + \partial_{x_2} f_2(u^\varepsilon) = r^\varepsilon, \quad r^\varepsilon \rightharpoonup 0. \tag{3.13}$$

Here, r^ε is the local residual, measuring the amount by which u^ε fails to satisfy (3.12), *with the following H^{-1}-compact entropy production,*

$$\{\eta'(u^\varepsilon)r^\varepsilon\} \in L^p([0,T];X) \quad \text{with } X \hookrightarrow H^{-1}_{\text{loc}}(\mathbb{R}^2_x) \text{ and } p \geq 1, \quad \text{for } \eta = f_1, f_2. \tag{3.14}$$

Finally, assume the time regularity bound, $\{\mathcal{A}3\}$ holds, i.e. there exists $q > 1$ such that $\partial_t u^\varepsilon \in L^q_{\text{loc}}(\mathbb{R}_t; \mathcal{M}(\mathbb{R}^2_x))$. Then a subsequence of u^ε converges, $\lim u^\varepsilon = \overline{u}$, to a weak solution of (3.12).

Remark 3.2. The entropy production bound (3.14) is a realization of hypothesis $\{\mathcal{A}2\}$ in the prototype case of vanishing viscosity which led to the above theorem; compare (3.3), (3.4).

Remark 3.3. The nonlinearity assumption (3.11) can be found in the study of Engquist and E [10] on the large time-behavior of 2D conservation laws. It is the 2D extension of one-dimensional notion of nonlinearity in (2.6). In its slightly stronger version, the 2D nonlinearity assumption requires that ${f_1}'$ and ${f_2}'$ are almost everywhere linearly independent in the sense that their linear combinations satisfy

$$\text{meas}\{v \,\big|\, |s(\xi, v)| = 0\} = 0, \quad \forall |\xi| = 1, \quad s(\xi, v) := \xi_1 {f_1}'(v) + \xi_2 {f_2}'(v). \tag{3.15}$$

This notion of nonlinearity can be found in the study of [16] on kinetic formulations for conservation laws; consult (4.5) below for the corresponding multidimensional analogue.

Theorem 3.1 can be recast in terms of the general compensated compactness framework which allows to relax the time regularity assumption $\{\mathcal{A}3\}$.

Theorem 3.4. *Let u^ε be a family of uniformly bounded solutions of the nonlinear, approximate 2D conservation law* (3.13), (3.11). *Assume it has H^{-1}_{loc}-compact entropy production in the sense of* (3.14). *Finally, assume that $\{\partial_t u^\varepsilon(\cdot,\cdot)\}$ is H^{-1}_{loc}-compact. Then a subsequence of u^ε converges, $\lim u^\varepsilon = \overline{u}$, to a weak solution of* (3.12).

We note in passing that by Murat lemma, $\{\mathcal{A}3\}$ implies the H^{-1}-compactness of $\{\partial_t u^\varepsilon\}$. The proof is based on classical Tartar–Murat compensated compactness theory [33, Theorem 11], [20, Theorem 3.2]. Let $\mathcal{V}$ denote the set

$$\mathcal{V} := \big\{(\lambda, \xi) \in \mathbb{R}^4 \times \mathbb{R}^3 - \{0\} \,\big|\, \text{s.t. } \lambda_1\xi_1 + \lambda_2\xi_2 = 0; \\ \lambda_3\xi_1 + \lambda_4\xi_2 = 0; \lambda_1\xi_0 = 0; \lambda_3\xi_0 = 0\big\}.$$

Arguing along the lines of Theorem 3.1, our assumptions imply the $H^1_{\text{loc}}(\mathbb{R}^2_x, \mathbb{R}_t)$-compactness of the four terms,

$$\partial_{x_1}F_{11}(u^\varepsilon) + \partial_{x_2}F_{12}(u^\varepsilon), \quad \partial_{x_1}F_{12}(u^\varepsilon) + \partial_{x_2}F_{22}(u^\varepsilon), \quad \partial_t F_{11} \quad \text{and} \quad \partial_t F_{22}.$$

It follows that $Q(F_{ij}(u^\varepsilon))$ is weakly continuous for any quadratic $Q(F_{11}, F_{12}, F_{12}, F_{22})$ which vanishes on the projection, $\Lambda = \{\lambda \in \mathbb{R}^4 \mid \text{s.t. } (\lambda, \xi) \in \mathcal{V}\}$. A straightforward computation shows that the latter is given by the cone $\lambda_1\lambda_4 - \lambda_2\lambda_3 = 0$, i.e. (3.7) or equivalently, (3.8) hold. Expressed in terms of the Young measures $\nu_{x,t}(\cdot)$ associated with $\{u^\varepsilon\}$, (3.8) recast into the form

$$\langle \nu_{x,t}(\lambda), (F_{11}(\lambda) - \overline{F_{11}}) \cdot (F_{22}(\lambda) - \overline{F_{22}}) - (F_{12}(\lambda) - \overline{F_{12}})^2 \rangle = 0.$$

One concludes with the proof of Theorem 3.1.

3.2. *2D examples*

Example 3.5 (Vanishing viscosity). The bound (3.14) can be viewed as a consistency condition for general residual terms, which enable us to convert entropy production bound into a compactness statement. As an example we consider the possibly nonlinear vanishing viscosity approximation

$$\partial_t u^\varepsilon + \partial_{x_1} f_1(u^\varepsilon) + \partial_{x_2} f_2(u^\varepsilon) = \varepsilon \nabla_x \cdot \mathbf{c}(u^\varepsilon, \nabla_x u^\varepsilon), \quad \varepsilon \mathbf{c}(u^\varepsilon, \nabla_x u^\varepsilon) \rightharpoonup 0. \tag{3.16}$$

It follows that if $\varepsilon \|\mathbf{c}(u^\varepsilon, \nabla_x u^\varepsilon)\|_{L^2_{\text{loc}}(\mathbb{R}^2_x, \mathbb{R}_t)} \to 0$ and $\varepsilon \|\mathbf{c}(u^\varepsilon, \nabla_x u^\varepsilon) \cdot \nabla_x u^\varepsilon\|_{L^1_{\text{loc}}(\mathbb{R}_t, \mathcal{M}(\mathbb{R}^2_x))} \leq \text{Const.}$ then (3.14) holds. The special case, $\mathbf{c}(u, \mathbf{p}) = b(u)\mathbf{p}$ with $0 \leq b(\cdot) \in L^\infty$ corresponds to vanishing viscosity with the H^{-1}-entropy bound $\varepsilon \|b(u^\varepsilon)|\nabla_x u^\varepsilon|^2\|_{L^1_{\text{loc}}(\mathbb{R}_t, \mathcal{M}(\mathbb{R}^2_x))} \leq \text{Const.}$ L^1 contraction and translation invariance in time implies that $\{\mathcal{A}3\}$ holds for regular initial data (3.2) and Theorem 3.1 implies that u^ε converges strongly to a weak solution, $u^\varepsilon \to \bar{u}$.

Example 3.6 (Kinetic BGK approximation). Let $\chi_w(c)$ denote the indicator function $\chi_w(c) := \left\{ \begin{smallmatrix} \text{sgn}(w), & \text{if } (w-c)c \geq 0, \\ 0 & \text{otherwise} \end{smallmatrix} \right\}$. We consider the BGK kinetic approximation of (3.12), e.g. [24],

$$\partial_t n^\varepsilon + {f_1}'(c)\partial_{x_1} n^\varepsilon + {f_2}'(c)\partial_{x_2} n^\varepsilon = \frac{1}{\varepsilon}(\chi_{u^\varepsilon} - n^\varepsilon), \tag{3.17}$$

where n^ε is a microscopic distribution function depending on the additional kinetic variable c with macroscopic average $u^\varepsilon := \int n^\varepsilon(x, t, c)dc$, so that integration over phase space yields

$$\partial_t u^\varepsilon + \partial_{x_1} \int_c {f_1}'(c) n^\varepsilon dc + \partial_{x_2} \int_c {f_2}'(c) n^\varepsilon dc = 0.$$

We rewrite this as

$$\partial_t u^\varepsilon + \partial_{x_1} f_1(u^\varepsilon) + \partial_{x_2} f_2(u^\varepsilon) = r^\varepsilon, \quad r^\varepsilon = \nabla_x \cdot \mathcal{F},$$

where $\mathcal{F} \equiv (\mathcal{F}_1, \mathcal{F}_2) = \int_c ({f_1}'(c), {f_2}'(c))(\chi_{u^\varepsilon}(c) - n^\varepsilon)dc$. If we prevent initial layers by preparing consistent initial data so that $\|n^\varepsilon(\cdot, 0) - \chi_{u_0^\varepsilon(\cdot)}\|_{\mathcal{M}(\mathbb{R}^2_x; \mathbb{R}_c)} \to 0$, then $\|n^\varepsilon(\cdot, t) - \chi_{u^\varepsilon(\cdot, t)}\|_{\mathcal{M}(\mathbb{R}^2_x; \mathbb{R}_c)} \to 0$, hence $\|\mathcal{F}\|_{L^2_{\text{loc}}(\mathbb{R}^2_x, \mathbb{R}_t)} \to 0$ and H^{-1}-compactness of r^ε follows. We note that the last argument, due to [24, Theorem 3.7], depends on the translation invariance in *time* of (3.17) which is responsible for the $\mathcal{M}$ bound, $\|\partial_t n^\varepsilon(\cdot, t)\|_{\mathcal{M}(\mathbb{R}^2_x; \mathbb{R}_c)} \leq \|\partial_t n^\varepsilon(\cdot, 0)\|_{\mathcal{M}(\mathbb{R}^2_x; \mathbb{R}_c)}$. The same argument implies the Lip-bound in time, i.e. $\{\mathcal{A}3\}$ holds with $q = \infty$ and strong convergence follows under the nonlinearity assumption (3.11).

4. Kinetic Formulation — The Multidimensional Case

How does the Theorem 3.1 compare with the compactness statement derived by the kinetic formulation arguments in [16]? We extend our discussion to the multidimensional conservation laws

$$\partial_t u^\varepsilon + \nabla_x \cdot \mathbf{f}(u) = r^\varepsilon, \quad \mathbf{f}(u) = \big(f_1(u), f_2(u), \ldots, f_d(u)\big). \tag{4.1}$$

The Krushkov entropy condition associated with approximate solutions of (4.1) reads

$$\partial_t \big[\eta(u^\varepsilon; c) - \eta(0;c)\big] + \sum_{j=1}^{d} \partial_{x_j}\big[q_j(u^\varepsilon;c) - q_j(0;c)\big] = \eta'(u^\varepsilon;c) r^\varepsilon =: -2m^\varepsilon. \tag{4.2}$$

Here, $\eta(u;c)$ is the family of Krushkov entropies, $\eta(u;c) = |u-c|$, where c is an arbitrary fixed contact at our disposal, q_j are the corresponding entropy fluxes, $q_j(u;c) = \text{sgn}(u-c)(f_j(u) - f_j(c))$ and $m^\varepsilon = m^\varepsilon(x,t;c)$ measures the corresponding entropy production (more precisely, $m^\varepsilon{}_+$ and respectively $m^\varepsilon{}_-$ measure the corresponding entropy production and entropy dissipation).

Differentiation of (4.2) with respect to c then yields the kinetic transport equation [16]

$$\partial_t \chi^\varepsilon + \sum_{j=1}^{d} {f_j}'(c)\partial_{x_j}\chi^\varepsilon = \partial_c m^\varepsilon, \quad \chi^\varepsilon(x,t;c) \equiv \chi_{u^\varepsilon(x,t)}(c). \tag{4.3}$$

In the present context we rewrite this as a multidimensional spatial kinetic formulation

$$\sum_{j=1}^{d} {f_j}'(c)\partial_{x_j}\chi^\varepsilon = \partial_c m^\varepsilon - \partial_t \chi^\varepsilon. \tag{4.4}$$

We seek the compactness of the averages, $\overline{\chi^\varepsilon} := \int \chi^\varepsilon_{u^\varepsilon}(c)dc = u^\varepsilon$. To apply the averaging lemma along the lines of [16], we introduce the notion of nonlinearity in the sense that the (linearized) symbol of the left-hand side is

$$\text{meas}\{v \,\big|\, |s(\xi,v)| = 0\} = 0, \quad \forall |\xi| = 1, \quad s(\xi,v) := \sum_{j=1}^{d} \xi_j f_j'(v). \tag{4.5}$$

This is the multidimensional generalization of the notion of 2D nonlinearity encountered earlier in (3.15), a slightly strengthened version of (3.11). Next, we ask the second term on the right of (4.4) to be a bounded measure, $\chi^\varepsilon_t \in \mathcal{M}(\mathbb{R}^d_x, \mathbb{R}_t; \mathbb{R}_c)$. If the approximate method (4.3) is L^1-contractive and translation invariant in time, then

$$\|\chi^\varepsilon_t(\cdot,t;c)\|_{\mathcal{M}(R^d_x;R_c)} \le \|\chi^\varepsilon_t(\cdot,0;c)\|_{\mathcal{M}(R^d_x;R_c)},$$

and the required bound follows for regular enough initial data. An example is provided by the BGK approximation 3.6, which prevents possible initial boundary layer if the initial data $u_0 \in BV$ so that $\nabla\chi^\varepsilon(\cdot,0;c) \in \mathcal{M}(R^d_x;R_c)$ and

$\chi_t^\varepsilon \in \mathcal{M}(\mathbb{R}_x^d, \mathbb{R}_t; \mathbb{R}_c)$, consult [24, Sec. 3, Remark 2]. Using the averaging lemma we conclude along the lines [16].

Theorem 4.1. *Consider the multidimensional scalar conservation law* (4.1), *and assume it is nonlinear in the sense that* (4.5) *holds. Let* $u^\varepsilon \in L^\infty_{\mathrm{loc}}(\mathbb{R}_x^2, \mathbb{R}_t)$ *be a family of uniformly bounded approximate solutions of* (4.1),

$$\partial_t u^\varepsilon + \sum_{j=1}^{d} \partial_{x_j} f_j(u^\varepsilon) = r^\varepsilon, \quad r^\varepsilon \rightharpoonup 0, \tag{4.6}$$

with a negative entropy production so that $\eta'(u^\varepsilon) r^\varepsilon \leq 0$ *for all convex* η*'s. Finally, assume the time regularity bound corresponding to* $\{\mathcal{A}3\}$ *holds,* $\partial_t \chi_{u^\varepsilon(x,t)}(c) \in L^q_{\mathrm{loc}}(\mathbb{R}_t; \mathcal{M}(\mathbb{R}_x^d; \mathbb{R}_c))$ *with* $q > 1$. *Then,* $\exists s \lim u^\varepsilon = \bar{u}$ *which is the unique entropy solution of* (4.1).

Remark 4.2. The last result brings closer the convergence statements based kinetic formulations and compensated compactness arguments. The kinetic formulation requires a stronger consistency condition with the whole family of Krushkov entropies (compared with the two entropies sought in (3.14)), and in return, it yields a stronger result of strong convergence towards entropy solution.

Remark 4.3. In this context we note that one can relax the negative entropy production assumption in Theorem 4.1, requiring that the analog of (3.14), $\eta'(u^\varepsilon) r^\varepsilon \in L^p([0,T]; X)$ with $X \hookrightarrow H^{-1}_{\mathrm{loc}}(\mathbb{R}_x^d)$, holds for *all* $C^2 - \eta$'s. The regularity of $\chi_{u^\varepsilon}(c)$ implies the $L^q(\mathbb{R}_t; L^1(\mathbb{R}_x^d))$ bound of $\partial_t u^\varepsilon$ and one concludes by the averaging lemma as in [16, Theorem B].

Remark 4.4. A kinetic formulation argument yields, in particular, a regularizing effect statement: quantifying the nonlinearity by requiring $\mathrm{meas}\{v \,|\, |s(\xi, v)| \leq \delta\} \leq$ Const. δ^α, is translated into a gain of regularity of the solution operator, $\mathcal{S}\colon L^\infty \mapsto B^s$ with order of regularity s depending on α (consult [16]). In the present context, however, the requirement of time regularity requires BV initial data to begin with. It would be desirable to utilize the present framework of compensated compactness in order to derive an alternative argument for the regularizing effect, independent of the averaging lemma.

5. Convergence of Multidimensional Finite Volume Schemes

We study the convergence of finite volume (FV) schemes for the approximate solution of the initial value problem associated with the nonlinear d-dimensional conservation law (3.12). The example of 2D convergence is brought up here as an application to demonstrate the compensated compactness arguments outlined in Sec. 3. In fact, the classes of FV schemes discussed below were shown to be entropic with *all* entropies which make the kinetic arguments of Sec. 4 apply in the multidimensional case. The question of convergence in the general multidimensional case based on kinetic formulation was already addressed in [36].

To begin, we let $\mathcal{T}$ be a finite volume mesh of $\mathbb{R}^d$ such that the common interface between two cells of $\mathcal{T}$ is included in a hyperplane of $\mathbb{R}^d$. We assume that there exist $h > 0$ and $\alpha > 0$ such that, for any control volume $p \in \mathcal{T}$:

$$\alpha h^d \leq |p|, \quad |\partial p| \leq \frac{1}{\alpha} h^{d-1}, \quad \delta(p) \leq h, \tag{5.1}$$

where $|p|$ denotes the d-dimensional Lebesgue measure of the cell p, $|\partial p|$ denotes the $(d-1)$-dimensional Hausdorff measure of its boundary and $\delta(p)$ denotes its diameter. With these notations, the parameter h defines the size of the mesh and α its regularity. We denote by $N(p)$ the set of the neighbors of a control volume p, and if $q \in N(p)$ then σ_{pq} is the common interface between p and q and $n_{p,q}$ stands for the unit normal vector to σ_{pq} oriented from p to q.

Next, we consider a general family of locally Lipschitz numerical fluxes, $g = g_{pq}(u, v) : \mathbb{R}^d \to \mathbb{R}$, satisfying the *conservation property*, $g_{pq}(u, v) = -g_{qp}(v, u)$ and the *consistency property*, $g_{pq}(u, u) = \mathbf{f}(u) \cdot n_{p,q}$. We assume these fluxes are *monotone*, in the sense

$$\frac{\partial g}{\partial u_j} \geq 0, \quad \frac{\partial g}{\partial v_j} \leq 0, \quad \forall\, u_j,\ v_j' s. \tag{5.2}$$

A larger class is provided by the *E-fluxes*, satisfying

$$\frac{g_{pq}(u, v) - \mathbf{f}(u) \cdot n_{p,q}}{u - v} \geq 0. \tag{5.3}$$

The Godunov and the Lax–Friedrichs are prototypes for monotone numerical fluxes. The finite volume approximation based on the above family of numerical fluxes leads to the following scheme

$$u_p^{n+1} = u_p^n - \frac{\Delta t |\partial p|}{|p|} \sum_{q \in N_p} g_{pq}(u_p^n, u_q^n). \tag{5.4}$$

Here, the constant u_p^n should be considered as an approximation of the mean value of u over the cell p at time $t^n := n\Delta t$, $u_p^n \cong \int_p u(x, t^n) dx / |p|$ and g_{pq} is an approximation of the (averaged values of the) flux across the interface σ_{pq}. The initial condition u_0 provides us with

$$u_p^0 = \frac{1}{|p|} \int_p u_0(x) dx. \tag{5.5}$$

The explicit finite volume scheme, (5.4) and (5.5), is augmented with a CFL condition,

$$\Delta t \sup_{p \in \mathcal{T}} \frac{|\partial p|}{|p|} \|f'\|_\infty \leq 1, \quad \|f'\|_\infty = \max_j \big(|f_j'|_{L^\infty} \big). \tag{5.6}$$

It follows, given (5.2) and (5.6), that the discrete solution operator, $\{u_p^n\} \mapsto \{u_p^{n+1}\}$ is monotone and hence, by conservation, it is an L^1 contraction. If in addition, the discrete solution operator is translation invariant is space then convergence follows from BV compactness. This line of argument applies to uniform grids. In the present context, however, the possibly unstructured grid lacks spatial translation invariance and convergence arguments based on BV bound break down. Instead, we appeal to compensated compactness arguments in the 2D case and to the kinetic arguments in the general multidimensional case.

The underlying approximation u^h takes the piecewise-constant form

$$u^h(x,t) = \sum_{p\in\mathcal{T}} u_p^n \mathbb{I}^n(t)\mathbb{I}_p(x), \tag{5.7}$$

where $\mathbb{I}^n(t)$ and $\mathbb{I}_p(x)$ are respectively the characteristic function of $[t^n, t^{n+1})$ and p. We revisit the three standard assumptions. Monotonicity implies u^h is uniformly bounded. Moreover, the so called "weak-BV" estimates [11, Theorem 4.1] imply that the H^{-1}-entropy production bound (3.14) (and even a stronger $W^{-1}(L^\infty)$-bound) holds; consult also [2, 3] for example. Finally, comparing the two discrete solutions $\{u^{n+1}\}$ and $\{u^n\}$, their L^1 contraction implies the Lip-time bound (consult [11, Lemma 3.2]),

$$\begin{aligned}\|\partial_t u^h(\cdot,t)\|_{\mathcal{M}(\mathbb{R}^2)} &= \sum_{p\in\mathcal{T}} |p|\frac{|u_p^{n+1}-u_p^n|}{\Delta t} \leq \sum_{p\in\mathcal{T}} |p|\frac{|u_p^1-u_p^0|}{\Delta t}\\ &= \sum_p |\partial p| \sum_{q\in N_p} g_{pq}(u_p^0, u^0, q) \leq \text{Const.}, \quad \forall\, n,\end{aligned} \tag{5.8}$$

so that $\{\mathcal{A}3\}$ with $q=\infty$ holds for smooth enough initial data, (3.2). Theorem 3.1 applies and we conclude

Theorem 5.1. *Consider the 2D scalar conservation law* (3.12) *subject to BV-bounded initial data and assume it is nonlinear in the sense that* (3.11) *holds. Let* $u^h = \sum_{p\in\mathcal{T}} u_p^n\mathbb{I}^n(t)\mathbb{I}_p(x)$ *be a family of consistent, conservative finite volume approximation,* (5.4), (5.5), *with monotone numerical flux,* (5.2)*. Then,* $\exists s \lim u^h = \bar{u}$ *which is a weak solution of* (3.12)*.*

The key for the convergence statement of Theorem 5.1 hinges on the H^{-1}-compactness of entropy production. Our compensated compactness arguments require such entropic bounds for only two preferred entropies. In fact, in the present context of FV schemes, such entropic bounds hold to *all* convex entropies, consult [21, 22, 36] and hence the kinetic arguments apply in the general multidimensional setup of E-fluxes (and in fact higher order cases [22]). We conclude by quoting

Theorem 5.2 ([36]). *Consider the multidimensional scalar conservation law* (4.1) *subject to BV-bounded initial data and assume it is nonlinear in the sense that* (4.5) *holds. Let* $u^h = \sum_{p\in\mathcal{T}} u_p^n\mathbb{I}^n(t)\mathbb{I}_p(x)$ *be a family of consistent, conservative finite volume approximation,* (5.4), (5.5), *with E-numerical flux,* (5.3)*. Then,* $\exists s \lim u^h = \bar{u}$ *which is the entropy solution of* (4.1)*.*

Appendix A

We use the time regularity assumption $\{\mathcal{A}3\}$, in order to "raise" 2D spatial compensated compactness arguments to handle the time dependent 2D conservation laws. To this end we prove the following "time-dependent" generalization of Murat lemma [19].

Lemma A.1. *Consider the family* $\{\phi^\varepsilon\}$ *which admits the following bounds*

$$\|\phi^\varepsilon\|_{L^\infty([0,T],W^{-1}(L^\infty(\Omega)))} + \|\phi^\varepsilon\|_{C^\lambda([0,T],W^{-1}(L^1(\Omega)))} \leq \text{Const.}, \quad \lambda > 0,\ \Omega \text{ bounded}.$$

Assume that ϕ^ε *can be expressed as* $\phi^\varepsilon = \chi^\varepsilon + \psi^\varepsilon$, *where* $\{\chi^\varepsilon\}$ *bounded in* $L^p([0,T],\mathcal{X})$ *with* $\mathcal{X} \hookrightarrow H^{-1}(\Omega)$ *while* $\{\psi^\varepsilon\}$ *is bounded in* $L^1([0,T],\mathcal{M}(\Omega))$. *Then* (*a subfamily of*) $\{\phi^\varepsilon\}$ *is compact in* $L^\infty([0,T],H^{-1}(\Omega))$.

Proof. We start by noting that an $L^p[0,T]$-bound of $\|w^\varepsilon(\cdot,t)\|_X$ implies — cf. [17, Theorem 3], that there exists a denumerable dense set of points, $\mathcal{T} := \{t_k\}$, such that $\|w^\varepsilon(\cdot,t_k)\|_X$ is bounded. Thus, there exists such a denumerable dense set such that the classical Murat lemma [19] applies to $\phi^\varepsilon(\cdot,t_k)$ and diagonalization process enables us to extract a subsequence such that $\{\phi^\varepsilon(\cdot,t)\}$ is compact in $H^{-1}(\mathbb{R}^2(\Omega))$ for all $t \in \{t_k\}$. We want to show that in fact, $\{\phi^\varepsilon(\cdot,t)\}$ contains an H^{-1}_{loc}-Cauchy sequence uniformly for all t's. To this end we estimate

$$\begin{aligned}\|\phi^\varepsilon(\cdot,t) - \phi^\delta(\cdot,t)\|_{H^{-1}(\Omega)} \leq\ & \|\phi^\varepsilon(\cdot,t) - \phi^\varepsilon(\cdot,t_k)\|_{H^{-1}(\Omega)} \\ & + \|\phi^\varepsilon(\cdot,t_k) - \phi^\delta(\cdot,t_k)\|_{H^{-1}(\Omega)} \\ & + \|\phi^\delta(\cdot,t) - \phi^\delta(\cdot,t_k)\|_{H^{-1}(\Omega)}. \qquad (A.1)\end{aligned}$$

By our assumption of time regularity, the ϕ's are in $C^\lambda([0,T],W^{-1}(L^1(\Omega)))$. This, together with the interpolation bound $\|w\|_{H^{-1}} \leq \|w\|^{1/2}_{W^{-1}(L^1)}\|w\|^{1/2}_{W^{-1}(L^\infty)}$ imply

$$\|\phi^\varepsilon(\cdot,t) - \phi^\varepsilon(\cdot,t_k)\|_{H^{-1}(\Omega)} \leq \text{Const.}\|\phi^\varepsilon(\cdot,t)\|^{1/2}_{W^{-1}(L^\infty(\Omega))} \cdot |t-t_k|^{\lambda/2}.$$

A similar bound holds for $\phi^\delta(\cdot,t) - \phi^\delta(\cdot,t_k)$ and hence the first and third terms on the right of (A.1) can be made arbitrarily small since $\{t_k\}$ is dense. The second term is made arbitrarily small for proper (ε,δ) by the H^{-1}-compactness of $\phi^\varepsilon(\cdot,t_k)$ and we are done. □

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